Introduction to CAT(0) spaces

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1. Geodesics and angles
2. CAT(0) spaces
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Geodesics

Let (X, d) be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping $\gamma : [0, d(x, y)] \to X$ such that

- $\gamma(0) = x$,
- $\gamma(d(x, y)) = y$,
- $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for any $t_1, t_2 \in [0, d(x, y)]$.

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Comparison triangle

Let (X,d) be a geodesic metric space. A *geodesic triangle* consists of three point $p,q,r\in X$ and three geodesics $[p,q],[q,r],[r,p]$. Denote $\triangle\left([p,q],[q,r],[r,p]\right).$

For such a triangle, there is a *comparison triangle* $\overline{\triangle}(\overline{p},\overline{q},\overline{r})\subset\mathbb{R}^2$:

- $d(p,q)=d(\overline{p},\overline{q})$
- $d(q,r)=d(\overline{q},\overline{r})$
- $d(r,p)=d(\overline{r},\overline{p})$

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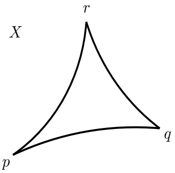

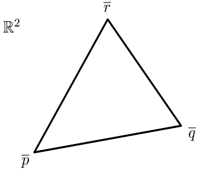

Length space

Let (X, d) be a metric space. A *curve* is a continuous mapping from a compact interval to X.

The *length* of a curve $\gamma : [a, b] \to X$ is

$$\ell(\gamma) = \sup_{\mathcal{P}} \sum_{i=1}^{n-1} d\left(\gamma(t_i), \gamma(t_{i+1})\right),$$

where $\mathcal{P}$ stands for the set of partitions of $[a, b]$.

Definition

(X, d) is a *length space* if for any $x, y \in X$ we have

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Angles

Definition

Let X be a geodesic space. We define the *Alexandrov angle* between two geodesics $\gamma_1 : [0, t_1] \to X$ and $\gamma_2 : [0, t_2] \to X$ with $\gamma_1(0) = \gamma_2(0)$ by

$$\alpha\left(\gamma_1, \gamma_2\right) = \limsup_{t_1, t_2 \to 0} \measuredangle\left(\gamma_1(t_1), \gamma_1(0), \gamma_2(t_2)\right).$$

So, the angle is a number from $[0, \pi]$. In CAT(0) spaces:

- one can take $\lim$ in place of $\limsup$,
- $\alpha\left(\gamma_1, \gamma_2\right) = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d\left(\gamma_1(t), \gamma_2(t)\right)$,
- for a fixed $p \in X$ the function $\alpha(\cdot, p, \cdot)$ is continuous on X^2,
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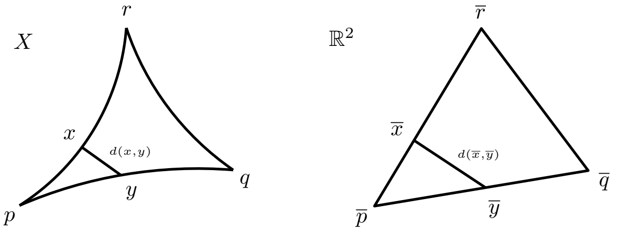
Definition of CAT(0) space

Definition (CAT(0) space)

Let (X,d) be a geodesic space. It is a CAT(0) space if for any geodesic triangle $\triangle \subset X$ and $x,y \in \triangle$ we have $d(x,y) \le d(\overline{x},\overline{y})$, where $\overline{x},\overline{y} \in \overline{\triangle}$.



Basic properties

Let X be a CAT(0) space. Then we have

1. For each $x, y \in X$ there is a unique geodesic connecting x, y.
2. Geodesics vary continuously with their end points.
3. X is Ptolemaic, i.e. the Ptolemy inequality holds:

$$d(x,y)d(u,v) \leq d(x,u)d(y,v) + d(x,v)d(y,u).$$

4. X is Busemann convex, i.e. for geodesics $\gamma_1, \gamma_2 : [a,b] \to X$ the function $t \mapsto d\left(\gamma_1(t), \gamma_2(t)\right),\ t \in [a,b]$ is convex.

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Equivalent conditions

Proposition

Let X be a complete metric space. Then X is a length space if and only if for any $x, y \in X$ and $\delta > 0$ there is $m \in X$ such that

$$\max\{d(x,m), d(y,m)\} \leq \frac{1}{2}d(x,y) + \delta.$$

Proposition (Menger)

Let X be a complete metric space. Then X is geodesic if and only if for any $x, y \in X$ there exists $m \in X$ such that

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Proposition

Let X be a complete metric space. The following conditions are equivalent.

1. X *is a CAT(0) space.*
2. *For any* $a, b \in X$ *and* $\delta > 0$ *there is* $m \in X$ *such that* $\max\{d(a,m), d(b,m)\} \leq \frac{1}{2}d(a,b) + \delta$, *and for any* $x_1, x_2, y_1, y_2 \in X$ *there exist* $\bar{x_1}, \bar{x_2}, \bar{y_1}, \bar{y_2} \in \mathbb{R}^2$ *such that* $d(x_i, y_j) = d(\bar{x_i}, \bar{y_j})$ *for* $i, j \in \{1, 2\}$, *and* $d(x_1, x_2) \leq d(\bar{x_1}, \bar{x_2})$ *and* $d(y_1, y_2) \leq d(\bar{y_1}, \bar{y_2})$.

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Let X be a geodesic space. TFAE

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3. *For every triangle $\triangle([p,q],[q,r],[r,p]) \subset X$ and every $x \in [p,q], y \in [p,r]$ with $x \neq p$ and $y \neq p$, we have*
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Equivalent conditions

Proposition (...continued)

4. *The angle between the sides of any geodesic triangle in X with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle.*
5. *For every triangle $\triangle([p,q],[q,r],[r,p]) \subset X$ with $p \neq q$ and $p \neq r$, if $\triangle([a,b],[b,c],[c,a]) \subset \mathbb{R}^2$ is a triangle with $d(p,q) = d(a,b), d(p,r) = d(a,c)$ and $\measuredangle(b,a,c) = \alpha(q,p,r)$, then $d(q,r) \geq d(b,c)$*
6. *For any $x,y,z \in X$ and $m \in X$ with $2d(y,m) = 2d(m,z) = d(y,z)$ we have*

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A geodesic space X is CAT(0) if and only if it is Ptolemaic and Busemann convex.

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Proposition (Berg, Nikolaev, (pf: Sato))

A geodesic space X is CAT(0) if and only if for any $x, y, u, v \in X$ we have

$$d(x,u)^2 + d(y,v)^2 \leq d(x,y)^2 + d(y,u)^2 + d(u,v)^2 + d(v,x)^2$$

Remark

- Answers a question of Gromov
- Roundness 2 (Enfo)
- The inequality holds for instance for the metric space $(Y, \sigma^{1/2})$ where (Y, σ) is an arbitrary metric space.

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Examples

1. Hilbert spaces – the only Banach spaces which are CAT(0)
2. $\mathbb{R}$-trees: a metric space T is an $\mathbb{R}$-tree if
   - for $x, y \in T$ there is a unique geodesic $[x, y]$
   - if $[x, y] \cap [y, z] = \{y\}$, then $[x, z] = [x, y] \cup [y, z]$
3. Classical hyperbolic spaces $\mathbb{H}^n$
4. Complete simply connected Riemannian manifolds with nonpositive sectional curvature
5. Euclidean buildings, . . .

Examples

1. Hilbert spaces – the only Banach spaces which are CAT(0)

2. $\mathbb{R}$-trees: a metric space T is an $\mathbb{R}$-tree if
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1 Geodesics and angles

2 CAT(0) spaces

3 Examples

4 Metric projections

5 Open problems

Projections

Definition

Let X be a uniquely geodesic space. A set $M \subset X$ is *convex* if, given $x, y \in M$, we have $[x, y] \subset M$.

Let (X, d) be a complete CAT(0) space and $C \subset X$ be a convex closed set. Define the *distance function* by

$$d(x, C) = \inf_{c \in C} d(x, c), \quad x \in X.$$

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Let $C \subset X$ be closed and convex. Then

1. *for every $x \in X$, there exists a unique point $P_C(x) \in C$ such that $d\left(x, P_C(x)\right) = d(x, C)$.*
2. *For any $y \in [x, P_C(x)]$ we have $P_C(y) = P_C(x)$.*
3. *For any $x \in X \setminus C$ and $y \in C \setminus \{P_C(x)\}$ we have*

$$\alpha\left(x, P_C(x), y\right) \geq \frac{\pi}{2}.$$

4. *P_C is a nonexpansive retraction onto C.*

The mapping $P_C : X \to X$ is called a *projection* onto C.

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Nice projections on geodesics

Definition

We shall say that X has the *property (N)* if, given a geodesic γ and $x, y \in X$, we have that $P_\gamma(m)$ lies on the geodesic from $P_\gamma(x)$ to $P_\gamma(y)$, for any $m \in [x, y]$.

Do all complete CAT(0) spaces have the property (N)?

A non-empty intersection

. . . on the blackboard

Suggestions for our next talks

- Examples (hyperbolic spaces, manifolds, buildings,. . .)
- Connections to Banach space geometry
- Topologies on CAT(0) spaces
- Alternating projections
- Fixed point theory
- Applications (phylogenetic trees)

Metric projections onto convex sets

Miroslav Bačák

CARMA, University of Newcastle

30 March 2010

Contents of the talk

1. Warm-up
   - Cosine rule
   - Inversion in metric spaces
2. Metric projections
   - Definitions
   - Main theorem(s)
   - Weak convergence
3. Final remarks

Cosine rule

Recall:

Proposition

Let X be a geodesic space. TFAE

1. X *is CAT(0).*
2. *For every triangle* $\triangle([p,q],[q,r],[r,p]) \subset X$ *with* $p \neq q$ *and* $p \neq r$, *if* $\triangle([a,b],[b,c],[c,a]) \subset \mathbb{R}^2$ *is a triangle with* $d(p,q) = d(a,b), d(p,r) = d(a,c)$ *and* $\measuredangle(b,a,c) = \alpha(q,p,r)$, *then* $d(q,r) \geq d(b,c)$.

Equivalently:

$$w^2 \geq u^2 + v^2 - 2uv\cos\gamma.$$

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Ptolemy inequality

A metric space is Ptolemaic if the Ptolemy inequality holds:

$$d(x,y)d(u,v) \leq d(x,u)d(y,v) + d(x,v)d(y,u).$$

A geodesic space is Busemann convex if for any $\gamma_1, \gamma_2 : [a,b] \to X$ the function $t \mapsto d\left(\gamma_1(t), \gamma_2(t)\right),\ t \in [a,b]$ is convex.

Proposition

A geodesic space X is CAT(0) if and only if it is Ptolemaic and Busemann convex.

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A geodesic space X is CAT(0) if and only if it is Ptolemaic and Busemann convex.

Inversion about sphere

Let (X, d) be a metric space. Fix $p \in X$. Define

$$i_p(x,y) = \frac{d(x,y)}{d(x,p)d(p,y)} \qquad x, y \in X \setminus \{p\}.$$

It is not a metric in general.

Proposition

Let X be Ptolemaic, then i_p is a metric on $X \setminus \{p\}$.

Inversion: nearest point mapping $\leftrightsquigarrow$ farthest point mapping.

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For any $x \in X$ denote its *projection* onto C by

$$P_C(x) = \{c \in C : d(x, c) = d_C(x)\}.$$

If the set $P_C(x)$ is a singleton, for every $x \in X$, we say C is Čebyšev.

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Convexity of d_C

Proposition

Let X be a CAT(0) space and $C \subset X$ convex complete. Then:

1. *d_C is convex.*
2. *For all x, y we have $|d_C(x) - d_C(y)| \leq d(x, y)$.*

Proof.

1. By convexity of d.
2. $d_C(x) \leq d\left(x, P_C(y)\right) \leq d\left(x, y\right) + d\left(y, P_C(y)\right) = d(x, y) + d_C(y)$.

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$\square$

Proposition

Let X be a CAT(0) space and $C \subset X$ a Čebyšev set. If P_C is nonexpansive, then C is convex.

Proof.

By contradiction, suppose there are $x, y \in C$ such that the point $m \in [x, y]$ with $d(x, m) = d(m, y)$ is not in C. If both $d\left(x, P_C(m)\right)$ and $d\left(y, P_C(m)\right)$ were less than or equal to $d(x, m)$, we would have another geodesic from x to y distinct from $[x, y]$, namely $[x, P_C(m)] \cup [P_C(m), y]$. Without loss of generality, let $d\left(x, P_C(m)\right) > d(x, m)$. But this yields a contradiction, since $P_C(x) = x$ and P_C is nonexpansive. □

Weak convergence

Suppose $(x_n) \subset X$ is a bounded sequence and define its *asymptotic radius* about a given point $x \in X$ as

$$r(x_n, x) = \limsup_{n\to\infty} d(x_n, x),$$

and the *asymptotic radius* as

$$r(x_n) = \inf_{x\in X} r(x_n, x).$$

Further, we say that a point $x \in X$ is the *asymptotic center* of (x_n) if

$$r(x_n, x) = r(x_n).$$

Recall that the asymptotic center of (x_n) exists and is unique, if X is a complete CAT(0) space.

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Weak convergence

Definition

We shall say that $(x_n) \subset X$ *weakly converges* to a point $x \in X$ if x is the asymptotic center of each subsequence of (x_n). We use the notation $x_n \xrightarrow{w} x$.

Clearly, if $x_n \to x$, then $x_n \xrightarrow{w} x$.

Lemma

Let X be a CAT(0) space and $(x_n) \subset X$ a bounded sequence. Then there is a subsequence (x_{n_k}) of (x_n) and a point $x \in X$ such that $x_n \xrightarrow{w} x$.

Lemma

Let X be a CAT(0) space and $C \subset X$ closed convex. If $(x_n) \subset C$ and $x_n \xrightarrow{w} x \in X$, then $x \in C$.

Lemma

Let X be a CAT(0) space and $C \subset X$ closed convex. The distance function d_C is weakly (sequentially) lsc, i.e., for any $x_n \xrightarrow{w} x$,

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An alternative proof

Theorem (Projection theorem revisited)

Let X be a CAT(0) space and $C \subset X$ complete convex. Then, for any $x \in X$, there exists a point $c \in C$ such that $d_C(x) = d(c, x)$.

Proof.

Let $x \in X$. There exists $(c_n) \subset C$ such that $d(c_n, x) \to d_C(x)$. It is bounded, so a subsequence (c_{n_k}) weakly converges to some $c \in X$. Since C is convex, $c \in C$. Now,

$$d_C(x) \leq d(x, c) \leq \liminf_{n \to \infty} d(x_n, c) = d_C(x).$$

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Final remarks

- Projections are nonexpansive even in CAT(1) spaces.
- Our assumptions: X a CAT(0) space and $C \subset X$ complete convex.
- Still things to do: e.g., are weakly closed Čebyšev sets convex?

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Euclidean buildings

Miroslav Bačák

CARMA, University of Newcastle

13 April 2010

Contents of the talk

1. Simplicial complexes
   - Gluing definition
   - Examples
   - Metrizing definition
2. Euclidean buildings
   - Definition
   - Examples
   - Euclidean buildings are CAT(0)
3. Final remarks

Piecewise Euclidean simplicial complex

Definition

Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of simplices $S_\lambda \subset \mathbb{R}^{n_\lambda}$. Let $X = \bigcup_{\lambda\in\Lambda} (S_\lambda \times \{\lambda\})$. Let $\sim$ be an equivalence relation and $K = X/\sim$. Let $p : X \to K$ be the projection and define $p_\lambda : S_\lambda \to K$ by $p_\lambda = p(\cdot, \lambda)$. Then K is a *piecewise Euclidean simplicial complex* if

1. the map p_λ is injective for every $\lambda \in \Lambda$,
2. if $p_\lambda(S_\lambda) \cap p_{\lambda'}(S_{\lambda'}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset S_\lambda$ onto a face $T_{\lambda'} \subset S_{\lambda'}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

Piecewise Euclidean simplicial complex

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Intrinsic metric

K comes equipped with the quotient pseudometric, which coincide with so-called intrinsic pseudo metric.

An m-string in K from x to y is a sequence $\sigma = (x_0, \ldots, x_m) \subset K$ such that $x = x_0$, $y = x_m$ and for each $i = 0, \ldots, m-1$, there is a simplex $S(i)$ containing x_i and x_{i+1}.

Define the *length* of σ by

$$\ell(\sigma) = \sum_{i=0}^{m-1} d_{S(i)}(x_i, x_{i+1}).$$

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Definition

The *intrinsic pseudometric* on K is defined by

$$d(x,y) = \inf \{\ell(\sigma) : \sigma \text{ a string from } x \text{ to } y\}.$$

Let $x \in K$. For a simplex S containing x, define

$$\varepsilon(x,S) = \inf \{d_S(x,T) : T \text{ a face of } S \text{ and } x \notin T\}$$

and

$$\varepsilon(x) = \inf \{\varepsilon(x,S) : S \subset K \text{ simplex containing } x\}.$$

If $\varepsilon(x) > 0$ for all $x \in K$, then d is a metric and (K,d) is a length space.

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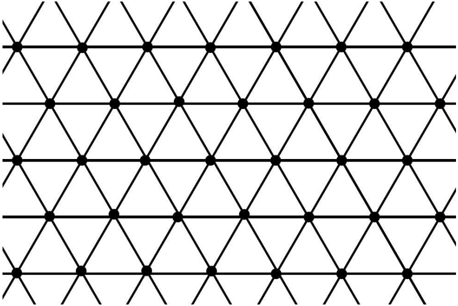
Examples of simplicial complexes



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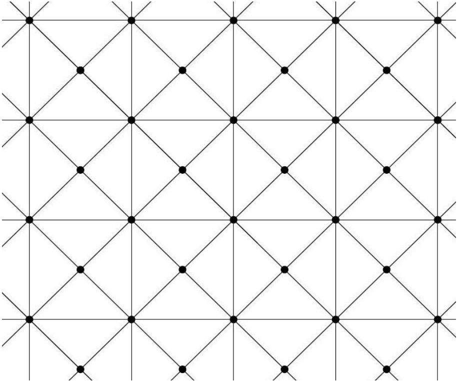

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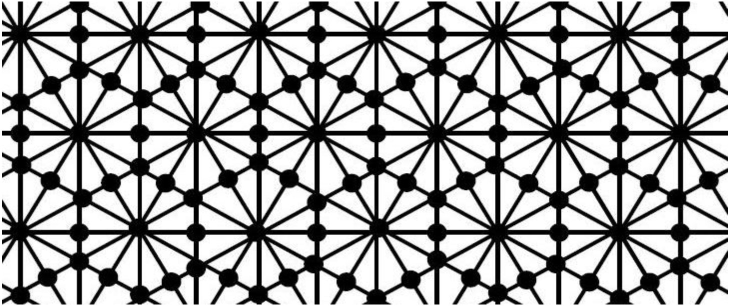

Abstract simplicial complex

Definition

An *abstract simplicial complex* consists of of a set V and a collection $\mathcal{S}$ of (nonempty) finite subsets of V, such that

- $\{v\} \in \mathcal{S}$ for all $v \in V$,
- if $S \in \mathcal{S}$, then any nonempty subset T of S belongs to $\mathcal{S}$.

We call elements of V vertices and elements of $\mathcal{S}$ simplices.

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Affine realization

Let K be an abstract simplicial complex with vertex set V. Let W be a real vector space with basis W. The *affine realization* $|S|$ of a simplex $S \subset K$ is the convex hull of S in W.

$|S|$ inherits the Euclidean topology.

The affine realization of K is

$$|K| = \bigcup \{|S| : S \subset K\}.$$

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Metrizing affine realization

Alternative definition of piecewise Euclidean simplicial complex:

Definition

A piecewise Euclidean simplicial complex consists of:

- an abstract simplicial complex,
- a set $\mathrm{Shapes}(K)$ of simplices $S_i' \subset \mathbb{E}^{n_i}$
- for any simplex S in the affine realization of K, an affine isomorphism $f_s : S' \to S$, where $S' \in \mathrm{Shapes}(K)$. If T is a face of S, then $f_S^{-1} \circ f_S$ is required to be an isometry from T' onto a face of S.

Using piecewise linear path, we define the intrinsic pseudometric.

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Euclidean building

Definition

A *Euclidean building* of dimension n is piecewise Euclidean simplicial complex X such that

1. X is a union of a collection $\mathcal{A}$ of subcomplexes E, called *apartments*, such that d_E makes (E, d_E) isometric to $\mathbb{E}^n$ and induces the given Euclidean metric on each simplex.
2. Any two simplices A, B are contained in an apartment.
3. Given two apartments E, E' containing A and B, there exists a simplicial isometry from (E, d_E) onto $(E', d_{E'})$ which leaves A and B fixed.

The n-simplices are called *chambers*.

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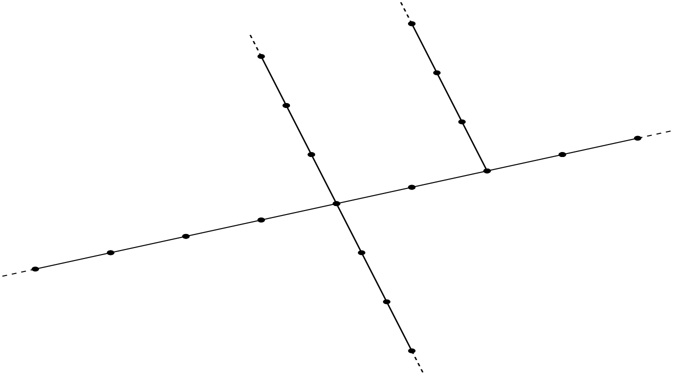

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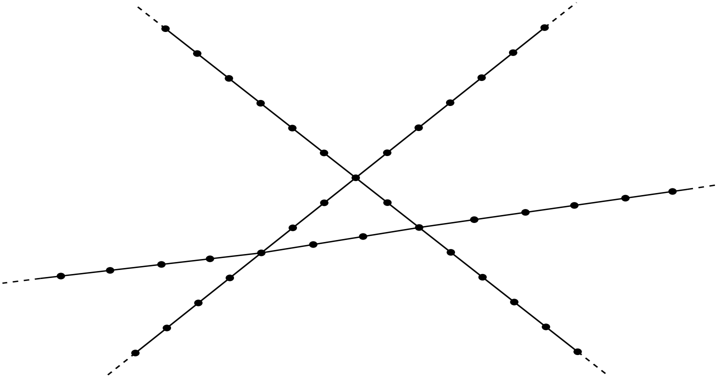

Main theorem

Theorem

Let X be a Euclidean building. Then X is a complete CAT(0) space.

Let C be a chamber in an apartment $E \subset X$. Define a retraction $\rho_{C,E} : X \to E$ by

$$\rho_{C,E}(x) = \phi_{E,E'}(x)$$

where E' is an apartment containing both x and C, and $\phi_{E,E'} : E' \to E$ is the unique isometry between E' and E.

Then $\rho_{C,E} : X \to E$ is a nonexpansive simplicial retraction.

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Given $x, y \in X$, choose an apartment $E \subset X$ containing them and let $[x, y]$ be the line segment joining them in E. Choose $p = p_t \in [x, y]$, where $0 \leq t \leq 1$, choose $C \subset E$ be a chamber containing p, and let $\rho = \rho_{C,E}$. Take any $z \in X$.

We must verify:

$$d^2(z,p) \leq (1-t)d^2(z,x) + td^2(z,y) - t(1-t)d^2(x,y).$$

But it follows from:

$$d^2\left(\rho(z),p\right) \leq (1-t)d^2\left(\rho(z),x\right) + td^2\left(\rho(z),y\right) - t(1-t)d^2(x,y).$$

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- M. Davis (1998) showed that **all** buildings are CAT(0).
- More general definition of buildings: a **non**-simplicial complex.
- Modern definition of buildings: W-metric spaces.
  This approach does not use apartments.

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Modern definition of buildings

We start with a Coxeter system (W, S), where

- W is a (reflection) group
- S is a set of generators of W.

A building is a pair $(\mathcal{C}, \delta)$ where $\mathcal{C}$ is a nonempty set (of chambers) and a 'distance' function $\delta : \mathcal{C} \times \mathcal{C} \to W$ such that

1. $\delta(C, D) = 1$ if and only if $C = D$.
2. If $\delta(C, D) = w$, and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) = sw$ or w. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(C', D) = sw$.
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Suggestions for our next talks

- Examples (hyperbolic spaces, Riemannian manifolds,...)
- Connections to Banach space geometry
- Topologies on CAT(0) spaces
- Alternating projections
- Fixed point theory
- Groups and CAT(0) spaces
- Applications (phylogenetic trees, robotics,...)

The weak topology on CAT(0) spaces

Miroslav Bačák

CARMA, University of Newcastle

20 April 2010

Overview

- 1976: Lim defined Δ-convergence in metric spaces
- 2004: Sosov defined Φ-convergence in metric spaces
- 2008: Kirk and Panyanak used Δ-convergence in CAT(0), and asked for topology
- 2009: Espínola and Fernández-León modified Φ-convergence to get equivalent condition for Δ-convergence in CAT(0)
- 2009: (M.B.) definition of a topology that corresponds to the above convergence

Weak convergence

Let (X, d) be a metric space. Suppose $(x_\nu) \subset X$ is a bounded net and define its *asymptotic radius* about a given point $x \in X$ as

$$r(x_\nu, x) = \limsup_\nu d(x_\nu, x),$$

and the *asymptotic radius* as

$$r(x_\nu) = \inf_{x \in X} r(x_\nu, x).$$

Further, we say that a point $x \in X$ is the *asymptotic center* of (x_ν) if

$$r(x_\nu, x) = r(x_\nu).$$

Recall that the asymptotic center of (x_ν) exists and is unique, if X is a complete CAT(0) space.

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Weak convergence

Definition (Lim)

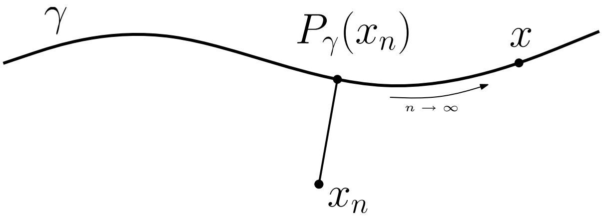
We shall say that $(x_\nu) \subset X$ *weakly converges* to a point $x \in X$ if x is the asymptotic center of each subnet of (x_ν). We use the notation $x_\nu \xrightarrow{w} x$.

Clearly, if $x_\nu \to x$, then $x_\nu \xrightarrow{w} x$.

Weak convergence

Proposition (Espínola, Fernández-León)

Let (X, d) be a complete CAT(0) space, $(x_n) \subset X$ be a bounded sequence and $x \in X$. Then $x_n \xrightarrow{w} x$ if and only if, for any geodesic γ through x we have $d(x, P_\gamma(x_n)) \to 0$.



Weak topology

Definition (M.B. 2009)

Let X be a complete CAT(0) space. A set $M \subset X$ is *open* if, for any $x_0 \in M$, there are $\varepsilon > 0$ and a finite family of nontrivial geodesics $\gamma_1, \ldots, \gamma_N$ through x_0 such that

$$U_{x_0}(\varepsilon, \gamma_1, \ldots, \gamma_N) = \{x \in X : d\left(x_0, P_{\gamma_i}(x)\right) < \varepsilon,\ i = 1, \ldots, N\}$$

is contained in M. Denote τ the collection of all open sets of X.

The sets $U_{x_0}(\varepsilon, \gamma_1, \ldots, \gamma_N)$ are convex iff X has the property (N).

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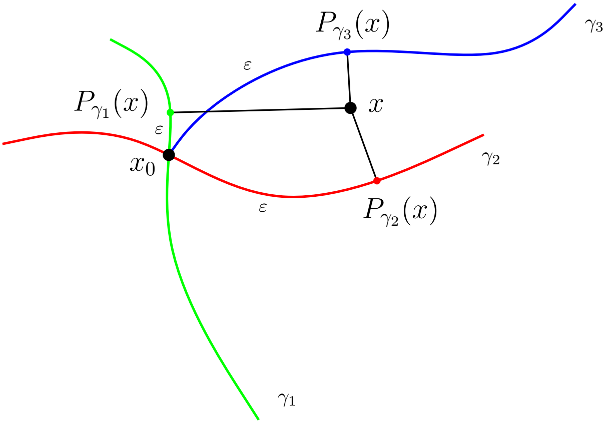
Weak topology

$U_{x_0}(\varepsilon, \gamma_1, \gamma_2, \gamma_3)$



Weak topology

Theorem (M.B. 2009)

Let (X, d) be a complete CAT(0) space and τ as above. Then

1. *τ is a Hausdorff topology on X,*
2. *$x_\nu \xrightarrow{\tau} x$ if and only if $x_\nu \xrightarrow{w} x$, for $(x_\nu) \subset X$ a bounded net and $x \in X$.*
3. *τ is weaker than the topology induced by the metric d,*
4. *τ is the $\sigma(X, X^*)$-topology when X is a Hilbert space.*
5. *τ is not metrizable in general.*

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5. *τ is not metrizable in general.*

Properties of the weak topology

The weak topology in Banach spaces:

compactness = sequential compactness = countable compactness

But not in CAT(0) spaces!

Example

Consider a countable set $\{x_1, x_2, \ldots, x_\infty\}$, and for every $n \in \mathbb{N}$, join x_∞ with x_n by a geodesic of length n. Then $x_n \xrightarrow{w} x_\infty$, but is unbounded. X is sequentially w-compact, but not (countably) $w-$compact.

Let C be a *convex* set in a complete CAT(0) space. Then $\overline{C} = \overline{C}^w$.

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Open problems

Let (X, d) be a complete CAT(0) space.

1. Let $(x_n) \subset X$ be a bounded sequence weakly converging to a point $x \in X$. Is then the case that

$$\{x\} = \bigcap_{n \in \mathbb{N}} \overline{\mathrm{co}}\,\{x_n, x_{n+1}, \dots\}?$$

   Note: "$\subset$" is known. The converse is true if we assume the property (N).

2. Suppose $C \subset X$ is compact. Is $\overline{\mathrm{co}}\,C$ compact?
3. Is the weak topology restricted on balls metrizable?