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The Four Theorems by Lawrence M. Graves

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The four theorems

- The Hildebrand-Graves theorem (1927)
- The (Lyusternik-) Graves theorem (1932,1950)
- The Bartle-Graves theorem (1952)
- The Karush-Kuhn-Tucker theorem (1939)?

Hildebrand–Graves inverse function theorem (TAMS 1927)

Lipschitz modulus

$$\operatorname{lip}(f;\bar{x}) := \limsup_{\substack{x',x\to\bar{x},\\ x\neq x'}} \frac{\|f(x')-f(x)\|}{\|x'-x\|}.$$

Theorem (Hildebrand-Graves slightly extended)

Let X be a Banach space and consider a continuous function $f : X \to X$ and a linear bounded mapping $A : X \to X$ which is invertible. Suppose that

$$\operatorname{lip}(f - A;\bar{x}) \cdot \|A^{-1}\| < 1.$$

Then the inverse f^{-1} has a single-valued **localization** around $f(\bar{x})$ for $\bar{x}$ which is Lipschitz continuous.

Proof of HG Theorem

Main step: the inverse f^{-1} has a nonempty-valued localization.

Show that for any y near $f(\bar{x})$ the function

$$x \mapsto \bar{x} + A^{-1}(y - f(x) + A(x - \bar{x}))$$

has a fixed point in a neighborhood of $\bar{x}$.

The Banach open mapping principle

Given a bounded linear mapping A acting between Banach spaces X and Y, the following three conditions are equivalent:

(i) A is surjective;

(ii) A is open at any $x \in X$, meaning that for every neighborhood U of x, AU is a neighborhood of Ax;

(iii) there exists a constant $\tau > 0$ such that

$$d(x, A^{-1}(y)) \leq \tau \|y - Ax\| \quad \text{for all } x \in X, \; y \in Y.$$

Condition (iii) is a prototype of Metric Regularity

A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in F(\bar{x})$, $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$ and there is a constant $\tau \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x, F^{-1}(y)\big) \leq \tau d\big(y, F(x)\big) \quad \text{for every} \quad (x, y) \in U \times V.$$

The infimum of all constants $\tau \geq 0$ for which this inequality holds is the regularity modulus of F at $\bar{x}$ for $\bar{y}$ denoted by $\operatorname{reg}(F; \bar{x} \,|\, \bar{y})$.

F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F^{-1} has the Aubin property at $\bar{y}$ for $\bar{x}$: there is a constant $\tau \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\sup_{x \in F^{-1}(y') \cap U} d(x, F^{-1}(y)) \leq \tau \|y - y'\| \quad \text{for every} \quad (y', y) \in V.$$

(Lyusternik-) Graves theorem (1950)

Theorem.

Consider a function $f : X \to Y$ along with a bounded linear mapping $A : X \to Y$ which is surjective, such that

$$\operatorname{lip}(f - A; \bar{x}) \cdot \operatorname{reg}(A) < 1.$$

Then f is metrically regular at $\bar{x}$ for $f(\bar{x})$.

The IFT paradigm

Theorem.

Let X be a complete metric space, Y be a linear normed space
1) κ and μ positive constants with $\kappa\mu < 1$.
2) $F : X \rightrightarrows Y$ is [strongly] metrically [sub-]regular at $\bar{x}$ for $\bar{y}$ with $\mathrm{reg}(F; \bar{x} \,|\, \bar{y}) \leq \kappa$.
3) $g : X \to Y$ and $\mathrm{lip}(g; \bar{x}) \leq \mu$.
Then $g + F$ is [strongly] metrically [sub-]regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with

$$\mathrm{reg}(g + F; \bar{x} \,|\, \bar{y}) \leq (\kappa^{-1} - \mu)^{-1}.$$

Bartle-Graves theorem (1952)

Theorem (Bartle–Graves theorem).

Let X and Y be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the derivative $Df(\bar{x})$ is surjective. Then there is a neighborhood V of $f(\bar{x})$ along with a constant $\gamma > 0$ such that f^{-1} has a continuous selection s on V with the property

$$\|s(y) - \bar{x}\| \leq \gamma \|y - f(\bar{x})\| \quad \text{for every} \quad y \in V.$$

Corollary. Let $A \in \mathcal{L}(X, Y)$ be surjective. Then A^{-1} has a continuous selection (which does not need to be linear!).

Extended Bartle-Graves theorem

Theorem.

Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and suppose that for some $c > 0$ the mapping $B_c(\bar{y}) \ni y \mapsto F^{-1}(y) \cap B_c(\bar{x})$ is closed-convex-valued. Consider also a function $g : X \to Y$ with $\bar{x} \in \operatorname{int} \operatorname{dom} g$. Let κ and μ be nonnegative constants such that

$$\kappa\mu < 1, \quad \operatorname{reg}(F; \bar{x} \,|\, \bar{y}) \leq \kappa \quad \text{and} \quad \operatorname{lip}(g; \bar{x}) \leq \mu.$$

Then for every $\gamma > \kappa/(1 - \kappa\mu)$ the mapping $(g + F)^{-1}$ has a continuous local selection s around $g(\bar{x}) + \bar{y}$ for $\bar{x}$ with the property

$$\|s(y) - \bar{x}\| \leq \gamma \|y - \bar{y}\| \quad \text{for every } \; y \in V.$$

Applications in Optimization

$f : \boldsymbol{R}^n \to \boldsymbol{R}$, $g : \boldsymbol{R}^n \to \boldsymbol{R}^m$ are C^2, (p, q) is a parameter

$$\min[f(x) + \langle p, x \rangle] \quad \text{subject to} \quad g(x) + q \leq 0$$

Solution mapping $(p, q) \mapsto S(p, q)$

Lagrangian

$$L(x, \lambda) = f(x) + \langle p, x \rangle + \langle \lambda, g(x) + q \rangle$$

KKT system (under a constraint qualification condition)

$$\begin{aligned} &L_x(x, \lambda, p, q) = 0 \\ &-L_\lambda(x, \lambda, p, q) + N_{\boldsymbol{R}^m_+}(\lambda) \ni 0 \end{aligned}$$

The (normal) Lagrange multiplier mapping $(p, q) \mapsto \Lambda(p, q)$
The composite mapping $(p, q) \mapsto (S, \Lambda)(p, q)$

IFT for optimization problem

Theorem (AD, R. T. Rockafellar 1996).

The mapping $(p,q) \mapsto (S,\Lambda)(p,q)$ has a Lipschitz continuous single-valued localization at $(0,0)$ for $(\bar{x},\bar{\lambda})$ with $\bar{x}$ being an optimal solution if and only if:
(a) the strong second-order sufficient optimality condition holds;
(b) the gradients of the active constraints at $\bar{x}$ are linearly independent

IFT for Newton's method

Newton method for a parameterized VI

$$x_0 = a, \quad f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p$$

Consider the mapping

$$\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (a,p) \mapsto \Xi(a,p) = \left\{ \{x_k\} \in l_\infty(\boldsymbol{R}^n) \,\middle|\, x_0 = a, \right.$$

$$\left. f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p, \quad k = 1, 2, \ldots \right\}$$

Theorem (Aragon, AD, Geoffroy, Gaydu and Veliov (2011)).

Let $f(\bar{x}) + N_C(\bar{x}) \ni 0$; then $\{\bar{x}\} \in \Xi(\bar{x}, 0)$. The mapping Ξ has a Lipschitz continuous single-valued localization around $(\bar{x}, 0)$ for $\{\bar{x}\}$ each value of which is a superlinearly convergent sequence to a solution $x(p)$ of $f(x) + N_C(x) \ni p$ if and only if $f + N_C$ is strongly metrically regular at $\bar{x}$ for 0.

IFT for nonsmooth functions

The Hildebrand-Graves theorem

Theorem (Clarke 1976).

Consider a function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ which is Lipschitz continuous around $\bar{x}$ and suppose that all matrices in Clarke's generalized Jacobian $\partial f(\bar{x})$ are nonsingular. Then f^{-1} has a Lipschitz continuous single-valued localization around $f(\bar{x})$ for $\bar{x}$.

IFT for nonsmooth VI (A. Izmailov; R. Cibulka and AD)

Theorem (finite dimensions).

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be Lipschitz continuous around $\bar{x}$, let $F : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^n$, and let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose for every $A \in \partial f(\bar{x})$ the mapping

$$y \mapsto (f(\bar{x}) + A(\cdot - \bar{x}) + F(\cdot))^{-1}(y)$$

has a Lipschitz continuous localization at $\bar{y}$ for $\bar{x}$. Then the mapping $(f + F)^{-1}$ has a Lipschitz continuous localization at $\bar{y}$ for $\bar{x}$.

For $F \equiv 0$ reduces to Clarke's IFT. For f smooth reduces to Robinson's theorem in finite dimensions.
Extension to Banach spaces

A nonsmooth Graves theorem (R. Cibulka, AD and V. Veliov)

Theorem (finite dimensions).

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be Lipschitz continuous around $\bar{x}$, let $F : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^n$ have closed graph, and let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose for every $A \in \partial f(\bar{x})$ the mapping

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x)$$

is metrically regular at $\bar{x}$ for $\bar{y}$. Then the mapping $f + F$ has the same property.

The case $F \equiv 0$ due to Pourciau (1977)

A nonsmooth Bartle-Graves theorem ?

Conjectured theorem.

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be Lipschitz continuous around $\bar{x}$ and suppose that every $A \in \partial f(\bar{x})$ is surjective. Then f^{-1} has a continuous selection around $(f(\bar{x}), \bar{x}))$ which is calm at $f(\bar{x})$.

THANK YOU!