Metric Diophantine approximation: an introduction

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  - ⋆ What happens if the denominators of rational numbers are equal to some integer value N?

  Ans: The best approximation rate we can guarantee is $\frac{1}{2N}$. Indeed, the distance between two consecutive rational numbers with the same denominator equals to $\frac{1}{N}$.

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Theorem

For any real number α and any positive integer N, there exists a rational p/q with positive denominator $q \leq N$, such that

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Corollary

For any irrational $\alpha \in \mathbb{R}$ there exists infinitely many rationals p/q such that

$$\left|\alpha - \frac{p}{q}\right| < q^{-2}. \tag{1}$$

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Theorem (Hurwitz 1891)

For any irrational real number α there exist infinitely many integers p and $q > 0$ such that

$$|\alpha - p/q| \leq 1/\sqrt{5}q^2 .$$

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A real number α is said to be *badly approximable* if there exists a constant $c = c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2},$$

for all integers p and $q > 0$. Let **Bad** denote the set of badly approximable numbers.

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How big are the sets $W(\tau)$ for $\tau > 2$, Bad and $\mathcal{L}$?

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Theorem (Khintchine(24, 25))

Let ψ be an approximating function. Then

$$|W(\psi)| = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q\psi(q) < \infty \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} q\psi(q) = \infty \quad \psi \textit{ is decreasing}. \end{cases}$$

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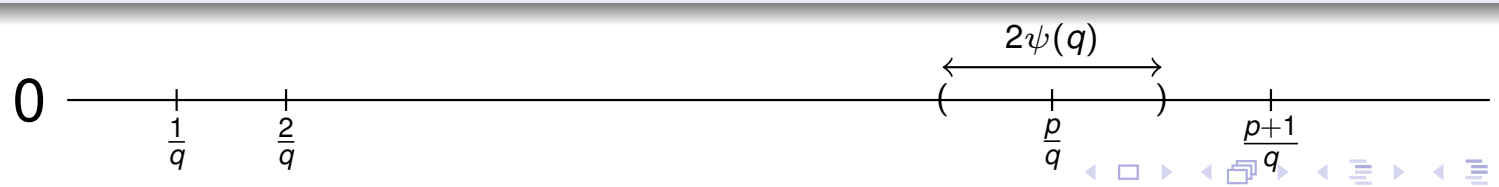

Consequences of Khintchine's theorem

In particular, Khintchine's theorem says that the probability of a randomly chosen number α to be $(q \log q)^{-1}$–approximable and not to be $(q \log^2 q)^{-1}$– approximable equals to 1.

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- But $W(\tau_1) \subseteq W(\tau_2)$ for $\tau_1 \geq \tau_2 > 2$?
- Jarnik (29)–Besicovitch(34) theorem implies that $\dim_H W(\tau) = \frac{2}{\tau}$.

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***Duffin-Schaeffer* Conjecture**: *For any function* $\psi : \mathbb{N} \mapsto \mathbb{R}^+$

$$\left| W'(\psi) \right| = 1 \quad \textit{if} \quad \sum_{q=1}^{\infty} \varphi(q)\psi(q) \ = \ \infty.$$

Where, ϕ denote the *Euler function*.

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- Recently, I have worked on the Khintchine and Jarnik type results in higher dimensions, that is, to linear forms setup and analogues of these results in the projective space and on the manifolds (e.g. a parabola).
- I also enjoy working on the applications of metric Diophantine approximations. For instance, in signal processing 'MIMO X-channels', the Khitnchine–type theorem provides an optimal way of finding the degrees of freedom.

Thank you