A lower bound theorem for general polytopes

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So it is interesting to bound the lengths of paths in a polytopes, and to estimate the total number of edges, in terms of the number of vertices or facets.
It is also interesting to bound the total number of k-dimensional faces. In this presentation we will concentrate only graph theoretic properties of a polytope (or its 1-skeleton). In particular, we want to estimate the number of edges, given the number of vertices.

Precise upper bounds for the numbers of edges are easy to obtain. If $d = 3$, a polyhedron with v vertices has at most $3v - 6$ edges, with equality iff every face is a triangle. Examples with $3v - 6$ edges are easy to construct.

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If $d \geq 4$, the cyclic polytope $C(v, d)$ has precisely $\binom{v}{2}$ edges. Clearly this is the maximum possible.
McMullen (1970) established the corresponding conclusion for k-dimensional faces for all k; this is known as the Upper Bound Theorem.

Lower bounds are not so easy to obtain. The following result of Barnette (1973) was considered a major breakthrough at the time. A polytope is *simplicial* if every facet (maximal proper face) is a simplex.

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A d-dimensional simplicial polytope v vertices has at least $dv - \binom{d}{2}$ edges; and there exist simplicial polytopes, namely the stacked polytopes, with precisely this many edges.

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There are some estimates for cubical polytopes, but little seems to be known for general polytopes.

We denote by $\phi(v, d)$ the minimum possible number of edges, over all d-polytopes with v vertices.
It is well known that $\phi(v, 3)$ is either $3v/2$ or $\frac{1}{2}(3v + 1)$ depending on the parity of v. Examples achieving these bounds are easily constructed by successively slicing corners off a tetrahedron or a pyramid.

The 4-dimensional case was solved by Grünbaum in 1967. He showed that $\phi(6,4) = 13$, $\phi(7,4) = 15$, $\phi(10,4) = 21$, and that $\phi(v,4) = 2v$ for all other values of v.

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Slicing an edge from a simple 4-polytope gives another simple polytope with four more vertices and eight more edges. Thus we obtain simple polytopes with $v = 9, 13, \cdots, 12, 16, \cdots$.

Simple polytopes in higher dimensions

A d-dimensional polytope is *simple* if every vertex has degree d. For any polytope, the sum of the degrees of the vertices is equal to twice the number of edges. So in general $\phi(v, d) \geq \frac{1}{2}dv$, with equality only if there exists a simple polytope with v vertices. For which values of v do we find simple polytopes?

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For $1 \leq k < d$, slicing off a $(k-1)$-dimensional face from a d-dimensional simple polytope will give another simple polytope with $kd - k^2$ more vertices. If d is even, then $d-1$ and $2d-4$ are relatively prime. Hence the following observation.

Theorem

If d is even, there is an integer K such that, for all $v > K$, $\phi(v, d) = \frac{1}{2}dv$ (i.e. there is a simple d-polytope with v vertices).

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Even if d is odd, $d - 1$ and $2d - 4$ have no odd common prime factors.

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So the problem of calculating $\phi(v, d)$ is more interesting for small values of v.

A simplex shows that $\phi(d+1, d) = \binom{d+1}{2}$. A prism based on a $(d-1)$-dimensional simplex shows that $\phi(2d, d) = d^2$.

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We also obtain precise values for $\phi(2d+1,d)$ and $\phi(2d+2,d)$.

Theorem

Let P be a d-dimensional polytope with $d+k$ vertices, where $0 < k \leq d$.
(i) If P is $(d-k)$-fold pyramid over the k-dimensional prism based on a simplex, then P has $\binom{d}{2} - \binom{k}{2} + kd$ edges.
(ii) Otherwise the numbers of edges is $> \binom{d}{2} - \binom{k}{2} + kd$.

Slicing one corner from the base of a square pyramid yields a polyhedron with 7 vertices and 6 faces, one of them a pentagon. We call this a *pentasm*.
We will use the same name for the higher-dimensional version, obtained by slicing one corner from the quadrilateral base of a $(d-2)$-fold pyramid. It has $2d+1$ vertices and can also be represented as the Minkowski sum of a d-dimensional simplex, and a line segment which lies in the affine span of one 2-face but is not parallel to any edge.

Theorem

Let P be a d-dimensional polytope with $2d+1$ vertices.
(i) If P is d-dimensional pentasm, then P has d^2+d-1 edges.
(ii) Otherwise the numbers of edges is $> d^2+d-1$, or P is the sum of two triangles.

This shows that the pentasm is the unique minimiser if $d \geq 5$.
If $d = 4$, the sum of two triangles has 9 vertices, and is the unique minimiser, with only 18 edges.
If $d = 3$, the sum of two triangles can have 7, 8 or 9 vertices; the example with $v = 7$ has 11 edges, the same as the pentasm.
Summarising, $\phi(9,4) = 18$, and $\phi(2d+1,d) = d^2+d-1$ for all $d \neq 4$.

Slicing one corner from the apex of a square pyramid yields a polyhedron combinatorially equivalent to the cube. Slicing one corner from 3-prism yields a polyhedron combinatorially equivalent to the 5-wedge. Of all the polyhedra with 8 vertices, these are the only two with 12 edges.

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We show that for $d \neq 5$, analogues of these polyhedra minimise the number of edges, amongst polytopes with $2d + 2$ vertices.

Consider first the polytope obtained by slicing one corner from the apex of a $(d - 2)$-fold pyramid on a square base. It has $2d + 2$ vertices, $(d + 1)^2 - 4$ edges and can also be represented as the Minkowski sum of a $(d - 3)$-fold pyramid on a square base, and a line segment in the other dimension.

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Consider next a $(d-3)$-fold pyramid whose base is a 3-prism, then slice one corner off. This example also has $2d+2$ vertices and $(d+1)^2-4$ edges.

Theorem

Let P be a d-dimensional polytope with $2d+2$ vertices, where $d \geq 6$ or $d = 3$.
(i) If P is one of the two polytopes just described, then P has $d^2 + 2d - 3$ edges.
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If $d = 4$, there is a third minimising polytope with 10 vertices and 21 edges.
If $d = 5$, the unique minimiser is the sum of a tetrahedron and triangle; this clearly has 12 vertices and 30 edges; $30 < 32$.
Summarising, $\phi(12, 5) = 30$, and $\phi(2d+2, d) = d^2 + 2d - 3$ for all $d \neq 5$.

The case of $2d+3$ vertices appears to be difficult.

Theorem

If $0 \leq k < d$, then

$$d^2 + \frac{1}{2}kd \leq \phi(2d+k, d) \leq d^2 + kd - \binom{k+1}{2}.$$

The upper bound is the exact value if $k = 1, 2$ (unless $d = 4$ or 5). The lower bound is the exact value if $k = 0, d-3$. Being equal, both are correct if $k = d-1$.

It is well known that there is no polyhedron with 7 edges. More generally a d-polytope cannot have between $\frac{1}{2}(d^2+d-2)$ and $\frac{1}{2}(d^2+3d-4)$ vertices, inclusive. Grünbaum [p 188] discusses gaps in the possible number of edges, pointing that a second gap opens when $d=6$ and a third gap opens when $d=11$. Our main theorem shows that there are infinitely many gaps.
More precisely, in dimension n^2+2, there is no polytope with $\frac{1}{2}(n^4+2n^3+4n^2+3n+4)$ edges.
The cyclic polytope $C(n^2+n+2, n^2+2)$ has one edge less, and the free join of an (n^2-n)-dimensional simplex and an $(n+1)$-prism has one edge more.
For example: a 27-dimensional polytope cannot have 497 edges. But there is a cyclic polytope with 496 edges, and a multiplex with 498 edges.

Finally, instead of just asking for upper and lower bounds, one may ask for the complete range of values. So for fixed d,
can we describe for exactly which values of e, v there exists a d-polytope with v vertices and e edges?
If $d = 3$, the answer is well known: if and only if $\frac{3}{2}v \leq e \leq 3v - 6$.
For $d = 4$, the complete answer was given by Grünbaum: iff $2v \leq e \leq \binom{v}{2}$ **and** (v, e) is not one of the pairs (6, 12), (7, 14), (10, 20) or (8, 17).
The first three exceptions are clear: the number of vertices of a simple d-polytope cannot be between d and $2d$, and it can be $2d + 2$ only if $d = 5$.

The fourth case seemed like an oddity, but it is part of a general pattern.

Theorem

If there is a d-polytope with $2d$ vertices and $d^2 + 1$ edges, then $d = 3$.

The following is the simplest open case: is there a 5-polytope with 9 vertices and 25 edges?