EXPECTATIONS OVER ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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Abstract. Motivated by the need for new mathematical tools applicable to the study of fractal point-cloud distributions, expectations of complex-valued functions defined over general 'deterministic' fractal domains are considered, following the development of a measure-theoretic foundation for their analysis. In particular, we wish to understand and evaluate separation moments as given by integrals of the form

$$B_n(s) = \int_{\mathbb{R}^n} |x|^s \mathrm{d}\mu(x) \quad \text{and} \quad \Delta_n(s) = \int_{\mathbb{R}^n} |x-y|^s \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

in the case where μ is a normalized Borel measure supported on a self-similar subset of $\mathbb{R}^n$. Previous work concerning such integrals supported over the special class of String-generated Cantor Set (SCS) fractals (see [5]) is generalised to encompass all fractal sets that can be expressed as the attractor of an Iterated Function System (IFS). The development of a generalised functional equation for expectations over IFS attractors (Proposition 3.2) enables the symbolic evaluation of certain even-order separation moments over attractors of affine IFSs, including such celebrated fractal sets as the von Köch Snowflake and Sierpiński Triangle—and more generally, any IFS attractor generated from real-world data by means of the Collage Theorem.

1. Introduction

The following mathematical considerations regarding expectations of functions defined over fractal domains were principally motivated by recent breakthroughs in the nano-scale imaging of biological structures. In particular, a team of scientists led by Steven Smith (at Smithlab, Stanford Medical School) have pioneered new array tomography techniques within the last decade, enabling the measurement of three-dimensional spatial coordinates for over one million mouse-brain synapses[1] at a resolution on the order of 10^{-8} metres (see [37], [38] and [39]).

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[1]An entry-level overview of neurology can be found in [34], and can be briefly summarized as follows: the human central nervous system primarily comprises over 100 billion highly-interconnected neurons. The junction points between these neurons are small physical spaces called synapses; different types of neurons can have anywhere from hundreds to tens of thousands of synaptic connections, each of which selectively transmits, blocks, amplifies or redirects signals via chemical neurotransmitters. The spatial distribution of these synapses reflects the spatial distribution of the associated neurons within the brain.

The existence of these large-scale empirical point-cloud neural distributions raises many interesting mathematical questions concerning the precise details of their structure. The arrangement of the synapses displays clear fractal signatures, with an average Minkowski (box-counting) dimension of 2.8 ± 0.05 over the entire imaged volume and sharp changes in the localized Minkowski dimension across biologically-predicted neural layers [25]; yet mathematical analysis based purely on the investigation of fractal dimensions can only partially resolve the full complexity of the structure.[1] Thus, mathematical methods that can symbolically and numerically evaluate, over a wide range of fractal sets, the same statistical measures that have been empirically determined for the neural data-sets—such as average neural separation—are of considerable interest, not least for their role in enabling the selection of suitable fractal models that can reproduce the measured distributions in detail.

The foundations for a theory of fractal separation expectations, or generalised fractal box integrals, were established by Crandall in 2012 immediately following the first analysis of the Smithlab data-sets from a mathematical standpoint [25]. To faciliate a comparison between the empirically-measured synapse separation moments and the statistics of a random point-cloud set, Crandall drew upon previous work with Bailey and Borwein regarding classical box integrals, which encapsulate separation moments between points randomly distributed throughout a unit hypercube (see for instance [3], [4], [21] and [26]). Subsequent attempts to fit the experimentally-observed fractal properties of the synapse distributions to an appropriate mathematical model raised the question of how classical box integral theory might be extended to provide a means of calculating separation moments of points restricted to lie within an arbitrary fractal set, and particular consideration of a special class of fractals known as String-generated Cantor Sets (SCSs) lead to a collaboration between Bailey, Crandall and the present authors [5].

Having developed a mathematical theory of fractal expectations (with a particular focus on biologically-relevant separation expectations) for the special class of SCS fractals, this sequel examines the extension of the theory to encapsulate the much more general class of so-called 'deterministic' fractal sets—namely, the class of fractals that can be expressed as the attractor of an appropriate Iterated Function System (IFS). In the remainder of this introductory section we review the main results from the prior analysis of expectations over SCS fractals and the relevant pieces of IFS theory that are needed to extend the analysis. In Section 2 the fundamental definitions concerning expectations of complex-valued functions over IFS attractors are

[1] Unsurprising, given that we are considering but one measurable parameter of an infinitely intricate set. Classical geometry provides many instances where geometric objects of vastly different character nonetheless have a common (topological) dimension, and the same is true in fractal geometry—as a case in point, both the unit square and the boundary of the Mandelbrot set share a Hausdorff dimension of 2.

established, proceeding from the generalization of the fundamental definitions from the SCS setting into a measure-theoretic framework. This enables the development of a functional equation (Proposition 3.2) in Section 3 that encapsulates the self-similarity of the fractal sets under consideration. Echoing the central role played by the equivalent SCS relation, this provides the key to establishing closed-form results and algorithms for the symbolic computation of fractal expectations in certain special cases. This functional equation is also used to develop several results concerning the complex poles of fractal separation moments. Section 4 concerns the exact evaluation of special cases—in particular, for even-order separation moments over affine IFSs, which are completely resolved by means of a readily-automated symbolic algorithm. Of particular interest is the existence of well-established algorithms [45] for encoding a wide variety of empirical data (particularly from digital images) into the IFS attractor framework via the Collage Theorem [6]. The symbolic evaluation of fractal expectations over affine IFS attractors thus has important implications regarding the analysis and modeling of real-world data.

We note that the functional equation of Proposition 3.2 is foreshadowed in the literature by [14], [16], [17], [18] and [22].

1.1. **Box Integrals.**

One of the measures most relevant to the present study of neurological data-sets are the separation moments over a fractal set embedded in the unit hypercube. Separation moments over the full unit hypercube were first investigated in 1976 by Anderssen, Brent, Daley and Moran [1] and subsequently developed over the last decade into the modern theory of box integrals by Bailey, Borwein, Crandall and their colleagues (particularly in [3], [4] and [21]).

A general box integral X is formally defined as the expectation of the (order-s) distance from a fixed point to a point equidistributed randomly over the unit hypercube in n-dimensions. The canonical definition, established in [4], is as follows:

Definition 1.1 (**Box Integral).** Given dimension n, complex parameter s and a fixed point q in the unit n-cube, the *box integral* $X_n(s,q)$ is defined as the expectation of a certain norm $|r-q|^s$, with q fixed and r chosen at random from a uniform distribution over the unit n-cube. That is,

$$X_n(s,q) := \langle |r-q|^s \rangle_{r\in[0,1]^n} = \int_{r\in[0,1]^n} |r-q|^s \mathrm{D}r \tag{1.1}$$

where $\mathrm{D}r := dr_1 \dots dr_n$ is the n-space volume element.

Of particular interest are the B and Δ box integrals, developed as functionals of the X-integrals. These are tailored to the analysis of expected norm and separation, respectively, of points uniformly distributed through unit hypercubes à la [1].

Definition 1.2 (**Classical Instances of Box Integrals**)**.** The *B box integral* $B_n(s)$, the order-s moment of separation between a random point and a vertex of the unit n-cube (such as the origin), is:

$$B_n(s) := X_n(s,0) = \langle |r|^s \rangle_{r\in[0,1]^n} = \int_{r\in[0,1]^n} |r|^s \mathrm{D}r; \tag{1.2}$$

The Δ *box integral* $\Delta_n(s)$, the order-s moment of separation between two random points in the unit n-cube, is:

$$\Delta_n(s) := \langle X_n(s,q) \rangle_{q\in[0,1]^n} = \langle |r-q|^s \rangle_{r,q\in[0,1]^n} = \int_{r,q\in[0,1]^n} |r-q|^s \mathrm{D}r\mathrm{D}q. \tag{1.3}$$

The $\Delta_n(s)$ box integrals—particularly $\Delta_3(1)$, the expected Euclidean distance between two points in the unit cube—are the measures whose generalisation is most relevant to the analysis of empirical synapse distributions. Thus, while this paper is concerned with expectations of arbitrary functions defined over fractal domains, particular attention will be given to the generalisation of the classical B and Δ box integrals into the fractal setting.

1.2. **String-generated Cantor Sets (SCSs).** As a first step towards a complete theory of expectations of general functions (and separation moments in particular) over arbitrary fractal sets, the class of String-generated Cantor Sets (SCSs) was selected for consideration in [5]. This class of sets, which aimed to capture the intuitive notion of 'Cantor-like structure',[1] allowed for fine-control over the fractal dimension[2] of a selected representative and struck a balance between the simplicity required to facilitate a first analysis of fractal expectations and the complexity required to produce interesting results.

As the name suggests, each SCS is uniquely determined by an associated generating string. For a given embedding dimension n, let $P = P_1P_2\dots P_p$ denote a *periodic* string of digits with period p, some positive integer, satisfying the restriction that $P_i \le n$ for all i. For example, with $n = 1$, $P = 01$ denotes the period-2 string $010101\dots$.

The string supplies parameters to a generating procedure that is, in essence, an extension of the classic representation of the Cantor middle-thirds set as those points in the interval $[0,1]$ with a ternary expansion that is entirely devoid of the digit 1. Consider the ternary expansion for

[1]Characterised by hypercubic symmetry in the unit n-cube and deterministic self-similarity. The class of SCSs includes the Cantor middle-thirds set, Cantor dust (embedded in arbitrary dimension) and the Menger sponge, among others of the same flavour.

[2]The phrase 'fractal dimension' here refers to both the Minkowski and Hausdorff dimensions, which are equivalent for any given SCS.

coordinates of an arbitrary point $x = (x_1, \ldots, x_n) \in [0,1]^n$:

$$\begin{array}{rcl} x_1 & = & 0\,.\,x_{11}\; x_{12}\; x_{13}\ldots \\ x_2 & = & 0\,.\,x_{21}\; x_{22}\; x_{23}\ldots \\ & \vdots & \\ x_n & = & 0\,.\,x_{n1}\; x_{n2}\; x_{n3}\ldots \\ & & \quad\;\uparrow\quad\uparrow\quad\uparrow \\ & & \quad\; c_1\quad c_2\quad c_3\quad\ldots \end{array}$$

with every digit $x_{jk} \in \{0,1,2\}$ and the vectors $c_k = (x_{1k}, \ldots, x_{nk})$ comprising respective columns of kth ternary digits. A given periodic string defines a unique SCS by providing a criterion for selecting points with 'admissible' ternary expansions; the associated SCS is simply the collection of such admissible points. In a given string P, the value of P_k determines the maximum number of coordinates of x that are permitted to take the digit 1 in the kth (and $(k+p)$th, $(k+2p)$th, ...) place of the ternary expansion.

For the purpose of enumerating the digits that are restrained by the generating strings, it is useful to define two counting functions: the *unit counter*, appropriate for standard ternary vectors c having all elements $\in \{0,1,2\}$:

$$U(c) \;:=\; \#\{1\text{'s in standard ternary vector } c\};$$

and for use with 'balanced' ternary vectors b having all elements $\in \{-1,0,1\}$ (obtained from a standard ternary vector c by the shift $b = c - \mathbf{1}_n$), the *zero counter*:

$$Z(b) \;:=\; \#\{0\text{'s in balanced-ternary vector } b\}.$$

The class of String-generated Cantor Sets can then be formally defined in the following manner:

Definition 1.3 (**String-generated Cantor Set**). Fix positive integers n and p. Given an embedding space $[0,1]^n$ and an entirely-periodic string $P = P_1P_2\ldots P_p$ of non-negative integers with $P_i \leq n$ for all $i = 1,2,\ldots,p$, the associated *String-generated Cantor Set (SCS)*, denoted $C_n(P)$, is the set:

$$\{x \in [0,1]^n :\; U(c_k) \leq P_k \quad \text{for all } k \in \mathbb{N}\} \tag{1.4}$$

with notational periodicity assumed: $P_{k+p} := P_k$ for all $k \in \mathbb{N}$.

The following column-counting formula occurs commonly in calculations involving SCSs:

Lemma 1.4. *For an SCS $C_n(P)$ with periodic generating string P, the associated set of admissible columns c_k is enumerated by:*

$$N_k(P,n) := N_k \;=\; \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}. \tag{1.5}$$

Lemma 1.4 follows directly from the observation that, in the *kth* digit of the ternary expansion of a given point, $j \leq P_k$ coordinates are permitted to take the value 1, leaving $n - j$ coordinates free to take either values 0 or 2.

Every SCS fractal encapsulated by this framework has a straightforward representation as the attractor of an appropriate IFS consisting entirely of similarity mappings that share an identical contraction factor (see Section 1.4 for the relevant definitions), as established by the following:

Proposition 1.5 (IFS representation of a given SCS). *The SCS $C_n(P)$ is the unique attractor of the IFS:*

$$\{[0,1]^n \subset \mathbb{R}^n; f_1, f_2, \ldots, f_m\}$$

where

$$f_i\left(x = (x_1, x_2, \ldots, x_n)\right) = \left(\frac{1}{3}\right)^p x + \left(\frac{1}{3}\right) c_{1_i} + \left(\frac{1}{3}\right)^2 c_{2_i} + \ldots + \left(\frac{1}{3}\right)^n c_{n_i}$$

for $i \in \{1, 2, \ldots, m\}$ ranging over all admissible columns c_k (as defined in Definition 1.3), where $m = \prod_{k=1}^{p} N_k$ and $N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$. Each mapping f_i is a similarity mapping with contraction factor $c_i = 3^{-p}$.

Proof. The SCS string $P = P_1 P_2 \ldots P_p$ encodes a structure that repeats after a resolution of 3^{-p}. Overlay the unit n-cube with 3^{pn} hypercubes of side length 3^{-p} and consider the similarities f_i that map the unit n-cube into these hypercube subsets. All such similarities have contraction factor 3^{-p}, and the set $C_n(P)$ can be identified with the attractor of the IFS consisting of those 'admissible' similarities S_i which map the unit n-cube into a hypercube subset that intersects $C_n(P)$. These admissible similarities are enumerated by the column-counting formula 1.5 of Lemma 1.4.

Consider the pre-fractal approximation to $C_n(P)$ obtained by truncation of the periodic string P to its first p digits (its first complete period). This set is identical to $C_n(P)$ on scales greater than 3^{-p} but contains no fine structure below this limit. Equivalently, an equivalence relation can be established on points in $[0,1]^n$ with classes containing those points in the unit n-cube whose coordinate ternary expansions are equal up to and including the p-th digit. Each equivalence class (represented by a coordinate ternary expansion consisting of all 0's after the p-th place) can be injectively mapped onto its own 3^{-pn}-scaled hypercube. The number of such hypercubes that intersect $C_n(P)$ is therefore equal to the number of admissible equivalence classes.

Each admissible equivalence class corresponds to a ordered concatenation of admissible c_k columns for $k = 1, \ldots, p$, so the admissible subsets are enumerated by the column-counting formula (1.5):

$$N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}.$$

For the kth ternary place there are N_k admissible columns c_k, so the number of admissible similarities is enumerated by:

$$m = \prod_{k=1}^{p} N_k$$

and these similarities can be expressed in terms of column concatenation via:

$$f_i\left(x = (x_1, x_2, \ldots, x_n)\right) = \left(\frac{1}{3}\right)^p x + \left(\frac{1}{3}\right) c_{1_i} + \left(\frac{1}{3}\right)^2 c_{2_i} + \ldots + \left(\frac{1}{3}\right)^n c_{n_i}.$$

□

This IFS representation of an arbitrary SCS $C_n(P)$ enables all subsequent developments of the theory to be checked against the established analogous statements of [5] concerning expectations over SCSs.

1.3. **Expectations over SCS Fractals.** Self-similarity considerations played a central role in unlocking all subsequent results in the special case of SCS fractal analysis. In particular, the following functional equation from [5] is the precursor of the generalised functional equation of Proposition 3.2:

Proposition 1.6 (**Functional Equations for Expectations**). *For $x, y \in \mathbb{R}^n$ and a complex-valued function $F : \mathbb{R}^n \to \mathbb{C}$:*

$$\langle F(x)\rangle_{x\in C_n(P)} = \frac{1}{\prod_{j=1}^{p} N_j} \sum_{U(c_k)\leq P_k} \langle F(x/3^p + c_1/3 + \cdots + c_p/3^p)\rangle \tag{1.6}$$

$$\langle F(d := x - \mathbf{1/2})\rangle_{x\in C_n(P)} = \frac{1}{\prod_{j=1}^{p} N_j} \sum_{Z(b_k)\leq P_k} \langle F(d/3^p + b_1/3 + \cdots + b_p/3^p)\rangle \tag{1.7}$$

$$\langle F(d := x - y)\rangle_{x,y\in C_n(P)} = \frac{1}{\prod_{j=1}^{p} N_j^2} \sum_{\substack{Z(b_k)\leq P_k \\ Z(a_k)\leq P_k}} \langle F(d/3^p + \sum_{j=1}^{p}(b_j - a_j)/3^j)\rangle \tag{1.8}$$

where $N_k(P, n) := N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$ is the number of admissible columns c_k for the given generating string P.

Exploiting the self-similarity fundamentally encoded by the functional equation of Proposition 1.6 led to a number of closed-form special cases for SCS expectations; we conclude this section by presenting a relevant selection. First, application of Proposition 1.6 to the standard Cantor middle-thirds set $C_1(0)$, followed by invoking the linearity properties of the expectations, established the following relation for the B box integral of order-s:

$$B(s, C_1(0)) = \frac{1}{2 \cdot 3^s - 1} \langle (x+2)^s \rangle . \tag{1.9}$$

This expression reveals a pole in the s-plane for $B(s, C_1(0))$ at $s = -\log_3 2$. That the pole happens to be located at the negated fractal dimension of the Cantor middle-thirds set is highly suggestive; indeed the self-similarity leveraged by Proposition 1.6 implies the following [5]:

Theorem 1.7 (**Pole of** $B(s, C_n(P))$). *For any SCS $C_n(P)$, the analytically-continued box integral $B(s, C_n(P))$ has a single pole on the real axis at*

$$s = -\delta(C_n(P)). \tag{1.10}$$

This result demonstrates consistency with the classical theory of box integrals on unit hypercubes. Over the full unit n-cube $C_n(n)$, the analytically-continued box integral $B(s, C_n(n))$ has precisely one complex pole at $s = -n$ [4]. Though it is known that $\Delta_n(s, C_n(n))$ has precisely $(n+1)$ complex poles [3], the pole structure for integrals $\Delta(s, C_n(P))$ with arbitrary $C_n(P)$ remains unresolved—no evidence of multiple Δ poles for any fractal has yet been encountered, aside from the trivial hypercube cases $C_n(n)$.

Besides establishing the B box integral poles, the functional expectation relations of Proposition 1.6 directly yield closed forms for all second-order separation expectations $B(2, C_n(P))$ and $\Delta(2, C_n(P))$. In the $s = 2$ case both the B and Δ box integrals evaluate as rational numbers, depending only on the defining string P and embedding dimension n, as follows:

Theorem 1.8 (**Closed Form for Second-Order Moments** $B(2, C_n(P))$). *For any embedding dimension n and SCS $C_n(P)$, the box integral $B(2, C_n(P))$ is given by the rational closed form:*

$$B(2, C_n(P)) \;=\; \frac{n}{4} + \frac{1}{1 - 9^{-p}} \sum_{k=1}^{p} \frac{1}{9^k} \frac{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j} (n-j)}{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}}. \tag{1.11}$$

Theorem 1.9 (**Closed Form for Second-Order Moments** $\Delta(2, C_n(P))$). *For any embedding dimension n and SCS $C_n(P)$, the box integral $\Delta(2, C_n(P))$ is given by the rational closed form:*

$$\Delta(2, C_n(P)) \;=\; 2B(2, C_n(P)) - \frac{n}{2}, \tag{1.12}$$

with $B(2, C_n(P))$ given by the closed form of Equation (1.11).

Theorems 1.8 and 1.9 immediately imply the following rationality result:

Corollary 1.10 (**Rationality of Second-Order SCS Box Integrals**). *For any SCS $C_n(P)$, the moments $B(2, C_n(P))$, $\Delta(2, C_n(P))$ are rational.*

Closed-form results such as these will be explored in the IFS attractor setting in Sections 3 and 4.

1.4. **Iterated Function Systems.** The concept of an Iterated Function System (IFS) was first introduced by Hutchinson in 1981 (see [35]). Hutchinson used IFSs to develop a rigorous framework in support the pioneering ideas of Mandelbröt, as put forth in the seminal essay [36]. Many subsequent developments of IFS theory, including the random iteration algorithm (or Chaos Game) and the Collage Theorem, were developed by Barnsley in [8]. A comprehensive recent survey of IFS theory can be found in [9], from which most of the following standard definitions and notation are taken.

Definition 1.11 (**Iterated Function System**). An *iterated function system (IFS)* $\mathcal{F}$ is a complete metric space X together with a finite set of continuous functions $f_i : X \to X$, $i = 1, 2, \dots, m$ (with $m \geq 2$), denoted by:

$$\mathcal{F} = \{X;\ f_1, f_2, \dots, f_m\} \tag{1.13}$$

An *iterated function system with probabilities (pIFS)* is an iterated function system for which every f_i has an associated $p_i \in (0, 1)$, subject to the restriction that $\sum_{i=1}^{m} p_i = 1$.

TThe following convention is important to note in the context of fractal expectations: in the absence of any explicit probability assignment, an IFS shall be considered to have a uniform probability distribution assigned to its mappings; that is, $p_i = \frac{1}{m}$ for all i. This ensures that the measure is uniform across the corresponding fractal attractor, which is required for considerations of standard separation moments. Adjustments to probability assignments away from this default are typically used to optimise the efficiency of the Chaos Game algorithm (see Section 1.6) when plotting graphical representations of IFS attractors;[1] however, care must be taken to note that assigning non-uniform probabilities to IFS mappings in an algorithm will alter the underlying measure (discussed in Section 1.8).

The objects of primary interest to this paper are closely linked to Iterated Function Systems, which play an analogous role to the generating string of an SCS in encoding the fractal structure. The sets regarded here as 'deterministic fractals'—the most general class of objects over which expectations will be defined—are the attractors each IFS uniquely defines. The introduction of the notion of IFS attractors requires several preliminary definitions.

Definition 1.12 (**Hutchinson Operator**). Let $\mathcal{H}(X)$ denote the collection of nonempty compact subsets of a metric space X. Given an IFS $\mathcal{F} = \{X;\ f_1, f_2, \dots, f_m\}$, the associated *Hutchinson operator* $F : \mathcal{H}(X) \to \mathcal{H}(X)$ is defined for all $S \in \mathcal{H}(X)$ by:

$$F(S) := \bigcup_{f \in \mathcal{F}} f(S)$$

where $f(S)$ denotes the set-valued mapping $f(S) = \{f(x) :\ x \in S\}$.

[1] In each iteration the Chaos Game algorithm selects one of the IFS mappings at random, with the mapping f_i chosen with probability p_i. By adjusting the probabilities p_i so they are in proportion to the invariant measure of the set-valued mapping $f_i(A)$, where A is the fractal attractor, the Chaos Game algorithm will effectively allocate time to plotting points in each piece of the fractal proportional to the size of the piece.

The mappings of an IFS each represent one of the self-similar pieces of the associated fractal attractor; the (set-valued) Hutchinson operator unites these pieces to create the entire fractal object. Notions of convergence upon iterating the Hutchinson operator first require the introduction of an appropriate metric—the Hausdorff metric.

Definition 1.13 (**Hausdorff Metric).** Let d_X be the metric on the metric space X. The corresponding *Hausdorff metric* on $\mathcal{H}(X)$ is:

$$d_{\mathcal{H}}(S_1, S_2) = \min\{\varepsilon \geq 0 : |\ S_1 \subset B(S_2, \varepsilon) \text{ and } S_2 \subset B(S_1, \epsilon)\} \tag{1.14}$$

for all $S_1, S_2 \in \mathcal{H}(X)$, where $B(S, \varepsilon)$ is the *dilation of S by ε*:

$$B(S, \varepsilon) = \{x \in X : \ d_X(s, x) \leq \varepsilon \text{ for some } s \in S\} \tag{1.15}$$

The Hausdorff metric $d_{\mathcal{H}}$ can be intuitively regarded as the maximum distance (in the sense of the metric d_X) that one would possibly have to travel when starting at an arbitrary point in one of the sets and then moving to the other set by taking the shortest path to the closest possible point. Note that the completeness of (X, d_X) implies the completeness of $(\mathcal{H}, d_{\mathcal{H}})$, and likewise for compactness [6].

One final notational convention is required: for $S \in \mathcal{H}(X)$, define $F^0(S) := S$ and denote by $F^k(S)$ the k-fold composition :

$$F^k(S) := \underbrace{F \circ F \circ \cdots F}_{\text{k times}}(S)$$

Equivalently, $F^k(S)$ comprises the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(S)$ over all finite words $i_1 i_2 \ldots i_k$ of length k, where $i_j \in \{1, 2, \ldots, m\}$ for all j (see Section 1.5).

The attractor of an IFS can now be defined in the following manner:

Definition 1.14 (**Attractor of an Iterated Function System).** The associated *attractor* A of the IFS $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ is the unique set $A \in \mathcal{H}(X)$ such that there exists an open set $U \subset X$ satisfying $A \subset U$ and

$$\lim_{k \to \infty} F^k(S) = A \tag{1.16}$$

for all $S \in \mathcal{H}(X)$ with $S \subset U$, where the limit is with respect to the Hausdorff metric on $\mathcal{H}(X)$.

Initial considerations of the existence of attractors used the Banach fixed-point theorem[1] as a natural starting point, from which the uniqueness of IFS attractors follows. Consequently the use of contraction mappings—functions whose image sets would shrink upon successive iterations and thereby converge to a fixed point—was paramount. The following relevant definitions are taken from [2].

[1] The Banach fixed-point theorem states that, given a closed subset $S \subset X$ of a complete metric space X and a contraction mapping $f : S \to S$, there exists a unique element $z \in S$ such that $f(z) = z$.

Definition 1.15 (Contraction and Similarity Mappings). Given a metric space X with associated metric d_X, a mapping $f_i : X \to X$ is a *contraction* if there exists a *contraction factor* $0 < c_i < 1$ such that $d_X(f_i(x) - f_i(y)) \leq c_i \cdot d_X(x - y)$ for all $x, y \in X$. If equality holds for all x and y the mapping is said to be a *similarity*.

Definition 1.16 (Contractive Iterated Function System). Given a metric space X with associated metric d_X, an IFS $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ is *contractive* if each function f_i is a contraction (of contraction factor c_i) with respect to a metric that induces the same topology on X as the metric d_X. The *contraction factor of a contractive IFS*, c, is given by:

$$c := \max\{c_1, \ldots, c_m\}. \tag{1.17}$$

Definition 1.17 (Hyperbolic Iterated Function System). Given a metric space X with associated metric d_X, an IFS $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ is *hyperbolic* if there is a metric on X, Lipschitz-equivalent to d_X, with respect to which each function f_i is a contraction mapping.

Unless otherwise stated, all IFSs considered herein are assumed to be hyperbolic IFSs. The following pivotal theorem by Hutchinson [35] establishes the existence and uniqueness of attractors for contractive IFSs:

Theorem 1.18 (The Contraction Mapping Theorem). *Let $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ be a contractive IFS on a nonempty complete metric space (X, d_X) with contraction factor c. Then $\mathcal{F}$ has a unique attractor $A \in \mathcal{H}(X)$. That is, the transformation defined by the Hutchinson operator F in Equation 1.16 is a contraction mapping on $\mathcal{H}(X)$ (with respect to the Hausdorff metric) with contraction factor c and unique fixed point A.*

Though the class of IFS attractors captures a rich diversity of fractal objects, all such fractal sets share the property of 'deterministic self-similarity'; that is, there are no probabilistic methods employed in their definition. Although IFS attractors are typically extremely complicated in a geometric sense, despite their simple definition, the information needed to describe such geometrical complexity in its exact details is fully encapsulated by the IFS, enabling complete reproducibility of the fractal set. Statistically-self-similar sets and random fractals are not covered by this current framework; such sets represent a next logical step in the generalisation of the theory.

1.5. **Code Space.** The IFS framework provides a natural addressing structure for points in an IFS attractor that is often far more useful than the standard Cartesian system of coordinates; namely, the code space associated with the IFS. The application of an infinite ordered sequence of IFS mappings to any starting point in the embedding metric space will converge to a precise point within the attractor of the IFS, hence a collection of infinite strings listing the order in which IFS mappings are to be applied can serve as coordinates tailor-made for a particular IFS attractor. The following definitions are taken from [12].

Definition 1.19 (**Code Space**)**.** Let $\mathcal{F} = \{X;\ f_1, f_2 \ldots, f_m\}$ be a hyperbolic IFS with attractor A. The associated *alphabet* $\mathcal{A}$ is the nonempty finite set of integer symbols $\mathcal{A} := \{1, 2, \ldots, m\}$. Given such an alphabet $\mathcal{A}$, the *code space* $\Omega_{\mathcal{A}}$ is the set of all infinite strings of symbols from the alphabet $\mathcal{A}$. That is, $\omega \in \Omega_{\mathcal{A}}$ if and only if it can be written, for $\omega_k \in \mathcal{A}$ for all $k \in \mathbb{N}$, as:

$$\omega = \omega_1\omega_2 \cdots \omega_k \ldots \tag{1.18}$$

The elements ω of the code space are known as *addresses*.

Definition 1.20 (**Finite Code Space**)**.** The *finite code space* $\Omega^N_{\mathcal{A}}$ denotes the set of all finite strings of symbols from the alphabet $\mathcal{A}$ of length N. That is, $\omega \in \Omega^N_{\mathcal{A}}$ if and only if it can be written as:

$$\omega = \omega_1\omega_2 \cdots \omega_N \tag{1.19}$$

Finite code spaces are useful in the context of pre-fractal approximations to given attractors, whereas considerations of a full fractal attractor set employ the associated infinite code space.

Definition 1.21 (**Code Space Metric**)**.** Given an IFS $\mathcal{F} = \{X; f_1, \ldots, f_m\}$ with associated code space $\Omega_{\mathcal{A}}$, the *code space metric* on addresses in $\Omega_{\mathcal{A}}$ is defined by:

$$d_\Omega(\omega, \sigma) = d_\Omega(\omega_1\omega_2 \ldots, \sigma_1\sigma_2 \ldots) := \sum_{k=1}^{\infty} \frac{|\omega_k - \sigma_k|}{(m+2)^k} \tag{1.20}$$

for all $\omega, \sigma \in \Omega_{\mathcal{A}}$.

The *finite code space metric* on addresses in $\Omega^N_{\mathcal{A}}$ is defined by:

$$d'_\Omega(\omega, \sigma) = d'_\Omega(\omega_1\omega_2 \ldots \omega_N, \sigma_1\sigma_2 \ldots \sigma_N) := \sum_{k=1}^{N} \frac{|\omega_k - \sigma_k|}{(m+1)^k} \tag{1.21}$$

Both $(\Omega_{\mathcal{A}}, d_\Omega)$ and $(\Omega^N_{\mathcal{A}}, d'_\Omega)$ are metric spaces. Combining the framework of code space with the Chaos Game algorithm, discussed in the next section, immediately provides the following addressing structure that will be employed in the definition of expectations over IFS attractors in Section 2.

Definition 1.22 (**Address Function**)**.** A mapping $\phi : \Omega_{\mathcal{A}} \to X$ is an *address function* for X, and any point $\omega \in \Omega_{\mathcal{A}}$ such that $\phi(\omega) = x$ is called an *address of* $x \in X$

In particular, the mapping defined in the following Theorem 1.23, known as the *code-space mapping*, is of particular interest.

Theorem 1.23 (**Well-defined mapping from code-space to points** [6])**.** *Let (X, d) be a complete metric space. Let $\{X; f_1, f_2, \ldots, f_m\}$ be a hyperbolic IFS with attractor A and associated* code space $\Omega_{\mathcal{A}}$. *For each $\omega \in \Omega$, $n \in \mathbb{N}$ and $x \in X$ let*

$$\phi(\omega, n, x) := f_{\omega_1} \circ f_{\omega_2} \circ \ldots \circ f_{\omega_n}(x).$$

Then

$$\phi(\omega) := \lim_{n\to\infty} \phi(\omega, n, x)$$

exists, belongs to A *and is independent of* $x \in X$*. If* K *is a compact subset of* X *then the convergence is uniform over* $x \in K$*. The function* $\phi : \Omega_A \to A$ *thus provided is continuous and onto.*

The code-space mapping furnishes IFS attractors with their natural address structure.

Definition 1.24 (**Addresses** [6]). Let $\{X; f_1, f_2 \ldots, f_m\}$ be a hyperbolic IFS with attractor A and associated *code space* $\Omega_{\mathcal{A}}$. Let $\phi : \Omega_A \to A$ be the code-space mapping. An *address* of a point $a \in A$ is any member of the set

$$\phi^{-1}(a) := \{\omega \in \Omega_A \ : \ \phi(\omega) = a\}$$

This set is called the *set of addresses* of $a \in A$.

Note that points in A can have multiple addresses if the IFS is not totally disconnected - for example, consider the unit interval $[0, 1]$ as represented by the attractor A of the IFS

$$\left\{[0,1] \subset \mathbb{R};\ f_1(x) = \frac{1}{2}x,\ f_2(x) = \frac{1}{2}x + \frac{1}{2}\right\}.$$

In this case, the point $\frac{1}{2} \in A$ can be represented using either of the infinite strings 1222... or 2111...

1.6. The Random Iteration (or ‘Chaos Game’) Algorithm.

One straightforward way of visualising the attractor of a given iterated function system is by direct application of the associated Hutchinson operator (Equation 1.16) to an arbitrary compact embedding space taken as a level-0 pre-fractal—the mappings of the IFS are first applied to the entirety of the embedding space, then to the resulting union of images, and so on for the desired amount of iterations; the fractal attractor thus being approximated by the pre-fractal set comprising the union of the final collection of image sets. However, this approach is relatively slow and time-consuming compared to the standard means of generating visual representations of IFS attractors, which forms the focus of this section.

The pictures of IFS attractors in Section 4 were produced using the Random Iteration Algorithm, also known as the Chaos Game, a fast and memory-efficient algorithm introduced by Barnsley in [8]. Beyond the visual approximation of IFS attractors, the Chaos Game has found applications in a wide variety of fractal contexts, particularly with regards to transformations between fractal sets (see for example [13]). The fundamental definition of expectation over IFS attractors in Section 2 has a straightforward link to the Chaos Game, which leads to an immediate Monte Carlo algorithm for numerical expectation of said expectations.

At the core of the Chaos Game is the idea of a Chaos Game orbit.

Definition 1.25 (**Chaos Game Orbit**)**.** Given an IFS $\mathcal{F} = \{X;\ f_1, f_2 \ldots, f_m\}$ and address $\omega = \omega_1\omega_2 \cdots \omega_n \cdots \in \Omega_{\mathcal{A}}$, the *chaos game orbit* of a point $x_0 \in X$ with respect to ω is the sequence $(x_n)_{n=0}^{\infty}$ where

$$x_n = f_{\omega_n}(x_{n-1}) \quad \text{for } n = 1, 2, \ldots \tag{1.22}$$

The chaos game orbit $(x_n)_{n=0}^{\infty}$ is a *random orbit* of x_0 if there exists $p \in (0, 1/m]$ such that for each $k \in \{1, 2, \ldots\}$,

$$P\left(w_k = n \mid x_0, \omega_1, \omega_2, \ldots, \omega_{k-1}\right) > p \tag{1.23}$$

The use of random orbits—the weighted selection of functions by their associated probabilities at each step of the orbit—is used to improve the efficiency of the algorithm with regards to 'filling out' the attractor. Broadly speaking, probabilities are weighted towards those mappings with larger image sets (more precisely, the probabilities are closely aligned with the invariant measure of the image sets, discussed in Section 1.8)—hence the choice of probabilities in the Barnsley Fern pIFS in Section 4.

The convergence of the chaos game orbits to the attractor of an IFS is guaranteed by the following theorem [11]:

Theorem 1.26 (**Convergence of Chaos Game Orbits**)**.** *Let X be a proper complete metric space and let $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ be a hyperbolic IFS with attractor A and basin B. If $(x_n)_{n=0}^{\infty}$ is a random orbit of $x_0 \in B$ under $\mathcal{F}$, then with probability 1*

$$\lim_{j\to\infty} \{x_n \mid \ n \geq j\} = \bigcap_{j\geq 1} \overline{\{x_n \mid \ n \geq j\}} = A \tag{1.24}$$

where convergence is with respect to the Hausdorff metric.

The Chaos Game algorithm operates as follows: given an iterated function system, choose an arbitrary starting point and generate a random orbit on the attractor (with probabilities weighted to the invariant measure of the image sets of the mappings, for extra efficiency) by selecting a mapping f_i from the IFS (with probability p_i) and applying it the the current point to obtain a new point. Repeat for the desired number of iterations and discard the first N points that were obtained before sufficient convergence to the attractor had occurred (in practice, N can usually be taken as small as 10).

Once a large number of points of an attractor have been generated by the Chaos Game, they can be plotted to visualise the attractor. They can also be used to sample points from the attractor in an efficient manner.

A more general theorem concerning conditions under which the chaos game algorithm yields the attractor of an iterated function system 'almost surely' (with probability 1) is provided in [19]. There it is shown that the condition of contractive functions may be relaxed to requiring continuous functions over a proper metric space (in which closed balls are compact).

1.7. **The Open Set Condition.** A particularly nice class of IFSs that will concern us are those satisfying the *open set condition*, which captures the notion of the fractal being built up from distinct 'non-overlapping' pieces. For the particular class of fractals satisfying the open set condition, the Hausdorff-Besicovitch and Minkowski (Box-Counting) fractal dimensions can be shown to be equivalent, thus opening the way for the Hausdorff-Besicovitch dimension to be computed in an accessible manner. Further, the open set condition allows the equivalent fractal dimensions to be tied to the contraction factors of the IFS mappings. This has important implications for the investigation of box-integral poles in Section 3.1.

Definition 1.27 (The Open Set Condition). Let $\mathcal{F} = \{X; f_1, \ldots, f_m\}$ be a hyperbolic IFS with attractor A. The IFS is said to obey the *open set condition*[1] if the attractor A contains a non-empty set $O \subset A$, which is open in the metric space A and satisfies

(1) $f_i(O) \cap f_j(O) = \emptyset$ for all $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$; and
(2) $F(O) = \bigcup_{i=1}^{m} f_i(O) \subset O$.

It is worth noting that IFS attractors can be classed according to whether the individual self-similar pieces (the images of the mappings applied to the entire set) are entirely disconnected, overlap 'non-trivially', or overlap only at their boundaries, as follows:

Definition 1.28 (Classification of IFS attractors [6]). Let $\{X; f_1, f_2 \ldots, f_m\}$ be a hyperbolic IFS with attractor A. The IFS is said to be *totally disconnected* if each point in the attractor possess a unique address, *just-touching* if it is not totally disconnected, but satisfies the open-set condition, and *overlapping* if it is neither just-touching nor disconnected.

The definition of totally disconnected IFS attractors is equivalent to:

Theorem 1.29 (Total disconnection [6]). *Let $\{X; f_1, f_2, \ldots, f_m\}$ be a hyperbolic IFS with attractor A. Then the IFS is totally disconnected if and only if $f_i(A) \cap f_j(A) = \emptyset$ for all $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$.*

The open set condition leads to the aforementioned fundamental result concerning the fractal dimension of an IFS.

Theorem 1.30 (Fractal Dimension with the Open Set Condition). *[35] Let $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ be an IFS satisfying the open-set condition, where the mappings f_i are similarity mappings with associated contraction factors $\{c_1, c_2, \ldots, c_m\}$. Then the Hausdorff-Besicovitch dimension*

[1] Strictly speaking, it is the 'intrinsic' open set condition that is defined by Definition 1.27. The intrinsic open set condition is implied by the 'strong' open set condition, in which the subset O must also, with probability 1, have non-empty intersection with A. On the other hand, it is more restrictive than the standard open set condition, in which the subset O is open in a topological sense.

and Minkowski dimension of the attractor of the IFS are equal and take the value δ, where:

$$\sum_{i=1}^{m} (c_i)^{\delta} = 1. \tag{1.25}$$

It is important to reiterate that there are many different definitions of 'fractal dimension' to which the phrase can refer. In light of Theorem 1.30, throughout this work the phrase 'fractal dimension' will be used interchangeably for both the Minkowski (box-counting) and Hausdorff-Besicovitch dimensions (and will thus assume that the open set condition holds for the fractal attractor of interest).

Note the following result from [5] concerning fractal dimensions of String-generated Cantor Sets:

Proposition 1.31 (**Fractal dimension of an SCS**). *The fractal dimension (in both the Hausdorff and box-counting sense) $\delta(C_n(P))$ of the SCS $C_n(P)$ is given by the closed form*

$$\delta\left(C_n(P)\right) \;=\; \frac{\log \prod_{k=1}^{p} N_k(P,n)}{p \log 3}.$$

The proof of the above proposition essentially relies on first encoding an SCS as the attractor of an IFS satisfying the open set condition, then appealing to Theorem 1.30. Consequently, though a closed-form for fractal dimension has been exhibited for SCS fractals—and perhaps might be discovered for attractors of graph-directed similitude IFSs—no further generalisation of this dimensional closed form into the IFS attractor setting beyond Theorem 1.30 seems feasable, aside from the following result established in [14]:

Theorem 1.32 (**Fractal dimension bounds for a hyperbolic IFS**). *Suppose that the open set condition holds for the hyperbolic IFS $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ (with associated contraction factors $\{c_1, c_2, \ldots, c_m\}$). If there exist numbers l_i, u_i such that $l_i|x-y| \le |f_i(x) - f_i(y)| \le u_i|x-y|$ for all $x, y \in \mathbb{R}^n$ and $i = 1, \ldots, m$, then the Hausdorff-Besicovitch dimension $\delta(A)$ of the IFS attractor (A) is bounded by:*

$$\min\{n, L\} \le \delta(A) \le U \tag{1.26}$$

where L and U are the positive solutions of

$$\sum_{i=1}^{m} (l_i)^{L} = 1 \quad \textit{and} \quad \sum_{i=1}^{m} (u_i)^{-U} = 1 \tag{1.27}$$

If the open set condition does not hold, the upper bound remains valid.

1.8. **Invariant Measure of an IFS.** Central to our analysis of expectations over fractal sets is the application of an appropriate measure defined over the fractal of interest. The following definitions are taken from [29].

Definition 1.33 (**Normalised and Invariant Measures**)**.** A measure μ on X is *normalised* if $\mu(X) = 1$ and is *invariant* for a mapping $f : X \to X$ if for every subset $A \subset X$ we have

$$\mu\left(f^{-1}(A)\right) = \mu\left(A\right)$$

A measure μ is *ergodic* for f if every measurable set A such that $A = f^{-1}(A)$ has $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

It can be shown (see [6] Chapter 9) that every IFS attractor supports a unique normalised invariant measure, known as the residence measure, which is defined in [6] as follows:

Definition 1.34 (**Residence Measure**)**.** Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a hyperbolic IFS with attractor $A \in \mathcal{H}(X)$, with all mappings being similarity mappings of identical contraction factor. Let $\{x_k\}_{k=0}^{\infty}$ denote a chaos game orbit of the IFS starting at $x_0 \in A$. Let B be a Borel subset of X with $\mu(B') = 0$ (where B' is the boundary of B). The *residence measure* μ on A is defined by the almost-sure limit:

$$\mu(B) := \lim_{n\to\infty} \frac{\#\{x_0, x_1, \ldots, x_n\} \cap B}{n+1} \tag{1.28}$$

for all $x_0 \in A$.

It is this residence measure with respect to which functions will be integrated in order to compute expectations via the definition of expectations over IFS attractors. The residence measure of a Borel set B is the limiting proportion of points produced by the Chaos Game that lie within B; accordingly, the residence measure may be visualised by terminating the Chaos Game after a relatively small number of iterations and examining the approximate mass distribution of points on the attractor.

That the residence measure μ is normalised follows immediately from its definition, as all points in the chaos game orbit lie on the attractor (provided $x_0 \in A$). The invariant nature of μ is proven in [29]; moreover, ergodic theory shows that the limit in Equation 1.34 exists and is identical for μ-almost all points in the basin of attraction. The residence measure μ is supported by an attractor of $\mathcal{F}$, since the measure is concentrated on the set of points to which $f^k(x)$ comes arbitrarily close to infinitely often [29].

Helpfully, Definition 1.34 immediately lends itself to a simple Chaos-Game algorithm that may be employed for numerical estimation of the residence measure of Borel sets. The Chaos Game algorithm for Monte-Carlo-style estimation of fractal expectations in Section 5 is in very much the same spirit.

Finally, the following theorem has important implications for linking the residence measure to fractal expectations.

Theorem 1.35 (**Existence of Limit** [30])**.** *Let $f : X \to X$, let μ be a finite measure on X that is invariant under f and let $\phi \in L^1(\mu)$). Then the* *limit*

$$\Phi(x) := \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \phi(f^j(x)) \tag{1.29}$$

exists for μ-almost all x. Moreover, if μ is ergodic then

$$\Phi(x) = \frac{1}{\mu(X)} \int_X \phi(y)\mathrm{d}\mu(y) \tag{1.30}$$

1.9. **Affine Iterated Function Systems.** An important subclass of iterated function systems are the *affine iterated function systems*, being IFSs for which the mappings f_i in Definition 1.11 are all affine mappings:

Definition 1.36 (**Affine Iterated Function System).** An *affine iterated function system (affine IFS)* $\mathcal{F}$ is a complete metric space $X = \mathbb{R}^n$ along with a finite set of affine functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, m$, denoted:

$$\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\} \tag{1.31}$$

where the action of each f_i can be expressed in the form:

$$f_i(x) = L_i(x) + T_i \tag{1.32}$$

where L_i is a linear mapping (corresponding to multiplication of x by an $n \times n$ matrix) and T_i is a translation (corresponding to addition by an $n \times 1$ matrix), for all $x \in \mathbb{R}^n$.

Many celebrated fractal sets can be represented as attractors of appropriate affine IFSs—in particular, every SCS and every example presented in Section 4 arises in this manner. The subclass of affine IFSs is of interest to the theory of fractal expectations for two reasons: first, the affine structure of the mappings enables additional theoretical results, such as Proposition 3.3, to be obtained; second, the approximation of digital images via the Collage Theorem always generates affine mappings (as discussed in Section 1.10) and consequently affine IFSs are closely tied to the fractal modeling of real-world image data. In particular, even-order box integrals can always be symbolically evaluated over affine IFSs by means of Algorithm 4.1.

Aside from their emergence from the Collage Theorem, perhaps the most important result concerning affine IFSs is the classification theorem of Atkins et al. in [2]. This theorem requires the establishment of the following definitions (also taken from [2]):

Definition 1.37 (**Coding Map).** A continuous map $\pi : \Sigma \to \mathbb{R}^n$ is a *coding map* for the IFS $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ if, for each $i = 1, 2, \ldots, m$, the following diagram commutes (where $s_i : \Sigma \to \Sigma$ denotes the inverse shift map $s_i(\sigma) = i\sigma$):

$$\begin{array}{ccc} \Sigma & \xrightarrow{s_i} & \Sigma \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathbb{R}^n & \xrightarrow{f_i} & \mathbb{R}^n \end{array}$$

Definition 1.38 (**Point-Fibred IFS**)**.** An IFS $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ is *point-fibred* if for each $\sigma = \sigma_1\sigma_2\sigma_3\cdots \in \Sigma$, the limit

$$\pi(\sigma) := \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x) \tag{1.33}$$

exists and is independent of $x \in \mathbb{R}^n$ for fixed σ, and the map $\pi : \Sigma \to \mathbb{R}^n$ is a coding map.

Though not every affine IFS is hyperbolic on all of $\mathbb{R}^n$, it can be shown that if $\mathcal{F}$ has a coding map then $\mathcal{F}$ is always hyperbolic on some affine subspace of $\mathbb{R}^n$ [2] [44]. Note that this is not true in general [20].

Theorem 1.1 (**Classification for Affine Hyperbolic IFSs** [2])**.** *If $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ is an affine iterated function system, then the following statements are equivalent: (i.) $\mathcal{F}$ has an attractor. (ii.) $\mathcal{F}$ is hyperbolic. (iii.) $\mathcal{F}$ is point-fibred. (iv.) $\mathcal{F}$ is a topological contraction with respect to some convex body $K \subset \mathbb{R}^n$. (v.) $\mathcal{F}$ is non-antipodal with respect to some convex body $K \subset \mathbb{R}^n$.*

The classification Theorem 1.1 demonstrates the tight link between several fundamental concepts in IFS theory when in the context of affine IFSs.

1.10. **The Collage Theorem.** One striking application of IFS theory is the encoding of digital images as the attractors of appropriate Iterated Function Systems, facilitated by Barnsley's Collage Theorem. A famous instance of such an application was the compression of the over 7000 photographs of the original Microsoft Encarta encyclopedia in order to permit their storage on a single CD-ROM [10].

The Chaos Game algorithm described in Section 1.6 permits efficient computation of the unique attractor of a given IFS. The inverse problem—namely, of finding a given IFS whose attractor approximates a given subset of a compact metric space—was effectively solved by the introduction of the Collage Theorem [15].[1]

Theorem 1.39 (**The Collage Theorem** [6])**.** *Let (X, d) be a complete metric space. Given an target image $L \in \mathcal{H}(X)$ and an $\epsilon \geq 0$, choose an IFS $\mathcal{F} = \{\mathbb{R}^n;\ f_1, f_2, \ldots, f_m\}$ with contractivity factor $0 \leq c < 1$ such that*

$$h\left(L, \bigcup_{i=1}^{m} f_i(L)\right) \leq \epsilon \tag{1.34}$$

where h is the Hausdorff metric. Then

$$h(L, A) \leq \frac{\epsilon}{1-c} \tag{1.35}$$

where A is the attractor of the IFS $\mathcal{F}$.

[1]The construction of the Barnsley Fern by means of the Collage Theorem was first presented in the same paper

The Collage Theorem leads to the simple algorithm that, given an input target set, generates an affine IFS 'by hand', whose attractor approximates the target set to within an error tolerance (as measured by the Hausdorff metric) that can be made arbitrarily small [10]. The algorithm is presented below for encoding two-dimensional digital images, though it is easily extended to higher dimensions.

Algorithm 1.40 (Generating an IFS attractor approximation [15]). *Given a digital target image, the following procedure generates an affine IFS whose attractor approximates the target image to within an arbitrary accuracy.*

- *Given a (binary) digital target image, rescale coordinates so as to embed the target image in the unit square.*
- *Overlay a smooth boundary curve around the target image.*
- *Construct a 'collage' of the target image by overlaying m smaller copies of the boundary curve, each transformed via a contractive affine mapping A_i, so as to approximately neatly cover the original boundary curve and its interior.*
- *Use the collection of affine mappings to form the encoding IFS $\{[0,1]^2; \mathcal{A}_1, A_2 \ldots, A_m\}$.*
- *Construct the attractor of the encoding IFS via the Chaos Game and compare to the target image. If the difference exceeds error tolerances, repeat steps 3-4 with a refined collage.*

The process of constructing the collage for a particular target image (of a fern), and the resulting attractor of the encoding IFS, is illustrated in [15].

The above algorithm relies on the property that small changes to the particular affine mappings used in the encoding IFS will lead to small, controllable changes in the resulting IFS attractor. This is guaranteed by the following theorem.

Theorem 1.41 (Continuous dependence of an IFS attractor on the mappings [6]). *Let $\mathcal{F} = \{X;\ f_1, f_2, \ldots, f_m\}$ be a hyperbolic IFS. For all $i = 1, 2, \ldots, m$, let the mapping f_i depend continuous on a parameter $p \in P$, where P is a compact metric space. Then the IFS attractor $A(p) \in \mathcal{H}(X)$ depends continuously on $p \in P$, with respect to the Hausdorff metric.*

The algorithm presented above for applying the Collage Theorem to generate an encoding IFS has been facilitated by many software packages that greatly ease the process of creating the collage and extracting the associated affine mappings. In particular, the process of constructing a collage of the target set has been automated by algorithms which search through the parameter space of all possible affine mappings and evaluate the constructed approximation by computing the Hausdorff distance between the target set and the attractor of the encoding IFS [45].

Note the exciting corollary that the Collage Theorem provides a means by which any real-world image of interest can be well approximated by considering only attractors of affine IFSs. The class of affine IFSs is therefore of particular practical interest, as well as being theoretically accessible. Results pertaining to affine IFSs will be developed in the following sections.

2. Fundamental Definition of IFS Expectations

The fundamental definition of the expectation of a complex-valued function $F : \mathcal{R}^n \to \mathcal{C}$ over a fractal SCS $C_n(P)$ arose from considerations of the discrete expectation of the (finitely-many) evaluations of F at every admissible point in successively-finer pre-fractal approximations of the SCS, as follows:

$$\langle F(x)\rangle_{x\in C_n(P)} := \lim_{j\to\infty} \frac{1}{N_1\cdots N_j} \sum_{U(c_i)\le P_i} F(c_1/3 + c_2/3^2 + \cdots + c_j/3^j), \tag{2.1}$$

$$\langle F(x-y)\rangle_{x,y\in C_n(P)} := \lim_{j\to\infty} \frac{1}{N_1^2\cdots N_j^2} \sum_{\substack{U(c_i)\le P_i \\ U(d_i)\le P_i}} F((c_1-d_1)/3 + \cdots + (c_j-d_j)/3^j), \tag{2.2}$$

where c_i and d_i range over the set of admissible columns as per Definition 1.3.

The extension of the fundamental definition of fractal expectation to encompass attractors of iterated function systems follows along precisely the same lines. The expectation may be approximated by examining the discrete expectation over a pre-fractal approximation to the attractor, where only finitely-many evaluations of F are required. The true expectation is thus defined by evaluating the limit of the discrete expectation approximations as the pre-fractal resolution is increased without limit—that is, as the transition from the pre-fractal approximations to the full fractal attractor is made. The fundamental definition thus adopted is most naturally stated in the language of code-space, as follows:

Definition 2.1 (**IFS Fractal Expectation**)**.** Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive IFS with attractor $A \in \mathcal{H}(X)$ and associated code space $\Omega_{\mathcal{A}}$. Let $\Omega_{\mathcal{A}}^j$ denote the set of finite codes of length j. Given a complex-valued function $F : X \to \mathbb{C}$, the *expectation of F over A* is defined as:

$$\langle F(x)\rangle_{x\in A} := \lim_{j\to\infty} \frac{1}{m^j} \sum_{\sigma\in\Omega_{\mathcal{A}}^j} F(\phi(\sigma)) \tag{2.3}$$

when the limit exists.

The question of precisely when these limits exist will be addressed in Section 2.2 after an equivalent formulation of the above definitions has been developed in Section 2.1. This reformulation enables the use of the Ergodic

Theorem (Theorem 1.35) to prove that the limit in Definition 2.1 (and the limits in Definitions 2.2 and 2.3 below) exist almost always in a measure-theoretic sense.

Definition 2.1 can be readily extended to multivariable functions in the following manner:

Definition 2.2 (**Multivariable IFS Fractal Expectation**)**.** Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive IFS with attractor $A \in \mathcal{H}(X)$ and associated code space $\Omega_{\mathcal{A}}$. Let $\Omega^j_{\mathcal{A}}$ denote the set of finite codes of length j. Given a complex-valued function $F : X^n \to \mathbb{C}$, the *expectation of* F *over* A is defined as:

$$\begin{aligned}&\langle F(\mathbf{x} = (x_1, x_2, \ldots, x_n))\rangle_{\mathbf{x}\in A^n}\\ &:= \lim_{j\to\infty} \frac{1}{m^{nj}} \sum_{\sigma_1\in\Omega^j_{\mathcal{A}}} \sum_{\sigma_2\in\Omega^j_{\mathcal{A}}} \cdots \sum_{\sigma_n\in\Omega^j_{\mathcal{A}}} F\left(\phi(\sigma_1), \phi(\sigma_2), \ldots, \phi(\sigma_n)\right)\end{aligned} \tag{2.4}$$

when the limit exists.

Consequently, separation expectations arise as a two-variable special case of Definition 2.2 as follows:

Definition 2.3 (**IFS Fractal Separation Expectation**)**.** Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive IFS with attractor $A \in \mathcal{H}(X)$. Given a complex-valued function $F : X \to \mathbb{C}$, the *separation expectation of* F *over* A is defined as:

$$\langle F(x-y)\rangle_{x,y\in A} := \lim_{j\to\infty} \frac{1}{m^{2j}} \sum_{\sigma_j\in\Omega^J_{\mathcal{A}}} \sum_{\tau_j\in\Omega^J_{\mathcal{A}}} F\left(\phi(\sigma_j) - \phi(\tau_j)\right) \tag{2.5}$$

when the limit exists.

Note that these definitions preserve the familiar classical properties of expectations: the IFS fractal expectations so-defined are both *linear*, as $\langle F(x) + \alpha G(x)\rangle = \langle F(x)\rangle + \alpha \langle G(x)\rangle$, and *monotonic*, as $F(x) \le G(x)$ for all $x \in A$ implies $\langle F(x)\rangle \le \langle G(x)\rangle$. Further, applying Definition 2.1 to an IFS with an SCS attractor immediately recovers the prior definitions of fractal expectations over SCSs (Equations 2.1 and 2.2). To see this, recall from Proposition 1.5 the formulation of any given SCS as the attractor of the associated IFS:

$$\{[0,1]^n \subset \mathbb{R}^n; f_1, f_2, \ldots, f_m\}$$

where

$$f_i\left(x = (x_1, x_2, \ldots, x_n)\right) = \left(\frac{1}{3}\right)^p x + \left(\frac{1}{3}\right) c_{1_i} + \left(\frac{1}{3}\right)^2 c_{2_i} + \ldots + \left(\frac{1}{3}\right)^n c_{n_i}$$

for $i \in \{1, 2, \ldots, m\}$ ranging over all admissible columns c_k, where $m = \prod_{k=1}^p N_k$ and $N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$.

Now the address mappings $\phi(\sigma)$ on codes of length 1 in Definition 2.1 correspond to a sample point in the image of each permutation of admissible columns, up to the resolution defined by the period p. For codes of length j we have j-fold repetition, so the definition

$$\langle F(x)\rangle_{x\in A} := \lim_{j\to\infty}\frac{1}{m^j}\sum_{\sigma\in\Omega^j_{\mathcal{A}}} F\left(\phi(\sigma)\right)$$

translates to:

$$\langle F(x)\rangle_{x\in A} = \lim_{j\to\infty}\frac{1}{(N_1\cdots N_p)^j}\sum_{U(c_i)\le P_i} F(c_1/3+c_2/3^2+\cdots+c_p/3^p+\cdots+c_{jp}/3^{jp})$$

where an additional p terms are added each time the length of the code strings is extended by 1. This can be recast as:

$$\langle F(x)\rangle_{x\in A} = \lim_{k\to\infty}\frac{1}{N_1\cdots N_k}\sum_{U(c_i)\le P_i} F(c_1/3+c_2/3^2+\cdots+c_k/3^k)$$

which is precisely Equation 2.1.

2.1. **Equivalent Formulation via the Random Iteration Algorithm.** A useful reformulation of Definition 2.1 in terms of the Chaos Game (of Section 1.6) is presented below, in which the expectations are reinterpreted as the limiting value of the arithmetic mean of function values evaluated at points chosen by a random-iteration sampling process over the IFS attractor. Note that the use of this particular definition regarding expectations over IFS attractors is well-established in the literature - see [30], for example.

Definition 2.4 (**IFS Fractal Expectation - Chaos Game Definition**). Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive IFS with attractor $A \in \mathcal{H}(X)$. Let $\{x_n\}_{n=0}^{\infty}$ denote a chaos game orbit of the IFS starting at $x_0 \in X$, that is, $x_n = f_{\sigma_n}\circ\cdots\circ f_{\sigma_1}(x_0)$ where the maps are chosen independently for all $n \in \mathbb{N}$ according to the uniform probabilities $p_i = 1/m$ for $i = 1, \ldots, p_m$. Given a complex-valued function $F : X \to \mathbb{C}$, define the *expectation of F over A* by:

$$\langle F(x)\rangle_{x\in A} := \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} F(x_k) \tag{2.6}$$

independently of $x_0 \in A$, where the limit exists.

A pictorial illustration of Definition 2.1 is shown in Figure 1.

The equivalence of Definitions 2.1 and 2.4 will be shown in Section 2.2. The alternate form of the fundamental definition of expectation in 2.4 has two advantages. First, it allows for expectations so-defined to be connected with a formulation of expectations as integrals with respect to an appropriate measure in Section 2.2. Secondly, Definition 2.4 leads immediately to a simple algorithm for numerical computation of expectations in Section 5.

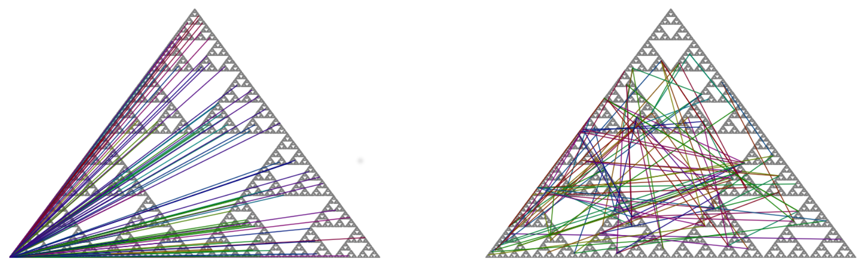

FIGURE 1. Representation of the box integrals $B_2 = \left\langle |x|^2 \right\rangle_{x=A}$ *(left)* and $\Delta_2 = \left\langle |x-y|^2 \right\rangle_{x,y=A}$ *(right)* over the unit Sierpiński Triangle, showing the first 100 randomly-sampled points within the attractor generated by the chaos-game definition 2.4. The colours represent sampled function values, with colours further towards the red end of the spectrum indicating larger magnitudes. The respective box integrals are the expectation of these function values.

Also, the fundamentally important functional equations of Proposition 3.1 in Section 3 stem from the code-space form of Definition 2.1.

2.2. **Measure-Theoretic Considerations.** Paralleling the development of SCS fractal expectation theory, the next objective, after having established a fundamental definition of expectation over attractors of iterated function systems, is to connect the fractal expectations to the notion of integration over an IFS attractor A by determining a measure μ such that:

$$\langle F(x) \rangle_{x \in A} = \int_X F(x) \mathrm{d}\mu(x)$$

It turns out that the unique residence measure of Equation (1.34) associated with an IFS (see Section 1.8) allows a straightforward connection between Definition 2.4 and a suitable integral, as would be expected for an appropriately-defined notion of expectation. Recall the residence measure μ on the attractor A of an iterated function system is defined as:

$$\mu(B) := \lim_{n \to \infty} \frac{\#\{x_0, x_1, \ldots, x_n\} \cap B}{n+1} \tag{2.7}$$

for all Borel subsets B of X with boundary of measure 0, where $\{x_k\}_{k=0}^{\infty}$ denote a chaos game orbit of the IFS starting at $x_0 \in A$.

The residence measure can be essentially regarded as following an infinite chaos-game orbit on the attractor and tracking the proportion of points that fall inside the particular Borel set under consideration. That the residence measure is normalised follows directly from its definition, as all points in the chaos game orbit lie on the attractor, and its invariant nature is proven in [29]. For the remainder of this paper μ will refer to the residence measure of Equation (1.34).

Using the residence measure, the existence of the limits involved in the fundamental definitions 2.1 and 2.4 can be justified by invoking the following ergodic theorem (see [29]):

Theorem 2.5 (**The Ergodic Theorem**). *Let $f : X \to X$, let μ be a finite measure on X that is invariant under f and let $\sigma \in L^1(\mu)$. Then the limit*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \sigma(f^j(x)) \tag{2.8}$$

exists for μ-almost all x. Moreover, if μ is ergodic then

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \phi(f^j(x)) = \frac{1}{\mu(X)} \int_X \sigma(y) \mathrm{d}\mu(y) \tag{2.9}$$

for μ-almost all x.

Thus, the limits in the fundamental definitions exist for μ-almost all x - that is, for all x excepting a set of μ-measure zero.

The equivalence of Definitions 2.1 and 2.4 follows from interpretation of the addressing function ϕ from definition 2.1 in terms of the composition of the IFS mappings, starting from an arbitrary point x_0 in the attractor, as follows:

$$\begin{aligned}\langle F(x)\rangle_{x\in A} &:= \lim_{j\to\infty} \frac{1}{m^j} \sum_{\sigma\in\Omega_{\mathcal{A}}^J} F\left(\phi(\sigma)\right) \\ &= \lim_{j\to\infty} \frac{1}{m^j} \sum_{\sigma_j=1}^{m} \cdots \sum_{\sigma_2=1}^{m} \sum_{\sigma_1=1}^{m} F\left(f_{\sigma_j} \circ \cdots f_{\sigma_2} \circ f_{\sigma_1}(x_0)\right)\end{aligned}$$

This definition considers the average of the function values evaluated simultaneously over all possible orbits of a fixed length, as the length of orbits tends to infinity. However, it follows from Theorem 2.5 that any infinite chaos-game orbit is ergodic and thus will trace out every point in the attractor. Hence, the summation over all deterministic chaos-game orbits of a fixed length can be replaced by a single sum over just one *random* chaos game orbit without affecting the limit.

The next theorem by Elton (see [28] and [33]) provides a direct connection between integration with respect to the residence measure and Definition 2.4.

Theorem 2.6 (**Elton's Theorem (Special Case)**). *Let (X, d) be a compact metric space and let $\{X; f_1, \ldots, f_m; p_1, \ldots, p_m\}$ be a hyperbolic IFS. Let $\{x_n\}_{n=0}^{\infty}$ denote a chaos game orbit of the IFS starting at $x_0 \in X$, that is, $x_n = f_{\sigma_n} \circ \ldots \circ f_{\sigma_1}(x_0)$ where the maps are chosen independently according to the probabilities $p_1, \ldots, p_m$ for $n \in \mathbb{N}$. Let μ be the unique invariant measure* *for the IFS. Then, with probability* 1 *(i.e. for all code sequences excepting a set having probability* 0*),*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}F(x_k)=\int_X F(x)\mathrm{d}\mu(x) \tag{2.10}$$

It immediately follows that the adopted definition of fractal expectation allows the expectation of a function F to be expressed as the integral of F with respect to the residence measure:

Corollary 2.7. *Let* $\mathcal{F}=\{X;f_1,f_2,\dots,f_m\}$ *be an IFS with attractor* $A\in\mathcal{H}(X)$*. Given a complex-valued function* $f:X\to\mathbb{C}$*, the* expectation of f over A *as defined by 2.1 or 2.4 is given by the integral:*

$$\langle F(x)\rangle_{x\in A}=\int_X F(x)\mathrm{d}\mu(x) \tag{2.11}$$

3. The Functional Equations for IFS Expectations

The functional equations for SCS expectations of Proposition 1.6 generalize to a powerful functional relation in the IFS attractor setting, shown below in Proposition 3.1. Closely shadowing the development of the SCS theory, the self-similarity encapsulated by such functional equations underpins all the results that follow, including the symbolic computation of separation expectations in special cases.

Proposition 3.1 (**Functional equation for expectations**)**.** *Let* $\mathcal{F}=\{X;f_1,f_2,\dots,f_m\}$ *be a contractive IFS with attractor* $A\in\mathcal{H}(X)$*. Then the expectation for a complex-valued function* $F:X\to\mathbb{C}$ *satisfies the functional equation:*

$$\langle F(x)\rangle_{x\in A}=\frac{1}{m}\sum_{j=1}^{m}\langle F\left(f_j(x)\right)\rangle \tag{3.1}$$

Proof. From the expectation definition 2.1, make the variable shift from $j\to j+1$, which preserves the limit, and then 'pull back' to level j—this has the effect of moving from the expectation over the attractor (at 'level 0') to the expectation of the fractal sets in the image of one application of the contraction mappings (at 'level 1'), as follows:

$$\begin{aligned}
\langle F(x)\rangle_{x\in A} := & \lim_{j\to\infty}\frac{1}{m^j}\sum_{\sigma\in\Omega_{\mathcal{A}}^J} F\left(\phi(\sigma)\right) \\
= & \lim_{j+1\to\infty}\frac{1}{m^{j+1}}\sum_{\sigma\in\Omega_{\mathcal{A}}^{J+1}} F\left(\phi(\sigma)\right) \\
= & \lim_{j\to\infty}\frac{1}{m^j}\frac{1}{m}\sum_{\sigma\in\Omega_{\mathcal{A}}^J} F\left(\phi(\sigma_1\sigma)\right)+F\left(\phi(\sigma_2\sigma)\right)+\cdots+F\left(\phi(\sigma_m\sigma)\right) \\
= & \frac{1}{m}\sum_{i=1}^{m}\left(\lim_{j\to\infty}\frac{1}{m^j}\sum_{\sigma\in\Omega_{\mathcal{A}}^J} F\left(\phi(\sigma_i\sigma)\right)\right) \\
= & \frac{1}{m}\sum_{i=1}^{m}\langle F\left(f_i(x)\right)\rangle
\end{aligned}$$

□

The proof immediately leads to other functional equations obtained by shifting to $j+k$ and pulling back, so moving to expectations at the 'level k' pre-fractal images of the fractal attractor. Such functional equations may find applications in development of numerical algorithms for expectations.

Proposition 3.1 is readily extended to multivariable functions as follows:

Proposition 3.2 (Multivariable functional equation for expectations). *Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive IFS with attractor $A \in \mathcal{H}(X)$. Then the expectation for a complex-valued function $F : X^n \to \mathbb{C}$ satisfies the functional equation:*

(3.2)

$$\langle F(x_1, x_2, \ldots, x_n)\rangle = \frac{1}{m^n}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_n=1}^{m}\langle F\left(f_{j_1}(x_1), f_{j_2}(x_2), \ldots, f_{j_n}(x_n)\right)\rangle$$

The proof of the multivariable functional equation of Proposition 3.2 is an immediate generalisation of the proof of Proposition 3.1.

Consequently, separation expectations satisfy the following functional equation arising as a two-variable special case of Proposition 3.2:

$$\langle F(x,y)\rangle_{x,y\in A} = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\langle F\left(f_j(x), f_k(y)\right)\rangle \tag{3.3}$$

$$\Rightarrow \langle F(x-y)\rangle_{x,y\in A} = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\langle F\left(f_j(x) - f_k(y)\right)\rangle \tag{3.4}$$

Note that these generalised functional equations do have some precedent in the literature - in particular, see [14], [16], [17], [18] and [22] .

3.1. **Pole Results.** Recall Theorem 1.7 concerning poles of B box integrals over SCSs: for any SCS $C_n(P)$, the (analytically-continued) box integral $B(s, C_n(P))$ has a pole at $s = -\delta(C_n(P))$. This theorem expanded upon the classical result of [3]: in the case of the unit n-cube, $B_n(s)$ contains a pole at $-n$, the negated dimension of the set.

The functional equation 3.1 immediately leads to a slight generalisation of Theorem 1.7 into the IFS attractor setting. However, the following propositions are only a partial result, requiring additional assumptions that are almost certainly too strict. In particular, Proposition 3.3 is only applicable to attractors of IFSs whose contraction mappings are affine mappings with the same contraction factor. Further, the open set condition must be be satisfied so as to obtain a dimensional relation via 1.30. Despite these restrictions, Proposition 3.3 is applicable to many well-known fractal sets including the Sierpiński triangle and von Köch snowflake.

Proposition 3.3 (**Pole of B Integrals over Uniform Affine IFSs**). *Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a contractive affine IFS satisfying the open set condition with uniform contraction factors; that is, $c_1 = c_2 = \ldots = c_m$. The (analytically continued) box integral $B(s, A)$ over the attractor $A \in \mathcal{H}(X)$ has a single pole on the real axis at*

$$s = -\delta(A) \tag{3.5}$$

where the fractal dimension $\delta(A)$ is established by Theorem 1.30.

Proof. Write each affine mapping as $f_j(x) = A_j x + T_j$, with linear component A_j and translation component T_j, under an appropriate selection of coordinates so that f_1 can be expressed as a pure scaling mapping; that is, $A_1 x = c_1 x$ and $T_1 = 0$. The functional relation of Proposition 1.30 yields

$$\begin{aligned} B(s, A) := \langle |x|^s \rangle_{x \in A} &= \frac{1}{m} \sum_{j=1}^{m} \langle |A_j(x) + T_j|^s \rangle \\ &= \frac{1}{m} c_1^s \langle |x|^s \rangle + \frac{1}{m} \sum_{j=2}^{m} \langle |A_j(x) + T_j|^s \rangle \end{aligned}$$

where the summation term in the right-hand-side is always finite (and nonzero) for any complex s, as it is a finite sum of expectations of nonzero vectors (by the open set condition).

Regrouping leads to:

$$\langle |x|^s \rangle_{x \in A} \left(1 - \frac{c^s}{m}\right) = \frac{1}{m} \sum_{j=2}^{m} \langle |A_j(x) + T_j|^s \rangle \tag{3.6}$$

$$B(s, A) = \frac{1}{1 - \frac{c^s}{m}} \frac{1}{m} \sum_{j=2}^{m} \langle |A_j(x) + T_j|^s \rangle \tag{3.7}$$

Now the companion factor to B in Equation (3.6), namely $(1 - \frac{c^s}{m})$, vanishes at the fractal dimension $s = -\delta(A)$ as $\sum_{j=1}^{m} (c_j)^\delta = 1 \Rightarrow mc^\delta = 1 \Rightarrow \frac{c^{-\delta}}{m} = 1$

while the right side remains bounded away from zero. It follows that B must have a pole at such s. □

Further generalisations of this result remain an active problem. In particular, the following is conjectured:

Conjecture 3.4 (**Pole of** B **Integrals over Contractive IFSs**). *Proposition 3.3 holds for all contractive IFSs.*

For separation expectations Δ, the following preliminary result establishes bounds on poles for $\Delta(s, A)$ in the case of an IFS of similarity mappings (though now the contraction factors need not be uniform):

Proposition 3.5 (**Bounds on Pole of** Δ **Integrals over Similarity IFSs**). *Let $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ be a similarity IFS (that is, that is, $|f_i(x) - f_i(y)| = c_i|x-y|$ for all i) satisfying the open set condition. Then, if the (analytically continued) box integral $\Delta(s, A)$ over the attractor $A \in \mathcal{H}(X)$ has a pole on the real axis, then this pole is bounded by:*

$$\frac{\log(m)}{\log(c_{\max})} \leq s \leq \frac{\log(m)}{\log(c_{\min})} \tag{3.8}$$

where $c_{\max} = c = \max\{c_1, \ldots, c_m\}$ and $c_{\min} = \min\{c_1, \ldots, c_m\}$.

Proof. Given $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$, the functional relation of Proposition 1.30 yields

$$\begin{aligned}
\Delta(s, A) := \langle |x-y|^s \rangle_{x,y\in A} &= \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} \langle |f_j(x) - f_k(y)|^s \rangle \\
&= \frac{1}{m^2} \sum_{j=1}^{m} \langle |f_j(x) - f_j(y)|^s \rangle + \frac{1}{m^2} \sum_{\substack{j=1 \\ j\neq k}}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \langle |f_j(x) - f_j(y)|^s \rangle
\end{aligned}$$

where the summation term has been split into cases where the indices do and do not match. In the former case, the similarity property $|f_i(x) - f_i(y)| = c_i|x-y|$ for all i yields:

$$\begin{aligned}
\langle |x-y|^s \rangle_{x,y\in A} \left(1 - \frac{1}{m^2} \sum_{j=1}^{m} c_j^s\right) &= \frac{1}{m^2} \sum_{\substack{j=1 \\ j\neq k}}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \langle |f_j(x) - f_j(y)|^s \rangle \\
\Rightarrow \Delta(s, A) &= \frac{1}{\left(1 - \frac{1}{m^2} \sum_{j=1}^{m} c_j^s\right)} \frac{1}{m^2} \sum_{\substack{j=1 \\ j\neq k}}^{m} \sum_{\substack{k=1 \\ k\neq j}}^{m} \langle |f_j(x) - f_j(y)|^s \rangle
\end{aligned}$$

The summation on the right-hand-side is strictly positive, so the companion factor to $\Delta(s, A)$, namely $(1 - \frac{1}{m^2} \sum_{j=1}^{m} c_j^s)$, yields existence of a pole when

$$\frac{1}{m^2}\sum_{j=1}^{m} c_j^s = 1. \tag{3.9}$$

Confining our attention to the real axis, first note that it cannot be the case that $s \geq 0$. Otherwise, the pole condition of Equation 3.9 contradicts the requirement that $0 < c_i \leq 1$, which in the case that $s \geq 0$ leads to:

$$\frac{1}{m^2}\sum_{j=1}^{m} c_j^s \leq \frac{1}{m^2}\sum_{j=1}^{m} c_j \leq \frac{1}{m} < 1.$$

Thus $s < 0$. Since the contraction factors satisfy $0 < c_i \leq 1$ for all i, it follows that

$$\frac{1}{m^2}c_{\max}^s \leq \frac{1}{m^2}\sum_{j=1}^{m} c_j^s \leq \frac{1}{m^2}c_{\min}^s$$

Consequently, the existence of a pole requires

$$\begin{aligned}\frac{1}{m^2}c_{\max}^s \leq 1 \leq \frac{1}{m^2}c_{\min}^s &\Rightarrow c_{\max}^s \leq m \leq c_{\min}^s\\ &\Rightarrow c_{\min} \leq \frac{1}{m^{-s}} \leq c_{\max}\\ &\Rightarrow \frac{\log(m)}{\log(c_{\max})} \leq s \leq \frac{\log(m)}{\log(c_{\min})}\end{aligned}$$

as required. □

4. Symbolic Evaluation of Box Integrals

While the functional equations of Theorems 3.1 and 3.2 hold generally, success in their application to resolve expectations symbolically depends heavily on the nature of both the function and the IFS under consideration. In certain special cases, such as separation moments of even degree over attractors of affine IFSs, the following procedure can be used to obtain exact symbolic evaluations. Note that the Collage Theorem process (discussed in Section 1.10) used to model a real-world image as an IFS attractor always produces affine IFSs, which are well-suited to this procedure.

Algorithm 4.1. ***Symbolic Computation for Special Expectations***

(1) *Given the attractor A of an IFS $\mathcal{F} = \{X; f_1, f_2, \ldots, f_m\}$ and a complex-valued function $F : A \to \mathbb{C}$, substitute the input data into the functional equations 3.1.*
(2) *Exploit the linear nature of the expectations to split the right-hand side into a sum of simpler expectations. Shift any resulting copies of the main expectation over the the left-hand side of the equation.*
(3) *Repeat steps 1-2 for each of the simpler expectations on the right-hand side*

(4) *Solve the resulting system of equations simultaneously to resolve all the simpler expectations, and hence the main expectation.*

To illustrate this algorithm, the problem of computing the second-order $B(2, A)$ and $\Delta(2, A)$ box integrals over a number of well-known fractal sets is examined in the remainder of this section. Limited numerical confirmation of the results so-obtained is presented in Section 5.

4.1. **The Unit Square.** Though not strictly classed as (non-trivial) fractal sets, nonetheless many classical objects, such as the unit square, can be represented in the framework of IFS attractors. This permits a direct connection between IFS fractal expectations and known results from classical box integral theory.

The unit square can be represented as the attractor of the IFS $\mathcal{F} = \left\{[0,1]^2;\ f_1, f_2, f_3, f_4\right\}$ where $f_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$, $f_3(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y + \frac{1}{2}\right)$ and $f_4(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y + \frac{1}{2}\right)$. Evaluation of the B_2 box integral (corresponding to the expectation $\sqrt{(x^2+y^2)^2}$) using the functional equation 3.1 leads to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{1}{3}\left(\langle x\rangle + \langle y\rangle + 1\right)$$

The simpler expectations can be re-evaluated as $\langle x\rangle = \frac{1}{2}$ and $\langle y\rangle = \frac{1}{2}$ leading to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{2}{3}$$

corresponding to the trivially-known classical value.

Similarly, evaluation of the Δ_2 box integral (corresponding to the expectation
$\sqrt{(x-x')^2 + (y-y')^2)^2}$) using the functional equation 3.1 leads to:

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1}{4}\left(\left\langle x^2\right\rangle - 2\langle xa\rangle + \left\langle a^2\right\rangle + \left\langle y^2\right\rangle - 2\langle yb\rangle + \left\langle b^2\right\rangle\right) + \frac{1}{4}$$

Upon factoring to quadratic form, we obtain:

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1}{4}\left\langle (x-a)^2 + (y-b)^2 \right\rangle + \frac{1}{4}$$

and hence

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1}{3}$$

in agreement with classical theory.

4.2. **The Cantor Middle-Thirds Set.** The simplest non-trivial fractal SCS is the middle-thirds Cantor Set $C_1(0)$, which can be represented as the attractor of the IFS $\mathcal{F} = \{[0,1];\ f_1, f_2\}$ with contraction mappings $f_1(x) = \frac{1}{3}$ and $f_2(x) = \frac{1}{3}x + \frac{2}{3}$.

To compute the expected distance of a point in $C_1(0)$ from the origin—that is, the $B(1, C_1(0))$ box integral—requires evaluation of the expectation of $|x|$ over the domain $C_1(0)$. Noting that $x \geq 0$, substitution of the function $F(x) = x$ and the contraction mappings f_1 and f_2 into 3.1 yields:

$$\begin{aligned}
\langle x \rangle_{x\in C} &= \frac{1}{2}\left(\langle f_1(x)\rangle + \langle f_2(x)\rangle\right) \\
&= \frac{1}{2}\left(\left\langle \frac{1}{3}x \right\rangle + \left\langle \frac{1}{3}x + \frac{2}{3}\right\rangle\right) = \frac{1}{3}\left\langle \frac{1}{3}x\right\rangle + \frac{1}{3}
\end{aligned}$$

upon exploiting the linearity of the expectation. Rearranging of the terms immediately yields:

$$\langle x \rangle_{x\in C} = \frac{1}{2}.$$

Similarly, for the expectation of the second-order moment of distance from the origin—that is, the $B(2, C_1(0))$ box integral—consider the expectation of x^2 over the domain $C_1(0)$. The functional relation 3.1 now leads to:

$$\begin{aligned}
\left\langle x^2 \right\rangle_{x\in C} &= \frac{1}{2}\left(\left\langle (f_1(x))^2\right\rangle + \left\langle (f_2(x))^2\right\rangle\right) \\
&= \frac{1}{2}\left(\left\langle \left(\frac{1}{3}x\right)^2\right\rangle + \left\langle \left(\frac{1}{3}x + \frac{2}{3}\right)^2\right\rangle\right) \\
&= \frac{1}{9}\left\langle x^2\right\rangle + \frac{2}{9}\langle x\rangle + \frac{2}{9}.
\end{aligned}$$

At this stage the nominal next step would be to compute the expectation $\langle x\rangle$ by restarting the calculation for this new function $F(x) = x$ that appeared in the calculation. Having thus computed $\langle x\rangle = 1/2$ in the preceding calculation, substitution of this expectation into the right-hand-side and rearrangement of terms leads to:

$$\langle x \rangle_{x\in C} = \frac{3}{8}.$$

Next, consider the expected square of separation between two points chosen randomly from $C_1(0)$—that is, the $\Delta(2, C_1(0)$ box integral. The relevant expectation is now that of $(x-y)^2$ over the domain $C_1(0)$. Applying the functional relation 3.1 produces:

$$\begin{aligned}
\left\langle (x-y)^2\right\rangle_{x,y\in C} &= \frac{1}{4}\left(\left\langle (f_1(x) - f_1(y))^2\right\rangle + \left\langle (f_1(x) - f_2(y))^2\right\rangle\right. \\
&\quad \left. + \left\langle (f_2(x) - f_1(y))^2\right\rangle + \left\langle (f_2(x) - f_2(y))^2\right\rangle\right) \\
&= \frac{1}{4}\left(\left\langle \left(\frac{1}{3}x - \frac{1}{3}y\right)^2\right\rangle + \left\langle \left(\frac{1}{3}x - \left(\frac{1}{3}y + \frac{2}{3}\right)\right)^2\right\rangle\right. \\
&\quad \left. + \left\langle \left(\frac{1}{3}x + \frac{2}{3} - \frac{1}{3}y\right)^2\right\rangle + \left\langle \left(\frac{1}{3}x + \frac{2}{3} - \left(\frac{1}{3}y + \frac{2}{3}\right)\right)^2\right\rangle\right) \\
&= \frac{1}{9}\left(\left\langle x^2\right\rangle + \left\langle y^2\right\rangle - 2\langle xy\rangle\right) + \frac{2}{9}.
\end{aligned}$$

At this point the calculation can be iterated twice to cover the new expectations that have appeared; namely, $\langle x^2 \rangle$ and $\langle xy \rangle$ (using the one- and two-variable forms of the functional expectation, respectively). Note that the expectations are invariant under exchange of x and y and so separate computation of the expectation $\langle y^2 \rangle$ is not required. Alternatively, the first term on the right-hand-side can be folded back into quadratic form, leaving:

$$\left\langle (x-y)^2 \right\rangle = \frac{1}{9} \left\langle (x-y)^2 \right\rangle + \frac{2}{9}$$

and hence

$$\left\langle (x-y)^2 \right\rangle_{x,y \in C} = \frac{1}{4}$$

These results are in agreement with prior symbolic calculations in the SCS framework [5].

4.3. **The Isosceles Sierpiński Triangle.** The subsequent examples lie outside the scope of the SCS framework. While exact theoretical verification is no longer available, the symbolic results of the following examples are supported by limited numerical computations as discussed in Section 5. The first calculation of an entirely new result considers the isosceles Sierpiński Triangle; a set which possesses the nice feature of expectations being identical in each coordinate.

The isosceles Sierpiński Triangle (of unit side length) can be represented as the attractor of the IFS $\mathcal{F} = \left\{[0,1]^2;\ f_1, f_2, f_3\right\}$ where $f_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$ and $f_3(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y + \frac{1}{2}\right)$. Evaluation of the B_2 box integral (corresponding to the expectation $\sqrt{(x^2+y^2)^2)}$ using the functional equation 3.1 leads to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{2}{9} \left(\langle x \rangle + \langle y \rangle + 1\right)$$

The simpler expectations can be re-evaluated as $\langle x \rangle = \frac{1}{3}$ and $\langle y \rangle = \frac{1}{3}$ leading to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{10}{27}.$$

Similarly, evaluation of the Δ_2 box integral (corresponding to the expectation $\sqrt{((x-x')^2 + (y-y')^2)^2)}$ using the functional equation 3.1 leads directly to:

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1}{4} \left\langle (x-a)^2 + (y-b)^2 \right\rangle + \frac{2}{9}$$

and hence

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{8}{27}.$$

4.4. **The Equilateral Sierpiński Triangle.** The next example considers a set for which expectations are no longer identical in each coordinate.

The equilateral Sierpiński Triangle (of unit side length) can be represented as the attractor of the IFS $\mathcal{F} = \left\{[0,1]^2;\ f_1, f_2, f_3\right\}$ where $f_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $f_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)$ and $f_3(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$. Evaluation of the B_2 box integral (corresponding to the expectation of $\sqrt{(x^2+y^2)^2}$) using the functional equation 3.1 leads to:

$$\begin{aligned}
&\left\langle x^2 + y^2 \right\rangle \\
&= \frac{1}{3}\left\langle F\left(\frac{1}{2}x, \frac{1}{2}y\right)\right\rangle + \frac{1}{3}\left\langle F\left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)\right\rangle + \frac{1}{3}\left\langle F\left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)\right\rangle \\
&= \frac{1}{3}\left\langle \frac{x^2}{2^2} + \frac{y^2}{2^2}\right\rangle + \frac{1}{3}\left\langle \left(\frac{x}{2} + \frac{1}{4}\right)^2 + \left(\frac{y}{2} + \frac{\sqrt{3}}{4}\right)^2\right\rangle + \frac{1}{3}\left\langle \left(\frac{x}{2} + \frac{1}{2}\right)^2 + \frac{y^2}{2^2}\right\rangle
\end{aligned}$$

Rearranging both sides and exploiting linearity of the expectation yields:

$$\left\langle x^2 + y^2 \right\rangle = \frac{1}{3^2}\left(3\langle x\rangle + \sqrt{3}\langle y\rangle + 2\right).$$

The simpler expectations re-evaluate as: $\langle x\rangle = \frac{1}{2}$ and $\langle y\rangle = \frac{4}{9}$ leading to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{4}{9}.$$

Similarly, evaluation of the Δ_2 box integral (corresponding to the expectation of $\sqrt{((x-a)^2 + (y-b)^2)^2}$) using the functional equation 3.1 leads directly to (after exploiting $(a-b)^2 + (a+b)^2 = 2(a^2+b^2)$:

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1}{4}\left\langle (x-a)^2 + (y-b)^2 \right\rangle + \frac{1}{6}$$

and hence

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{2}{9}.$$

These expectations for B and Δ box integrals are illustrated in Figure 2.

4.5. **The von Köch Curve.** The unit von Köch Curve (three copies of which can be arranged to form the celebrated von Köch Snowflake) can be

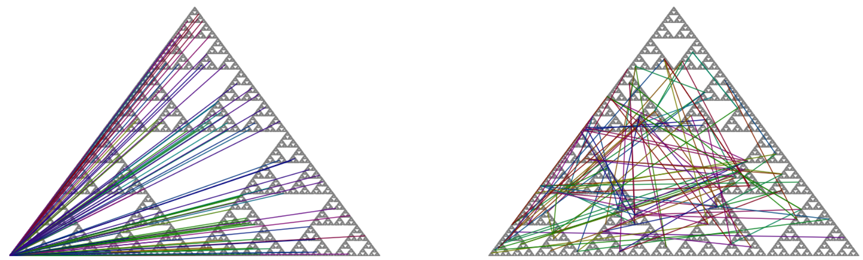

FIGURE 2. Representations of the box integrals $B_2 = \frac{4}{9}$ (left) and $\Delta_2 = \frac{2}{9}$ (right) over the unit equilateral Sierpiński triangle. Each set is overlayed with 100 random uniformly sampled points (or point-pairs), with distances indicated by colour—as distance increases, the displayed colour shifts further towards the violet end of the visible spectrum.

represented[1] as the attractor of the IFS $\mathcal{F} = \left\{[0,1]^2;\ f_1, f_2, f_3, f_4\right\}$ where

$$f_1(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right)$$
$$f_2(x,y) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right)$$
$$f_3(x,y) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6}\right)$$
$$f_4(x,y) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right)$$

Evaluation of the B_2 box integral (corresponding to the expectation $\sqrt{(x^2+y^2)^2}$ using the functional equation 3.1 leads to:

$$\left\langle x^2 + y^2 \right\rangle = \frac{5}{32}\left\langle x \right\rangle + \frac{\sqrt{3}}{32}\left\langle y \right\rangle + \frac{1}{4}$$

The simpler expectations can be re-evaluated as $\langle x \rangle = \frac{1}{2}$ and $\langle y \rangle = \frac{\sqrt{3}}{18}$ leading to:

$$B_2 = \frac{1}{3}.$$

[1]The von Köch Curve also has a representation as the attractor of an IFS with only two affine mappings, if the associated metric space on which the mappings are defined is changed to the complex plane (in place of $[0,1]^2$)—see for instance [43].

Similarly, evaluation of the Δ_2 box integral (corresponding to the expectation $\sqrt{((x-x')^2+(y-y')^2)^2}$ using functional equation 3.1 leads to:

$$\left\langle (x-a)^2+(y-b)^2\right\rangle = \frac{1}{9}\left(\left\langle x^2\right\rangle + \left\langle y^2\right\rangle + \left\langle a^2\right\rangle + \left\langle b^2\right\rangle\right) - \frac{7}{144}\left(\langle x\rangle + \langle a\rangle\right) + \frac{\sqrt{3}}{144}\left(\langle y\rangle + \langle b\rangle\right) - \frac{1}{8}\left(\langle xa\rangle + \langle yb\rangle\right) + \frac{11}{72}$$

By symmetry,

$$\left\langle (x-a)^2+(y-b)^2\right\rangle = \frac{2}{9}\left(\left\langle x^2\right\rangle + \left\langle y^2\right\rangle\right) - \frac{7}{72}\left(\langle x\rangle\right) + \frac{\sqrt{3}}{72}\left(\langle y\rangle\right) - \frac{1}{8}\left(\langle xa\rangle + \langle yb\rangle\right) + \frac{11}{72}.$$

The simpler expectations can be re-evaluated as:

$$\langle x\rangle = \frac{1}{2}, \quad \left\langle x^2\right\rangle = \frac{19}{60}, \quad \langle xa\rangle = \frac{1}{4},$$
$$\langle y\rangle = \frac{\sqrt{3}}{18}, \quad \left\langle y^2\right\rangle = \frac{1}{60}, \quad \langle yb\rangle = \frac{1}{108}$$

leading to:

$$\Delta_2 = \frac{4}{27}.$$

4.6. **The Barnsley Fern.** The final example illustrates how current theory of IFS fractal expectations allows certain expectations (in particular, second-order box integrals) to be computed over fractal sets generated from real-world image data via the Collage Theorem, a powerful tool by which an affine IFS may be generated with attractor lying within a pre-determined tolerance (in the sense of Hausdorff distance) of the input image (see Section 1.10). The well-known Barnsley Fern is taken as an illustrative example, though the algorithm is applicable to any image data that can be so-represented.

The Barnsley Fern can be represented as the attractor of the IFS $\mathcal{F} = \left\{[0,1]^2;\ f_1, f_2, f_3, f_4\right\}$ where

$$f_1(x,y) = (0, 0.16y)$$
$$f_2(x,y) = (0.85x + 0.04y, -0.04x + 0.85y + 1.6)$$
$$f_3(x,y) = (0.20x - 0.26y, 0.23x + 0.22y + 1.6)$$
$$f_4(x,y) = (-0.15x + 0.28y, 0.26x + 0.24y + 0.44)$$

Evaluation of the B_2 box integral (corresponding to the expectation $\sqrt{(x^2+y^2)^2}$) using the functional equation 3.1 leads to:

$$\left\langle x^2+y^2\right\rangle = \frac{10267}{40000}\left\langle x^2\right\rangle + \frac{10017}{40000}\left\langle y^2\right\rangle + \frac{523}{2500}\langle x\rangle + \frac{568}{625}\langle y\rangle - \frac{27}{10000}\langle xy\rangle + \frac{3321}{2500}$$

The simpler expectations can be re-evaluated by the functional equation, leading to the system of equations:

$$\langle x\rangle = \frac{13}{50}\langle x\rangle + \frac{3}{200}\langle y\rangle, \qquad \langle y\rangle = \frac{9}{80}\langle x\rangle + \frac{147}{400}\langle y\rangle + \frac{91}{100},$$

$$\langle x^2\rangle = \frac{4523}{20000}\langle x^2\rangle + \frac{369}{10000}\langle y^2\rangle - \frac{211}{5000}\langle xy\rangle,$$

$$\langle y^2\rangle = \frac{1221}{40000}\langle x^2\rangle + \frac{8541}{40000}\langle y^2\rangle + \frac{79}{2000}\langle xy\rangle + \frac{523}{2500}\langle x\rangle + \frac{568}{625}\langle y\rangle + \frac{3321}{2500},$$

$$\langle xy\rangle = -\frac{7}{5000}\langle x^2\rangle + \frac{11}{1000}\langle y^2\rangle + \frac{7979}{40000}\langle xy\rangle + \frac{919}{2000}\langle x\rangle - \frac{143}{2500}\langle y\rangle.$$

Solving this system yields:

$$\langle x\rangle = \frac{1092}{37309}, \qquad \langle y\rangle = \frac{53872}{37309}, \qquad \langle xy\rangle = -\frac{23413005490249872}{580160660775546421},$$

$$\langle x^2\rangle = \frac{94495707021238368}{580160660775546421}, \qquad \langle y^2\rangle = \frac{1954945096116443536}{580160660775546421}.$$

leading to

$$\langle x^2 + y^2\rangle = \frac{2049440803137681904}{580160660775546421}.$$

Similarly, evaluation of the Δ_2 box integral (corresponding to the expectation $\sqrt{((x-a)^2+(y-b)^2)^2)}$ using the functional equation 3.1 leads to:

$$\langle (x-a)^2 + (y-b)^2\rangle$$
$$= \frac{89}{10000}\langle x\rangle + \frac{4799}{10000}\langle y\rangle + \frac{10267}{20000}\langle x^2\rangle + \frac{10017}{20000}\langle y^2\rangle - \frac{27}{5000}\langle xy\rangle$$
$$- \frac{7239}{40000}\langle xb\rangle - \frac{12841}{80000}\langle xa\rangle - \frac{4329}{16000}\langle yb\rangle + \frac{5003}{5000}$$

The simpler expectations can be re-evaluated by the functional equation, leading to the additional set of equations (taken together with those derived from the B_2 computation):

$$\langle xb\rangle = \frac{1183}{5000}\langle x\rangle + \frac{273}{20000}\langle y\rangle + \frac{441}{80000}\langle yb\rangle + \frac{117}{4000}\langle xa\rangle + \frac{7779}{80000}\langle xb\rangle$$

$$\langle xa\rangle = \frac{9}{40000}\langle yb\rangle + \frac{169}{2500}\langle xa\rangle + \frac{39}{5000}\langle xb\rangle$$

$$\langle yb\rangle = \frac{819}{4000}\langle x\rangle + \frac{13377}{20000}\langle y\rangle + \frac{21609}{160000}\langle yb\rangle + \frac{81}{6400}\langle xa\rangle + \frac{1323}{16000}\langle xb\rangle + \frac{8281}{10000}$$

Solving this system yields:

$$\langle x\rangle = \langle a\rangle = \frac{1092}{37309}, \qquad \langle y\rangle = \langle b\rangle = \frac{53872}{37309},$$

$$\langle x^2\rangle = \langle a^2\rangle = \frac{94495707021238368}{580160660775546421}, \qquad \langle y^2\rangle = \langle b^2\rangle = \frac{1954945096116443536}{580160660775546421},$$

$$\langle xy\rangle = \langle ab\rangle = -\frac{23413005490249872}{580160660775546421}, \qquad \langle xb\rangle = \langle ay\rangle = \frac{58828224}{1391961481},$$

$$\langle xa\rangle = \frac{1192464}{1391961481}, \qquad \langle yb\rangle = \frac{2902192384}{1391961481}.$$

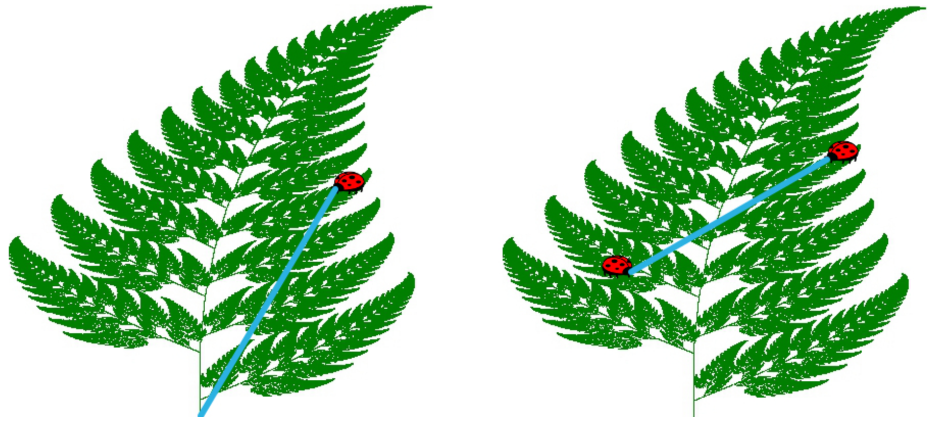

FIGURE 3. Representations of the box integrals $B_2 = \frac{2049440803137681904}{580160660775546421}$ (left) and $\Delta_2 = \frac{1561818604387599983932186}{541130352321871535527225}$ (right) over the Barnsley Fern. The B_2 integral represents the expected square of the distance between the origin (at the base of the fern's stem) and a random point chosen from within the fern (as represented by the ladybird), while the Δ_2 integral represents the expected square of the separation between the two random points chosen from within the fern (as represented by the two ladybirds).

leading to:

$$\left\langle (x-a)^2 + (y-b)^2 \right\rangle = \frac{1561818604387599983932186}{541130352321871535527225}.$$

These expectations for B and Δ box integrals are illustrated in Figure 3. Note that the second-order separation moments can be symbolically evaluated for *any* IFS attractor generated by means of the Collage Theorem on using the same sequence of computations.

5. Numerical Approximation of Box Integrals

5.1. **Monte Carlo Algorithm.** A simple and direct approximation to the expectations of Definition 2.4 can be obtained by truncating the right-hand side of Equation 2.6 after a finite number of steps in the Chaos-Game orbit. This leads to the following Chaos-Game algorithm for computing numerical approximations to expectations over IFS attractors.

Algorithm 5.1 (Chaos-Game Sampling over 1st-order IFS). *Given an arbitrary IFS, the following procedure computes a Monte-Carlo approximation of an expectation defined over the IFS attractor.*

(1) *Select a point x_0 in the attractor (typically $x_0 = \mathbf{0}$ for suitable choice of coordinates).*

(2) *For a fixed number of iterations N, compute an $(N+1)$-step Chaos-Game orbit (x_k) starting at x_0 via successive applications of the IFS mappings to the current point x_k, with the IFS mappings chosen uniformly at random at each step.*
(3) *Evaluate the function F at each sampled point in the Chaos-Game orbit.*
(4) *Compute the approximation to the expectation via truncation of Definition 2.4:*

$$\langle F(x)\rangle_{x\in A} = \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} F(x_k) \approx \frac{1}{N+1}\sum_{k=0}^{N} F(x_k)$$

Alternatively, if an explicit starting point within the attractor is unknown, the fixed-point nature of the attractor may be leveraged by selecting an arbitrary starting point and discarding a finite number of initial Chaos-Game orbit points (say 100), after which the orbit will be trapped within an acceptable distance of the attractor [6]. Table 1 shows a selection of approximations to various separation expectations over the fractal sets in Section 4 obtained via the Chaos Game algorithm.

While the Chaos Game algorithm is easy to implement and highly parallelisable, only a few digits can be accessed in a reasonable time-frame (a limitation suffered by all Monte-Carlo sampling algorithms).

In addition to Chaos Game sampling, a generalization of an algorithm employed for numerics over SCS fractals [5] lead to an approach analogous to the original expectation definition, in which points were systematically sampled over a level-q pre-fractal set by considering all q-fold compositions of mappings chosen from the m IFS contractions and sampling of one point from each of the resulting m^q image sets. More precisely:

Algorithm 5.2 (**Systematic Sampling over qth-order IFS).** *Given an arbitrary IFS, the following procedure computes an approximation of an expectation defined over the IFS attractor via considerations of the level-q pre-fractal set.*

(1) *For a fixed pre-fractal depth q, pre-compute the qth-level IFS by taking all m^q possible $q-fold$ compositions of the IFS mappings.*
(2) *Select a point x_0 in the attractor (typically $x_0 = \mathbf{0}$ for suitable choice of coordinates).*
(3) *Apply each qth-level IFS mapping to x_0 to generate the sequence (x_k) of points systematically sampled across the qth-level pre-fractal of the attractor.*
(4) *Evaluate the function F at each sampled point in the Chaos-Game orbit.*

Fractal	Function	Iterations	Value	Exact
Full Isosceles Triangle	$\Delta(1)$	10^9	0.414	
Full Isosceles Triangle	$\Delta(2)$	10^8	0.222	
Cantor Middle-Thirds Set	$B(1)$	2×10^5	0.500	0.5
Cantor Middle-Thirds Set	$B(2)$	2×10^5	0.375	0.375
Isosceles Sierpiński Triangle	$B(1)$	10^8	0.566	
Isosceles Sierpiński Triangle	$B(2)$	10^5	0.374	
Isosceles Sierpiński Triangle	$B(2)$	10^8	0.373	$0.\overline{370}$
Isosceles Sierpiński Triangle	$\Delta(1)$	10^9	0.481	
Isosceles Sierpiński Triangle	$\Delta(2)$	10^8	0.297	$0.\overline{296}$
Equilateral Sierpiński Triangle	$B(1)$	10^3	0.612	
Equilateral Sierpiński Triangle	$B(1)$	10^8	0.621	
Equilateral Sierpiński Triangle	$B(1)$	10^9	0.621	
Equilateral Sierpiński Triangle	$B(2)$	10^5	0.447	$0.\overline{4}$
Equilateral Sierpiński Triangle	$B(2)$	10^8	0.448	$0.\overline{4}$
Equilateral Sierpiński Triangle	$\Delta(1)$	10^3	0.436	
Equilateral Sierpiński Triangle	$\Delta(1)$	10^5	0.420	
Equilateral Sierpiński Triangle	$\Delta(1)$	10^{10}	0.423	
Equilateral Sierpiński Triangle	$\Delta(2)$	10^5	0.222	$0.\overline{2}$
von Köch Curve	$B(2)$	10^6	0.333	$0.\overline{3}$
von Köch Curve	$\Delta(2)$	10^6	0.148	$0.\overline{148}$
Barnsley Fern	$B(2)$	10^6	3.534	3.532540...
Barnsley Fern	$\Delta(2)$	10^6	2.886	2.886215...

TABLE 1. A selection of approximations to various separation expectations over the fractal sets in Section 4 obtained via the Chaos Game algorithm. The exact values were computed using Algorithm 4.1 as exhibited in in Section 4. Note that in the Δ calculations, 2 points were sampled per iteration. All sets are of unit length (with regards to the level-0 pre-fractal), excepting the Barnsley Fern.

(5) *Compute the approximation to the expectation via truncation of Definition 2.4:*

$$\langle F(x) \rangle_{x \in A} \approx \frac{1}{N+1} \sum_{k=0}^{N} F(x_k)$$

For attractors of affine IFSs, the first step of Algorithm 5.2 becomes an easily-computed exercise in matrix algebra. This approach was able to extract one extra correct digit in the same time-frame as the Chaos Game Algorithm 5.1, at the cost of losing parallelisability. Unfortunately the approach was unable to produce high-precision estimates, since the special additive structure of admissible SCS columns can no longer be exploited in the general setting. This approach does have the advantage of leading to bounds on the expectations through careful choice of the starting point x_0, similar to the related SCS algorithm of [5].

Further progress in the development of algorithms for high-precision numerical estimates of expectations over IFS attractors is discussed below.

6. Future Directions

In the general IFS setting, promising progress has been made regarding extreme-precision[1] evaluation of certain unresolved fractal box integrals in collaboration with Nathan Clisby, who has applied a generalised Richardson extrapolation technique (introduced in [41]; see also [42], [23] and [40]) to the problem of computing fractal expectations numerically.

In the first attempts at extreme-precision evaluation of fractal expectations over IFS attractors, the box integral has been considered as an analytic function of parameter k, where expectations are approximated by taking the finite mean of points uniformly sampled from the kth pre-fractal (or equivalently, uniformly sampled from Chaos Game orbits of length m^k), assuming the error behaves as a power series in $\frac{1}{k}$. Combining Richardson's deferred approach to the limit with sequence extrapolation techniques has enabled the $B_{s=1}$ integral over the equilateral unit Sierpiński triangle to be evaluated to the following 112 digits, computed in 4000 CPU-hours:

$$0.6180082171582247074177418624555167834492481641438960879796572765289499278172412596284649585736706991061075618 07$$

This computationally-intensive result was non-trivial, only being possible thanks to some subtle ideas from Nathan Clisby. These ideas originated from fruitful discussions concerning the numerical study of characteristics of self-avoiding random walks, via techniques that can efficiently sample such walks uniformly at random (see for example [24]).

Currently we are combining this result with the PSLQ integer relation detection algorithm (see [31], [32]) in an attempt to discover closed forms for such odd-order moments that can guide further theoretical developments. In addition, we are aiming to improve the numerical algorithm by employing an analogue of the Bulirsch-Stöer method, in which the expectations are fit to appropriate rational functions of k.

Our work in fractal expectations has also led to renewed interest in fractal quadrature techniques that may also be of use in obtaining precise numerics. In particular, the recent work of Dereich and Müller-Gronbach [27], which cites our work in [5], provides deterministic and random quadrature rules for self-similar probability distributions that perform asymptotically optimally.

Once the techniques herein are refined for deterministic fractal sets (beyond affine IFSs to all attractors of hyperbolic iterated function systems),

[1]'High-precision' being taken to mean at least 20 digits, whereas previous works have used the phrase 'extreme precision' to mean at least 100 digits, or certainly enough to discover identities via integer-relation detection.

the next logical progression would be to move beyond deterministic fractals and consider random fractal sets. In particular, the gap between deterministic and random fractals are bridged by V-variable fractals and superfractals (collections of V-variable random fractals), which can be described in a more generalised IFS framework [7]. Just as SCS fractal expectations were generalised to IFS fractal expectations, it seems only natural that IFS fractal expectations may in turn be generalised to encompass V-variable fractals and superfractals.

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