

Effective Laguerre asymptotics

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Abstract

It is known that the generalized Laguerre polynomials can enjoy subexponential growth for large primary index. In particular, for certain fixed parameter pairs (a, z) one has the large- n asymptotic behavior

$$L_n^{(-a)}(-z) \sim C(a, z)n^{-a/2-1/4}e^{2\sqrt{nz}}.$$

We introduce a computationally motivated contour integral that allows efficient numerical Laguerre evaluations, yet also leads to the complete asymptotic series over the full parameter domain of subexponential behavior. We present a fast algorithm for symbolic generation of the rather formidable expansion coefficients. Along the way we address the difficult problem of establishing effective (i.e. rigorous and explicit) error bounds on the general expansion. A primary tool for these developments is an “exp-arc” method giving a natural bridge between converging series and effective asymptotics.

1 The challenge of “effectiveness”

For primary integer indices $n = 0, 1, 2, \dots$ we define the Laguerre polynomial thus:

$$L_n^{(-a)}(-z) := \sum_{k=0}^n \binom{n-a}{n-k} \frac{z^k}{k!}. \quad (1)$$

Our use of negated parameters $-a, -z$ is intentional, for convenience in our analysis and in connection with related research, as we later explain. We shall work on the difficult problem of establishing asymptotics *with* effective error bounds for the two-parameter domain

$$\mathbb{D} := \{(a, z) \in \mathbb{C} \times \mathbb{C} : z \notin (-\infty, 0]\}.$$

That is, a is any complex number, while z is a complex number not on the negative-closed cut $(-\infty, 0]$.

Herein, we say a function $f(n)$ is of *subexponential* growth if $\log f(n) \sim Cn^D$ as $n \rightarrow \infty$ for positive constants C, D with $D < 1$. It will turn out that \mathbb{D} is the precise domain of subexponential growth of $L_n^{(-a)}(-z)$ as $n \rightarrow \infty$, with (a, z) fixed. The reason for the negative-cut exclusion on z is simple: For z negative real, the Laguerre polynomial exhibits oscillatory behavior in large n , and is not of subexponential growth. Note also, from the definition (1), that

$$L_n^{(-a)}(0) = \binom{n-a}{n}, \quad (2)$$

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which covers the case $z = 0$; again, not subexponential growth.

For fixed $(a, z) \in \mathbb{D}$ we shall have the large- n behavior (here and beyond we define $m := n + 1$ which will turn out to be a natural reassignment):

$$L_n^{(-a)}(-z) \sim S_n(a, z) \left(1 + O\left(\frac{1}{m^{1/2}}\right) \right), \quad (3)$$

where the subexponential term S_n is

$$S_n(a, z) := \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{mz}}}{z^{1/4-a/2} m^{1/4+a/2}}. \quad (4)$$

In such expressions, $\Re(\sqrt{mz})$ denotes $\sqrt{m|z|} \cos(\theta/2)$ where $\theta := \arg(z) \in (-\pi, \pi]$ (we hereby impose $\arg(-1) := \pi$), and so for $(a, z) \in \mathbb{D}$ the expression (4) involves genuinely diverging growth in n .

1.1 Research motives

There are many interdisciplinary applications of Laguerre asymptotics. The Laguerre functions appear standardly in the quantum theory of the hydrogen atom [48, Ch. 4] and in certain exactly solvable three-body problems of chemical physics [13][14]. The so-called WKB phase of a quantum eigenstate can, in such cases and for high quantum numbers, be calculated via Laguerre asymptotics. The Hermite polynomials $H_n(x)$ —closely related to the Laguerre polynomials $L_n^{(\pm 1/2)}(x^2)$ —also figure in quantum analyses; e.g. one may derive Hermite asymptotics from Laguerre asymptotics. In this context there is the fascinating López–Temme representation of *any* $L_n^{(-a)}(-z)$ as a finite superposition of Hermite evaluations [24].

Our own research motive for providing effective asymptotics involves not the Laguerre–Hermite connection, rather it begins with a beautiful link to the incomplete gamma function, namely [19]

$$\Gamma(a, z) = z^a e^{-z} \frac{1}{z + \frac{1-a}{1 + \frac{1}{z + \frac{2-a}{1 + \dots}}}} = z^a e^{-z} \sum_{n=0}^{\infty} \frac{(1-a)_n}{(n+1)!} \frac{1}{L_n^{(-a)}(-z) L_{n+1}^{(-a)}(-z)}, \quad (5)$$

where $(c)_n := c(c+1)\cdots(c+n-1)$ is the *Pochhammer symbol*. This series is valid whenever none of the Laguerre denominators has a zero. Thus an interesting sidelight is the research problem of establishing zero-free regions for Laguerre polynomials (see our Open problems section).

The problem, then, of general Laguerre asymptotics arises because there is a way to write Riemann zeta-function evaluations $\zeta(s)$ in terms of the incomplete gamma function, as was known to Riemann himself [42, p. 22]. But when s has large imaginary height, one becomes interested in the incomplete gamma's corresponding complex parameters (a, z) also of possibly large imaginary height [11][12]. For example, in the art of prime-number counting, say for primes $< 10^{20}$, one might need to know, for example, $\Gamma(3/4 + 10^{10}i, z)$, for various z also of large imaginary height, to good precision, in order to evaluate $\zeta(s)$ high up on the line $\Re(s) = 3/2$. The point is, exact computations on prime numbers must, of course, employ rigor—hence the need for effective error bounds.

Independent of Riemann- ζ considerations is the following issue: Continued-fraction theory is to this very day incomplete in a distinct sense. The vast majority of available convergence theorems are for Stieltjes, or S-fractions. Now the continued fraction in (5) is an S-fraction only when the a parameter is real [25, p.138]. Due to the relative paucity of convergence theorems outside the S-fraction class, one encounters great difficulty in estimating the convergence rate for arbitrary $\Gamma(a, z)$; this is what led to our focus on the Laguerre asymptotics. Incidentally, we are aware that subexponential convergence results *might* be attainable via the complicated and profound work of Jacobsen and Thron [21] on oval convergence regions. In any case, our effective-Laguerre approach proves subexponential convergence of $\Gamma(a, z)$ for any complex pairs $(a, z) \in \mathbb{D}$; moreover, this is done with effective constant factors. Separate research on these superexponential convergence issues for continued fractions is underway.

1.2 Historical results on Laguerre asymptotics

Laguerre asymptotics have long been established for certain restricted domains, and usually with noneffective asymptotics.¹ For example, in 1909 Fejér established that for z on the open cut $(-\infty, 0)$ and any real a , one has [38, Theorem 8.22.1]:

$$L_n^{(-a)}(-z) = \frac{e^{-z/2}}{\sqrt{\pi}(-z)^{1/4-a/2} m^{1/4+a/2}} \cos(2\sqrt{-mz} + a\pi/2 - \pi/4) + O(m^{-a/2-3/4}), \quad (6)$$

where we again use index $m := n + 1$, which slightly alters the coefficients in such classical expansions, said coefficients being for powers $n^{-k/2}$ but we are now using powers $m^{-k/2}$. By 1921 Perron [36] had generalized the Fejér series to arbitrary orders, then for $z \notin (-\infty, 0]$ established a series consistent with (3) and (4), in essentially the form: [38, Theorem 8.22.3]

$$L_n^{(-a)}(-z) = S_n(a, z) \left(\sum_{k=0}^{N-1} \frac{C_k}{m^{k/2}} + O(m^{-N/2}) \right), \quad (7)$$

although this was for *real* a and so not for our general parameter domain \mathbb{D} . Note that $C_0 = 1$, consistent with our (3) and (4); however, one should take care that because we are using index $m := n + 1$, the coefficients C_k in the above formula differ slightly from the historical ones.

A modern literature treatment that is again consistent with the heuristic (3)–(4), is given by Winitzki in [47], where one invokes a formal generating function to yield a contour integral for $L_n^{(-a)}(-z)$. Then a stationary-phase approach yields the correct first-asymptotic term, at least for certain subregions of \mathbb{D} . Winitzki’s treatment is both elegant and nonrigorous; there is no explicit estimate given on the $O(1/\sqrt{m})$ correction in (3).

There is an interesting anecdote that reveals the difficulty inherent in Laguerre asymptotics. Namely, W. Van Assche in a fine 1985 paper [43] used the expansion (7) for work on zero-distributions, only to find by 2001 that the C_1 term in that 1985 paper had been calculated incorrectly. The amended series is given in his correction note [44] as

$$L_n^{(-a)}(-z) = \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{nz}}}{z^{1/4-a/2} n^{1/4+a/2}} \cdot \left(1 + \left(\frac{3 - 12a^2 + 24(1-a)z + 4z^2}{48\sqrt{z}} \right) \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right) \right),$$

or in our own notation with $m := n + 1$,

$$= S_n(a, z) \left(1 + \left(\frac{3 - 12a^2 - 24(1+a)z + 4z^2}{48\sqrt{z}} \right) \frac{1}{\sqrt{m}} + O\left(\frac{1}{m}\right) \right). \quad (8)$$

Note the slight alteration used to obtain our C_1/\sqrt{m} term. Van Assche credits T. Müller and F. Olver for aid in working out the correct $O(1/\sqrt{n})$ component.² This story suggests that even a low-order asymptotic development is nontrivial.

The great classical analysts certainly knew in principle how to establish effective error bounds. The excellent treatment of effectiveness for Laplace’s method of steepest descent in [28] is a shining example. Also illuminating is Olver’s paper [27], which explains effective bounding and shows how unwieldy rigorous bounds can be obtained. However, efficient algorithms for generating *explicit* effective big-O constants have only become practicable in recent times, when computational machinery is prevalent.

2 Contour representation

In this section we develop an efficient—both numerically and analytically—contour-integral representation for $L_n^{(-a)}(-z)$.

¹Some researchers use the term “realistic error bound” for big- O terms that have explicit structure. We prefer “effective bound,” and when an expansion is bestowed with such a bound, we may say “effective expansion.”

²Accordingly, we hereby name the polynomial $C_1(a, z)$ the Perron–van Assche–Müller–Olver, or “PAMO” coefficient.

2.1 Development of a “keyhole” contour

A well-known integral [38, §5.4] has contour, Γ , encircling $s = 1$ and avoiding the branch cut $(-\infty, 0]$:

$$L_n^{(-a)}(-z) = \frac{e^{-z}}{2\pi i} \oint_{\Gamma} s^{-1-a} \left(1 - \frac{1}{s}\right)^{-n-1} e^{zs} ds. \quad (9)$$

This representation holds for all pairs $(a, z) \in \mathbb{C} \times \mathbb{C}$, n a non-negative integer.³

However—and this is important—we found via experimental mathematical techniques, e.g. extreme-precision evaluations, that a specific kind of contour allows very accurate, efficient, and well behaved *numerical* Laguerre evaluations. It is not completely understood why the “keyhole” contour we are about to define does so well in numerical Laguerre evaluations; we do know that this new contour consistently provides better numerics than, say, a simple circle surrounding $s = 1$. It may be simply the internal quirks of various modern numerical integrators at work, or it may be the smooth phase behavior along our keyhole’s perimeter.

For the desired contour, take $z \neq 0$, $m := n + 1$ and assume $r := \sqrt{m/z}$ has $|r| > 1/2$. Then, use a circular contour centered at $s = 1/2$ with radius $|r|$. This contour will encompass $s = 1$, so the remaining requirement is to avoid the cut $s \in (-\infty, 0]$ as can be done by cutting out a “wedge” from the negative-real arc of the circle, with apex at $s = 1/2$. We tried such schemes with high-precision integration, to settle finally on the contour of Figure 1, where the aforementioned wedge has evolved to a “keyhole” pattern consisting of cut-run D_1 and small, origin-centered circle E_1 of radius $1/2$.

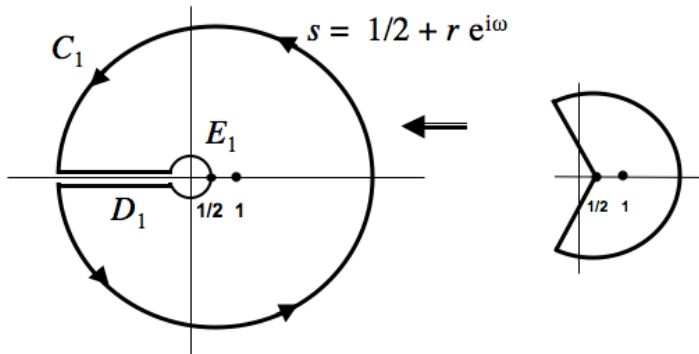


Figure 1: A numerically efficient “keyhole contour” for Laguerre evaluations $L_n^{(-a)}(-z)$, valid for all complex a, z with $z \neq 0$ and $n + 1 > |z|/4$. Wedges with center- $1/2$ —as pictured at right—are experimentally accurate, leading to a “keyhole” deformation avoiding the cut $s \in (-\infty, 0]$. It turns that for any $(a, z) \in \mathbb{D}$, the main arc C_1 gives the predominant contribution for large n , the D_1, E_1 components being subexponentially minuscule.

So, adopting constraints and nomenclature:

$$z \neq 0, \quad \theta := \arg(z), \quad \omega_{\pm} := \pm\pi + \theta/2,$$

$$m := n + 1, \quad r := \sqrt{m/z} := \sqrt{m/|z|} e^{-i\theta/2}, \quad R := |r| > 1/2,$$

but no other constraints, we have the following representation:

$$L_n^{(-a)}(-z) = c_1 + d_1 + e_1, \quad (10)$$

³The contour representation (9) can easily be continued to non-integer n , with care taken on the $(-n - 1)$ -th power, but our present treatment will only use nonnegative integer n .

where c_1, d_1, e_1 are the respective contributions from contour C_1 , cut-discontinuity D_1 , and contour E_1 from Figure 1. Exact formulae for said contributions are

$$c_1 = \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{\omega_-}^{\omega_+} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz} \cos \omega} d\omega, \quad (11)$$

$$d_1 = \frac{e^{-z}}{\pi} \sin(\pi a) \int_{1/2}^{R-1/2} T^{-1-a} \left(1 + \frac{1}{T}\right)^{-m} e^{-zT} dT, \quad (12)$$

$$e_1 = -\frac{e^{-z}}{4\pi} \int_{-\pi}^{\pi} (2e^{-i\omega})^{1+a} (1 - 2e^{-i\omega})^{-m} e^{i\omega + \frac{z}{2} e^{i\omega}} d\omega, \quad (13)$$

and when $m > \Re(a)$ we may write this last contribution by shrinking down the radius-1/2 contour segment to embrace the cut $(-1/2, 0]$, as

$$e_1 = \frac{e^{-z}}{\pi} \sin(\pi a) \int_0^{1/2} T^{-1-a} \left(1 + \frac{1}{T}\right)^{-m} e^{-zT} dT. \quad (14)$$

For the c_1 contribution above, we have used the function \mathcal{H}_m defined by

$$\mathcal{H}_m(a, z, v) := v^a \left(1 + \frac{v}{2r}\right)^{-1-a} \left[F\left(\frac{v}{r}\right)\right]^m, \quad (15)$$

$$F(t) := \left(\frac{1+t/2}{1-t/2}\right) e^{-t}, \quad (16)$$

which for small t can be written $F(t) = 1 + t^3/12 + t^5/80 + \dots = 1 + O(t^3)$.

In any event, e_1 , like d_1 , is subexponentially small relative to c_1 , for large n and $(a, z) \in \mathbb{D}$. It is to be stressed that decomposition (10) holds for all $(a, z) \in \mathbb{C} \times \mathbb{C}, z \neq 0$, as long as $m := n + 1 > |z|/4$. We remind ourselves of (2) for $z = 0$. This means that such contour calculus applies to both oscillatory (Fejér) cases, where z is negative real, and subexponential (Perron) cases for the stated parameters.

2.2 Numerical success on the keyhole contour

The triumvirate c_1, d_1, e_1 of integrals is suitable for accurate Laguerre computations, which computations do show that *integrals d_1, e_1 tend to be subexponentially small* relative to c_1 . This, of course, is our motive for so identifying the contour terms. An example, using the defining series (1) with $(a, z) := (-i, -1 - i)$ is:

$$L_8^{(i)}(1+i) = -\frac{137}{288} + \frac{53}{45}i \approx -0.47569444444444444444 + 1.17777777777777777777i,$$

while the contributions from the C_1, D_1, E_1 segments of the contour of Figure 1 are

$$c_1 \approx -0.44406762576110996056 + 0.81722282272705891241i,$$

$$d_1 \approx -0.03169598827425878852 + 0.36065591639721657587i,$$

$$e_1 \approx 0.00006916959092430464 - 0.00010096134649771051i,$$

with the sum $c_1 + d_1 + e_1$ giving $L_8^{(i)}(1+i)$ to the implied precision.

Remarkably, this “keyhole-contour” approach has the additional, unexpected feature that for some parameter regions the contour evaluation of $L_n^{(a)}(-z)$ is actually *faster* than direct summation of the defining series (1). A typical example is: For the 13-digit evaluation

$$L_{50000}^{(25i/2)}(-30 + 3i) \approx (0.9275136583293 + 1.7406691595239i) \times 10^{1056}$$

the defining series (1) was, in our trials, slower than the integral c_1 .⁴

3 Effective bounds on the contour components

3.1 Theta-calculus

We now introduce a notation useful in effective-error analysis. When two functions enjoy

$$|f(z)| \leq |g(z)|$$

over some relevant domain of z values, we shall say that

$$f(z) = \Theta(g(z)), \quad \text{or} \quad f(z) = \Theta : g(z)$$

on said domain. Thus, the Θ -notation is an effective replacement for big- O notation. The reason for allowing the notation $\Theta : \dots$ is that a long formula g can run arbitrarily to the right of the colon.

Incidentally there is a literature precedent for such a “theta-calculus.” Namely, in computational number theory treatments such as [10] one encounters terms such as, say, $\theta x / \log^2 x$, with a stated constraint $\theta \in [-10, 10]$. This means the term is $O(x / \log^2 x)$ but with effective big- O constant bounded in magnitude by 10. In our nomenclature we would use $\Theta : 10x / \log^2 x$.

3.2 Effective bound for e_1

First we address the integral e_1 as given in relation (13). We need some preliminary lemmas, starting with a collection of polynomial estimates to transcendental functions.

Lemma 1 *The following inequalities hold for the respective conditions on ω :*

$$\log(5 - 4 \cos \omega) \geq \frac{\log 9}{\pi^2} \omega^2, \quad \omega \in [-\pi, \pi];$$

$$\log(1 + \omega) \geq \frac{4}{5} \omega, \quad \omega \in [0, 1/2];$$

$$\arcsin \omega \leq \frac{\pi}{\sqrt{8}} \omega, \quad \omega \in [0, 1/\sqrt{2}];$$

$$\cos \omega \leq 1 - \frac{4}{\pi^2} \omega^2, \quad \omega \in [-\pi/2, \pi/2].$$

Proof. All four are straightforward calculus exercises. **QED**

Lemma 2 *Consider integrals of error-function class, specifically, for nonnegative real parameter μ ,*

$$V_\mu(\alpha, \beta, \gamma) := \int_\gamma^\infty (x^2)^\mu e^{2\alpha x - \beta x^2} dx,$$

⁴Of course, series acceleration as in [4] would give the direct series a “leg up.” Still, it is remarkable that contour integration is competitive. For the record, we compared *Mathematica*’s numerical integration to its own `LaguerreL[]` function.

where each of α, β, γ is real, with $\alpha, \beta > 0$. Then we always have the bound

$$V_0(\alpha, \beta, \gamma) = \Theta : \sqrt{\frac{\pi}{\beta}} e^{\frac{\alpha^2}{\beta}}.$$

If in addition $\gamma > 2\alpha/\beta$, we also have

$$V_\mu(\alpha, \beta, \gamma) = \Theta : \frac{1}{2} \frac{\Gamma(\mu + 1/2)}{(\beta - 2\alpha/\gamma)^{\mu+1/2}};$$

finally, if $\gamma > 4\alpha/\beta$ we have

$$V_\mu(\alpha, \beta, \gamma) = \Theta : 2^{\mu-1/2} \frac{\Gamma(\mu + 1/2)}{\beta^{\mu+1/2}}.$$

Remark: We use the double-exponential $(x^2)^\mu$ because the first case of the theorem allows $\gamma < 0$.

Proof. Completing the exponent's square gives the first bound easily, since

$$V_0 = e^{\alpha^2/\beta} \int_\gamma^\infty e^{-\beta(x-\alpha/\beta)^2} dx = \Theta : \sqrt{\pi/\beta}.$$

For the second bound, it is elementary that for $x \geq \gamma$ one has $2\alpha x - \beta x^2 \leq x^2(2\alpha/\gamma - \beta)$, so that

$$V_\mu \leq \int_0^\infty x^{2\mu} e^{-(\beta-2\alpha/\gamma)x^2} dx$$

and the resulting bound follows. The final bound follows immediately from the previous bound, because $\gamma > 4\alpha/\beta$ implies $\beta - 2\alpha/\gamma > \beta/2$. **QED**

Theorem 1 *The contour contribution e_1 defined by (13), under conditions $(a, z) \in \mathbb{D}$ and m sufficiently large in the explicit sense $m := n+1 > m_0 := |z|/4$, $m > m_1 := 5(|\Re(z)| + (|\Im(a)| + |\Im(z)|/2)^2)$, is bounded as*

$$e_1 = \Theta : e^{-z/2} 2^{\Re(a)+3} \frac{1}{\sqrt{m}}.$$

Moreover, under alternative conditions $(a, z) \in \mathbb{D}$ and $m > m_0$, $m > m_2 := \Re(a)$, $m > m_3 := -\frac{5}{4}|z|\cos\theta$ we have a bound

$$e_1 = \Theta : \frac{e^{-z}}{\pi} \sin(\pi a) \frac{2^{\Re(a)-m}}{m - \Re(a)}$$

Proof. The proof involves careful application of Lemmas 1, 2 to the integral forms (13), (14); for brevity we omit these details. **QED**

3.3 Effective bound for d_1

Again we need an opening lemma:

Lemma 3 *Consider integrals of the incomplete-Bessel class, specifically*

$$W(\alpha, \beta) := \int_0^1 e^{-\alpha x - \frac{\beta}{x}} \frac{dx}{x},$$

where each of α, β is real, with $\beta > 0$. If $\beta > \alpha$ we have a bound

$$W = \Theta : \frac{1}{2} e^{-\beta-\alpha} \sqrt{\frac{\pi}{\beta}}.$$

Proof. We write

$$W(\alpha, \beta) = \int_0^1 e^{\beta x - \alpha x} e^{-\beta x - \frac{\beta}{x}} \frac{dx}{x} \leq e^{\beta - \alpha} \int_0^1 e^{-\beta x - \frac{\beta}{x}} \frac{dx}{x}.$$

This last integral is a modified-Bessel term $K_0(2\beta)$, and has the required bound [2, Lemma 1]. **QED**

Theorem 2 *The contour contribution d_1 defined by (12) can be bounded, under conditions $(a, z) \in \mathbb{D}$ and $m > m_4 := 4|z|$, as*

$$d_1 = \Theta : \frac{e^{-z/2}}{\sqrt{\pi}} m^{|\Re(a)|/2-1/4} |z|^{-|\Re(a)|/2-1/4} \sin(\pi a) e^{-2\sqrt{m|z|} \cos^2 \frac{\theta}{2}}.$$

Proof. Starting from (12) one applies Lemma 3; again for brevity we omit the details. **QED**

3.4 Rigorous estimates on c_1

Having dispensed with d_1, e_1 , we next show the integration limits ω_-, ω_+ on the c_1 contribution can be changed—with a subexponentially small error penalty—to $-\pi/2, \pi/2$ respectively:

Lemma 4 *For any v on the unit circle $\{e^{i\phi} : \phi \in (-\pi, \pi)\}$, any complex a , and any real $R > (1+|a|)/2$ we have*

$$\left| \left(1 + \frac{v}{2R}\right)^{-1-a} \right| \leq \frac{1}{1 - \frac{1+|a|}{2R}}.$$

Proof. Via the binomial theorem,

$$\left| \left(1 + \frac{v}{2R}\right)^{-1-a} \right| \leq 1 + (1+|a|) \frac{1}{2R} + (1+|a|)^2 \left(\frac{1}{2R}\right)^2 + \dots = \frac{1}{1 - \frac{1+|a|}{2R}}.$$

QED

Lemma 5 *Let v be on the unit circle as in Lemma 4, let $R > 1$ be real, and let m be a positive integer. For the function F appearing in (16), we have the bound*

$$\left| F\left(\frac{v}{R}\right)^m \right| \leq e^{\frac{1}{6} \frac{m}{R^3}}.$$

Proof. From the definition (16) we have

$$F(v/R)^m = e^{\frac{1}{12} \frac{m}{R^3} (1+(3/5)/(2R)^2+(3/7)/(2R)^4+\dots)}.$$

For $R := 1$ the infinite sum is no larger than 1.18 and is monotonic decreasing in R . **QED**

Lemma 6 *Let v be on the unit circle as in Lemma 4 and define for nonnegative integer m*

$$K := \left(1 + \frac{v}{2R}\right)^{-1-a} F\left(\frac{v}{R}\right)^m.$$

For the assignments $(a, z) \in \mathbb{C} \times \mathbb{C}, z \neq 0, R := \sqrt{m/|z|} > 1, m > m_5 := |z|(1+|a|+|z|/2)^2$ we have

$$K = \Theta : 2.$$

Proof. From Lemmas 4, 5 we have

$$|K| \leq \frac{1}{1 - \frac{1+|a|}{2(1+|a|+|z|/2)}} e^{\frac{1}{6} \frac{|z|}{1+|a|+|z|/2}}.$$

The fact that the right-hand side is $\Theta(2)$ follows from the observation that the function

$$\frac{1}{1 - \frac{Q}{2Q+x}} e^{\frac{1}{3} \frac{x}{2Q+x}}$$

for $Q \geq 1, x \in [0, \infty)$ is itself $\Theta(2)$. This in turn follows easily on substituting $y := x/(2Q+x)$. Now the function to be bounded is $g(y) := (2/(1+y))e^{y/3}$ on $y \in [0, 1/2]$ — as differentiation settles. **QED**

These lemmas in turn allow us to contract the range on the c_1 contour integral:

Theorem 3 *Decompose c_1 as defined by (11) into two terms,*

$$c_1 := c_0 + c_2,$$

with c_0 involving the integral's range contracted to $[-\pi/2, \pi/2]$, namely

$$c_0 := \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz} \cos \omega} d\omega,$$

Then under conditions $(a, z) \in \mathbb{D}$ and $m > m_5 := |z|(1+|a|+|z|/2)^2$ we have a bound

$$c_2 = \Theta : r^{-a} e^{-z/2} e^{\frac{3}{2}\pi|\Im(a)|}.$$

Proof. It is evident that c_2 is obtained from the definition (11) but with the integral replaced via

$$\int_{\omega_-}^{\omega_+} \rightarrow \left\{ \int_{\omega_-}^{-\pi/2} + \int_{\pi/2}^{\omega_+} \right\}.$$

Over these domains of integration, we have $e^{2\sqrt{mz} \cos \omega} = \Theta(1)$. Thus, Lemmas 4, 5, 6 show

$$c_2 = \Theta : \frac{1}{2\pi} r^{-a} e^{-z/2} \left\{ \int_{\omega_-}^{-\pi/2} + \int_{\pi/2}^{\omega_+} \right\} 2 |e^{-i\omega a}| d\omega.$$

As the total support of the integrals cannot exceed π , the desired c_2 bound follows. **QED**

3.5 Summary of the contour decomposition for $L_n^{(-a)}(-z)$

The above manipulations lead to the main result of the present section, namely a formula that decomposes the Laguerre evaluation, as in

Theorem 4 (Contour decomposition) *Let $(a, z) \in \mathbb{C} \times \mathbb{C}$ be an arbitrary parameter pair with $z \neq 0$ (with $z = 0$ cases resolved exactly by (2)). If $m := n+1 = m_0 > |z|/4$, then the contour decomposition*

$$L_n^{(-a)}(-z) = c_0 + \mathcal{E}$$

holds, with c_0, c_2 as defined in Theorem 3 and $\mathcal{E} := c_2 + d_1 + e_1$. If an appropriate set of conditions on m across Theorems 1, 2, 3 holds, then we can write

$$L_n^{(-a)}(-z) = c_0 + S_n(a, z)\mathcal{E}_1,$$

where \mathcal{E}_1 is subexponentially small, in the sense that for any fixed positive ϵ the large- n behavior is

$$\mathcal{E}_1 = O\left(e^{-(2-\epsilon)\sqrt{m|z|} \cos \frac{\theta}{2}}\right).$$

Moreover, an effective big- O constant is available (the proof exhibits explicit forms).

Proof. The contour calculus holds for all complex pairs (a, z) with $z \neq 0$, $R := \sqrt{m/|z|} > 1/2$, which assures that the point 1 is contained in the contour. It remains to analyze \mathcal{E}_1 . Consider, then, an appropriate union of conditions from the cited theorems, say $(a, z) \in \mathbb{D}$ and $m > \max(m_0, m_1, m_2, m_3, m_4, m_5)$, in which case we have, from Theorems 1, 2, 3 (in that respective order of Θ terms):

$$\begin{aligned} \mathcal{E}_1 &= \Theta : \frac{e^{-z/2}}{\sqrt{\pi}} \sin(\pi a) \frac{2^{1+\Re(a)-m}}{m - \Re(a)} z^{1/4-a/2} m^{1/4+a/2} e^{-2\sqrt{m|z|} \cos \frac{\theta}{2}} \\ &+ \Theta : 2 \left(\frac{m}{|z|} \right)^{(\Re(a)+|\Re(a)|)/2} e^{\frac{3}{2}\pi|\Im(a)|} e^{-2\sqrt{m|z|}(\cos^2 \frac{\theta}{2} + \cos \frac{\theta}{2})} \\ &+ \Theta : 2\sqrt{\pi} |mz|^{1/4} e^{\frac{3}{2}\pi|\Im(a)|} e^{-2\sqrt{m|z|} \cos \frac{\theta}{2}}. \end{aligned}$$

This explicit bounding of the error term \mathcal{E}_1 proves the big- O statement of the theorem, while for any choice of ε an effective big- O constant can be read off at will. **QED**

4 Effective expansion for the \mathcal{H} -kernel

Theorem 4 shows the main contribution to $L_n^{(a)}(-z)$, for $(a, z) \in \mathbb{D}$, with $r := \sqrt{m/z}$, $|r| > 1/2$, is

$$c_0 := \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz} \cos \omega} d\omega, \quad (17)$$

with the integration kernel \mathcal{H}_m defined, see (15) and (16), as

$$\mathcal{H}_m(a, z, v) := v^a \left(1 + \frac{v}{2r}\right)^{-1-a} \left(\frac{1 + \frac{v}{2r}}{1 - \frac{v}{2r}}\right)^m e^{-mv/r}, \quad (18)$$

where in the integral we assign $v := e^{-i\omega}$. We need to obtain the growth properties of \mathcal{H}_m .

4.1 Exponential form for \mathcal{H}_m

Lemma 7 For $|v| = 1$ and $m > |z|/4$, the \mathcal{H} -kernel can be cast in the exponential form

$$\mathcal{H}_m := v^a \exp \left\{ \sum_{k \geq 1} \frac{a_k}{k} \frac{1}{m^{k/2}} \right\}, \quad (19)$$

where

$$a_k := (1+a)(-1)^k \left(\frac{v\sqrt{z}}{2}\right)^k + (1 - (-1)^k) \frac{k}{k+2} \left(\frac{v\sqrt{z}}{2}\right)^{k+2}. \quad (20)$$

Moreover, we have the general coefficient bound

$$|a_k| \leq \left(\frac{\sqrt{z}}{2}\right)^k (1 + |a| + |z|/2). \quad (21)$$

Proof. (18) can be recast, with $\rho := v/(2r)$, as

$$\mathcal{H}_m := v^a \exp \{ -(1+a) \log(1+\rho) - 2m\rho + m(\log(1+\rho) - \log(1-\rho)) \}. \quad (22)$$

Since $|r| > 1/2$ and $|v| = 1$, the logarithmic series converge absolutely and we have

$$\mathcal{H}_m := v^a \exp \left\{ \sum_{k \geq 1} \left[\frac{(1+a)(-1)^k}{k} \left(\frac{v\sqrt{z}}{2} \right)^k g^k + \frac{2}{2k+1} \left(\frac{v\sqrt{z}}{2} \right)^{2k+1} g^{2k-2} \right] \right\}, \quad (23)$$

where $g := 1/\sqrt{m}$, and the precise form (20) for the a_k follows immediately. The given bound on $|a_k|$ is also immediate from (20). **QED**

4.2 Exponentiation of series

Though Lemma 7 is progress, we still need to exponentiate a series, in the sense that we want to know, for the following expansion, given the sequence (a_k) ,

$$\exp \left\{ \sum_{k \geq 1} \frac{a_k}{k} x^k \right\} =: \sum_{h \geq 0} A_h x^h,$$

how the A_h depend on the a_k . The combinatorial answer is

$$A_h = \sum_{j=0}^h \frac{1}{j!} G_h(j; \vec{a}), \quad G_h(j; \vec{a}) := \sum_{h_1 + \dots + h_j = h} \frac{a_{h_1} \cdots a_{h_j}}{h_1 \cdots h_j},$$

with the understanding that $G_h(0, \vec{a}) := \delta_{0h}$ and that such combinatorial sums involve positive integer indices h_i . One useful result is:

Lemma 8 *If all coefficients $a_k = 1$, then for $j > 0$*

$$G_h(j; \vec{1}) = \sum_{h_1 + \dots + h_j = h} \frac{1}{h_1 \cdots h_j} = \Theta : \frac{1}{h} (2H_{h-j+1})^{j-1} = \Theta : \frac{1}{h} (2\gamma + 2 \log h)^{j-1}.$$

Here, $H_p := 1 + 1/2 + \dots + 1/p$ is the p -th harmonic number, $H_0 := 0$, and γ is the Euler constant.

Remark: It turns out that G here enjoys a closed form of sorts, namely

$$G_h(j; \vec{1}) = \frac{j!}{h!} (-1)^{h-j} \mathcal{S}_h^{(j)},$$

where \mathcal{S} denotes the Stirling number of the first kind, normalized via $x(x-1)\cdots(x-h+1) =: \sum_{j=0}^h \mathcal{S}_h^{(j)} x^j$. So one byproduct of our lemma is a rigorous bound on the growth of Stirling numbers; see [1, 24.1.3,III] and [39] for research on Stirling asymptotics.

Proof. The first Θ -estimate arises by induction. For notational convenience we omit the vector $\vec{1}$ and just use the symbol $G_h(j)$. Note $G_N(1) = 1/N$ and $G_N(2) = \frac{2}{N} H_{N-1}$. Generally we have

$$G_N(J) = \sum_{j=1}^{N-J+1} \frac{1}{j} G_{N-j}(J-1).$$

Now, assume by induction that $G_h(j) = \Theta : \frac{1}{h} (2H_{h-j+1})^{j-1}$, holds for all $j < J$. Then

$$G(N, J) \leq \sum_{j=1}^{N-j+1} \frac{2^{J-2} H_{N-j+2}^{J-2}}{j(N-j)} \leq \frac{2^{J-2}}{N} H_{N-J+1}^{J-2} \sum_{j=1}^{N-J+1} \left(\frac{1}{j} + \frac{1}{N-j} \right).$$

Now this last parenthetical term is $H_{N-J+1} + H_{N-1} - H_J$ which, because $H_a - H_b \leq H_{a-b}$ for any positive integer indices $a > b$, is bounded above by $2H_{N-J+1}$, which proves the first Θ -bound of the theorem. For the second θ -bound it suffices, since H_j is increasing, to show that $H_{n-1} > \gamma + \log(n)$, which is an elementary calculus problem. **QED**

Though we do not use Lemma 8 directly in what follows, it is useful in proving convergence for various sums $\sum A_h x^h$, and may well matter in future research along our lines.

Lemma 9 *Let $y \geq 1$ and $x \in (-1, 1)$ be real. Then in the expansion*

$$\exp \left\{ y \sum_{k \geq 1} \frac{x^k}{k} \right\} =: \sum_{h \geq 0} Y_h x^h$$

the coefficients Y_h enjoy the bound $Y_h = \Theta : y^h$.

Proof. First, $Y_0 = 1 \leq 1$. The left-hand side is $\exp(-y \log(1-x)) = (1-x)^{-y}$ whose binomial expansion has h -th coefficient ($h \geq 1$) equal to

$$\frac{y(y+1) \cdots (y+h-1)}{h!} \leq y^h \frac{1}{1} \frac{1+1/y}{2} \cdots \frac{1+(h-1)/y}{h} \leq y^h.$$

QED

4.3 Effective expansions of exponentiated series

We are now in a position to derive an effective expansion for an exponentiated series, starting with

Lemma 10 *Assume complex vector \vec{b} of defining coefficients b_k bounded as $|b_k| \leq cd^k$ for positive real c, d with $c \geq 1$. Assume also $|x| < 1/(2cd)$. Then for any order $N \geq 0$ we have an effective expansion*

$$\exp \left\{ \sum_{k \geq 1} \frac{b_k}{k} x^k \right\} = \sum_{h=0}^{N-1} B_h x^h + \Theta : 2c^N d^N x^N,$$

where

$$B_h = \sum_{j=0}^h \frac{G_h(j; \vec{b})}{j!}$$

are the usual coefficients of the full formal exponentiation.

Proof. Denoting $f(x) := \exp \left\{ \sum_{k \geq 1} \frac{b_k}{k} x^k \right\}$ we have

$$f(x) = \sum_{h=0}^{N-1} B_h x^h + T_N,$$

with remainder $T_N = \sum_{h \geq N} B_h x^h$, having B_h coefficients given by a G -sum as in Lemma 10. Now,

$$|B_h| \leq \sum_{j=0}^h \frac{|G_h(j; \vec{b})|}{j!} \leq \sum_{j=0}^h \frac{|G_h(j; \vec{f})|}{j!},$$

where $\vec{f} = (cd^k : k \geq 0)$. By Lemma 9 we know that $|B_h| \leq (cd)^h$. Therefore

$$|T_N| \leq \sum_{h \geq N} (cd)^h x^h = \frac{c^N d^N x^N}{1 - cdx} = \Theta : 2(cd x)^N.$$

QED

Finally we arrive at a general expansion—with effective remainder—for the \mathcal{H} -kernel:

Theorem 5 (Effective expansion for \mathcal{H}_m .) *For general complex $(a, z) \in \mathbb{C} \times \mathbb{C}$, assume $m > m_5 := |z|(1 + |a| + |z|/2)^2$ and $|v| = 1$. Then for any expansion order $N \geq 0$ we have*

$$\mathcal{H}_m(a, z, v) = v^a \left(\sum_{h=0}^{N-1} \frac{A_h}{m^{h/2}} + \Theta : 2 \left(\frac{m_5}{4m} \right)^{N/2} \right),$$

where, on the basis of the defining coefficients a_k given in (20),

$$A_h := \sum_{j=0}^h \frac{G_h(j; \vec{a})}{j!}, \quad G_h(j; \vec{a}) := \sum_{h_1 + \dots + h_j = h} \frac{a_{h_1} \cdots a_{h_j}}{h_1 \cdots h_j}.$$

Proof. The result follows immediately from Lemma 10, on assigning $\vec{b} = \vec{a}$, with $x := 1/\sqrt{m}$, $c := 1 + |a| + |z|/2$, $d := (1/2)\sqrt{|z|}$. QED

Note that Theorem 5 in the instance $N = 0$ implies our previous Lemma 6. It is interesting and suggestive that the threshold $m_5 := |z|(1 + |a| + |z|/2)^2$ appears in these results rather naturally.

4.4 Effective integral form for c_0

To obtain a useful form for c_0 , the dominant component of $L_n^{(-a)}(-z)$, we use Theorem 5 to obtain:

Theorem 6 *For $(a, z) \in \mathbb{D}$ and $m > m_5$, the dominant component of Theorems 3 and 4, namely*

$$c_0 := \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz} \cos \omega} d\omega,$$

can be given an effective form for any order $N \geq 0$, as

$$c_0 = \frac{1}{2\pi} r^{-a} e^{-z/2} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \int_{-\pi/2}^{\pi/2} e^{-i\omega a} A_h e^{2\sqrt{mz} \cos \omega} d\omega \\ + S_n(a, z) \mathcal{E}_{2,N},$$

where the error term is bounded as

$$\mathcal{E}_{2,N} = \Theta : \frac{\pi}{\sqrt{2}} \left(\frac{m_5}{4m} \right)^{N/2} \exp \left(\frac{\pi^2 \Im(a)^2 \sec \frac{\theta}{2}}{32\sqrt{m|z|}} \right) \sec^{1/2} \frac{\theta}{2}$$

and the A_h are to be calculated as the first N coefficients of

$$\sum_{h=0}^{\infty} A_h x^h := \exp \left\{ \sum_{k \geq 1} \frac{a_k}{k} x^k \right\},$$

via (20) with $v := e^{-i\omega}$.

Proof. Inserting the effective \mathcal{H} -kernel expansion from Theorem 5 directly into the c_0 integral gives the indicated sum over $h \in [0, N - 1]$ plus an error term

$$\Theta : \frac{1}{2\pi} r^{-a} e^{-z/2} 2 \left(\frac{m_5}{4m} \right)^{N/2} \int_{-\pi/2}^{\pi/2} e^{\omega \Im(a)} e^{-2\sqrt{m|z|} \cos(\theta/2) \cos \omega} d\omega.$$

Using Lemma 1 on $\cos \omega$ and the V_0 -part of Lemma 2 we obtain the $\mathcal{E}_{2,N}$ bound of the theorem. **QED**

With Theorem 6 we have come far enough to see that a Laguerre evaluation can be obtained—up to a subexponentially small relative error—via the A_h terms in said theorem. To this end, inspection of the defining relations reveals that in general we can decompose an A_h coefficient in terms of powers of $v := e^{-i\omega}$, namely we define $\alpha_{h,\mu}$ terms via

$$A_h =: \sum_{u=0}^h \alpha_{h,u}(a, z) v^{h+2u}. \quad (24)$$

For example,

$$\alpha_{00} = 1, \quad \alpha_{10} = -\frac{1+a}{2} z^{1/2}, \quad \alpha_{11} = \frac{z^{3/2}}{12}, \quad \alpha_{31} = \frac{1}{480} (5a^2 + 15a + 16) z^{5/2}.$$

The point being, we now have special formulae for the dominant contribution c_0 , namely

$$c_0 = \frac{1}{2\pi} r^{-a} e^{-z/2} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \sum_{u=0}^h \alpha_{h,u}(a, z) \mathcal{I}(2\sqrt{mz}, a + h + 2u) + S_n(a, z) \mathcal{E}_{2,N}, \quad (25)$$

where the integral

$$\mathcal{I}(p, q) := \int_{-\pi/2}^{\pi/2} e^{-iq\omega} e^{p \cos \omega} d\omega \quad (26)$$

thus emerges as a fundamental entity for the research at hand.

5 \mathcal{I} -integrals and an “exp-arc” method

Having reduced the problem of subexponential Laguerre growth to a study of the \mathcal{I} -integrals (26), we next develop a method that is effective for both their numerical and theoretical estimation. This method amounts to the avoidance of stationary-phase techniques, employing instead various forms of exponential-arcsine series, as we see shortly.

Let us first define, for any complex pair (p, q) and $\alpha, \beta \in (-\pi, \pi)$,

$$I(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-iq\omega} e^{p \cos \omega} d\omega, \quad (27)$$

so that our special case (26) is simply $\mathcal{I}(p, q) := I(p, q, -\pi/2, \pi/2)$.

Importantly, one may write the Bessel functions J_n of integer order n in terms of \mathcal{I} -integrals:

$$J_n(z) = \frac{1}{2\pi} \left(e^{-i\pi n/2} \mathcal{I}(iz, n) + e^{i\pi n/2} \mathcal{I}(-iz, n) \right), \quad (28)$$

and the modified Bessel function, again of integer order n :

$$I_n(z) = \frac{1}{2\pi} (\mathcal{I}(z, n) + (-1)^n \mathcal{I}(-z, n)), \quad (29)$$

about which representations we shall have more to say in a later section.⁵

⁵The \mathcal{I} integral can also be written in compact terms of an *Anger function* \mathbf{J} and *Weber function* \mathbf{E} —see [1].

5.1 Essentials of the exp-arc method

Now we investigate what we call *exponential-arcsine* (“exp-arc”) series. First, for any complex τ and $x \in [-1, 1]$, one has a remarkable, absolutely convergent expansion (see [5]):

$$e^{\tau \arcsin x} = \sum_{k=0}^{\infty} r_k(\tau) \frac{x^k}{k!}, \quad (30)$$

where the coefficients depend on the parity of the index:

$$r_{2m+1}(\tau) := \tau \prod_{j=1}^m (\tau^2 + (2j-1)^2), \quad r_{2m}(\tau) := \prod_{j=1}^m (\tau^2 + (2j-2)^2).$$

By differentiating with respect to x we obtain

$$\frac{e^{\tau \arcsin x}}{\sqrt{1-x^2}} = \frac{1}{\tau} \sum_{k=0}^{\infty} r_{k+1}(\tau) \frac{x^k}{k!},$$

valid for $x \in (-1, 1)$. In particular, we have the important expansion (using a function G , in passing)

$$G(\tau, x) := \frac{\cosh(\tau \arcsin x)}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} g_k(\tau) \frac{x^{2k}}{(2k)!}, \quad (31)$$

where

$$g_k(\tau) := \prod_{j=1}^k ((2j-1)^2 + \tau^2).$$

These exp-arc expansions may be applied to the $I(p, q, \alpha, \beta)$ integrals as follows.⁶ From (27) and its subsequent manipulations we have

$$\begin{aligned} I(p, q, \alpha, \beta) &= e^p \int_{\alpha}^{\beta} e^{-iq\omega} e^{-2p \sin^2(\omega/2)} d\omega = 2e^p \int_{\sin \frac{\alpha}{2}}^{\sin \frac{\beta}{2}} \frac{e^{-2iq \arcsin x}}{\sqrt{1-x^2}} e^{-2px^2} dx \\ &= i \frac{e^p}{q} \sum_{k=0}^{\infty} \frac{r_{k+1}(-2iq)}{k!} \int_{\sin \frac{\alpha}{2}}^{\sin \frac{\beta}{2}} x^k e^{-2px^2} dx. \end{aligned} \quad (32)$$

Even though we eventually consider asymptotic expansions, convergence of such I -series is the rule:

Lemma 11 *For $\alpha, \beta \in (-\pi, \pi)$ and any complex pair (p, q) , the series (32) converges absolutely.*

Proof. Define $\delta := \max(|\sin(\alpha/2)|, |\sin(\beta/2)|)$, so that the integral in (32) is bounded in magnitude by $2 \exp(2\Re(p)) \delta^{k+1}/(k+1)$. But this means the k -th summand in (32) is, up to a k -independent multiplier, bounded in magnitude by $r_{k+1}(2|q|) \delta^{k+1}/(k+1)!$. This summand is of the same form as that in the defining series (30) so absolute convergence is assured. **QED**

Likewise, from (31), (32) we have an absolutely convergent expansion for the special-case \mathcal{I} integral:

$$\mathcal{I}(p, q) = 4e^p \int_0^{1/\sqrt{2}} G(-2iq, x) e^{-2px^2} = 4e^p \sum_{k=0}^{\infty} \frac{g_k(-2iq)}{(2k)!} B_k(p), \quad (33)$$

⁶An equivalent approach to expanding I uses a series for $e^{iq\omega} \sec(\omega/2)$ in terms of $x := \sin(\omega/2)$, as in [20, p.276].

where B_k is an error-function-class integral

$$B_k(p) := \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2} dx = \frac{1}{2} \frac{1}{(2p)^{k+1/2}} \{\Gamma(k+1/2) - \Gamma(k+1/2, p)\}. \quad (34)$$

A computationally key recurrence relation accrues: with $B_0(p) = \sqrt{\frac{\pi}{8p}} \operatorname{erf}(\sqrt{p})$ and $k > 0$, we have

$$B_k(p) := \frac{2k-1}{4p} B_{k-1} - \frac{2^{-k-3/2}}{p} e^{-p}. \quad (35)$$

5.2 Effective expansion for the cosh-arc G -series

Our asymptotic analysis of representation (33) begins with a lemma that reveals how the G function (31) has an attractive self-similarity property. Namely, the function appears naturally in modified form within its own error terms.

Lemma 12 *For any complex τ the G function (31) can be given an effective expansion to any integer order $N \geq 0$, as*

$$G(\tau, x) := \frac{\cosh(\tau \arcsin x)}{\sqrt{1-x^2}} = \sum_{k=0}^{N-1} g_k(\tau) \frac{x^{2k}}{(2k)!} + g_N(\tau) \frac{x^{2N}}{(2N)!} (1 + T_N(\tau, x)), \quad (36)$$

with the error term T_N conditionally bounded over real $x \in [0, 1/\sqrt{2}]$ in the form

$$\begin{aligned} T_N &= \Theta : 1 && , \quad \text{if } N \geq 2x^2|\tau|^2 - 1; \\ &= \Theta : (\sqrt{2} x^2 + x|\tau|)e^{|\tau| \arcsin x} && , \quad \text{otherwise.} \end{aligned}$$

Proof. The error term is, by the definition of the $g_k(\tau)$, given by the absolutely convergent sum

$$T_N = h_N + h_N h_{N+1} + h_N h_{N+1} h_{N+2} + \dots,$$

where

$$h_k := \frac{(2k+1)^2 + \tau^2}{(2k+1)(2k+2)} x^2.$$

When $|\tau|^2 x^2 \leq (N+1)/2$, and since $x^2 \leq 1/2$, it is immediate that for $k \geq N$ we have a bound:

$$|h_k| \leq \frac{(4k^2 + 4k + 1)/2 + (N+1)/2}{4k^2 + 5k + 2} \leq 1/2,$$

whence $T_N = \Theta : 1/2 + 1/4 + 1/8 + \dots$, settling the first conditional bound of the theorem. In any case—i.e. any complex τ and any $x \in [0, 1/\sqrt{2}]$, we have for $j \geq 0$:

$$h_N h_{N+1} \cdots h_{N+j} = x^{2j} \prod_{k=0}^j \frac{(2N+2k+1)^2 \left((2k+1)^2 + \tau^2 \frac{(2k+1)^2}{(2N+2k+1)^2} \right)}{(2N+2k+1)(2N+2k+2)(2k+1)^2} = \Theta : \frac{g_{j+1}(|\tau|)}{(2j+1)!!^2}.$$

However, it is elementary that $(2j+1)!!^2 \geq (2j+1)!$ by simple factor-tallying, so

$$|T_N| \leq \frac{(1^2 + |\tau|^2) \cdot 2}{2!} x^2 + \frac{(1^2 + |\tau|^2)(3^2 + |\tau|^2) \cdot 4}{4!} x^4 + \dots = x \frac{\partial}{\partial x} G(|\tau|, x),$$

where we have noticed that the right-hand series here is itself a differentiated “cosh-arc” series. Thus

$$T_N = \Theta : x \frac{\partial}{\partial x} \frac{\cosh(|\tau| \arcsin x)}{\sqrt{1-x^2}} = \Theta : \frac{x^2}{2(1-x^2)^{3/2}} (e^u + e^{-u}) + \frac{x|\tau|}{2(1-x^2)} (e^u - e^{-u})$$

where $u := |\tau| \arcsin x$. Now by excluding $2x^2|\tau|^2 \leq N+1$ for the second conditional bound of the lemma, we have $|\tau| \geq 1$, whence the e^{-u} terms can be ignored over $x \in [0, 1/\sqrt{2}]$, and the second conditional bound follows. **QED**

5.3 Effective expansion of the \mathcal{I} -integral

We first invoke a classical lemma that bounds the incomplete gamma function for certain parameters:

Lemma 13 *For integer $M \geq 0$, $\Re(z) \geq 0$, $z \neq 0$ and $|z| \geq 2M - 1$, we have*

$$\Gamma(M + 1/2, z) = \Theta : 2z^{M-1/2}e^{-z}.$$

Proof. Proofs of results such as these on incomplete-gamma bounds appear in various texts on special functions, e.g., [28, §. 2.2, p. 110]. **QED**

The results of the present section may now be applied to a general, effective expansion of the \mathcal{I} -integral whenever $\Re(p)$ is sufficiently positive.

Theorem 7 (Effective \mathcal{I} expansion) *For the integral*

$$\mathcal{I}(p, q) := I(p, q, -\pi/2, \pi/2) = \int_{-\pi/2}^{\pi/2} e^{-iq\omega} e^{p \cos \omega} d\omega,$$

assume an integer expansion order $N \geq 1$. Assume $\phi := \arg(p) \in (-\pi/2, \pi/2)$ and conditions

$$\Re(p) \geq 2N + 1, \quad \Re(p) \geq 2\pi|q|^2.$$

Then we have an effective expansion

$$\begin{aligned} \mathcal{I}(p, q) = & \sqrt{\frac{2\pi}{p}} e^p \left\{ \sum_{k=0}^{N-1} \frac{g_k(-2iq)}{k! 8^k} \frac{1}{p^k} + \Theta : \sqrt{\frac{8}{\pi p}} e^{-p} \cosh\left(\frac{\pi}{2}|q|\right) \right. \\ & \left. + \frac{g_N(-2iq)}{N! 8^N} \frac{1}{p^N} \left(1 + \Theta : u_N \sec^{N+1/2} \phi\right) \right\}, \end{aligned} \quad (37)$$

where the g_k are as in the cosh-arc expansion (31), and we may take

$$u_N := 1 + 2^N + \frac{2^{3N+1}}{\pi \binom{2N}{N}}; \quad (38)$$

however, on the extra condition $N \geq 4|q|^2 - 1$, taking $u_N := 1$ suffices.

Proof. Insertion of the series of Lemma 12 into representation (33) results in

$$\mathcal{I}(p, q) = 4e^p \left\{ \sum_{k=0}^N \frac{g_k(-2iq)}{(2k)!} B_k(p) + \frac{g_N(-2iq)}{(2N)!} \int_0^{1/\sqrt{2}} x^{2N} T_N(-2iq, x) e^{-2px^2} dx \right\},$$

where the $B_k(p)$ are given by (34). The sum over $k \in [0, N]$ here is thus

$$\frac{1}{2} \sum_{k=0}^N \frac{g_k(-2iq)}{(2k)!} \frac{\Gamma(k + 1/2)}{(2p)^{k+1/2}} + \Theta : e^{-p} \sum_{k=0}^N \frac{|g_k(-2iq)|}{(2k)!} \frac{1}{2^{k+1/2} p},$$

where the Θ -term here follows from Lemma 13 on our condition $\Re(p) \geq 2N + 1$. But this very Θ -term is bounded above by

$$\frac{e^{-p}}{p\sqrt{2}} \sum_{k=0}^{\infty} \frac{g_k(2|q|)}{(2k)!} \frac{1}{2^k} = \frac{e^{-p}}{p} \cosh\left(2|q| \arcsin(1/\sqrt{2})\right),$$

so we have settled the summation and the $\cosh(\pi|q|/2)$ term in (37). Now consider the integral term

$$I_0 := \int_0^{1/\sqrt{2}} x^{2N} T_N(-2iq, x) e^{-2px^2} dx.$$

Define $\gamma := |q|^{-1} \sqrt{(N+1)/8}$. If $\gamma \geq 1/\sqrt{2}$ then by Lemma 12 we know $T_N = \Theta(1)$ and our theorem follows in the $u_N := 1$ case. Otherwise, $\gamma < 1/\sqrt{2}$ and we bound I_0 using two integrals

$$|I_0| \leq \int_0^{1/\sqrt{2}} x^{2N} e^{-2\Re(p)x^2} dx + \int_\gamma^{1/\sqrt{2}} \left(\sqrt{2} x^{2N+2} + 2|q| x^{2N+1} \right) e^{2|q| \arcsin x - 2\Re(p)x^2} dx.$$

From Lemma 1 we know that the exponent here can be taken to be $\pi|q|x/\sqrt{2} - 2\Re(p)x^2$. For the assignments $\alpha := \pi|q|/\sqrt{8}$, $\beta := 2\Re(p)$ we have $\beta > 4\alpha/\gamma$ so that by Lemma 2 there is a bound

$$\begin{aligned} I_0 &\leq \frac{1}{2} \frac{\Gamma(N+1/2)}{(2\Re(p))^{N+1/2}} + \sqrt{2} V_{N+1}(\alpha, \beta, \gamma) + 2|q| V_{N+1/2}(\alpha, \beta, \gamma) \\ &\leq \frac{1}{2} \frac{\Gamma(N+1/2)}{(2\Re(p))^{N+1/2}} + 2^N \frac{\Gamma(N+3/2)}{(2\Re(p))^{N+3/2}} + 2^{N+1} |q| \frac{\Gamma(N+1)}{(2\Re(p))^{N+1}}. \end{aligned}$$

Using $|q|^2 \leq \Re(p)/(2\pi)$, $\Re(p) \geq 2N+1$ with $\Re(p) = |p| \cos \phi$ yields the N -dependent u_N form of the theorem. **QED**

6 Effective asymptotics for $L_n^{(-a)}(-z)$

At last we may provide explicit terms for Laguerre expansions in the subexponential-growth regime, which regime turns out to be precisely characterized by the parameter-pair requirement: $(a, z) \in \mathbb{D}$.

First, for convenience we recapitulate the thresholds for sufficiently large $m := n+1$ from our previous theorems:

$$\begin{aligned} m_0 &:= |z|/4, \quad m_1 := 5 \left(|\Re(z)| + (|\Im(a)| + |\Im(z)|/2)^2 \right), \\ m_2 &:= \Re(a), \quad m_3 := -(5/4)|z| \cos \theta, \quad m_4 = 4|z|, \quad m_5 := |z|(1 + |a| + |z|/2)^2, \\ m_6 &:= (2N+1)^2 \frac{\sec^2(\theta/2)}{|z|}, \quad m_7 := 4\pi^2 \frac{\sec^2(\theta/2)}{|z|} (|a| + 3N - 3)^4. \end{aligned}$$

Here, $\theta := \arg(z)$ as before, while m_6, m_7 involve an asymptotic expansion order $N \geq 1$. We are aware that the m_i bounds here are interdependent, and some are masked by others (e.g., m_0 is masked by m_4). The important quantity in our central result (Theorem 8, below) is simply $\max_{i \in [0,7]} m_i$.

Before delving into the central result, let us remind ourselves of previous nomenclature:

$$g_k(\tau) := \prod_{j=1}^k ((2j-1)^2 + \tau^2), \tag{39}$$

$$a_k := (1+a)(-1)^k \left(\frac{v\sqrt{z}}{2} \right)^k + (1 - (-1)^k) \frac{k}{k+2} \left(\frac{v\sqrt{z}}{2} \right)^{k+2}, \tag{40}$$

$$\sum_{h=0}^{\infty} A_h x^h := \exp \left\{ \sum_{k \geq 1} \frac{a_k}{k} x^k \right\}, \tag{41}$$

$$A_h := \sum_{u=0}^h \alpha_{h,u}(a, z) v^{h+2u}. \tag{42}$$

Note that the $\alpha_{h,u}$ coefficients are thus implicitly defined in terms of the original a_k functions.

6.1 The general subexponential expansion

Using Theorem 7 with $p := 2\sqrt{mz}$, and inserting this into formula (25), we arrive at our desired effective Laguerre expansion.

Theorem 8 (Effective Laguerre expansion) *Assume $(a, z) \in \mathbb{D}$. For asymptotic expansion order $N \geq 1$, and $m := n + 1$ sufficiently large in the sense $m > \max_{i \in [0,7]} m_i$, we have the expansion*

$$L_n^{(-a)}(-z) = \frac{1}{2\sqrt{\pi}} \frac{e^{-z/2} e^{2\sqrt{mz}}}{z^{1/4-a/2} m^{1/4+a/2}} \left\{ \sum_{j=0}^{N-1} \frac{C_j}{m^{j/2}} + \frac{\bar{C}_N}{m^{N/2}} + \mathcal{E}_1 + \mathcal{E}_{3,N} \right\}, \quad (43)$$

where the expansion coefficients C_j are given in finite form:

$$C_j := \sum_{k=0}^j \frac{1}{16^k} \frac{1}{k!} \frac{1}{z^{k/2}} \sum_{u=0}^{j-k} \alpha_{j-k,u}(a, z) g_k(-2i(a + j - k + 2u)), \quad (44)$$

while the error term \bar{C}_N is bounded as

$$\begin{aligned} \bar{C}_N = \Theta : \sum_{v=1}^N \frac{1}{|z|^{v/2} 16^v v!} \left(1 + u_v \sec^{v+1/2} \frac{\theta}{2} \right) \sum_{u=0}^{N-v} |\alpha_{N-v,u}(a, z) g_v(-2i(a + N - v + 2u))| \\ + \Theta : 4(m_5/4)^{N/2} \sec^{1/2} \frac{\theta}{2}, \end{aligned}$$

with u_v taking the v -dependent form of (38) in Theorem 7 (with $q := a$ there). Finally, the term \mathcal{E}_1 is subexponentially small (from Theorem 4), as is

$$\mathcal{E}_{3,N} = \Theta : \sqrt{\frac{4}{\pi\sqrt{mz}}} e^{-2\sqrt{mz}} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \sum_{u=0}^h |\alpha_{h,u}(a, z)| \cosh\left(\frac{\pi}{2}|a + h + 2u|\right).$$

Proof. For brevity we leave out the details—all of which are straightforward, if tedious applications of the previous theorems and formulae. **QED**

We now have a resolution of the domain of subexponential growth, as:

Corollary 1 *For $(a, z) \in \mathbb{D}$, the Laguerre polynomial grows subexponentially, in the sense that for order $N \geq 1$, and any $\varepsilon > 0$,*

$$L_n^{(-a)}(-z) = S_n(a, z) \left\{ \sum_{j=0}^{N-1} \frac{C_j}{m^{j/2}} + O\left(\frac{1}{m^{N/2}}\right) + O\left(e^{-(2-\varepsilon)\sqrt{m|z|\cos(\theta/2)}}\right) \right\},$$

with all coefficients and the implied big- O constant effectively bounded via our previous theorems. Moreover, for $(a, z) \notin \mathbb{D}$, the large- n growth is not subexponential. Thus, the precise domain of subexponential growth is characterized by $(a, z) \in \mathbb{D}$.

Proof. The given subexponential formula is a paraphrase of Theorem 8. Now assume $z \in (-\infty, 0]$. We already know $z = 0$ does not yield such growth (see (2)). Now for z negative real, note that the integrals in (12, 14) are both decaying in large m . Finally, the integral in (11) has phase factor $|\exp(2\sqrt{mz} \cos \omega)| \leq 1$, and Lemma 6 show that c_1 also cannot grow subexponentially in m . **QED**

This corollary echoes, of course, the classical Perron result (7), and we again admit that historical efforts derived the C_j coefficients in principle. The new aspects are a) we have asymptotic coefficients and effective bounds for general $(a, z) \in \mathbb{D}$ parameters, and b) we can develop in a natural way an algorithm for symbolic generation of said coefficients.

6.2 Algorithm for explicit asymptotic coefficients

Theorem 8 indicates that, to obtain actual C_j coefficients, we need the cosh-arc numbers $g_k(\tau)$ and the $\alpha_{h,u}(a, z)$ coefficients. Observe that the chain of relations starting with (39) is a prescription for generation of the C_k . All of this can proceed via symbolic processing, noting that v is simply a place-holder throughout.

Remarkably, there is a fast algorithm that bypasses much of the symbolic tedium. First, we have an explicit recursion for A_h , with $A_0 := 1$, as

$$A_k = \frac{1}{k} \sum_{j=0}^{k-1} A_j a_{k-j} \quad (45)$$

as follows from differentiating (41) logarithmically, and then comparing terms. Second, when we use (42) together with (45), we obtain a recursion *devoid* of the symbolic placeholder v , as

$$\alpha_{k,u} = \frac{1}{k} \sum_{j=0}^{k-1} (\alpha_{j,u} b_{k-j} + \alpha_{j,u-1} d_{k-j}), \quad (46)$$

where these new recursion coefficients are

$$b_h := (-1)^h (1+a) \left(\frac{\sqrt{z}}{2} \right)^h, \quad d_h := (1 - (-1)^h) \frac{h}{h+2} \left(\frac{\sqrt{z}}{2} \right)^{h+2}.$$

In practice we define $\alpha_{0,0} := 1$ and force any $\alpha_{j,u}$ with $u > j$ or $u < 0$ to vanish. In this sense, the collection of $\alpha_{k,u}$ make up a *lower-triangular matrix*, e.g. the entries for $k \leq 3$ appear thus:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{(a+1)}{2} z^{1/2} & \frac{1}{12} z^{3/2} & 0 & 0 \\ \frac{a^2+3a+2}{z} & -\frac{(a+1)}{24} z^2 & \frac{1}{288} z^3 & 0 \\ -\frac{a^3+6a^2+11a+6}{48} z^{3/2} & \frac{5a^2+15a+16}{480} z^{5/2} & -\frac{a+1}{576} z^{7/2} & \frac{1}{10368} z^{9/2} \end{pmatrix},$$

where $\alpha_{3,3}$ is the lower-right element here.

These observations lead to a fast algorithm for computation of the asymptotic coefficients:

Algorithm 1 (Fast computation of Laguerre asymptotic coefficients): For given $(a, z) \in \mathbb{D}$ and desired expansion order N , this algorithm returns the asymptotic coefficients $(C_k : k \in [0, N])$ of relation (44), Theorem 8.

1) Set $\alpha_{0,0} := 1$ and for desired order N , calculate the lower-triangular matrix elements $(\alpha_{k,u} : 0 \leq u \leq k \leq N)$ via a recursion such as

$$\alpha(k, u) \{ \begin{array}{l} \text{if}(k == 0) \text{ return } \delta_{0,u}; \\ \text{return } \frac{1}{k} \sum_{j=0}^{k-1} (\alpha_{j,u} b_{k-j} + \alpha_{j,u-1} d_{k-j}); \end{array} \}$$

or use an unrolled, equivalent loop (i.e., one may generate the left-hand column of the α -matrix, then fill in one row at a time, lexicographically).

2) Use the recursion $g_k(\tau) = ((2k+1)^2 + \tau^2) g_{k-1}(\tau)$ and the lower-triangular matrix of α values to generate the C coefficients via (44). \blacklozenge

We observe that the expensive sums in this algorithm are all *acyclic convolutions*. Thus, for numerical input (a, z) the algorithm complexity turns out to be $O(N^{2+\epsilon})$ arithmetic operations, with the “2” part of the complexity power arising from the area of the lower-triangular sector.⁷

We employed the algorithm to generate exact asymptotic coefficients as follows:

$$C_0 = 1, \quad C_1 = \frac{1}{48\sqrt{z}} (-12a^2 - 24za + 4z^2 - 24z + 3),$$

$$C_2 = \frac{1}{4608z} (144a^4 + 576za^3 + 480z^2a^2 + 1728za^2 - 360a^2 - 192z^3a + 1152z^2a + 1584za + 16z^4 - 192z^3 + 312z^2 + 432z + 81),$$

and so on. Note that the C_1 form here agrees with the PAMO coefficient in (8). We were able to generate the full, symbolic $C_{64}(a, z)$ in about one minute of CPU on a typical desktop computer. To aid future researchers, we report that the numerator of C_{64} has degree 128 in both a, z , while the denominator is an integer divisible by every prime not exceeding 64, namely

$$2^{319} \cdot 3^{94} \cdot 5^{25} \cdot 7^{13} \cdot 11^7 \cdot 13^5 \cdot 17^4 \cdot 19^3 \cdot 23^3 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61.$$

While formidable, the bivariate $C_k(a, z)$ coefficients evidently have intricate structure and patterning. Future research into said structure would be of interest.

6.3 Generating and verifying effective bounds

For the first nontrivial effective bound, we can employ the rigorous bound \bar{C}_1 in Theorem 8, with $N = 1$, to get an effective version of the original asymptotic (3), as

$$L_n^{(-a)}(-z) = S_n(a, z) \left(1 + \frac{\bar{C}_1}{\sqrt{m}} + \mathcal{E}_1 + \mathcal{E}_{3,1} \right),$$

with

$$\bar{C}_1 = \Theta : \frac{|1 - 4a^2|}{16|z|^{1/2}} \left(1 + 6 \sec^{3/2} \frac{\theta}{2} \right) + 2|z|^{1/2}(1 + |a| + |z|/2) \sec^{1/2} \frac{\theta}{2},$$

and we remind ourselves that $\mathcal{E}_{3,1}$ (Theorem 8) and \mathcal{E}_1 (Theorem 4) are both subexponentially small.

At last we have an effective numerator, then, for the $1/\sqrt{m}$ asymptotic term. Though this effective numerator is almost surely nonoptimal, we are evidently on the right track, because the exact C_1 asymptotic coefficient above (see (8)) has very much the same form as does the Θ -expression for \bar{C}_1 here (i.e., same degrees of appearance for a, z , and similar coefficients). And, it is easy to see that the Θ -expression here is an upper bound on $|C_1|$ itself, as must of course be true; it is necessary that $\bar{C}_N \geq |C_N|$ for every asymptotic order N .

Table 1 shows numerical instances of our asymptotic series from Theorem 8. It displays various indices n and parameters a, z , together with corresponding asymptotic orders N from the theorem, using the following nomenclature. Denote, for asymptotic order N ,

$$T_{n,N} := 1 + \frac{C_1}{m^{1/2}} + \cdots + \frac{C_{N-1}}{m^{(N-1)/2}},$$

so for S_n being the assignment (4) we have an approximation $\mathcal{L}_{n,N}$ defined by

$$\mathcal{L}_{n,N} := S_n T_{n,N} \approx L_n^{(-a)}(-z).$$

⁷E.g., floating-point FFT-based convolutions of length L require $O(L \log L)$ operations, of complexity less than $O(L^{1+\epsilon})$.

Thus, $\mathcal{L}_{n,1}$ is simply S_n , as $T_{n,1} = 1$ always. How good are these $\mathcal{L}_{n,N}$ approximations? The important right-most columns of Table 1 report relative error (call it RE), with

$$\text{RE} := \left| \frac{\mathcal{L}_{n,N} - L_n^{(-a)}(-z)}{L_n^{(-a)}(-z)} \right|,$$

and a *rigorous* upper bound on relative error (call it RRE), expressed in terms of Theorem 8’s various error bounds, with

$$\text{RRE} := \left| \frac{\epsilon}{T_{n,N}} \right|,$$

where $\epsilon > 0$ can be taken to be any number not exceeding $|\overline{C}_N/m^{N/2} + \mathcal{E}_1 + \mathcal{E}_{3,N}|$. Note that the three error components in this last expression are all magnitude-bounded via Theorem 8; this is how the RRE table entry was constructed.

A convenient way to view RRE is to realize that $L_n^{(-a)}(-z)$ is rigorously known to lie in the interval $\mathcal{L}_{n,N} \cdot (1 - \text{RRE}, 1 + \text{RRE})$. The decimal representations in Table 1 are reflexive; i.e., there is ambiguity only in the respective final digits. (When a reported value is $(a + ib) \cdot 10^c$, then both a and b are correct to within ± 1 in their respective last digits.)

One might ask about the “first-missing” asymptotic term $C_N/m^{N/2}$ —a term of interest in many asymptotic theories. This term is typically close to, but often greater than, the RE. As is common to many asymptotic developments, rules about the behavior of these first-missing terms are hard to establish when parameters (in our case a, z) roam over the complex plane. In any case, the first-missing term is often an order of magnitude or more greater than the RRE. Yet this discrepancy between first-missing and RRE terms is not as harmful as might appear, as we discuss next.

6.4 Example application of effective bounds: Rigor for incomplete gamma

We have mentioned research motives in Section 1.1. A worked example of *rigorous* incomplete gamma evaluations is the following. Let us use our very first entry from Table 1, namely

$$L_{1300}^{(0)}(1 - 4i) \approx (7 - i) \cdot 10^{37},$$

with a rather large RRE = 0.43. Though this is an order of magnitude larger than the relative error, the penalty paid is only a few digits of precision for the relevant incomplete gamma. Indeed, from (5),

$$\Gamma(0, -1 + 4i) = e^{1-4i} \sum_{n=0}^{1299} \frac{1}{n+1} \frac{1}{L_n^{(0)}(1-4i)L_{n+1}^{(0)}(1-4i)} + \delta,$$

where the error δ is bounded, using RRE = 0.43 and (4), by

$$|\delta| < \frac{|e^{1-4i}|}{(1 - \text{RRE})^2} \sum_{n \geq 1300} \frac{1}{n+1} \frac{1}{|S_n S_{n+1}|} < 10^{-76}, \quad (47)$$

the last bound using (4) and careful estimates on the summation over $n \geq 1300$. Thence the sum of $n \in [1, 1299]$ gives a rigorous, 76-digit-accurate value for $\Gamma(0, -1 + 4i)$, ($\approx 0.497 + 0.415 i$). Correspondingly, the actual absolute error in using the sum over $n \in [0, 1299]$ is about 10^{-78} , so the apparently poor RRE bound brings only a 2-digit penalty.

An additional application of Theorem 8 would be to use effective bounds to rule out zeros of L_n in the crossed (a, z) -plane, that is, to establish an m_0 such that the term $1 + \overline{C}_1/\sqrt{m}$ must be positive for all $m > m_0$ (see our Open problems section).

n	a, z	N	Approximation $\mathcal{L}_{n,N}$ to $L_n^{(-a)}(-z)$	RE	RRE
1300	$0, -1 + 4i$	1	$(7 - i) \cdot 10^{37}$	0.0038	0.43
10^7	$0, 1$	1	$1.59 \cdot 10^{2744}$	0.00011	0.00099
		2	$1.595190 \cdot 10^{2744}$	$1.4 \cdot 10^{-9}$	$2.8 \cdot 10^{-7}$
		3	$1.5951907764 \cdot 10^{2744}$	$5.4 \cdot 10^{-13}$	$9.2 \cdot 10^{-11}$
10^6	$\frac{1}{2} + i, \frac{1}{2} + 4i$	1	$(-1.5 - 2.0i) \cdot 10^{1303}$	0.0016	0.021
		2	$(-1.565 - 2.041i) \cdot 10^{1303}$	$1.6 \cdot 10^{-6}$	$9.8 \cdot 10^{-5}$
		3	$(-1.562555 - 2.041423i) \cdot 10^{1303}$	$1.4 \cdot 10^{-9}$	$4.2 \cdot 10^{-7}$
		4	$(-1.56255567 - 2.041423269i) \cdot 10^{1303}$	$1.3 \cdot 10^{-12}$	$1.4 \cdot 10^{-9}$
		5	$(-1.56255567881 - 2.04142326969i) \cdot 10^{1303}$	$1.3 \cdot 10^{-15}$	$5.9 \cdot 10^{-12}$
10^7	$-\frac{3}{2}, 1 - 100i$	1	$(2 + i) \cdot 10^{19520}$	0.027	0.41
		2	$(1.8 + 0.9i) \cdot 10^{19520}$	0.00035	0.034
		3	$(-1.85 + 0.95i) \cdot 10^{19520}$	$3.1 \cdot 10^{-6}$	0.0028
		4	$(-1.851 + 0.957i) \cdot 10^{19520}$	$2.1 \cdot 10^{-8}$	0.00023
		5	$(-1.8510 + 0.9575i) \cdot 10^{19520}$	$1.1 \cdot 10^{-10}$	$2.0 \cdot 10^{-5}$
		10	$(-1.8509993203 + 0.9575616603i) \cdot 10^{19520}$	$7.1 \cdot 10^{-23}$	$7.6 \cdot 10^{-11}$

Table 1: **Examples of rigorous accuracy vs. asymptotic order.** For the indices n and parameters $(a, z) \in \mathbb{D}$, numbers $\mathcal{L}_{n,N}$ are computed via the asymptotic series through term $C_{N-1}/(n+1)^{(N-1)/2}$ as in Theorem 8. The relative error (RE) and rigorous upper bound (RRE) on the relative errors in taking these terms is reported. All decimal values are reported as correct to ± 1 in the last displayed digit of \Re, \Im parts. Such rigorous bounds can be used to establish indisputable accuracies for various other functions.

7 Brief remarks on oscillatory regimes

The exp-arc method has led to rigorous asymptotics for subexponential growth, but not for the Fejér form (6) for $z \neq 0$ on the cut $(-\infty, 0]$. Such oscillatory behavior can be dealt with, but other techniques come into play. For one thing, contour integrals must be handled differently.

7.1 When z is on the negative real cut

Even on $z \in (-\infty, 0)$ the contour prescription of Figure 1 is valid, and the Laguerre polynomial is exactly the sum $c_1 + d_1 + e_1$, with $R := \sqrt{m/|z|} > 1/2$ being the only requirement for contour validity. However—and this is important—the dominant contribution (17) has to change, to involve an expanded integration interval; in fact, now we must use the contour integral c_1 itself as the main contribution. For reasons of brevity, we simply state the complete asymptotic result stemming from contour integral c_1 , as

$$L_n^{(-a)}(-z) \sim \frac{e^{-z/2}}{\sqrt{\pi}(-z)^{1/4-a/2} m^{1/4+a/2}} \times \left\{ \left(\sum_{k=0}^{\infty} \frac{A_k}{m^k} \right) \cos(2\sqrt{-mz} + a\pi/2 - \pi/4) + \left(\sum_{k=0}^{\infty} \frac{B_k}{m^{k+1/2}} \right) \sin(2\sqrt{-mz} + a\pi/2 - \pi/4) \right\}, \quad (48)$$

where these oscillatory-series coefficients are directly related to the coefficients in Theorem 8 by

$$A_k := C_{2k}(a, z), \quad B_k := iC_{2k+1}(a, z).$$

It transpires that for a real, every A_k, B_k is real, whence the asymptotic has all real terms. The Fejér–Perron–Szegő expansion in [38, Theorem 8.22.2] for the oscillatory Laguerre mode is stated there in a fashion structurally different from our asymptotic (48); notwithstanding this Szegő’s own $A_{\text{odd}}, B_{\text{even}}$ —also not defined quite like ours—vanish.⁸

7.2 Brief remarks on Bessel functions I_n, J_n

There is a vast literature on Bessel asymptotics [46] [41], resulting in the *Hankel asymptotic series* [1, §9.2.5] and effective bounds on error terms (for certain parameter domains). In many cases these restricted error bounds are nevertheless optimal [46]. It is instructive to explore, at least briefly, the application of our exp-arc method to Bessel expansions. We recall (29) which leads to

$$I_n(z) = \frac{2}{\pi} \sum_{k \geq 0} \frac{g_k(-2in)}{(2k)!} \{e^z B_k(z) + (-1)^n e^{-z} B_k(-z)\} \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{g_k(-2in)}{k! 8^k} \frac{1}{z^k}, \quad (49)$$

(again we omit proofs) where the first sum is convergent, exact, and the second sum agrees with the classical Hankel asymptotic [46] [1]. For $\arg(z) \in (-\pi/2, \pi/2)$, at least, one may derive effective bounds on I_n error terms using the exp-arc techniques.

Computationalists have known for decades that one way to evaluate Bessel functions uniformly in the argument z is to use the standard ascending series for small $|z|$, but an asymptotic series for large $|z|$. However, via the exp-arc method one can establish a converging series whose evaluation only involves a *single error-function evaluation*, followed by recursion and elementary algebra. In fact, the relations (34), (35) can be used to calculate $J_n(z)$ from relation (28) via the sum

$$J_n(z) = \frac{2}{\pi} \sum_{k=0}^{\infty} g_k(-2in) (b_k \cos \chi - c_k \sin \chi), \quad (50)$$

⁸Our oscillatory asymptotic (48) has been verified to several terms by N. Temme; also he verifies our claim that the classical Szegő coefficients do vanish for the parities indicated [40].

with

$$\chi := z - \pi n/2 - \pi/4, \quad b_k := B_k(iz)e^{i\pi/4} + B_k(-iz)e^{-i\pi/4}, \quad ic_k := B_k(iz)e^{i\pi/4} - B_k(-iz)e^{-i\pi/4}.$$

Note that if z is real then each b_k, c_k is real, whence our series here has all real terms. Note that our recursion (35) likewise ignites a recursion amongst the b_k, c_k .

Note that (50) is actually the Hankel asymptotic if we replace B_k by its first term in (34), namely $(1/2)(2iz)^{-k-1/2}\Gamma(k+1/2)$; however, we already know that the sum (50) is always convergent. It is remarkable that we are using the same structure as the classical asymptotic, yet convergence for all complex z is guaranteed. Moreover, the $B_k(iz)$ are independent of the order n and so can be re-used if multiple $J_n(z)$ are desired for fixed z .

Those acquainted with the intricacies of Bessel theory may observe that our convergent expansion (49) is at least reminiscent of the convergent *Hadamard expansion* found in [46, p.204] for the modified Bessel function I_ν . Though both expansions are absolutely convergent, there are some important differences between this Hadamard expansion and our exp-arc forms (49, 50). For example we have given our convergent sum only for integer ν . Moreover, the exp-arc expansion is geometrically convergent, while the Hadamard expansion is genuinely slower.

The research area of convergent expansions related to classical, asymptotic ones has been pioneered in large part by R. Paris, whose works cover real and complex domains, saddle points, and the like [30] [31] [32] [33] [34] [35]. We should point out that Paris was able to develop within the last decade some similar, linearly convergent Bessel series by modifying the “tails” of Hadamard-class series. Finally, we note that the problem of generalizing such unconditionally convergent series as our (50) to non-integer indices—and with comparison to the recent work of Paris—is analyzed in a new work [3].

8 Open problems

- How might one proceed with the exp-arc theory to obtain effective error bounds for oscillatory Laguerre modes, and-or oscillatory Bessel modes? We know that previous researchers have described how to give effective bounds in these cases (e.g., to our asymptotic (48), as in [38, §8.72]) but once again we stress: How can this be done explicitly, and for full parameter ranges?
- Where are the zeros in the complex z -plane—for fixed a —of $L_n^{(-a)}(-z)$? Are “most of” the zeros along some a -dependent ray, in some sense? There is a considerable literature on this zero-free-region topic, especially for polynomials in real variables. For example, with $a := 0$ the Laguerre zeros are all real and negative; see [25, Ch. X] and references therein. There is also an interesting connection between Laguerre zeros and eigenvalues of certain (large) matrices [15]. Certainly the theorems of the Saff–Varga type are relevant in this context [25, Th. 11].
- How can Laguerre asymptotics be gleaned from standard recurrence relations amongst the $L_n^{(-a)}(-z)$? One may ask the same question for the Laguerre differential equation as starting point. A promising research avenue for a discrete-iterative approach to asymptotics is [45]; see also [8] for the asymptotic analysis of certain complex continued fractions. As for differential-equation approaches, there is the classical work of Erdélyi and Olver, plus modern work on combinations of differential, discrete, and saddle-point theory [16][18].
- The efficient “keyhole” contour of Figure 1 was discovered experimentally. What other analytical problems might be approached in this fashion? For that matter, how might one properly use the celebrated Watson loop-integral lemma with error term [28, Theorem 5.1] on our keyhole contour to obtain similar effective asymptotics?
- Here is a scenario suggested by N. Temme [40]. When n is small (in some sense not made rigorous here), and say $|z| \ll |a|$, one should expect an asymptotic behavior

$$L_n^{(-a)}(-z) \sim \binom{n-a}{n} \left(1 + \frac{z}{1-a}\right)^n.$$

The problem is: How does one make this rigorous, with effective error bounds?

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