Lattice Sums and Variants on the Riemann Hypothesis

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1. Introduction

These research notes were written to complement the discussion of angular lattice sums and their zeros in Chapter 3 of Borwein *et al* (2013). There it was mentioned that McPhedran *et al* (2013) had proved that the basic double lattice sum $\mathcal{C}(0,1;s)$ obeyed the Riemann hypothesis if and only if any of the angular lattice sums $\mathcal{C}(1,4m;s)$ obeyed the Riemann hypothesis. Professor Heath-Brown has recently communicated to the author the fact that there is a gap in the graphical proof of this result. This communication provided a stimulus towards the exploration of algebraic methods of investigation of the distribution of the zeros of the sums just mentioned. These methods have met with some success, chiefly through the use of a proposition put forward in the paper of Lagarias and Suzuki (2006).

The research notes here presented are somewhat discursive, containing a range of results which in some cases contrast with, and in others complement, the main results: Theorems 4.5, 7.1 and 8.1. It is planned to publish in more concise form the main results and the elements required to arrive at them, but it is felt that the extra elements given here may be of some use in related investigations.

2. Formulae for Macdonald Function Sums

We recall the definition from McPhedran *et al*, 2008, hereafter referred to as (I), of two sets of angular lattice sums for the square array:

$$\mathcal{C}(n,m;s)=\sum_{p_1,p_2}{}' \frac{\cos^n(m\theta_{p_1,p_2})}{(p_1^2+p_2^2)^s},\quad \mathcal{S}(n,m;s)=\sum_{p_1,p_2}{}' \frac{\sin^n(m\theta_{p_1,p_2})}{(p_1^2+p_2^2)^s}, \tag{2.1}$$

where $\theta_{p_1,p_2}=\arg(p_1+\mathrm{i}p_2)$, the prime denotes the exclusion of the point at the origin, and the complex number s is written in terms of real and imaginary parts as $s=\sigma+it$. The sum independent of the angle θ_{p_1,p_2} was evaluated by Lorenz (1871) and Hardy(1920) in terms of the product of Dirichlet L functions:

$$\mathcal{C}(0,m;s)=\mathcal{S}(0,m;s)\equiv\mathcal{C}(0,1;s)\equiv\mathcal{C}(1,0;s)=4L_1(s)L_{-4}(s)=4\zeta(s)L_{-4}(s). \tag{2.2}$$

A useful account of the properties of Dirichlet L functions such as $L_{-4}(s)$ has been given by Zucker & Robertson (1976).

It is convenient to use a subset of the angular sums (2.1) as a basis for numerical evaluations. We note that the sums $\mathcal{C}(n,1;s)$ are zero if n is odd. We have for the non-zero sums $\mathcal{C}(2n,1;s)$:

$$\sum_{(p_1,p_2)}{}' \frac{p_1^{2n}}{(p_1^2+p_2^2)^{s+n}} = \mathcal{C}(2n,1;s) = \frac{2\sqrt{\pi}\Gamma(s+n-1/2)\zeta(2s-1)}{\Gamma(s+n)}$$
$$+\frac{8\pi^s}{\Gamma(s+n)}\sum_{p_1=1}^{\infty}\sum_{p_2=1}^{\infty}\left(\frac{p_2}{p_1}\right)^{s-1/2} p_1^n p_2^n \pi^n K_{s+n-1/2}(2\pi p_1 p_2), \qquad (2.3)$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind, or Macdonald function, with order ν and argument z. The general form (2.3) may be derived following Kober (1936), in the way described in McPhedran *et al* (2010), hereafter referred to as II. A variant of (2.3) occurs for $n=0$, when an extra term occurs, arising from an axial term from $p_1=0$:

$$\mathcal{C}(0,1;s) = 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)$$
$$+\frac{8\pi^s}{\Gamma(s)}\sum_{p_1=1}^{\infty}\sum_{p_2=1}^{\infty}\left(\frac{p_2}{p_1}\right)^{s-1/2} K_{s-1/2}(2\pi p_1 p_2). \qquad (2.4)$$

Hejhal (1987,1990) has analysed the distribution of zeros of Macdonald function sums akin to those occurring in (2.3) and (2.4). He comments on the numerical difficulties associated with their accurate evaluation when t is large. In general, the double sums can be re-expressed as single sums over the variable $l=p_1p_2$, and the value of $2\pi l$ must range up to around $1.5t$ for adequate accuracy in the production of graphs and the evaluation of location of zeros. Note that this estimate of the summation region required is based on the transition being completed from the small argument form of the Macdonald function:

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{2}{z}\right)^\nu, \qquad (2.5)$$

to the large argument form

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}. \qquad (2.6)$$

The evaluation of lattice sums can also be made more efficient using the standard recurrence relations

$$K_{\nu-1}(z) - K_{\nu+1}(z) = \frac{-2\nu}{z}K_\nu(z), \quad K_{\nu-1}(z) + K_{\nu+1}(z) = -2K_\nu'(z), \qquad (2.7)$$

to reduce the Madonald functions needed. We will use the abbreviated notations for complex orders $s_- = s-1/2$ and $s_+ = s+1/2$. One of our goals is to derive functional equations for angular lattice sums of the type (2.1), and to investigate what these functional equations say about the location of zeros of the sums. We will follow the approach in McPhedran *et al* (2007), and consider lattice sums grouped into systems by their order.

3. The Systems of Order 0,2

The system of order 0 contains the single element $C_{0,1}$. The system of order 2 can be constructed from that of order 0, since $\mathcal{C}(2,1;s) = (1/2)\mathcal{C}(0,1;s)$ and from (2.3)

$$2\mathcal{C}(2,1;s) = \frac{4\sqrt{\pi}\Gamma(s_+)\zeta(2s_-)}{\Gamma(s+1)} + \frac{16\pi^s}{\Gamma(s+1)} \sum_{p_1,p_2=1}^{\infty} \left(\frac{p_2}{p_1}\right)^{s_-} p_1 p_2 \pi K_{s_+}(2\pi p_1 p_2). \tag{3.1}$$

We solve (2.4) and (3.1) for the first two Macdonald functions sums we consider. Let us introduce a general notation for these sums:

$$\mathcal{K}(n,m;s) = \sum_{p_1,p_2=1}^{\infty} \left(\frac{p_2}{p_1}\right)^{s_-} (p_1 p_2 \pi)^n K_{m+s_-}(2\pi p_1 p_2). \tag{3.2}$$

By interchanging p_2 and p_1 and using the relation $K_{-\nu}(z) = K_\nu(z)$, we see that $\mathcal{K}(n,0;s)$ is symmetric under the substitution $s \to 1-s$:

$$\mathcal{K}(n,0;1-s) = \mathcal{K}(n,0;s), \tag{3.3}$$

while $\mathcal{K}(n,1;s)$ is not symmetric. Note that from (3.3) the sum $\mathcal{K}(n,0;s)$ is real on the critical line. The set of sums $\mathcal{K}(n,0;s)$ is privileged numerically, since the same set of Macdonald function evaluations is required for each. Using the divisor function $\sigma_a(n)$, the sum of the ath power of the divisors of n, we may write

$$\mathcal{K}(n,m;s) = \sum_{p=1}^{\infty} \sigma_{2s_-}(p)(p\pi)^n K_{m+s_-}(2\pi p). \tag{3.4}$$

From the first of the relations (2.7), we find the following recurrence relation for the sums $\mathcal{K}(n,m;s)$

$$\mathcal{K}(n,m;s) = (s_- + m - 1)\mathcal{K}(n-1,m-1;s) + \mathcal{K}(n,m-2;s). \tag{3.5}$$

We have the following general symmetry property of the sums $\mathcal{K}(n,m;s)$:

$$\mathcal{K}(n,m;1-s) = \mathcal{K}(n,-m;s), \tag{3.6}$$

which is proved using the symmetry of the square lattice, employed using am interchange of the summation indices p_2 and p_1, together with that of $K_\nu(z)$ under the change of sign of ν. The special form of (3.6) relating to the critical line is

$$\overline{\mathcal{K}(n,m;\frac{1}{2}+it)} = \mathcal{K}(n,-m;\frac{1}{2}+it). \tag{3.7}$$

The sum of the left and right-hand sides of (3.7) is thus real, and the difference pure imaginary.

An interesting special case of (3.5) is obtained by putting $m = 1$:

$$\mathcal{K}(n,1;s) = s_-\mathcal{K}(n-1,0;s) + \mathcal{K}(n,-1;s). \tag{3.8}$$

From this and (3.6) we find

$$\mathcal{K}(n-1,0;s) = \frac{\mathcal{K}(n,1;s) - \mathcal{K}(n,1;1-s)}{s_-} = \frac{\mathcal{K}(n,1;s) - \mathcal{K}(n,-1;s)}{s_-}. \tag{3.9}$$

Thus, the zeros of $\mathcal{K}(n-1,0;s)$ on the critical line are those of $\Im\mathcal{K}(n,1;s)$. More generally, all zeros of the even part of $\mathcal{K}(n,1;s)$ with respect to the variable $s-1/2$ are also zeros of $\mathcal{K}(n-1,0;s)$.

The lowest symmetric sum is

$$\mathcal{K}(0,0;s)=\frac{\Gamma(s)}{8\pi^s}\mathcal{C}(0,1;s)-\left[\frac{\Gamma(s)\zeta(2s)}{4\pi^s}+\frac{\Gamma(s_-)\zeta(2s_-)}{4\pi^{s_-}}\right]. \tag{3.10}$$

Each of the two terms on the right-hand side of (3.10) is unchanged by the substitution $s\to 1-s$. (That the first term is unchanged follows from the functional equation for $\mathcal{C}(0,1;s)$; that the second is unchanged then follows from the preceding fact and the symmetry of $\mathcal{K}(0,0;s)$.) They are therefore real on the critical line $\sigma=1/2$.

The lowest non-symmetric sum we consider is

$$\mathcal{K}(1,1;s)=s\left[\frac{\Gamma(s)}{16\pi^s}\mathcal{C}(0,1;s)\right]-s_-\left[\frac{\Gamma(s_-)\zeta(2s_-)}{4\pi^{s_-}}\right]. \tag{3.11}$$

We now introduce two symmetrised functions, the first of which occurred in equation (3.10):

$$\mathcal{T}_+(s)=\frac{\Gamma(s)\zeta(2s)}{4\pi^s}+\frac{\Gamma(s_-)\zeta(2s_-)}{4\pi^{s_-}}, \tag{3.12}$$

so that

$$\mathcal{K}(0,0;s)=\frac{\Gamma(s)}{8\pi^s}\mathcal{C}(0,1;s)-\mathcal{T}_+(s). \tag{3.13}$$

Its antisymmetric counterpart is

$$\mathcal{T}_-(s)=\frac{\Gamma(s)\zeta(2s)}{4\pi^s}-\frac{\Gamma(s_-)\zeta(2s_-)}{4\pi^{s_-}}. \tag{3.14}$$

This is equivalent to a function considered by P.R. Taylor (1945):

$$\xi_1(s+1/2)-\xi_1(s-1/2),\quad \text{where}\quad \xi_1(s)=\frac{\Gamma(s/2)\zeta(s)}{\pi^{(s/2)}}, \tag{3.15}$$

after his variable s is replaced by our $2s-1/2$. Taylor proved in fact that his function obeys the Riemann hypothesis, as must then $\mathcal{T}_-(s)$. We have:

$$\xi_1(2s)=2[\mathcal{T}_+(s)+\mathcal{T}_-(s)],\quad \xi_1(2s-1)=2[\mathcal{T}_+(s)-\mathcal{T}_-(s)]. \tag{3.16}$$

We further define

$$\mathcal{V}(s)=\frac{\mathcal{T}_+(s)}{\mathcal{T}_-(s)}=\frac{1+\mathcal{U}(s)}{1-\mathcal{U}(s)},\quad \mathcal{U}(s)=\frac{\xi_1(2s-1)}{\xi_1(2s)}. \tag{3.17}$$

It has since been proved (see Ki, 2006, Lagarias and Suzuki 2006, McPhedran & Poulton, 2013) that $\mathcal{T}_+(s)$ has all its zeros on the critical line. Furthermore, it is real-valued and monotonic increasing there, while $\mathcal{T}_-(s)$ is pure imaginary, and its imaginary part is again monotonic increasing. The zeros of both functions are all of first order, and alternate on the critical line.

Now, from Titchmarsh (1987), the Riemann zeta function has its zeros confined to the region $0 < \sigma < 1$, from which we immediately see that $\mathcal{T}_+(s)$ and $\mathcal{T}_-(s)$ cannot have coincident zeros. We can combine (3.10) and (3.11) to give

$$\mathcal{K}(1,1;s) = \frac{\Gamma(s)}{32\pi^s}\mathcal{C}(0,1;s) + \frac{s_-}{2}\mathcal{T}_-(s) + \frac{s_-}{2}\mathcal{K}(0,0;s), \tag{3.18}$$

or, with the parts of (3.18) with even and odd symmetry under $s \to 1-s$ separated:

$$\mathcal{K}(1,1;s) = \frac{\Gamma(s)}{32\pi^s}\mathcal{C}(0,1;s) + \frac{1}{2}(s-\frac{1}{2})\mathcal{T}_-(s) + \frac{1}{2}(s-\frac{1}{2})\left[\frac{\Gamma(s)}{8\pi^s}\mathcal{C}(0,1;s) - \mathcal{T}_+(s)\right]. \tag{3.19}$$

This may be written in the form of two functional equations for $\mathcal{K}(1,1;s)$:

$$\mathcal{K}(1,1;s) - \mathcal{K}(1,1;1-s) = s_-\mathcal{K}(0,0;s). \tag{3.20}$$

and

$$\mathcal{K}(1,1;s) + \mathcal{K}(1,1;1-s) = \frac{\Gamma(s)}{16\pi^s}\mathcal{C}(0,1;s) + s_-\mathcal{T}_-(s). \tag{3.21}$$

In Fig. 1 we show the variation along the critical line of the functions $\mathcal{T}_+(s)$, $\mathcal{T}_-(s)$

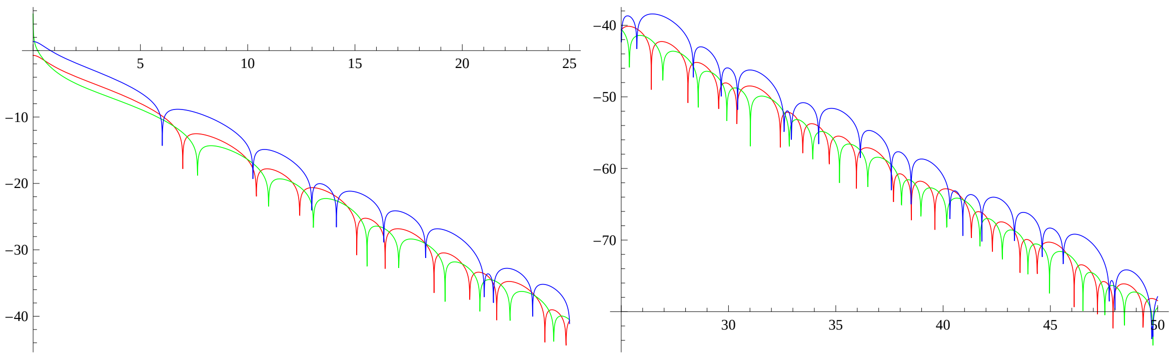


Figure 1. Plots of the logarithmic modulus of the functions $\mathcal{T}_+(s)$ (red), $\mathcal{T}_-(s)$ (green) and $\mathcal{C}(0,1;s)*\Gamma(s)/\pi^s$ (blue) along the critical line.

and $\mathcal{C}(0,1;s)*\Gamma(s)/\pi^s$. Each has a similar number of zeros in the interval shown. In Fig.2 (left) we further show the behaviour on the critical line of the modulus of the functions $\mathcal{K}(0,0;s)$, $\mathcal{K}(1,1;s)$ and $\mathcal{C}(0,1;s)*\Gamma(s)/\pi^s$. It can be seen that the first of these has significantly fewer zeros on the critical line than the third, while the second has none (for a related proposition, see Theorem 2.2 below). Weak minima of $|\mathcal{K}(1,1;s)|$ are correlated with the zeros of $\mathcal{K}(0,0;s)$, in keeping with a remark made after equation (3.9). Note that $\mathcal{K}(0,0;s)$ has approximately the same number of zeros (in complex conjugate pairs) off the critical line as on it (see Fig. 2 (right): of the zeros shown, 38 are off the critical line, and 39 on it).

Using the relations (2.7) we can obtain the value of $\mathcal{K}$ sums with m negative. For example,

$$\mathcal{K}(1,-1;s) = \frac{-(s-1)\Gamma(s)}{16\pi^s}\mathcal{C}(0,1;s) + s_-\left[\frac{\Gamma(s)\zeta(2s)}{4\pi^s}\right] = \mathcal{K}(1,1;1-s), \tag{3.22}$$

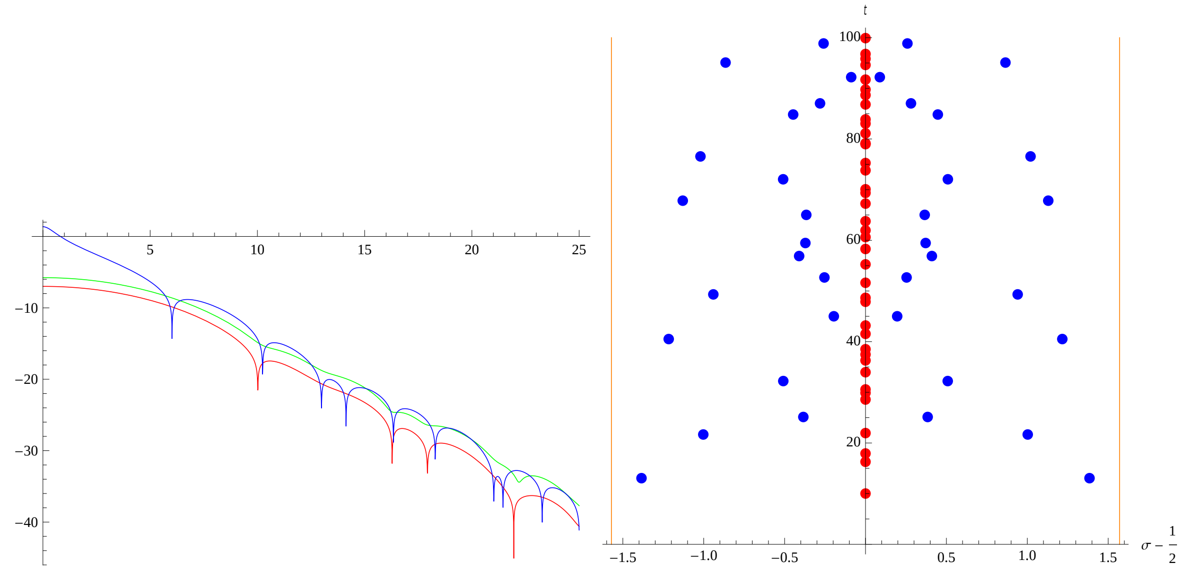


Figure 2. (Left) Plots of the logarithmic modulus of the functions $\mathcal{K}(0,0;s)$ (red), $\mathcal{K}(1,1;s)$ (green) and $\mathcal{C}(0,1;s)*\Gamma(s)/\pi^s$ (blue) along the critical line. Right: The lowest zeros of $\mathcal{K}(0,0;s)$ in the complex plane of s_-.

using (3.6). We can also define a class of sums which involve the derivative of the Macdonald function with respect to its argument

$$\mathcal{L}(n,m;s)=\sum_{p_1,p_2=1}^{\infty}\left(\frac{p_2}{p_1}\right)^{s_-}(p_1p_2\pi)^n K'_{m+s_-}(2\pi p_1p_2). \tag{3.23}$$

Using (2.7) we obtain

$$\mathcal{L}(n,m;s)=-\frac{1}{2}[\mathcal{K}(n,m+1;s)+\mathcal{K}(n,m-1;s)]. \tag{3.24}$$

Using (3.18) and (3.22) we find

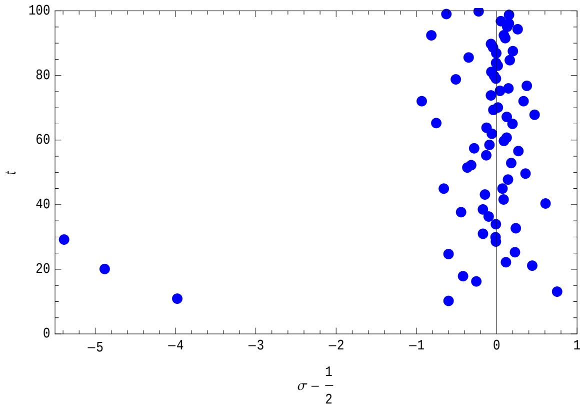
$$\mathcal{L}(1,0;s)=-\frac{1}{2}\left[\frac{\Gamma(s)}{16\pi^s}\mathcal{C}(0,1;s)+s_-\mathcal{T}_-(s).\right] \tag{3.25}$$

From their definitions and the symmetry of the square lattice, $\mathcal{K}(1,1;s)$ and $\mathcal{K}(1,-1;s)$ are conjugates of each other on the critical line, and $\mathcal{L}(1,0;s)$ is real there. More generally, using (3.6),

$$\mathcal{L}(n,0;s)=-\frac{1}{2}[\mathcal{K}(n,1;s)+\mathcal{K}(n,1;1-s)]. \tag{3.26}$$

Thus, the zeros of $\mathcal{L}(n,0;s)$ are those of the even part of $\mathcal{K}(n,1;s)$ with respect to the variable $s-1/2$. Combining this remark with one made after equation (3.9), we see that zeros of $\mathcal{K}(n,1;s)$ on the critical line require $\mathcal{L}(n,0;s)$ and $\mathcal{K}(n-1,0;s)$ to be zero simultaneously.

In Fig. 3 we show the distribution of the lowest zeros of $\mathcal{K}(1,-1;s)$. All the 71 zeros shown lie off the critical line. Apart from the first 3 zeros of $\mathcal{K}(1,-1;s)$, the scatter of the plots in Fig. 3 and Fig. 2 do not seem to differ greatly.



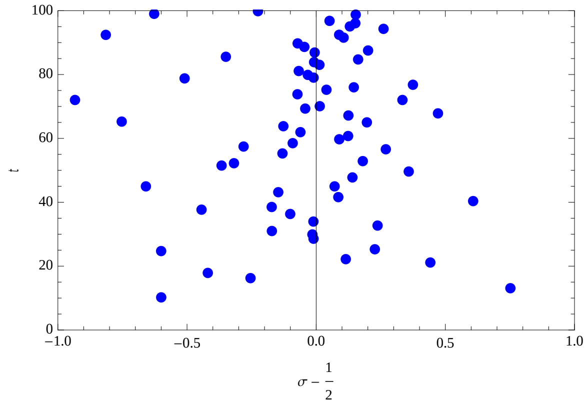


Figure 3. Plots of the lowest zeros of $\mathcal{K}(1,-1;s)$ in the complex plane of s_-.

If we consider the zeros s_0 of $\mathcal{K}(0,0;s)$, we find that

$$\frac{\Gamma(s_0)}{\pi^{s_0}}\mathcal{C}(0,1;s)=\mathcal{T}_+(s),\quad \mathcal{K}(1,1;s_0)=\frac{1}{2}\left[\mathcal{T}_+(s_0)+2(s_0-\frac{1}{2})\mathcal{T}_-(s_0)\right]. \tag{3.27}$$

The expression for $\mathcal{K}(1,1;s_0)$ can only be zero on the critical line. Indeed, if we assume in fact this term is zero, we obtain

$$\xi_1(2s_0)+\xi_1(2s_0-1)+(2s_0-1)\left[\xi_1(2s_0)-\xi_1(2s_0-1)\right]=0. \tag{3.28}$$

Hence,

$$\mathcal{U}(s_0)=\frac{\xi_1(2s_0-1)}{\xi_1(2s_0)}=\frac{s_0}{s_0-1}=\frac{\sigma_0+it_0}{\sigma_0-1+it_0}. \tag{3.29}$$

This yields a contradiction, since the modulus of the fourth expression in (3.29) is larger than unity if $\sigma_0>1/2$, and smaller than unity if $\sigma_0<1/2$, whereas the function $\mathcal{U}(s)$ has the opposite properties. This proposition was proved in Theorem 1 of Lagarias & Suzuki (2006).

On the critical line, we have from equations (3.12) and (3.14) that

$$\mathcal{T}_+(1/2+it)=\frac{1}{2}|\xi_1(1+2it)|\cos[\arg[\xi_1(1+2it)]],\ \ \mathcal{T}_-(1/2+it)=\frac{i}{2}|\xi_1(1+2it)|\sin[\arg[\xi_1(1+2it)]]. \tag{3.30}$$

Hence, we see that

$$\mathcal{V}(1/2+it)=-i\cot[\arg\xi_1(1+2it)],\ \ \mathcal{U}(1/2+it)=\exp[-2i\arg\xi_1(1+2it)]. \tag{3.31}$$

From equations (3.20) and (3.13),

$$2i\Im[\mathcal{K}(1,1;1/2+it)]=it\left[\frac{\Gamma(1/2+it)\mathcal{C}(0,1;1/2+it)}{8\pi^{1/2+it}}-\frac{1}{2}|\xi_1(1+2it)|\cos[\arg[\xi_1(1+2it)]]\right]. \tag{3.32}$$

Also, from equation (3.21),

$$2\Re[\mathcal{K}(1,1;1/2+it)]=\frac{\Gamma(1/2+it)\mathcal{C}(0,1;1/2+it)}{16\pi^{1/2+it}}-\frac{t}{2}|\xi_1(1+2it)|\sin[\arg[\xi_1(1+2it)]]. \tag{3.33}$$

We use quantities $\mathcal{V}_\mathcal{K}(1,1;s)$ and $\mathcal{U}_\mathcal{K}(1,1;s)$ defined by analogy to $\mathcal{V}(s)$, $\mathcal{U}(s)$ of (3.17):

$$\mathcal{V}_\mathcal{K}(1,1;s)=\frac{\mathcal{K}(1,1;s)-\mathcal{K}(1,1;1-s)}{\mathcal{K}(1,1;s)+\mathcal{K}(1,1;1-s)},\ \ \mathcal{U}_\mathcal{K}(1,1;s)=\frac{\mathcal{K}(1,1;s)}{\mathcal{K}(1,1;1-s)}. \qquad (3.34)$$

Hence,

$$\arg[\mathcal{U}_\mathcal{K}(1,1;1/2+it)] = \arg\left[i\left(\frac{2\Gamma(1/2+it)\mathcal{C}(0,1;1/2+it)}{8\pi^{1/2+it}|\xi_1(1+2it)|}-\cos[\arg[\xi_1(1+2it)]]\right)+\left(\frac{\Gamma(1/2+it)\mathcal{C}(0,1;1/2+it)}{8\pi^{1/2+it}|\xi_1(1+2it)|t}-\sin[\arg[\xi_1(1+2it)]]\right)\right]. \qquad (3.35)$$

Theorem 3.1. *$\mathcal{K}(1,1;s)$ and $\mathcal{K}(1,1;1-s)$ cannot be simultaneously zero for s off the critical line.*

Proof. We consider the situation where s_0 lies off the critical line, and is such that $\mathcal{K}(1,1;s_0)=0$ and $\mathcal{K}(1,1;1-s_0)=0$. From the first of these and (3.11),

$$\frac{\Gamma(s_0)\mathcal{C}(0,1;s_0)}{16\pi^{s_0}}=\frac{(s_0-1/2)}{s_0}\frac{\Gamma(s_0-1/2)\zeta(2s_0-1)}{4\pi^{s_0-1/2}}, \qquad (3.36)$$

while from the second and (3.22),

$$\frac{\Gamma(s_0)\mathcal{C}(0,1;s_0)}{16\pi^{s_0}}=\frac{(s_0-1/2)}{s_0-1}\frac{\Gamma(s_0)\zeta(2s_0)}{4\pi^{s_0}}, \qquad (3.37)$$

If both these hold then $\mathcal{C}(0,1;s_0)$ cannot be zero, since $\zeta(2s_0)$ and $\zeta(2s_0-1)$ cannot be zero simultaneously. Further, comparing (3.36) and (3.37),

$$\mathcal{U}(s_0)=\frac{\xi_1(2s_0-1)}{\xi_1(2s_0)}=\frac{s_0}{s_0-1}=\frac{\sigma_0+it_0}{\sigma_0-1+it_0}. \qquad (3.38)$$

We see from (3.38) that $|\mathcal{U}(s_0)|>1$ if $\sigma_0>1/2$, while $|\mathcal{U}(s_0)|<1$ if $\sigma_0<1/2$. This in fact contradicts the known behaviour of $|\mathcal{U}(s_0)|$, given that its zeros lie in $\sigma_0>1/2$ and its poles in $\sigma_0<1/2$. □

Corollary 3.2. *$\mathcal{L}(1,0;s)$ and $\mathcal{K}(0,0;s)$ cannot be simultaneously zero for s off the critical line.*

Proof. For any integer n, we can write

$$\mathcal{K}(n,1;s)=-\mathcal{L}(n,1;s)+\frac{1}{2}\left(s-\frac{1}{2}\right)\mathcal{K}(n-1,0;s). \qquad (3.39)$$

If $\mathcal{L}(n,1;s)$ and $\mathcal{K}(n-1,0;s)$ are both zero, then $\mathcal{L}(n,1;1-s)$ and $\mathcal{K}(n-1,0;1-s)$ are also both zero, the two functions being even. Thus, $\mathcal{K}(n,1;s)$ and $\mathcal{K}(n,1;1-s)$ are both zero. For $n=1$, this cannot occur by the Theorem just proved, giving the stated result. □

Theorem 3.3. *For $\sigma_0\leq 1/2$, $\mathcal{C}(0,1;s_0)$ and $\mathcal{K}(1,1;s_0)$ cannot be simultaneously zero.*

Proof. Let s_0 be such that $\mathcal{C}(0,1;s_0)=0$. Consider first the case when $s_0=1/2+it_0$. Then if also $\mathcal{K}(1,1;s_0)=0$, its conjugate $\mathcal{K}(1,1;1-s_0)=0$. From equations (3.20,3.21) this requires $\mathcal{T}_+(s_0)=0=\mathcal{T}_-(s_0)$, a contradiction.

Consider next the case when s_0 lies to the left of the critical line. Then from (3.20,3.21) ,

$$\mathcal{K}(1,1;s_0)=-\frac{1}{2}(s_0-\frac{1}{2})[\mathcal{T}_+(s_0)-\mathcal{T}_-(s_0)]=-(s_0-\frac{1}{2})\frac{\Gamma(s_0-\frac{1}{2})\zeta(2s_0-1)}{4\pi^{s_0-1/2}}, \quad (3.40)$$

which is zero only for s_0 to the right of the critical line, a contradiction. □

Theorem 3.4. *For any zero s_0 of $\mathcal{K}(0,0;s)$ off the critical line, $\mathcal{C}(0,1;s_0)$ is non-zero, and vice versa.*

Proof. If $\mathcal{C}(0,1;s_0)=0=\mathcal{K}(0,0;s_0)$, then from equation (3.13), $\mathcal{T}_+(s_0)$ iz zero, which requires s_0 to lie on the critical line– a contradiction. □

4. Alternative Symmetrization of $\mathcal{K}(1,1;s)$

The symmetrisation of $\mathcal{K}(1,1;s)$ embodied in equations (3.20) and (3.21) can be improved. A defect of the previous choice (3.20) is that the right-hand side has zeros both off and on the critical line. The alternative symmetrisation we now investigate is less obvious than its predecessor, but overcomes this deficiency. Let us define

$$\mathcal{K}_-(1,1;s)=(1-s)\mathcal{K}(1,1;s)-s\mathcal{K}(1,1;1-s) \quad (4.1)$$

and

$$\mathcal{K}_+(1,1;s)=(1-s)\mathcal{K}(1,1;s)+s\mathcal{K}(1,1;1-s). \quad (4.2)$$

Theorem 4.1. *The symmetrised functions are given by*

$$\mathcal{K}_-(1,1;s)=-\frac{(s-1/2)}{2}[\mathcal{T}_+(s)+(2s-1)\mathcal{T}_-(s)], \quad (4.3)$$

and

$$\mathcal{K}_+(1,1;s)=\frac{s(1-s)}{8\pi^s}\Gamma(s)\mathcal{C}(0,1;s)+\frac{(s-1/2)}{2}[(2s-1)\mathcal{T}_+(s)+\mathcal{T}_-(s)]. \quad (4.4)$$

The former has all its zeros on the critical line, as does $\mathcal{K}_+(1,1;s)-[s(1-s)/(8\pi^s)]\Gamma(s)\mathcal{C}(0,1;s)$. The distribution functions of the zeros are the same, and agree with those of $\mathcal{T}_+(s)$ and $\mathcal{T}_-(s)$.

Proof. The equations (4.3) and (4.4) follow easily from equation (3.11). The assertion that the zeros of the right-hand side of (4.3) all lie on the critical line has already been proved above (see the discussion around equations (3.28) and (3.29)). The proof that the zeros of $[(2s-1)\mathcal{T}_+(s)+\mathcal{T}_-(s)]$ lie on the critical line follows in a similar fashion. Indeed,

$$(2s-1)\mathcal{T}_+(s)+\mathcal{T}_-(s) \iff 2s\xi_1(2s)+(2s-2)\xi_1(2s-1)=0 \quad (4.5)$$

or

$$\frac{\xi_1(2s-1)}{\xi_1(2s)}=-\frac{s}{s-1} \text{ or } \mathcal{U}(s)=-\left(\frac{\sigma+it}{\sigma-1+it}\right). \quad (4.6)$$

Given $|\mathcal{U}(s)| < 1$ to the right of the critical line, and larger than it to the left, the equation (4.6) can only hold on the critical line.

With regard to the location of those zeros on the critical line, using equation (3.30) we find that these are given by

$$(2s-1)\mathcal{T}_+(s) + \mathcal{T}_-(s) = 0 \iff \tan[\arg \xi_1(1+2it)] = -2t \tag{4.7}$$

and

$$\mathcal{T}_+(s) + (2s-1)\mathcal{T}_-(s)] = 0 \iff \tan[\arg \xi_1(1+2it)] = \frac{1}{2t}. \tag{4.8}$$

Given $\arg \xi_1(1+2it)$ is monotonic increasing with t beyond $t \simeq 2.94$, these zeros alternate and have the same distribution function. As t increases, the zeros of (4.7) tend towards the zeros of $\mathcal{T}_+(s)$, and the zeros of (4.8) tend towards the zeros of $\mathcal{T}_-(s)$. □

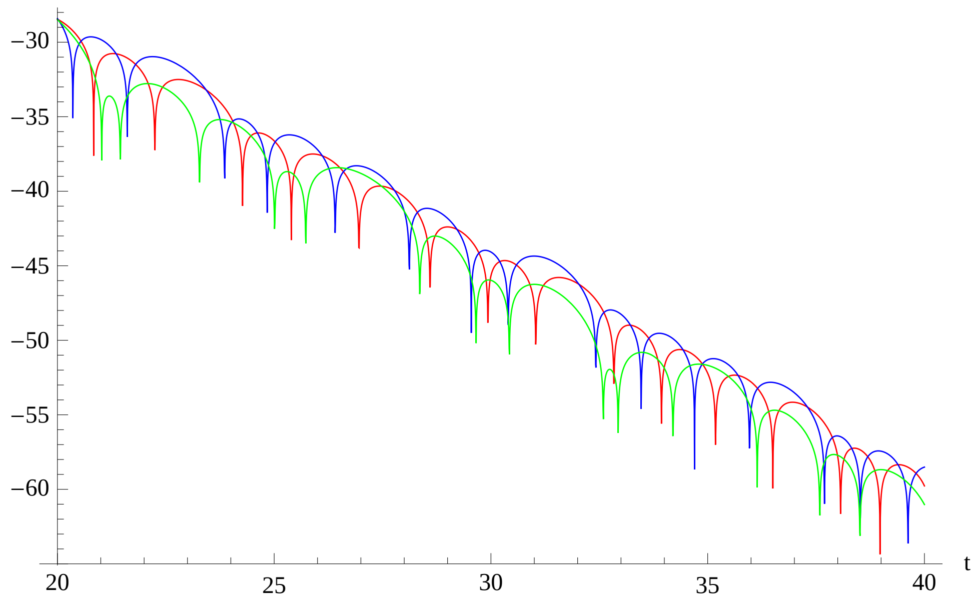


Figure 4. Left: Plot of the logarithmic modulus of the functions $\mathcal{T}_+(s) + (2s-1)\mathcal{T}_-(s)$ (red), $(2s-1)\mathcal{T}_+(s) + \mathcal{T}_-(s)$ (blue) and $\mathcal{C}(0,1;s)\Gamma(s)/\pi^s$ (green) along the critical line.

In Fig. 4 we show as an illustration of the results of Theorem 4.1 the variation along the critical line of the logarithmic moduli of $\mathcal{T}_+(s) + (2s-1)\mathcal{T}_-(s)$, $(2s-1)\mathcal{T}_+(s) + \mathcal{T}_-(s)$, and $\mathcal{C}(0,1;s)\Gamma(s)/\pi^s$. It can be seen that all three functions have similar exponential decay, and similar distributions of zeros, with for example the numbers of zeros of each in the range of t from 0 to 100 being 79. The alternating property of the zeros of the first two functions is also evident. In Fig. 5 we illustrate the last remark in Theorem 4.1: the alignment of zeros between the pair $\mathcal{T}_+(s) + (2s-1)\mathcal{T}_-(s)$, $\mathcal{T}_-(s)$, and between the second pair $(2s-1)\mathcal{T}_+(s) + \mathcal{T}_-(s)$, $\mathcal{T}_+(s)$ is clear.

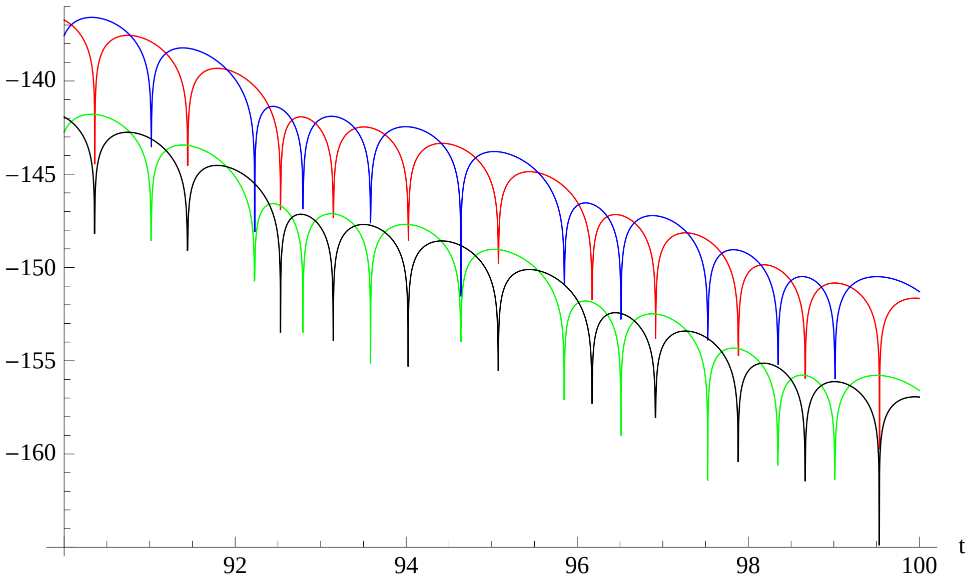


Figure 5. Left: Plot of the logarithmic modulus of the functions $\mathcal{T}_+(s) + (2s-1)\mathcal{T}_-(s)$ (red), $(2s-1)\mathcal{T}_+(s) + \mathcal{T}_-(s)$ (blue), $\mathcal{T}_+(s)$ (green) and $\mathcal{T}_-(s)$ (black) along the critical line.

Theorem 4.1 gives an example of a Macdonald function series, all of whose zeros lie on the critical line:

$$(1-s)\mathcal{K}(1,1;s) - s\mathcal{K}(1,1;1-s) = \sum_{p_1,p_2=1}^{\infty} \left(\frac{p_2}{p_1}\right)^{s-1/2} (p_1p_2\pi)[(1-s)K_{s+1/2}(2\pi p_1p_2) - sK_{3/2-s}(2\pi p_1 p_2)]. \tag{4.9}$$

Corollary 4.2. *If* $\mathcal{V}_{\mathcal{K}}(1,1;s_0) = -\mathcal{V}(s_0)$ *then* $\mathcal{C}(0,1;s_0)\mathcal{K}_-(1,1;s_0) = 0$.

Proof. We start with equations (3.20, 3.21) and obtain two expressions for $\mathcal{C}(0,1,s)$:

$$\frac{\mathcal{K}(1,1;s) - \mathcal{K}(1,1;1-s)}{s_-\mathcal{T}_+(s)} = \frac{\Gamma(s)\mathcal{C}(0,1;s)}{8\pi^s\mathcal{T}_+(s)} - 1, \tag{4.10}$$

and

$$\frac{\mathcal{K}(1,1;s) + \mathcal{K}(1,1;1-s)}{s_-\mathcal{T}_-(s)} = \frac{\Gamma(s)\mathcal{C}(0,1;s)}{16\pi^s s_-\mathcal{T}_-(s)} + 1. \tag{4.11}$$

Dividing these,

$$\mathcal{V}_{\mathcal{K}}(1,1;s)\frac{\mathcal{T}_-(s)}{\mathcal{T}_+(s)} = \frac{\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{8\pi^s\mathcal{T}_+(s)} - 1\right]}{\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{16\pi^s s_-\mathcal{T}_-(s)} + 1\right]} \tag{4.12}$$

We now use the first element of equation (3.34) applying at $s = s_0$, to give

$$-1\left[\frac{\Gamma(s_0)\mathcal{C}(0,1;s_0)}{16\pi^{s_0}(s_0-1/2)\mathcal{T}_-(s_0)} + 1\right] = \left[\frac{\Gamma(s_0)\mathcal{C}(0,1;s_0)}{8\pi^{s_0}\mathcal{T}_+(s_0)} - 1\right] \tag{4.13}$$

This leads to

$$\frac{\Gamma(s_0)\mathcal{C}(0,1;s_0)}{8\pi^{s_0}}\left[\frac{\mathcal{T}_+(s_0) + 2(s_0-1/2)\mathcal{T}_-(s_0)}{2\mathcal{T}_+(s_0)(s_0-1/2)\mathcal{T}_-(s_0)}\right] = 0, \tag{4.14}$$

or $\mathcal{C}(0,1;s_0)\mathcal{K}_-(1,1;s_0)=0$ as asserted. □

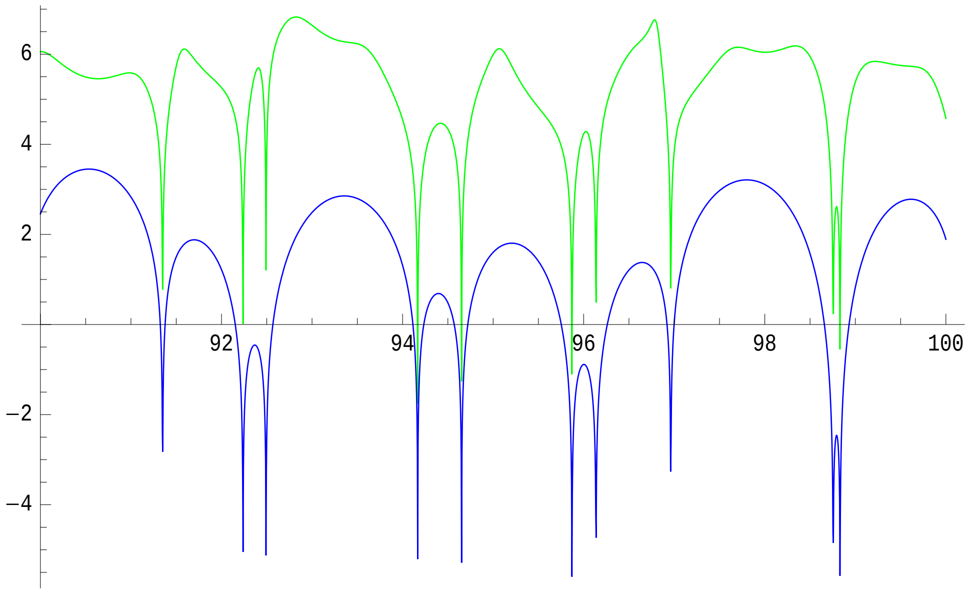


Figure 6. Left: Plot of the logarithmic modulus of the functions $\mathcal{C}(0,1;s)$ (blue) and $[\mathcal{U}_\mathcal{K}(1,1;s)+\mathcal{U}(s)]/[10^{-6}\exp(\pi t/2)\mathcal{K}_-(1,1;s)]$ (green) along the critical line.

We illustrate Corollary 4.2 in Fig. 6. Here we divide out a suitably scaled factor of $\mathcal{K}_-(1,1;s)$ from $\mathcal{U}_\mathcal{K}(1,1;s)+\mathcal{U}(s)$ and compare with $\mathcal{C}(0,1;s)$. The two functions have clearly the same zeros, but differ between them.

Theorem 4.3. *The Riemann Hypothesis for $\mathcal{C}(0,1;s)$ holds if and only if $\mathcal{V}_\mathcal{K}(1,1;s)$ is pure imaginary at every zero.*

Proof. From equation (3.11), if at a point s_0 we have $\mathcal{C}(0,1;s_0)=0$, then

$$\mathcal{K}(1,1;s_0)=-\frac{(s_0-1/2)}{4}\xi_1(2s_0-1)=\frac{(s_0-1/2)}{2}[\mathcal{T}_-(s_0)-\mathcal{T}_+(s_0)], \tag{4.15}$$

and, using the functional equation for ξ_1,

$$\mathcal{K}(1,1;1-s_0)=\frac{(s_0-1/2)}{4}\xi_1(2s_0)=\frac{(s_0-1/2)}{2}[\mathcal{T}_-(s_0)+\mathcal{T}_+(s_0)]. \tag{4.16}$$

Dividing (4.15) by (4.16) and using (3.34), we find

$$\mathcal{U}_\mathcal{K}(1,1;s_0)=-\frac{\xi_1(2s_0-1)}{\xi_1(2s_0)}=-\mathcal{U}(s_0), \tag{4.17}$$

and its equivalent equation

$$\mathcal{V}_\mathcal{K}(1,1;s_0)=\frac{\mathcal{K}(1,1;s_0)-\mathcal{K}(1,1;1-s_0)}{\mathcal{K}(1,1;s_0)+\mathcal{K}(1,1;1-s_0)}=\frac{\mathcal{U}_\mathcal{K}(1,1;s_0)-1}{\mathcal{U}_\mathcal{K}(1,1;s_0)+1}=-\mathcal{V}(s_0). \tag{4.18}$$

Suppose now the Riemann Hypothesis holds for $\mathcal{C}(0,1;s)$. Then s_0 lies on the critical line, so that $|\mathcal{U}(s_0)| = 1$, and by (4.17), $|\mathcal{U}_\mathcal{K}(1,1;s_0)| = 1$. Now,

$$\mathcal{U}_\mathcal{K}(1,1;s) = \frac{1+\mathcal{V}_\mathcal{K}(1,1;s)}{1-\mathcal{V}_\mathcal{K}(1,1;s)}. \tag{4.19}$$

Hence, $|\mathcal{U}_\mathcal{K}(1,1;s_0)| = 1$ if and only if $\mathcal{V}_\mathcal{K}(1,1;s_))$ is pure imaginary.

Conversely, if $\mathcal{V}_\mathcal{K}(1,1;s_0)$ is pure imaginary and $\mathcal{C}(0,1;s_0) = 0$, $|\mathcal{U}(s_0)| = 1$ and s_0 must lie on the critical line. □

For the alternative symmetrised $\mathcal{K}$'s, we have at a zero s_0 of $\mathcal{C}(0,1,s)$,

$$\mathcal{K}_+(1,1;s_0) = (s_0 - 1/2)^2\left[\mathcal{T}_+(s_0) + \frac{\mathcal{T}_-(s_0)}{2(s_0-1/2)}\right], \tag{4.20}$$

and

$$\mathcal{K}_-(1,1;s_0) = -(s_0 - 1/2)^2\left[\mathcal{T}_-(s_0) + \frac{\mathcal{T}_+(s_0)}{2(s_0-1/2)}\right]. \tag{4.21}$$

Hence,

$$\frac{\mathcal{K}_+(1,1;s_0)}{\mathcal{K}_-(1,1;s_0)} = -\left\{\frac{\mathcal{V}(s_0)+1/[2(s_0-1/2)]}{1+\mathcal{V}(s_0)/[2(s_0-1/2)]}\right\}. \tag{4.22}$$

We can strengthen Theorem 4.3 using results proved by Lagarias and Suzuki (2006), in conjunction with equation (4.17). This relation applying when $\mathcal{C}(0,1;s_0) = 0$ can be written

$$\xi_1(2s_0) + \mathcal{U}_\mathcal{K}(1,1;1-s_0)\xi_1(2-2s_0) = 0. \tag{4.23}$$

We express this in a more generic form:

$$\xi_1(2s_0) + \mathcal{F}(s_0)\xi_1(2-2s_0) = 0, \tag{4.24}$$

where $\mathcal{F}(s)$ is a function obeying the equation $\mathcal{F}(1-s) = 1/\mathcal{F}(s)$. Lagarias and Suzuki (2006) prove that if (4.23) holds then s_0 must lie on the critical line, for three examples of $\mathcal{F}(s) : \mathcal{F}(s) = (1-s)/s$ (their Theorem 1), $\mathcal{F}(s) = (1-s)T^{(1-2s)}/s$ with $T \geq 1$ (their Theorem 2), and $\mathcal{F}(s) = -y^{(1-2s)}$ with $y \geq 1$ (their Theorem 3). In the third case, two real zeros can exist off the critical line, if $y > 7.055507+$. The real quantities T and y are parameters lending generality to the following result.

Theorem 4.4. *If s_0 is such that $\mathcal{C}(0,1;s_0) = 0$, then s_0 lies on the critical line if*

$$\frac{1}{2s_0-1}\log\left[\frac{(1-s_0)\mathcal{U}_\mathcal{K}(1,1;s_0)}{s_0}\right] \geq 1, \quad \text{or} \quad \frac{1}{2s_0-1}\log\left[-\mathcal{U}_\mathcal{K}(1,1;s_0)\right] \geq 1. \tag{4.25}$$

In the latter case, there can occur two exceptional zeros on the real line.

Proof. The proof follows immediately using equation (4.17) and Theorem 2 and 3 of Lagarias and Suzuki (2006). The first of (4.25) gives a quantity required to be real, denoted T, while the second gives a quantity required to be real, denoted by y. The result is only meaningful if s_0 is such that the quantities are in fact real. (This occurs on specific contours in the complex plane, but not in general.) □

While the Theorem 4.4 is of interest, in fact the conditions specified in equation (4.25) only apply on specific arcs in the complex plane. Lagarias and Suzuki (2006) also comment on an alternate method which gives information on the zeros of the functions

$$H(y;s)=p(s)\xi_1(2s)y^s+p(1-s)\xi_1(2-2s)y^{1-s}, \tag{4.26}$$

for $y\geq 1$, provided that $p(s)$ is a polynomial with real coefficients. (Note that the notation in equation (4.26) has been changed to conform with that of this work, and a typographic error has been corrected.) The alternate method shows that all but finitely many zeros of $H(y;s)$ lie on the critical line, that the zeros off the critical line are confined to a compact set independent of $y\geq 1$ and that their number is uniformly bounded for all $y\geq 1$. The method is not presented in their paper, but the wish to publish it elsewhere is expressed. The power and utility of this result is made manifest in the following result.

Theorem 4.5. *Assuming the result of Lagarias and Suzuki (2006) concerning the zeros of the function (4.26), the Riemann Hypothesis holds for the function $\mathcal{C}(0,1;s)$, and so for the functions $\zeta(s)$ and $L_{-4}(s)$, possibly with a finite number of exceptions lying in a compact set.*

Proof. We have from the proofs of Corollary 4.2 and Theorem 4.3 that a zero s_0 of $\mathcal{C}(0,1;s)$ can lie off the critical line if and only if $\mathcal{U}(s_0)=-\mathcal{U}_\mathcal{K}(1,1;s_0)$. From the definition (3.17), this is equivalent to

$$\xi_1(2s_0-1)+\mathcal{U}_\mathcal{K}(1,1;s_0)\xi_1(2s_0)=0. \tag{4.27}$$

We can apply the result of Lagarias and Suzuki (2006) then if we can establish that, if, for any candidate zero s_0 located (without loss of generality) to the right of the critical line,

$$\mathcal{U}_\mathcal{K}(1,1;s_0)=\frac{p_N(s_0)}{p_N(1-s_0)}y^{2s_0-1}, \tag{4.28}$$

for some polynomial $p_N(s)$ of finite degree N with real coefficients, and for some real $y\geq 1$. Note that $\mathcal{U}(s_0)$ is neither zero nor infinity, and its modulus is less than unity.

For our purpose, it is sufficient to take the case $N=2$, where we require two real coefficients α and β to be determined, or equivalently one complex zero s_1, with the scaling parameter y also to be chosen. We suppress arguments to render expressions more compact, and let

$$\tilde{\mathcal{U}_\mathcal{K}}=\mathcal{U}_\mathcal{K}(1,1;s_0)y^{1-2s_0}, \tag{4.29}$$

with the equation(4.28) becoming

$$\tilde{\mathcal{U}_\mathcal{K}}=\frac{[s_0^2+\alpha s_0+\beta]}{[(1-s_0)^2+\alpha(1-s_0)+\beta]}=\frac{(s_0-s_1)(s_0-\bar{s}_1)}{(1-s_0-s_1)(1-s_0-\bar{s}_1)}. \tag{4.30}$$

Let

$$\mathcal{L}_\mathcal{K}=(1-\tilde{\mathcal{U}_\mathcal{K}})s_0^2+2\tilde{\mathcal{U}_\mathcal{K}}s_0-\tilde{\mathcal{U}_\mathcal{K}}. \tag{4.31}$$

Then (4.30) becomes

$$\mathcal{L}_\mathcal{K}=\alpha[\tilde{\mathcal{U}_\mathcal{K}}(1-s_0)-s_0]+\beta[\tilde{\mathcal{U}_\mathcal{K}}]. \tag{4.32}$$

We combine (4.32) with its conjugate

$$\bar{\mathcal{L}}_{\mathcal{K}} = \alpha[\bar{\tilde{\mathcal{U}}}_{\mathcal{K}}(1-\bar{s}_0) - \bar{s}_0] + \beta[\bar{\tilde{\mathcal{U}}}_{\mathcal{K}}]. \tag{4.33}$$

We solve the two linear equations (4.32) and (4.33) to obtain the real coefficients:

$$\alpha = \frac{\Re[\mathcal{L}_{\mathcal{K}}]\Im[\tilde{\mathcal{U}}_{\mathcal{K}}-1] - \Im[\mathcal{L}_{\mathcal{K}}]\Re[\tilde{\mathcal{U}}_{\mathcal{K}}-1]}{\Re[\tilde{\mathcal{U}}_{\mathcal{K}}(1-s_0)-s_0]\Im[\tilde{\mathcal{U}}_{\mathcal{K}}-1] - \Im[\tilde{\mathcal{U}}_{\mathcal{K}}(1-s_0)-s_0]\Re[\tilde{\mathcal{U}}_{\mathcal{K}}-1]}, \tag{4.34}$$

and

$$\beta = \frac{\mathcal{L}_{\mathcal{K}} - \alpha[\tilde{\mathcal{U}}_{\mathcal{K}}(1-s_0)-s_0]}{\tilde{\mathcal{U}}_{\mathcal{K}}-1}. \tag{4.35}$$

For $\tilde{\mathcal{U}}_{\mathcal{K}} << 1$, it is easy to see that the denominators in (4.34) and (4.34) are non-zero. The two zeros of the numerator in (4.30) are given by the complex conjugate pair

$$s_1 = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}. \tag{4.36}$$

The real factor y is required to be not less than unity. It serves to scale the distance between s_0 and s_1: the bigger y, the closer s_1 is to s_0. For small $|\tilde{\mathcal{U}}_{\mathcal{K}}|$, the simple first-order estimate for s_1 is $s_0 + (1-2\sigma_0)\tilde{\mathcal{U}}_{\mathcal{K}}$. If we prescribe the value of $|\tilde{\mathcal{U}}_{\mathcal{K}}|$, then y is given by

$$y = \left[\frac{|\tilde{\mathcal{U}}_{\mathcal{K}}|}{|\mathcal{U}_{\mathcal{K}}|}\right]^{\frac{1}{1-2\sigma_0}} \tag{4.37}$$

□

As an illustration of the concluding remarks in Theorem 4.5, we show in Fig. 7 a plot of $s - s_1$, for $\Re(s_0) = 0.51$ and t varying between 90 and 100. The parameter y has been chosen using (4.37) to keep $|\tilde{\mathcal{U}}_{\mathcal{K}}|$ fixed at the value 0.0108 as t varies. The first order estimate for $s_0 - s_1$ is then $0.02 \times 0.0108 = 2.16 \times 10^{-4}$ independent of t_0; the red circle shows the accurate values for $s_0 - s_1$ calculated from (4.35) lie close to a circle of radius 2.170×10^{-4}, independent of t_0.

5. The Sums Z

It is useful to introduce a new notation for certain lattice sums which involve mixtures of powers of both sin and cos of the angle θ_{p,p_2}:

$$\mathcal{Z}(2n,2m;s) = \sum_{p_1,p_2}{}' \frac{p_1^{2n} p_2^{2m}}{(p_1^2+p_2^2)^{n+m+s}}. \tag{5.1}$$

Then from (2.3)

$$\mathcal{Z}(2n,0;s) = \mathcal{C}(2n,1;s) = \frac{\pi^s}{\Gamma(s+n)}\left[2\frac{\Gamma(s+n-1/2)}{\Gamma(s-1/2)}\xi_1(2s-1) + 8\mathcal{K}(n,n;s)\right], \tag{5.2}$$

or

$$\frac{\Gamma(s+n)}{\pi^s}\mathcal{Z}(2n,0;s) = 2\frac{\Gamma(s+n-1/2)}{\Gamma(s-1/2)}\xi_1(2s-1) + 8\mathcal{K}(n,n;s). \tag{5.3}$$

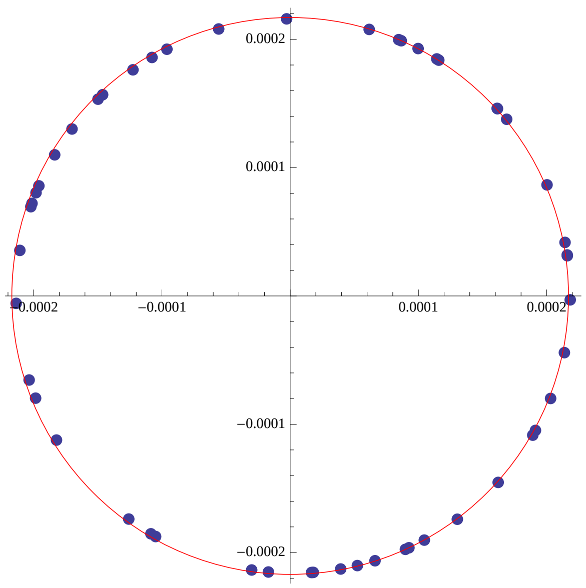


Figure 7. Plot of $s_0 - s_1$, for $\sigma_0 = 0.51$ and t_0 ranging from 90 to 100 in steps of 0.2. The red line is a circle of radius 2.170×10^{-4}.

The lattice sums $\mathcal{Z}(2n, 2m; s)$ obey the recurrence relation

$$\mathcal{Z}(2n, 2m; s) = \mathcal{Z}(2n+2, 2m; s) + \mathcal{Z}(2n, 2m+2; s), \tag{5.4}$$

or a more general form which expresses any $\mathcal{Z}(2n, 2m; s)$ in terms of $\mathcal{Z}(2l, 0; s)$:

$$\mathcal{Z}(2n, 2m; s) = \sum_{l=0}^{m} {}^{m}C_l (-1)^{m-l} \mathcal{Z}(2n+2m-2l, 0; s). \tag{5.5}$$

We can start a recurrent solution for the sums $\mathcal{Z}(2n, 2m; s)$ using the starting values

$$\text{Orders } 0, 2: \mathcal{Z}(0, 0; s) = \mathcal{C}(0, 1; s), \quad \mathcal{Z}(0, 2; s) = \mathcal{Z}(2, 0; s) = \frac{1}{2}\mathcal{C}(0, 1; s). \tag{5.6}$$

The recurrent solution expresses each $\mathcal{Z}(2n, 2m; s)$ of order $2n+2m$ in terms of sums $\mathcal{Z}(4l, 0; s)$ of equal or lower order. Its results of low order will now be given.

$$\text{Order } 4: \mathcal{Z}(2, 2; s) = \frac{1}{2}\mathcal{C}(0, 1; s) - \mathcal{Z}(4, 0; s). \tag{5.7}$$

$$\text{Order } 6: \quad \mathcal{Z}(2, 4; s) = \frac{1}{2}\mathcal{Z}(2, 2; s) = \frac{1}{4}\mathcal{C}(0, 1; s) - \frac{1}{2}\mathcal{Z}(4, 0; s); \tag{5.8}$$

$$\mathcal{Z}(6, 0; s) = \frac{3}{2}\mathcal{Z}(4, 0; s) - \frac{1}{4}\mathcal{C}(0, 1; s). \tag{5.9}$$

$$\text{Order } 8: \quad \mathcal{Z}(2, 6; s) = \frac{3}{2}\mathcal{Z}(4, 0; s) - \mathcal{Z}(8, 0; s) - \frac{1}{4}\mathcal{C}(0, 1; s); \tag{5.10}$$

$$\mathcal{Z}(4, 4; s) = -2\mathcal{Z}(4, 0; s) + \mathcal{Z}(8, 0; s) + \frac{1}{2}\mathcal{C}(0, 1; s). \tag{5.11}$$

$$\text{Order 10}: \quad \mathcal{Z}(10,0;s) = \frac{5}{2}\mathcal{Z}(8,0;s) - \frac{5}{2}\mathcal{Z}(4,0;s) + \frac{1}{2}\mathcal{C}(0,1;s); \tag{5.12}$$

$$\mathcal{Z}(2,8;s) = -\frac{3}{2}\mathcal{Z}(8,0;s) + \frac{5}{2}\mathcal{Z}(4,0;s) - \frac{1}{2}\mathcal{C}(0,1;s); \tag{5.13}$$

$$\mathcal{Z}(4,6;s) = \frac{1}{2}\mathcal{Z}(8,0;s) - \mathcal{Z}(4,0;s) + \frac{1}{4}\mathcal{C}(0,1;s). \tag{5.14}$$

These results are sufficient to establish the pattern of the $\mathcal{Z}(2n,2m;s)$: a linear combination of all $\mathcal{Z}(4m,0;s)$ up to the order in question, followed by a term proportional to $\mathcal{C}(0,1;s)$. The $\mathcal{Z}(4m+2,0;s)$ can all be determined explicitly in terms of $\mathcal{Z}(4m,0;s)$ of lower order and $\mathcal{C}(0,1;s)$.

6. The System of Order 4

We start with equation (2.3) for $n=2$:

$$\sum_{(p_1,p_2)}{}' \frac{p_1^4}{(p_1^2+p_2^2)^{s+2}} = \mathcal{C}(4,1;s) = \frac{2\sqrt{\pi}\Gamma(s+3/2)\zeta(2s-1)}{\Gamma(s+2)} + \frac{8\pi^s}{\Gamma(s+2)}\mathcal{K}(2,2;s). \tag{6.1}$$

We use the first of the recurrence relations (2.7) to obtain

$$\mathcal{K}(2,2;s) = \mathcal{K}(2,0;s) + (s+1/2)\mathcal{K}(1,1;s), \tag{6.2}$$

We use (3.11) in (7.8) and simplify to obtain

$$\mathcal{C}(4,1;s) = \frac{(s+1/2)}{2(s+1)}\mathcal{C}(0,1;s) + \frac{8\pi^s}{\Gamma(s+2)}\mathcal{K}(2,0;s) = \mathcal{Z}(4,0;s). \tag{6.3}$$

The other elements of the system of order 4 can be evaluated from results given in McPhedran *et al* (2010). We have

$$\mathcal{C}(2,2;s) = \sum_{p_1,p_2}{}' \frac{(p_1^2-p_2^2)^2}{(p_1^2+p_2^2)^{s+2}} = 4\mathcal{C}(4,1;s) - \mathcal{C}(0,1;s), \tag{6.4}$$

where in the primed sum p_1 and p_2 run over all integers, excluding the origin (0,0). This gives

$$\mathcal{C}(2,2;s) = \left(\frac{s}{s+1}\right)\mathcal{C}(0,1;s) + \frac{32\pi^s}{\Gamma(s+2)}\mathcal{K}(2,0;s). \tag{6.5}$$

Also,

$$\mathcal{S}(2,2;s) = \sum_{p_1,p_2}{}' \frac{4p_1^2p_2^2}{(p_1^2+p_2^2)^{s+2}} = -4\mathcal{C}(4,1;s) + 2\mathcal{C}(0,1;s), \tag{6.6}$$

and

$$\mathcal{S}(2,2;s) = \left(\frac{1}{s+1}\right)\mathcal{C}(0,1;s) - \frac{32\pi^s}{\Gamma(s+2)}\mathcal{K}(2,0;s) = 4\mathcal{Z}(2,2;s). \tag{6.7}$$

The most important member of the family is

$$\mathcal{C}(1,4;s)=\sum_{p_1,p_2}{}' \frac{\cos 4\theta_{p_1,p_2}}{(p_1^2+p_2^2)^{s+2}}=\mathcal{C}(2,2;s)-\mathcal{S}(2,2;s), \tag{6.8}$$

where $\theta_{p_1,p_2}=\arg(p_1+ip_2)$. This gives

$$\mathcal{C}(1,4;s)=\left[\frac{(s-1)}{s+1}\right]\mathcal{C}(0,1;s)+\frac{64\pi^s}{\Gamma(s+2)}\mathcal{K}(2,0;s). \tag{6.9}$$

These sums may be re-expressed in terms of $\mathcal{C}(0,1;s)$ and $\mathcal{C}(1,4;s)$:

$$\mathcal{C}(4,1;s)=\frac{3}{8}\mathcal{C}(0,1;s)+\frac{1}{8}\mathcal{C}(1,4;s), \tag{6.10}$$

$$\mathcal{C}(2,2;s)=\frac{1}{2}\mathcal{C}(0,1;s)+\frac{1}{2}\mathcal{C}(1,4;s), \tag{6.11}$$

$$\mathcal{S}(2,2;s)=\frac{1}{2}\mathcal{C}(0,1;s)-\frac{1}{2}\mathcal{C}(1,4;s), \tag{6.12}$$

and

$$64\mathcal{K}(2,0;s)=s(1-s)\frac{\Gamma(s)}{\pi^s}\mathcal{C}(0,1;s)-\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(1,4;s). \tag{6.13}$$

The equation (6.13) reflects the symmetry of $\mathcal{K}(2,0;s)$ under $s\to 1-s$.

Functional equations for these sums may be derived using this symmetry property of $\mathcal{K}(2,0;s)$. For example, from (7.16) we obtain

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(1,4;s)=(s-1)s\,\frac{\Gamma(s)}{\pi^s}\mathcal{C}(0,1;s)+64\,\mathcal{K}(2,0;s). \tag{6.14}$$

The right-hand side of (6.14) is unaltered if s is replaced by $1-s$, giving a new derivation of the functional equation for $\mathcal{C}(1,4;s)$.

The equations corresponding to (6.14) for the other sums of order four consist of a coefficient of $\mathcal{C}(0,1;s)$ which is neither even nor odd under $s\to 1-s$. These are written below with the first two terms on the right-hand side being even, and the third odd.

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(2,2;s)=[(s-1/2)^2+1/4]\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]+32\mathcal{K}(2,0;s)+(s-1/2)\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]. \tag{6.15}$$

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{S}(2,2;s)=\frac{1}{2}\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]-32\mathcal{K}(2,0;s)+(s-\frac{1}{2})\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]. \tag{6.16}$$

$$\begin{aligned}\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(4,1;s) &= \frac{[(s-1/2)^2+1/2]}{2}\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]+8\mathcal{K}(2,0;s)+\frac{3(s-1/2)}{4}\left[\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right]\\ &= \frac{\Gamma(s+2)}{\pi^s}\mathcal{Z}(4,0;s).\end{aligned} \tag{6.17}$$

Theorem 6.1. *The Riemann hypothesis for $\mathcal{C}(0,1;s)$ is equivalent to the proposition that, for arbitrary complex numbers α, β, γ subject only to the restriction $\alpha+\beta+3\gamma/4\neq 0$, and for all s not on the critical line,*

$$\frac{\Gamma(s+2)}{\pi^s}[\alpha\mathcal{C}(2,2;s)+\beta\mathcal{S}(2,2;s)+\gamma\mathcal{C}(4,1;s)]\neq\frac{\Gamma(3-s)}{\pi^{1-s}}[\alpha\mathcal{C}(2,2;1-s)+\beta\mathcal{S}(2,2;1-s)+\gamma\mathcal{C}(4,1;1-s)]. \quad (6.18)$$

Proof. We have

$$\alpha\mathcal{C}(2,2;s)+\beta\mathcal{S}(2,2;s)+\gamma\mathcal{C}(4,1;s)=\left[\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{3}{8}\gamma\right]\mathcal{C}(0,1;s)+\left[\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{8}\gamma\right]\mathcal{C}(1,4;s). \quad (6.19)$$

We multiply this expression by $\Gamma(s+2)/\pi^s$, giving a second term on the right-hand side which is even under $s\to 1-s$, and so cancels out in (6.18). The first term has both an even part and an odd part, with only the odd part contributing to (6.18). We disregard the trivial cases $s=0$, and $s=-1$, and deduce that, for the left-hand and right-hand sides of (6.18) to be equal, it is necessary and sufficient that

$$\left(\alpha+\beta+\frac{3}{4}\gamma\right)\frac{\Gamma(s)}{\pi^s}\mathcal{C}(0,1;s)=0, \quad (6.20)$$

i.e, that $\mathcal{C}(0,1;s)=0$. □

Corollary 6.2. *The Riemann hypothesis for $\mathcal{C}(0,1;s)$ implies that for all s not on the critical line, the following three inequalities hold:*

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(2,2;s)\neq\frac{\Gamma(3-s)}{\pi^{1-s}}\mathcal{C}(2,2;1-s), \quad (6.21)$$

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{S}(2,2;s)\neq\frac{\Gamma(3-s)}{\pi^{1-s}}\mathcal{S}(2,2;1-s), \quad (6.22)$$

and

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(4,1;s)\neq\frac{\Gamma(3-s)}{\pi^{1-s}}\mathcal{C}(4,1;1-s). \quad (6.23)$$

If any of the three inequalities fails, the Riemann hypothesis for $\mathcal{C}(0,1;s)$ fails.

Proof. We recall the identities $\mathcal{C}(1,4;s)=\mathcal{C}(2,2;s)-\mathcal{S}(2,2;s)$, $\mathcal{C}(0,1;s)=\mathcal{C}(2,2;s)+\mathcal{S}(2,2;s)$. From Theorem 4.5, if we let $\gamma=0$ and $\alpha=-\beta+\delta$, and use the functional equation (6.14) for $\mathcal{C}(1,4;s)$, we obtain (6.21). Setting $\gamma=0$ and $\beta=-\alpha+\delta$, we obtain (6.22). Putting $\beta=-\alpha$ and using $\gamma\neq 0$, we obtain (6.23).

If any of the three inequalities is replaced by an equality, then from the vanishing of the third term on the right-hand sides of one of equations (6.15-6.17), $\mathcal{C}(0,1;s)=0$ for an s off the critical line. □

As a trivial extension of Corollary 6.2, we have:

Corollary 6.3. *The Riemann hypothesis for $\mathcal{C}(0,1;s)$ implies that for s_0 not on the critical line:*
if $\mathcal{C}(2,2;s_0)=0$, then $\mathcal{C}(2,2;1-s_0)\neq 0$;
if $\mathcal{S}(2,2;s_0)=0$, then $\mathcal{S}(2,2;1-s_0)\neq 0$;
if $\mathcal{C}(4,1;s_0)=0$, then $\mathcal{C}(4,1;1-s_0)\neq 0$.

If any of the three inequalities fails, the Riemann hypothesis for $\mathcal{C}(0,1;s)$ fails.

The previous Theorem and Corollaries have the following equivalents for $\mathcal{C}(1,4;s)$:

Theorem 6.4. *The Riemann hypothesis for $\mathcal{C}(1,4;s)$ is equivalent to the proposition that, for arbitrary complex numbers α', β', γ' subject only to the restriction $\alpha'-\beta'+\gamma'/4\neq 0$, and for all s not on the critical line,*

$$\frac{\Gamma(s)}{\pi^s}[\alpha'\mathcal{C}(2,2;s)+\beta'\mathcal{S}(2,2;s)+\gamma'\mathcal{C}(4,1;s)]\neq\frac{\Gamma(1-s)}{\pi^{1-s}}[\alpha'\mathcal{C}(2,2;1-s)+\beta'\mathcal{S}(2,2;1-s)+\gamma'\mathcal{C}(4,1;1-s)]. \tag{6.24}$$

Corollary 6.5. *The Riemann hypothesis for $\mathcal{C}(1,4;s)$ implies that for all s not on the critical line, the following three inequalities hold:*

$$\frac{\Gamma(s)}{\pi^s}\mathcal{C}(2,2;s)\neq\frac{\Gamma(1-s)}{\pi^{1-s}}\mathcal{C}(2,2;1-s), \tag{6.25}$$

$$\frac{\Gamma(s)}{\pi^s}\mathcal{S}(2,2;s)\neq\frac{\Gamma(1-s)}{\pi^{1-s}}\mathcal{S}(2,2;1-s), \tag{6.26}$$

and

$$\frac{\Gamma(s)}{\pi^s}\mathcal{C}(4,1;s)\neq\frac{\Gamma(1-s)}{\pi^{1-s}}\mathcal{C}(4,1;1-s). \tag{6.27}$$

If any of the three inequalities fails, the Riemann hypothesis for $\mathcal{C}(1,4;s)$ fails.

These are proved by multiplying (6.19) by $\Gamma(s)/\pi^s$ rather than by $\Gamma(s+2)/\pi^s$. This results in the cancellation of $\mathcal{C}(0,1;s)$ rather than $\mathcal{C}(1,4;s)$ when antisymmetric combinations are formed.

Corollary 6.6. *The Riemann hypothesis for $\mathcal{C}(1,4;s)$ implies that for s_0 not on the critical line:*
if $\mathcal{C}(2,2;s_0)=0$, then $\mathcal{C}(2,2;1-s_0)\neq 0$;
if $\mathcal{S}(2,2;s_0)=0$, then $\mathcal{S}(2,2;1-s_0)\neq 0$;
if $\mathcal{C}(4,1;s_0)=0$, then $\mathcal{C}(4,1;1-s_0)\neq 0$.
If any of the three inequalities fails, the Riemann hypothesis for $\mathcal{C}(1,4;s)$ fails.

The Corollaries 6.3 and 6.6 when combined give

Theorem 6.7. *If any of the lattice sums $\mathcal{C}(2,2;s)$, $\mathcal{S}(2,2;s)$, $\mathcal{C}(4,1;s)$ is zero for a non-trivial complex point s_0 and in addition is zero for $1-s_0$, then all are zero at both points, as are $\mathcal{C}(0,1;s)$, $\mathcal{K}(2,0;s)$, $\mathcal{C}(1,4;s)$, and all $\mathcal{Z}(2n,2m;s)$, with $2n+2m\leq 6$.*

In Fig. 8 we show plots of $\log|\mathcal{C}(2,2;s)|$ and $\log|\mathcal{C}(0,1;s)|$ in a region of the critical line. Note the correspondence between zeros of $\mathcal{C}(0,1;s)$ and near-zeros of $\mathcal{C}(1,4;s)$. We also show a contour plot of the argument of $\mathcal{C}(2,2;s)$. In the region shown there are zeros of $\mathcal{C}(2,2;s)$ both to the left and the right of the critical line- e.g, at $s=0.531596+12.43i$ and $s=0.467428+14.8151i$.

Theorem 6.7 is in keeping with an argument given in Section 9 of McPhedran *et al* (2008). Suppose one of the three sums is zero for s_0 and $1-s_0$. Then from the previous Theorem, all are, as are all sums $\mathcal{Z}$ up to order 6. Thus, the general sum with a sixth-order numerator is zero:

$$\sum_{p_1,p_2}{}' \frac{P_6(p_1,p_2)}{(p_1^2+p_2^2)^{s_0+2}}=0, \tag{6.28}$$

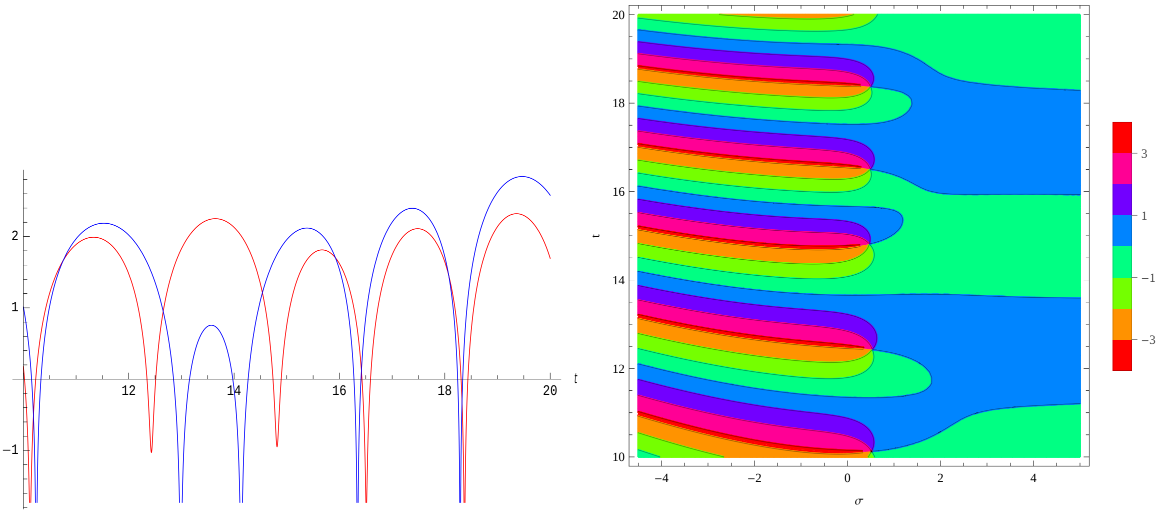


Figure 8. Left: Plot of the logarithmic modulus of the functions $\mathcal{C}(2,2;s)$ (red), and $\mathcal{C}(0,1;s)$ (blue) along the critical line. Right: Contour plot of the argument of $\mathcal{C}(2,2;s)$ as a function of σ and t.

where, with $c_{4,0}$, α_4, β_4 and γ_4 being arbitrary real or complex quantities,

$$P_6(p_1,p_2) = c_{6,0}p_1^6+c_{4,2}p_1^4p_2^2+c_{2,4}p_1^2p_2^4+c_{0,6}p_2^6 = c_{6,0}(p_1^2-\alpha_6 p_2^2)(p_1^2-\beta_6 p_2^2)(p_1^2-\gamma_6 p_2^2). \tag{6.29}$$

Note that this sum is uniquely defined by the Macdonald function expansion of $\mathcal{Z}(4,0;s)$ and by that of $\mathcal{Z}(2,0;s)$. We can choose the quantities α_6, β_6 and γ_6 at will to alter the contributions of any line running through the double array of points (p_1,p_2) without altering the value of the sum, with the sums over such lines being absolutely convergent for s to the right of the critical line. This argues against the existence of a point s_0 invalidating the Riemann hypotheses for both $\mathcal{C}(0,1;s)$ and $\mathcal{C}(1,4;s)$.

7. The System of Order 6

From McPhedran *et al* (2010), we have

$$\mathcal{C}(6,1;s) = \mathcal{Z}(6,0;s) = \frac{3}{2}\mathcal{C}(4,1;s) - \frac{1}{4}\mathcal{C}(0,1;s). \tag{7.1}$$

Using (6.17), we find

$$\mathcal{C}(6,1;s) = \left[\frac{(2s+1/2)}{4(s+1)}\right]\mathcal{C}(0,1;s) + \frac{12\pi^s}{\Gamma(s+2)}\mathcal{K}(2,0;s). \tag{7.2}$$

We can rewrite (7.2) in a symmetrized form suitable for constructing functional equations

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(6,1;s) = \left[\frac{4(s-1/2)^2+3/2}{8}\right]\left(\frac{\Gamma(s)\mathcal{C}(0,1;s)}{\pi^s}\right)+12\mathcal{K}(2,0;s)+\left[\frac{5(s-1/2)}{8}\right]\mathcal{C}(0,1;s). \tag{7.3}$$

The first two terms on the right-hand side of (7.3) are even under $s \to 1-s$, while the third is odd.

From McPhedran *et al* (2010) we have

$$\mathcal{Z}(4,2;s) = \frac{1}{8}\mathcal{S}(2,2;s), \quad \mathcal{C}(2,3;s) = \mathcal{S}(2,3;s) = \frac{1}{2}\mathcal{C}(0,1;s). \tag{7.4}$$

The first of these gives

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{Z}(4,2;s) = \frac{1}{16}\left[\frac{\mathcal{C}(0,1;s)\Gamma(s)}{\pi^s}\right] - 4\mathcal{K}(2,0;s) + \frac{(s-1/2)}{8}\left[\frac{\mathcal{C}(0,1;s)\Gamma(s)}{\pi^s}\right]. \tag{7.5}$$

Using the recurrence relation (2.7),

$$\mathcal{K}(2,2;s) = \mathcal{K}(2,0;s) + \frac{1}{4}\left[\mathcal{T}_+(s) + (1+2s_-^2)\mathcal{K}(0,0;s) + 2s_-\mathcal{T}_-(s))\right] + \frac{s_-}{4}\left[3\mathcal{K}(0,0;s) + \mathcal{T}_+(s) + 2s_-\mathcal{T}_-(s)\right]. \tag{7.6}$$

Theorem 7.1. *Assuming the result of Lagarias and Suzuki (2006) concerning the zeros of the function (4.26), the Riemann Hypothesis holds for the function $\mathcal{C}(1,4;s)$, possibly with a finite number of exceptions lying in a compact set.*

Proof. The first step in the proof is to construct a function which can play the role of $\mathcal{K}(1,1;s)$ in Theorem 4.5. We do this by combining the various expressions for $\mathcal{C}(4,1;s)$ and $\mathcal{C}(6,1;s)$ in such a way as to eliminate $\mathcal{C}(0,1;s)$. Now,

$$\mathcal{C}(4,1;s) = \frac{2\sqrt{\pi}\Gamma(s+3/2)}{\Gamma(s+2)}\zeta(2s-1) + \frac{8\pi^s}{\Gamma(s+2)}\mathcal{K}(2,2;s) = \frac{3}{8}\mathcal{C}(0,1;s) + \frac{1}{8}\mathcal{C}(1,4;s). \tag{7.7}$$

and

$$\mathcal{C}(6,1;s) = \frac{2\sqrt{\pi}\Gamma(s+5/2)}{\Gamma(s+3)}\zeta(2s-1) + \frac{8\pi^s}{\Gamma(s+3)}\mathcal{K}(3,3;s) = \frac{5}{16}\mathcal{C}(0,1;s) + \frac{3}{16}\mathcal{C}(1,4;s). \tag{7.8}$$

Taking the appropriate combination of equations (7.7) and (7.8) to eliminate $\mathcal{C}(0,1;s)$, we obtain

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(1,4;s) = 4\xi_1(2s-1)\left[\frac{6(s+3/2)(s+1/2)(s-1/2)}{s+2} - 5(s+1/2)(s-1/2)\right] + 16\mathcal{K}_{4,6}(s), \tag{7.9}$$

or

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(1,4;s) = 4\xi_1(2s-1)(s+1/2)(s-1/2)\left[\frac{s-1}{s+2}\right] + 16\mathcal{K}_{4,6}(s), \tag{7.10}$$

where

$$\mathcal{K}_{4,6}(s) = \frac{6\mathcal{K}(3,3;s)}{s+2} - 5\mathcal{K}(2,2;s). \tag{7.11}$$

Replacing s by $1-s$ in (7.10), the left-hand side is unaltered, with the equation becoming

$$\frac{\Gamma(s+2)}{\pi^s}\mathcal{C}(1,4;s)=4\xi_1(2s)(3/2-s)(1/2-s)\left[\frac{-s}{3-s}\right]+16\mathcal{K}_{4,6}(1-s). \quad (7.12)$$

The second step is to note that the antisymmetric combination $\mathcal{K}_{4,6}(s)-\mathcal{K}_{4,6}(1-s)$ obeys an identity from (7.10) and (7.12):

$$\mathcal{K}_{4,6}(s)-\mathcal{K}_{4,6}(1-s)=-\frac{s(3/2-s)(1/2-s)}{4(3-s)}\xi_1(2s)-\frac{(s-1)(s-1/2)(s+1/2)}{4(s+2)}\xi_1(2s-1). \quad (7.13)$$

This antisymmetric combination will be zero when

$$\frac{\xi_1(2s-1)}{\xi_1(2s)}\left[\frac{(s-3)(s-1)}{(s+2)s}\right]\left[\frac{(s+1/2)}{(s-3/2)}\right]=1. \quad (7.14)$$

The numerator in equation (7.14) has all its zeros to the right of the critical line, apart from the single zero at $s=-1/2$.The denominator has all its zeros to the left of the critical line, apart from the single zero at s=3/2. From the treatments given in Lagarias and Suzuki (2006) and McPhedran and Poulton (2013), the function on the left-hand side has modulus unity everywhere on the critical line, with the modulus being greater than unity to the left of the critical line, and smaller than unity to its right, with the exception of a region with $|t|$ small. In this case, the exceptional region runs up to $|t|\simeq 1.33634$, where the imaginary part of this function has a turning point on the critical line, with its modulus being around 0.68168. (Above this modulus, all other turning points of the real or imaginary parts correspond to the function reaching ± 1.)

The third step is to consider the zeros of $\mathcal{C}(1,4;s)$. If $\mathcal{C}(1,4;s_0)=0$, then from equation (7.10),

$$\xi_1(2s_0-1)(s_0+1/2)(s_0-1/2)\left[\frac{s_0-1}{s_0+2}\right]=-4\mathcal{K}_{4,6}(s_0), \quad (7.15)$$

while from equation (7.12),

$$\xi_1(2s_0)(3/2-s_0)(1/2-s_0)\left[\frac{s_0}{3-s_0}\right]=4\mathcal{K}_{4,6}(1-s_0). \quad (7.16)$$

Dividing (7.16) by (7.15) and rearranging gives

$$\xi_1(2s_0)+\left[\frac{\mathcal{K}_{4,6}(1-s_0)(s_0+1/2)(1-s_0)(3-s_0)}{\mathcal{K}_{4,6}(s_0)(3/2-s_0)s_0(s_0+2)}\right]\xi_1(2s_0-1)=0. \quad (7.17)$$

It may be verified that, if (7.17) holds, then we have, using equation (7.14),

$$[\mathcal{K}_{4,6}(s_0)-\mathcal{K}_{4,6}(1-s_0)]\left[\mathcal{K}_{4,6}(s_0)+\frac{\xi_1(2s_0-1)(s_0+1/2)(s_0-1/2)(s_0-1)}{4(s_0+2)}\right]=0, \quad (7.18)$$

i.e., either $\mathcal{K}_{4,6}(s_0)=\mathcal{K}_{4,6}(1-s_0)$ or $\mathcal{C}(1,4;s_0)=0$.

The fourth step is to adapt the method of Theorem 4.5 to the equation (7.17). We replace $\tilde{\mathcal{U}}$ by

$$\tilde{\mathcal{U}}_{\mathcal{K}46} = \frac{(3/2 - s_0)s_0(s_0+2)\mathcal{K}_{46}(s_0)y^{1-2s_0}}{(s_0+1/2)(1-s_0)(3-s_0)\mathcal{K}_{46}(1-s_0)}, \tag{7.19}$$

and

$$\mathcal{L}_{\mathcal{K}46} = (1 - \tilde{\mathcal{U}}_{\mathcal{K}46})s_0^2 + 2\tilde{\mathcal{U}}_{\mathcal{K}46}s_0 - \tilde{\mathcal{U}}_{\mathcal{K}46}. \tag{7.20}$$

The equations (4.34-4.36) may then be used to determine the parameters α, β and s_1 of the quadratic polynomial quotient, and equation (4.37) the scale parameter y. This enables the result of Lagarias and Suzuki (2006) to be applied, proving the stated result. □

The argument of the third step in Theorem is illustrated in Fig. 9: the zeros of the function corresponding to the red line are the unions of the zeros corresponding to $\mathcal{K}_{4,6}(s_0) - \mathcal{K}_{4,6}(1-s_0)$ (green line) or $\mathcal{C}(1,4;s_0)$ (blue line) . The argument of the fourth step is illustrated in Fig. 10, the equivalent of Fig. 7, but adapted to the different definition of $\tilde{\mathcal{U}}_{\mathcal{K}}$ occurring in equation (7.19). The two figures show the same close correspondence between the first order estimate (the red circle) and the accurate results (points), but it will be noticed that the points have a slightly different distribution around the circle circumference in the two cases.

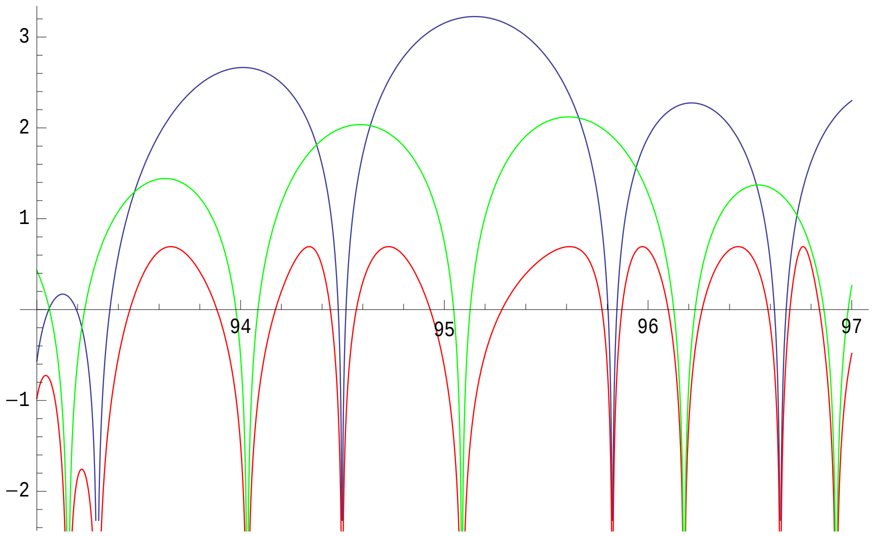


Figure 9. Plot of the logarithmic modulus of the functions $\mathcal{C}(1,4;s)$ (blue), $[\mathcal{K}_{4,6}(s) - \mathcal{K}_{4,6}(1-s)]\exp[\pi t/2]10^{-3}]$ (green) and $\mathcal{K}_{4,6}(s)/\mathcal{K}_{4,6}(1-s) + (\xi_1(2s-1)(s+1/2)(1-s)(3-s)/(\xi_1(2s)(3/2-s)s(s+2)))$ (red) along the critical line.

Remark: Theorems 6.1, 6.4 and Corollaries 6.2, 6.3, 6.5, 6.6 apply to $\mathcal{C}(6,1;s) = \mathcal{Z}(6,0;s)$ and $\mathcal{Z}(4,2;s)$, so that neither of these can be zero at both points s_0 and $1-s_0$ without violating the Riemann hypotheses for $\mathcal{C}(0,1;s)$ and $\mathcal{C}(1,4;s)$.

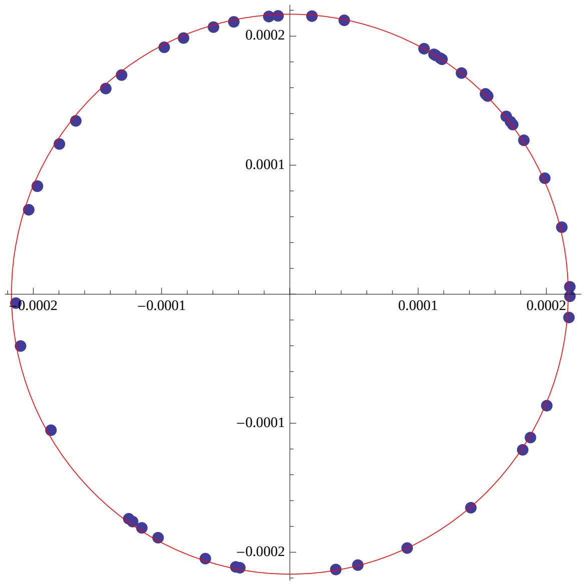


Figure 10. Plot of $s_0 - s_1$ in the context of the discussion of Theorem 7 , for $\sigma_0 = 0.51$ and t_0 ranging from 90 to 100 in steps of 0.2. The red line is a circle of radius 2.170×10^{-4}.

8. The System of Order Eight

The object of this section is restricted to obtaining the equivalent of Theorem 7 for the function $\mathcal{C}(1,8;s)$. The method used will be similar, so the exposition will be as brief as possible.

Theorem 8.1. *Assuming the result of Lagarias and Suzuki (2006) concerning the zeros of the function (4.26), the Riemann Hypothesis holds for the function* $\mathcal{C}(1,8;s)$*, possibly with a finite number of exceptions lying in a compact set.*

Proof. From the expansion of the Chebyshev polynomial of order eight, it follows that

$$\mathcal{C}(1,8;s) = 128\mathcal{C}(8,1;s) - 224\mathcal{C}(4,1;s) + 49\mathcal{C}(0,1;s). \tag{8.1}$$

We use equation (2.3) to expand (8.1), together with the relation $\mathcal{C}(0,1;s) = 2\mathcal{C}(2,0;s)$, giving in symmetrised form

$$\frac{\Gamma(s+4)\mathcal{C}(1,8;s)}{\pi^s} = 4\xi_1(2s-1)[(s-1)(s-1/2)(s^2-33s-78)] + 8\mathcal{K}_8(s), \tag{8.2}$$

where

$$\mathcal{K}_8(s) = 128\mathcal{K}(4,4;s) - 224(s+3)(s+2)\mathcal{K}(2,2;s) + 98(s+3)(s+2)(s+1)\mathcal{K}(1,1;s). \tag{8.3}$$

Hence, if $\mathcal{C}(1,8;s_0) = 0$,

$$\xi_1(2s_0-1)[(s_0-1)(s_0-1/2)(s_0^2-33s_0-78)] + 2\mathcal{K}_8(s_0) = 0. \tag{8.4}$$

Replacing s_0 by $1-s_0$, we also require

$$\xi_1(2s_0)[s_0(s_0-1/2)(s_0^2+31s_0-110)] + 2\mathcal{K}_8(1-s_0) = 0. \tag{8.5}$$

Dividing (8.4) by (8.5), we obtain

$$\frac{\xi_1(2s_0-1)[(1-s_0)(s_0^2-33s_0-78)]}{\xi_1(2s_0)[s_0(s_0^2+31s_0-110)]}+\frac{\mathcal{K}_8(s_0)}{\mathcal{K}_8(1-s_0)}=0. \tag{8.6}$$

This leads to the definition

$$\tilde{\mathcal{U}}_{\mathcal{K}8}=\frac{\mathcal{K}_8(s_0)[s_0(s_0^2+31s_0-110)]y^{1-2s_0}}{\mathcal{K}_8(1-s_0)(1-s_0)(s_0^2-33s_0-78)]}. \tag{8.7}$$

With $\tilde{\mathcal{U}}_{\mathcal{K}8}$ replacing $\tilde{\mathcal{U}}_{\mathcal{K}46}$, the proof is completed according to the argument of Theorem 7. □

The exceptional region for the discussion of Theorem 8.1 is shown in Fig. 11. This region is located below the contour of unit modulus of the first function on the left-hand side of equation (8.6), and as this line is crossed the region where its modulus is larger than unity switches sides of the critical line. For large t zeros are to the right of the critical line and poles to its left. The function has a zero at around $s=-2.21497$, and a pole at around $s=3.21497$ within the plot region; outside it there are a zero at $s\approx 35.215$ and a pole at $s\approx -34.215$.

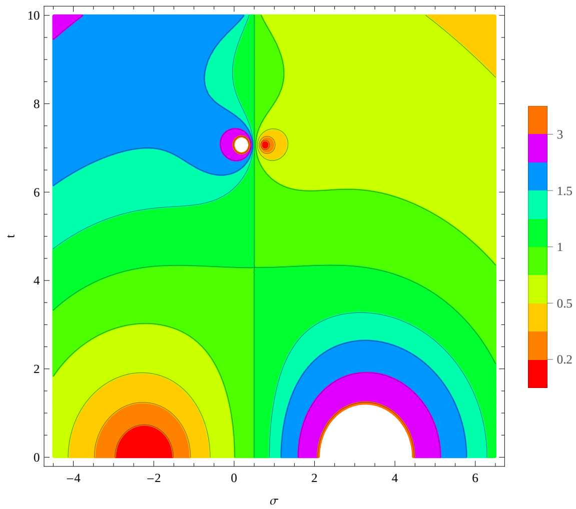


Figure 11. A contour plot of the function
$\xi_1(2s-1)[(1-s)(s^2-33s-78)/(\xi_1(2s)s(s^2+31s-110))$.

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