

Zeros of Epstein Zeta Functions off the Critical Line

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We consider zeros of two-dimensional sums over rectangular lattices, and in particular a sum first studied by Potter and Titchmarsh in 1935. They proved several properties of the zeros of sums over the rectangular lattice, and commented on the fact that a particular sum had zeros off the critical line. We investigate the behaviour of one such zero as a function of the ratio of the periods of the rectangular lattice, and show that it evolves continuously along a trajectory which approaches the critical line, reaching it at a point which is a second-order zero of the rectangular lattice sum.

I. INTRODUCTION

There has been considerable interest over around one hundred and fifty years in the properties of sums of analytic functions over lattices generated by variation of two integers over an infinite range. Many results connected with such sums have been collected in the recent book *Lattice Sums Then and Now*¹, hereafter denoted *LSTN*. These include analytic results concerning their factorisation into terms involving products of two Dirichlet L functions², and also some results on the distribution of zeros on and off the critical line. The latter are of particular interest in that they bear upon the question of whether the Riemann hypothesis that the non-trivial zeros of $\zeta(s) = \zeta(\sigma + it)$ are all located on the critical line $\sigma = 1/2$ can be generalised to certain types of double sum. This proposition reduces to the generalised Riemann hypothesis if the lattice sum can be expressed as a single term involving the product of two Dirichlet L functions, possibly times a prefactor whose zeros lie on the critical line. It is widely accepted that the generalised Riemann hypothesis holds, with strong numerical evidence supporting this, but a proof has long remained elusive.

Epstein zeta functions take the form of a double sum

$$\zeta(s, Q) = \sum_{(p_1, p_2) \neq (0, 0)} Q(p_1, p_2)^{-s}, \quad Q(p_1, p_2) = ap_1^2 + bp_1p_2 + cp_2^2 \quad (1)$$

being a positive-definite quadratic form with integer coefficients a, b, c and a fundamental discriminant $d = b^2 - 4ac$. Potter and Titchmarsh³ proved that $\zeta(s, Q)$ has an infinity of zeros on $\sigma = 1/2$ and exhibited a zero lying off the critical line for a particular choice of $\zeta(s, Q)$. Davenport and Heilbronn⁴ proved that, if the class number $h(d)$ is even, then $\zeta(s, Q)$ has an infinity of zeros in $\sigma > 1$. The condition $h(d)$ is even is satisfied unless $d = -4, -8$ or $-p$, p prime. They also proved⁵ that there are an infinity of zeros in $\sigma > 1$ for $h(d)$ odd and different from unity.

Numerical investigations of the distribution of zeros of Epstein zeta functions have been discussed by Hejhal⁶, including the statistics of the separation of zeros. Bogomolny and Leboeuf⁷ have also discussed the separation of zeros for the case $a = c = 1, b = 0$, finding that the known analytic form of this basic sum resulted in a distribution of zeros with higher probability of smaller gaps than for individual Dirichlet L functions.

McPhedran and coworkers⁸⁻¹⁰ considered a set of double sums incorporating a trigonometric function of p_1 and p_2 in the numerator, with the denominator $(p_1^2 + p_2^2)^s$. They

presented some numerical evidence that a particular group of sums, varying trigonometrically as $\cos(4\theta)$, had all zeros on the critical line, with gaps between the zeros behaving in the manner expected of Dirichlet L functions. An attempt¹⁰ to prove the equivalence of these sums with that for the Epstein zeta function with $a = c = 1, b = 0$ contained an error, as was pointed out to the author by Professor Heath-Brown in a private communication.

In this work, we consider the zero off the critical line identified by Potter and Titchmarsh³ for the Epstein zeta function with $a = 1, b = 0, c = 5$. We replace the integer c by the real λ^2 , so enabling the investigation of the movement of this zero as the ratio of the periods of the rectangular unit cell λ varies continuously. We show that the zero follows a smooth trajectory, with the trajectory to the right of the critical line mirrored by one to its left. The two trajectories of off-axis zeros (i.e., zeros off the critical line) tend to a common point, from which two zeros then migrate upwards and downwards on the critical line.

II. SOME PROPERTIES OF RECTANGULAR LATTICE SUMS

We consider the sum discussed in Section 1.7 of *LSTN*. This sum is:

$$S_0(\lambda, s) = \sum'_{p_1, p_2} \frac{1}{(p_1^2 + p_2^2 \lambda^2)^s}, \quad (2)$$

where the sum over the integers p_1 and p_2 runs over all integer pairs, apart from $(0, 0)$, as indicated by the superscript prime. The quantity λ corresponds to the period ratio of the rectangular lattice, and s is an arbitrary complex number. For λ^2 an integer, this is an Epstein zeta function, but for λ^2 non-integer we will refer to it as a lattice sum over the rectangular lattice.

Connected to this sum is a general class of MacDonalld function double sums for rectangular lattices:

$$\mathcal{K}(n, m; s; \lambda) = \pi^n \sum_{p_1, p_2=1}^{\infty} \left(\frac{p_2^{s-1/2+n}}{p_1^{s-1/2-n}} \right) K_{s-1/2+m}(2\pi p_1 p_2 \lambda). \quad (3)$$

For $\lambda \geq 1$ and the (possibly complex) number s small in magnitude, such sums converge rapidly, facilitating numerical evaluations. (The sum gives accurate answers as soon as the argument of the MacDonalld function exceeds the modulus of its order by a factor of 1.3 or so.) The double sums satisfy the following symmetry relation, obtained by interchanging p_1

and p_2 in the definition (3):

$$\mathcal{K}(n, -m; s; \lambda) = \mathcal{K}(n, m; 1 - s; \lambda). \quad (4)$$

The lowest order sum $\mathcal{K}(0, 0; s; \lambda)$ occurs in the representation of $S_0(\lambda, s)$ due to Kober¹²:

$$\lambda^{s+1/2} \frac{\Gamma(s)}{8\pi^s} S_0(\lambda, s) = \frac{1}{4} \frac{\xi_1(2s)}{\lambda^{s-1/2}} + \frac{1}{4} \lambda^{s-1/2} \xi_1(2s-1) + \mathcal{K}(0, 0; s; \frac{1}{\lambda}). \quad (5)$$

Here $\xi_1(s)$ is the symmetrised zeta function. In terms of the Riemann zeta function, (5) is

$$S_0(\lambda, s) = \frac{2\zeta(2s)}{\lambda^{2s}} + 2\sqrt{\pi} \frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\lambda} + \frac{8\pi^s}{\Gamma(s)\lambda^{s+1/2}} \mathcal{K}(0, 0; s; \frac{1}{\lambda}). \quad (6)$$

A fully symmetrised form of (5) is:

$$\lambda^s \frac{\Gamma(s)}{8\pi^s} S_0(\lambda, s) = \mathcal{T}_+(\lambda, s) + \frac{1}{\sqrt{\lambda}} \mathcal{K}(0, 0; s; \frac{1}{\lambda}), \quad (7)$$

where

$$\mathcal{T}_+(\lambda, s) = \frac{1}{4} \left[\frac{\xi_1(2s)}{\lambda^s} + \frac{\xi_1(2s-1)}{\lambda^{1-s}} \right]. \quad (8)$$

Note that $\mathcal{T}_+(\lambda, 1-s) = \mathcal{T}_+(\lambda, s)$ and $\mathcal{K}(0, 0; 1-s; \lambda) = \mathcal{K}(0, 0; s; \lambda)$, so that the left-hand side of equation (7) must then be unchanged under replacement of s by $1-s$. The left-hand side is also unchanged under replacement of λ by $1/\lambda$, so the same is true for the sum of the two terms on the right-hand side, although in general it will not be true for them individually. The symmetry relations for $S_0(\lambda, s)$ then are

$$\lambda^s \frac{\Gamma(s)}{8\pi^s} S_0(\lambda, s) = \frac{1}{\lambda^s} \frac{\Gamma(s)}{8\pi^s} S_0\left(\frac{1}{\lambda}, s\right) = \lambda^{1-s} \frac{\Gamma(1-s)}{8\pi^{(1-s)}} S_0(\lambda, 1-s) = \frac{1}{\lambda^{1-s}} \frac{\Gamma(1-s)}{8\pi^{(1-s)}} S_0\left(\frac{1}{\lambda}, 1-s\right). \quad (9)$$

From the equations (9), if s_0 is a zero of $S_0(\lambda, s)$ then

$$S_0(\lambda, s_0) = 0 \implies S_0(1/\lambda, s_0) = 0 = S_0(1/\lambda, 1-s_0) = S_0(\lambda, 1-s_0). \quad (10)$$

Another interesting deduction from (7) relates to the derivative of $S_0(\lambda, s_0)$ with respect to λ :

$$\begin{aligned} \lambda^s S_0(\lambda, s) = \frac{1}{\lambda^s} S_0\left(\frac{1}{\lambda}, s\right) &\implies \\ s\lambda^{s-1} S_0(\lambda, s) + \lambda^s \frac{\partial}{\partial \lambda} S_0(\lambda, s) &= \frac{-s}{\lambda^{s+1}} S_0\left(\frac{1}{\lambda}, s\right) - \frac{1}{\lambda^{s+2}} \frac{\partial}{\partial \lambda} S_0\left(\frac{1}{\lambda}, s\right), \end{aligned} \quad (11)$$

so that

$$\left. \frac{\partial}{\partial \lambda} S_0(\lambda, s) \right|_{\lambda=1} = -s S_0(1, s). \quad (12)$$

Thus, trajectories of $S_0(\lambda, s) = 0$ starting at a zero s_0 for $\lambda = 1$ will leave the line $\lambda = 1$ at right angles to it as λ varies. One such will exist for λ increasing, and another for λ decreasing.

Combining (7) and (9), we arrive at a general symmetry relationship for $\mathcal{K}(0, 0; s; \lambda)$:

$$\mathcal{T}_+(\lambda, s) - \mathcal{T}_+\left(\frac{1}{\lambda}, s\right) = \sqrt{\lambda}\mathcal{K}(0, 0; s; \lambda) - \frac{1}{\sqrt{\lambda}}\mathcal{K}\left(0, 0; s; \frac{1}{\lambda}\right), \quad (13)$$

or

$$\begin{aligned} & \frac{1}{4} \left[\xi_1(2s) \left(\frac{1}{\lambda^s} - \lambda^s \right) + \xi_1(2s-1) \left(\frac{1}{\lambda^{1-s}} - \lambda^{1-s} \right) \right] = \\ & \sqrt{\lambda}\mathcal{K}(0, 0; s; \lambda) - \frac{1}{\sqrt{\lambda}}\mathcal{K}\left(0, 0; s; \frac{1}{\lambda}\right). \end{aligned} \quad (14)$$

This identity holds for all values of s and λ . One use of it is to expand about $\lambda = 1$, which gives identities for the partial derivatives of $\mathcal{K}(0, 0; s; \lambda)$ with respect to λ , evaluated at $\lambda = 1$. The first of these is

$$s\xi_1(2s) + (1-s)\xi_1(2s-1) = -2\mathcal{K}(0, 0; s; 1) - 4\left.\frac{\partial}{\partial\lambda}\mathcal{K}(0, 0; s; \lambda)\right|_{\lambda=1}. \quad (15)$$

To go beyond first order with the identity (14), one needs to use the correct form for the expansion variable- rather than use $\lambda - 1$, one should expand using

$$\chi = \lambda - \frac{1}{\lambda}, \quad \lambda = \frac{\chi}{2} + \sqrt{1 + \frac{\chi^2}{4}}, \quad \frac{1}{\lambda} = -\frac{\chi}{2} + \sqrt{1 + \frac{\chi^2}{4}}. \quad (16)$$

$S_0(\lambda, s)$ has factorisations in terms of Dirichlet L functions for particular values of λ . We take from Table 1.6 in Chapter 1 of *LSTN* the first seven of these:

$$S_0(1, s) = 4\zeta(s)L_{-4}(s), \quad S_0(\sqrt{2}, s) = 2\zeta(s)L_{-8}(s), \quad (17)$$

$$S_0(\sqrt{3}, s) = 2(1 - 2^{1-2s})\zeta(s)L_{-3}(s), \quad S_0(\sqrt{4}, s) = 2(1 - 2^{-s} + 2^{1-2s})\zeta(s)L_{-4}(s), \quad (18)$$

$$S_0(\sqrt{5}, s) = \zeta(s)L_{-20}(s) + L_{-4}(s)L_{+5}(s), \quad S_0(\sqrt{6}, s) = \zeta(s)L_{-24}(s) + L_{-3}(s)L_{+8}(s), \quad (19)$$

$$S_0(\sqrt{7}, s) = 2(1 - 2^{1-s} + 2^{1-2s})\zeta(s)L_{-7}(s). \quad (20)$$

The expressions for $S_0(\sqrt{3}, s)$, $S_0(\sqrt{4}, s)$ and $S_0(\sqrt{7}, s)$ have prefactors whose zeros may be determined analytically. These are, for arbitrary integers n ,

$$S_0(\sqrt{3}, s) : s = \frac{1}{2} \left(1 + \frac{(2n+1)\pi i}{\ln 2} \right), \quad (21)$$

$$S_0(\sqrt{4}, s) : s = \frac{1}{2} \pm \frac{i \arctan \sqrt{7}}{\ln 2} + \frac{2n\pi i}{\ln 2}, \quad (22)$$

and

$$S_0(\sqrt{7}, s) : s = \frac{1}{2} + \frac{i\pi}{4 \ln 2} + \frac{2n\pi i}{\ln 2}. \quad (23)$$

There are no other factorisations in Table 1.6 of the form of $S_0(\lambda, s)$ containing only a single term. These results then show that the generalised Riemann hypothesis applies to the seven lattice sums of equations (17-20).

III. EXPANSIONS ABOUT $\lambda = 1$

We now expand the sum

$$\tilde{S}_0(\lambda, s) = \lambda^s \frac{\Gamma(s)}{8\pi^s} S_0(\lambda, s) = \frac{\Gamma(s)}{8\pi^s} \sum'_{p_1, p_2} \frac{1}{(p_1^2/\lambda + p_2^2\lambda)^s}. \quad (24)$$

This sum is symmetric under both operations $\lambda \rightarrow 1/\lambda$ and $s \rightarrow 1 - s$.

We use the expansion parameter χ of (16), but re-express it in trigonometric form:

$$\frac{\chi}{2} = \tan \phi, \quad \sqrt{1 + \frac{\chi^2}{4}} = \sec \phi, \quad (25)$$

where we have taken $\cos \phi > 0$. We then have:

$$\tilde{S}_0(\lambda, s) = \frac{\Gamma(s)}{8\pi^s (1 + \chi^2/4)^{s/2}} \sum'_{p_1, p_2} \frac{1}{(p_1^2 + p_2^2)^s} \left[1 - \left(\frac{\chi/2}{\sqrt{1 + \chi^2/4}} \right) \cos 2\theta_{1,2} \right]^{-s}, \quad (26)$$

where $\cos \theta_{1,2} = p_1/\sqrt{p_1^2 + p_2^2}$. We expand the last term in the double sum using the Binomial Theorem, and re-express even powers of $\cos 2\theta_{1,2}$ as combinations of $\cos 4m\theta_{1,2}$. (Odd powers of $\cos 2\theta_{1,2}$ sum to zero over the square lattice.) The χ -dependent term multiplying the sum in (26) is $(\cos \phi)^s$, which is expanded as $(1 - \sin^2 \phi)^{s/2}$. The double sums over the square lattice are then written⁸⁻¹⁰ in terms of

$$\tilde{C}(1, 4m; s) = \frac{\Gamma(2m + s)}{8\pi^s} \sum'_{p_1, p_2} \frac{\cos 4m\theta_{1,2}}{(p_1^2 + p_2^2)^s}, \quad (27)$$

which form is symmetric under $s \rightarrow 1 - s$. Note that $\tilde{C}(1, 0; s) = \tilde{C}(0, 1; s)$.

The result of this procedure is an expression which may be written as:

$$\tilde{S}_0(\lambda, s) = \tilde{C}(0, 1; s) + \sum_{m=1}^{\infty} \mathcal{S}_{2m}(s) \sin^{2m} \phi, \quad (28)$$

where the symmetry under $\lambda \rightarrow 1/\lambda$ is manifest in the presence of only even powers of $\sin \phi$ on the right-hand side of (28). The symmetry under $s \rightarrow 1 - s$ is evident in the form of the $\mathcal{S}_{2m}(s)$, the first few of which are:

$$\mathcal{S}_2(s) = -\frac{1}{4}s(1-s)\tilde{C}(0, 1; s) + \frac{1}{4}\tilde{C}(1, 4; s), \quad (29)$$

$$\mathcal{S}_4(s) = -\frac{1}{64}s(1-s)(10-s(1-s))\tilde{C}(0, 1; s) + \frac{1}{48}(6-s(1-s))\tilde{C}(1, 4; s) + \frac{1}{192}\tilde{C}(1, 8; s), \quad (30)$$

$$\begin{aligned} \mathcal{S}_6(s) = & -\frac{1}{2304}s(1-s)(264-46s(1-s)+s^2(1-s)^2)\tilde{C}(0, 1; s) + \\ & \frac{1}{1536}(120-38s(1-s)+s^2(1-s)^2)\tilde{C}(1, 4; s) + \\ & \frac{1}{3840}(20-s(1-s))\tilde{C}(1, 8; s) + \frac{1}{23040}\tilde{C}(1, 12; s), \end{aligned} \quad (31)$$

and

$$\begin{aligned} \mathcal{S}_8(s) = & -\frac{1}{147456}s(1-s)(13392-3132s(1-s)+124s^2(1-s)^2-s^3(1-s)^3)\tilde{C}(0, 1; s) + \\ & \frac{1}{92160}(5040-2292s(1-s)+112s^2(1-s)^2-s^3(1-s)^3)\tilde{C}(1, 4; s) + \\ & \frac{1}{184320}(12-s(1-s))(70-s(1-s))\tilde{C}(1, 8; s) + \frac{1}{654120}(42-s(1-s))\tilde{C}(1, 12; s) + \\ & \frac{1}{5160960}\tilde{C}(1, 16; s). \end{aligned} \quad (32)$$

Note that, apart from the numerical coefficients, each term in the expansions of the $\mathcal{S}_{2m}(s)$ has modulus for large $|s|$ of order $|s|^{2m}$ times a sum of a trigonometric term weighting $1/(p_1^2 + p_2^2)^s$.

The form established in equations (28-32) makes it easy to establish a useful result.

Theorem 1. *A trajectory $\tilde{S}_0(\lambda, s) = 0$ giving s as a function of λ which contains a point s_0 on the critical line at which $\partial\tilde{S}_0(\lambda, s)/\partial s \neq 0$ must include an interval around s_0 lying on the critical line. Furthermore, if s_* is a point on the critical line at which $\tilde{S}_0(\lambda, s) = 0$ and $\partial\tilde{S}_0(\lambda, s)/\partial s = 0$, then a trajectory $\tilde{S}_0(\lambda, s) = 0$ passing through s_* runs along the critical line along one side of t_* and at right angles to it on the other side.*

Proof. Let s_0 be a point on the critical line for which $\tilde{S}_0(\lambda, s) = 0$ and $\partial\tilde{S}_0(\lambda, s)/\partial s \neq 0$. Let $w = \sin(\phi)$. The differential equation for trajectories along which $\tilde{S}_0(\lambda, s)$ is constant is described by the equation

$$d\tilde{S}_0(\lambda, s) = 0 = \frac{\partial\tilde{C}(0, 1; s)}{\partial s}ds + \sum_{m=1}^{\infty} w^{2m} \frac{\partial\mathcal{S}_{2m}(s)}{\partial s}ds + \sum_{m=1}^{\infty} 2mw^{2m-1}\mathcal{S}_{2m}(s)dw. \quad (33)$$

We solve (33) for ds :

$$ds = \left\{ \frac{-[\sum_{m=1}^{\infty} 2mw^{2m-1}\mathcal{S}_{2m}(s)]}{\frac{\partial\tilde{C}(0,1;s)}{\partial s} + \sum_{m=1}^{\infty} w^{2m}\frac{\partial\mathcal{S}_{2m}(s)}{\partial s}} \right\}_{s=s_0} dw. \quad (34)$$

Using this to construct the trajectory from the point s_0 on the critical line, corresponding to w_0 , each term in the numerator is real, while each term in the denominator is pure imaginary. Thus, ds is pure imaginary, and the trajectory continues along the critical line in an interval surrounding s_0 . The proof applies to $\tilde{S}_0(\lambda, s)$ taking any real constant value, including of course zero. We can continue to enlarge the interval by considering successive points s_0 until we reach a point where $\partial\tilde{S}_0(\lambda, s)/\partial s = 0$.

For the second proposition, given that the first two terms in the Taylor series of $\tilde{S}_0(\lambda, s)$ about $s = s_*$ are zero, then the trajectory $\tilde{S}_0(\lambda, s) = 0$ is described by

$$ds^2 = \left\{ \frac{-[\sum_{m=1}^{\infty} 4mw^{2m-1}\mathcal{S}_{2m}(s)]}{\frac{\partial^2\tilde{C}(0,1;s)}{\partial s^2} + \sum_{m=1}^{\infty} w^{2m}\frac{\partial^2\mathcal{S}_{2m}(s)}{\partial s^2}} \right\}_{s=s_*} dw. \quad (35)$$

If the constant in the curly brackets in (35) is positive, then $ds^2 = d\sigma^2$ if $dw > 0$, with $d\sigma \propto \sqrt{dw}$ then, while $ds^2 = -dt^2$ and $dt \propto \sqrt{-dw}$ if $dw < 0$. If the constant in the curly brackets in (35) is negative, then $ds^2 = d\sigma^2$ if $dw < 0$, and $ds^2 = -dt^2$ if $dw > 0$. \square

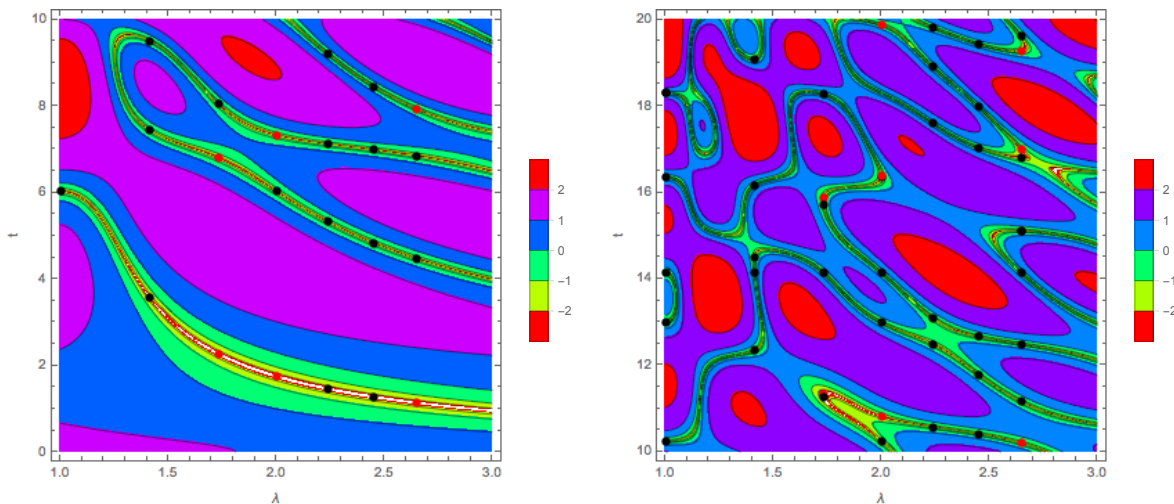


FIG. 1. Contours of $\log |\mathcal{S}_0(\lambda, 1/2 + it)|$ in the plane (λ, t) . Black dots and red dots correspond to zeros for which there is a factorization given the text, with the red dots being known analytically.

Figure 1 shows contours of $\log |\mathcal{S}_0(\lambda, 1/2 + it)|$ in the plane (λ, t) , calculated using numerical summation of the expression (5). Also indicated are positions of zeros of this function, calculated from the factorised forms (17-20).

The contours of zero amplitude of $S_0(\lambda, s)$ shown in Fig. 1 have a general trend of decreasing as λ increases away from unity, but may have intervals in which they increase. Some of the turning points in these curves are associated with prefactor and Dirichlet L function zeros being in close proximity.

Theorem 1 does not imply that all zeros of the lattice sums $S_0(\lambda, s)$ lie on the critical line. Indeed, it has been known since the work of Potter and Titchmarsh in 1935 that the sum $S_0(\sqrt{5}, s)$ has zeros off the critical line. The first such is illustrated in Fig. 2. In the next section, we will examine whether zeros off the critical line can be linked to factorised forms of $\mathcal{S}_0(\lambda, s)$, like those in (17-20). What is clear from Theorem 1 is that the turning points of contours of zero amplitude of $S_0(\lambda, s)$ evident in Fig. 1, where $\partial S_0(\lambda, 1/2 + it)/\partial t = 0$, should play an important role in any linkage between zeros off the critical line and those on the critical line.

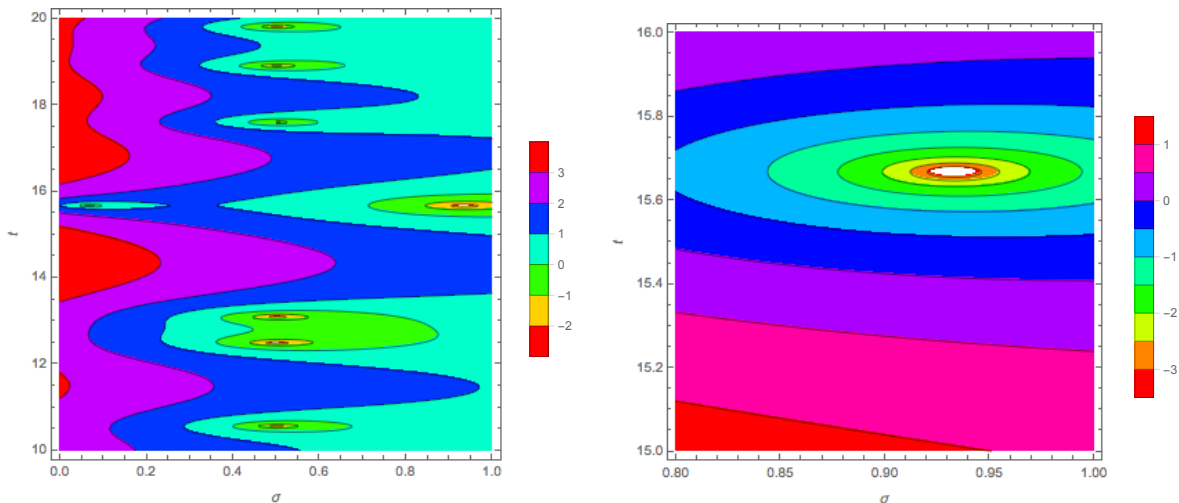


FIG. 2. Contours of $\log |\mathcal{S}_0(\sqrt{5}, \sigma + it)|$ in the plane (σ, t) . The first off-axis zeros of this sum are illustrated, which lie near $s_{PT} = 0.9329 + 15.6682i$ and $1 - \overline{s_{PT}}$.

The equation (7) gives $\tilde{S}_0(\lambda, s)$ as the sum of $\mathcal{T}_+(\lambda, s)$ and $\mathcal{K}(0, 0; s; \frac{1}{\lambda})/\sqrt{\lambda}$. We can readily obtain the expansion of $\mathcal{T}_+(\lambda, s)$ in powers of $\sin \phi$ if in (8) we replace λ by $(1 + \sin \phi)/\sqrt{1 - \sin^2 \phi}$. It is also useful to replace $\xi_1(2s)$ and $\xi_1(2s - 1)$ by superpositions of functions which are even and odd with respect to the transformation $s \rightarrow 1 - s$:

$$\xi_1(2s) = 2[\mathcal{T}_+(1, s) + \mathcal{T}_-(1, s)], \quad \xi_1(2s - 1) = 2[\mathcal{T}_+(1, s) - \mathcal{T}_-(1, s)]. \quad (36)$$

We then obtain:

$$\begin{aligned}
 \mathcal{T}_+(\lambda, s) = & \mathcal{T}_+(1, s) \left\{ 1 + 1/2(-1 + s)^2 \sin^2(\phi) + \frac{1}{24} [12 + s(-34 + s(32 + (-8 + s)s))] \sin^4(\phi) + \right. \\
 & \frac{1}{720} [360 + s(-1212 + s(1504 + s(-750 + s(205 + (-18 + s)s)))] \sin^6(\phi) + \dots \left. \right\} \\
 & + \mathcal{T}_-(1, s) \left\{ -\frac{1}{2} \sin(\phi) + \left[-\frac{1}{2} + s - \frac{3s^2}{4}\right] \sin^3(\phi) + \right. \\
 & \frac{1}{48} [-24 + s(68 + s(-76 + s(28 - 5s)))] \sin^5(\phi) + \\
 & \left. \frac{1}{1440} [(-720 + s(2424 + s(-3328 + s(1980 + s(-670 + s(96 - 7s)))))] \sin^7(\phi) + \dots \right\}.
 \end{aligned} \tag{37}$$

This expression contains both odd and even powers in $\sin(\phi)$, while the dependence of the coefficients of powers of $\sin(\phi)$ on s is of mixed parity under $s \rightarrow 1 - s$. The functions $\mathcal{T}_+(1, s)$ and $\mathcal{T}_-(1, s)$ are respectively even and odd under $s \rightarrow 1 - s$, all their zeros lie on the critical line and form distinct sets with the same distribution function, while all zeros are simple^{13,14}. From (8), the equation for zeros of $\mathcal{T}_+(\lambda, s)$ is

$$\frac{\xi_1(2s - 1)}{\xi_1(2s)} = \lambda^{1-2s}. \tag{38}$$

The left-hand side in (38) has modulus smaller than unity in $\sigma > 1/2$, and larger than unity in $\sigma < 1/2$. The opposite is true for the right-hand side if $\lambda < 1$. All zeros of $\mathcal{T}_+(\lambda, s)$ thus lie on the critical line if $\lambda < 1$.

IV. THE TRAJECTORY OF AN OFF-AXIS ZERO

Fig. 3 shows the trajectory in the σ, t plane of numerically-determined zeros of $\tilde{S}_0(\lambda, s)$, as λ varies. The trajectory curves upwards as λ decreases towards $\sqrt{4}$, and reaches the critical line at a point sandwiched between a prefactor zero of $S_0(\sqrt{4}, s)$ at $t \approx 16.384603$ and a zero of $L_{-4}(s)$ at $t \approx 16.342539$. The Potter-Titchmarsh zero is indicated by a point near the rightmost extremity of the trajectory. The trajectory curves down and back towards the critical line as λ increases towards 6.343472. This value of course does not correspond to a known factorisation of $\tilde{S}_0(\lambda, s)$.

In the vicinity of the upper intersection point, we illustrate the behaviour of $\tilde{S}_0(\lambda, s)$ in Figs. 4, 5. Fig 4 shows the endpoint chosen for a process of localising the λ value at which zeros transition from positions off the critical line (curves with a single central minimum) to

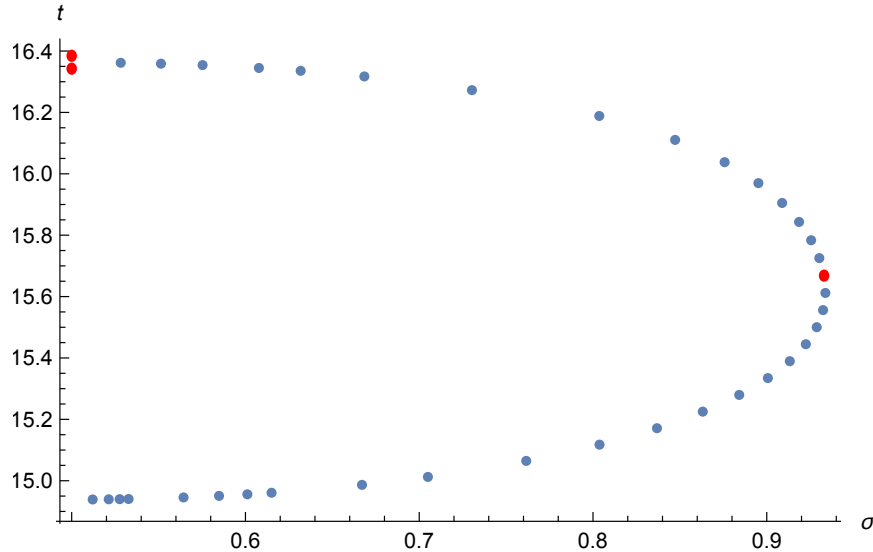


FIG. 3. The trajectory of a zero off the critical line of $\tilde{S}_0(\lambda, s)$, as λ varies, plotted in the σ, t plane. The red dots represent the zero off the critical line corresponding to $\lambda = \sqrt{5}$, and two zeros on the critical line corresponding to $\lambda = \sqrt{4}$.

on the critical line (curves with two negative approximate singularities symmetrically located about a local maximum). This transition value of λ is then between 4.0007109411 and 4.0007109410. Figure 5 shows the variation of the logarithmic modulus and the argument of $\tilde{S}_0(\lambda, s)$ in the σ, t plane for a value of λ just before the transition value, where the locations of two zeros to the left and right of the critical line are evident.

Similar figures for the lower intersection point are given in Figs. 6, 7. In this case, the transition value of λ lies between 6.343471 and 6.343472, with off-axis zeros on the low side of this value. This is clearly evident in the amplitude and argument plots of Fig. 7. The argument plots in Figs. 5 and 7 are both clearly in support of the behaviour at the exact transition value corresponding to a zero of multiplicity two on the critical line, although this cannot be proved numerically.

The conclusion of this work is that zeros of $\tilde{S}_0(\lambda, s)$ off the critical line can lie on constant-modulus trajectories reaching the critical line. Such trajectories behave in a way consistent with the generalised Riemann hypothesis. The point where they reach the critical line corresponds to a second-order zero of $\tilde{S}_0(\lambda, s)$, and after reaching the critical line the trajectory continues along the critical line for an interval. Further work on the properties of such trajectories would be of interest and value.

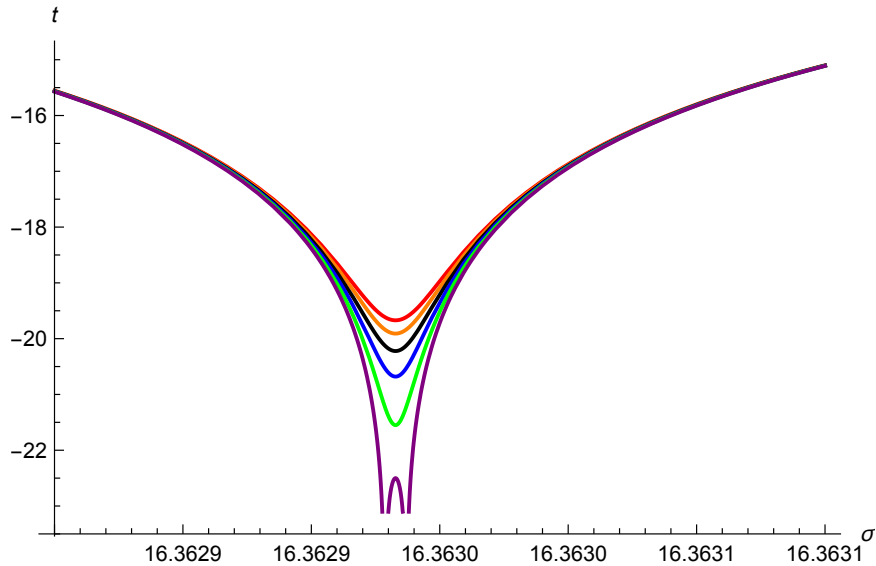


FIG. 4. Plots of $\log |\tilde{S}_0(\lambda, 1/2 + it)|$ as a function of t for λ ranging from 4.0007109415 to 4.0007109410 in equal decrements, for respective line colours: red, orange, black, blue, green, purple.

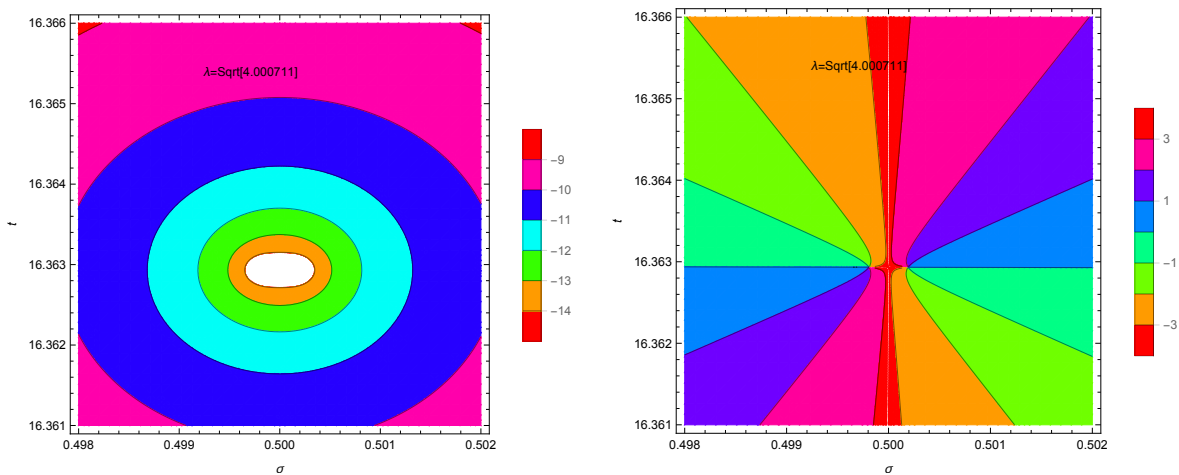


FIG. 5. Contour plots of the logarithmic modulus (left) and the argument (right) of $\tilde{S}_0(\lambda, \sigma + it)$ for $\lambda = 4.000711$.

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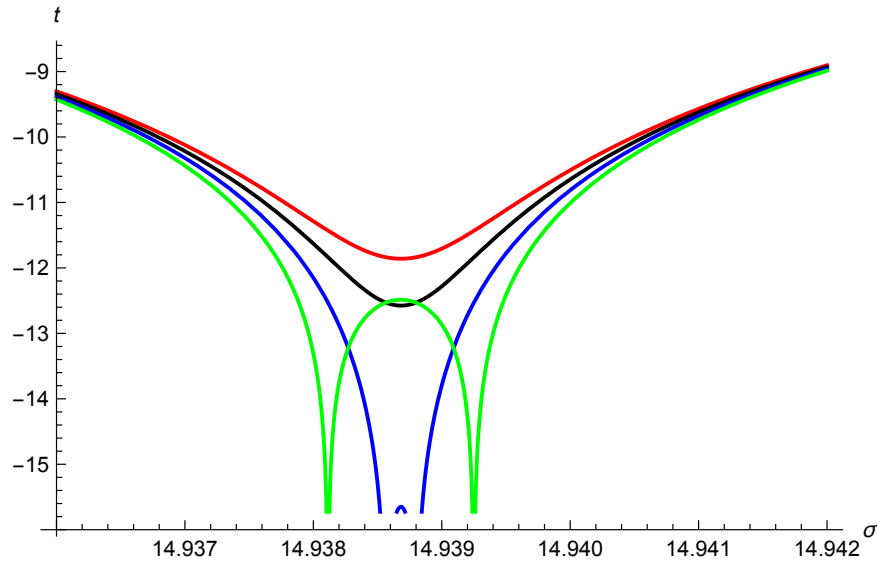


FIG. 6. Plots of $\log |\tilde{S}_0(\lambda, 1/2 + it)|$ as a function of t for λ ranging from 6.343470 to 6.343473 in equal increments, for respective line colours: red, black, blue, green.

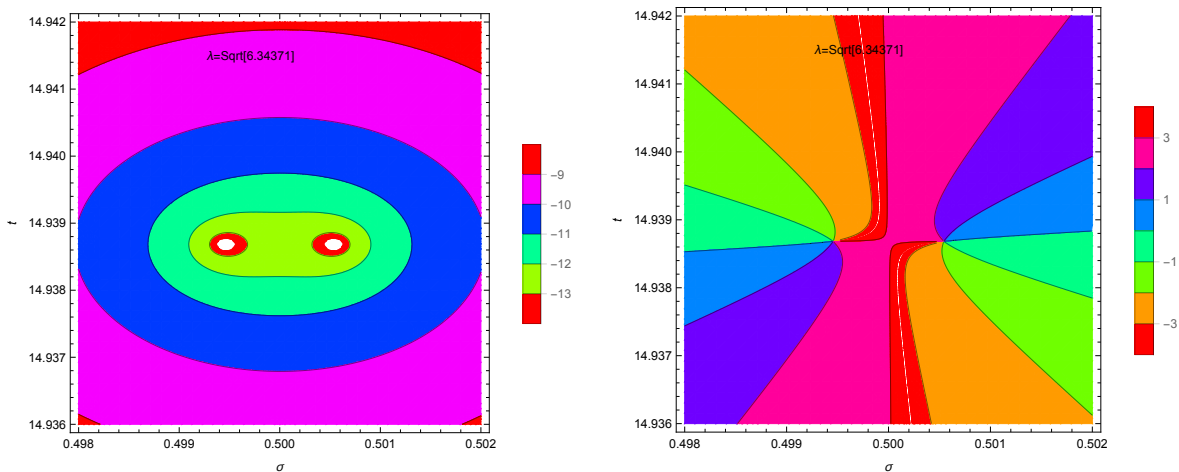


FIG. 7. Contour plots of the logarithmic modulus (left) and the argument (right) of $\tilde{S}_0(\lambda, \sigma + it)$ for $\lambda = 6.34371$.

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