Three Lectures on the Fractional Calculus

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Outline

- Motivation, history
- Fractional calculus of one variable
- Fourier transforms, Green's functions and distributions
- Laplacian in two dimensions to an arbitrary power

References

- *A child's garden of fractional derivatives*, Marcia Kleinz and Tom Osler
- *Mathematica for theoretical physics*, Gerd Baumann, Springer 2004, Chapter 7.
- *Fractional kinetics*, I.M .Solokov, J. Klafter and A. Blumen,Physics Today, November 2002, pp. 48-54
- *Electromagnetic processes in dispersive media*, D. B. Melrose and R.C. McPhedran, Cambridge University Press 2005, Chapters 4,5.

Motivation, History

- In the late 17th century calculus had transformed mathematics and physics- where were its boundaries?
- Letter from Leibnitz to l'Hospital: Can the meaning of derivatives with integral order *n* be transformed to non-integral, even complex, orders?
- Difficulties arose: Leibnitz: *Il y a de l'apparence qu'on tirera un jour des consequences bien utiles de ces paradoxes, car il n'y a gueres de paradoxes sans utilite*

Motivation, History (2)

- Initial motivation: curiosity. Major contributions from Liouville, Riemann, Laplace, Heaviside, Weyl, etc
- Well established mathematical framework now finding applications
- Differentiation makes functions nastier; integration makes them better; fractional differentiation can make them “just right”. See the Physics Today article.

Scope of Fractional Calculus

- Differentiation with respect to arbitrary powers: they can be negative
- Negative differentiation is integration
- Integration needs two limits to have a precise meaning
- Fractional derivatives in general need a lower limit, and a variable indicated $\mathcal{D}^{\alpha}_{a,x}$

  α- order of differentiation or integration

  a lower limit; x variable

Fractional Differentiation (1)

$$\frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}$$

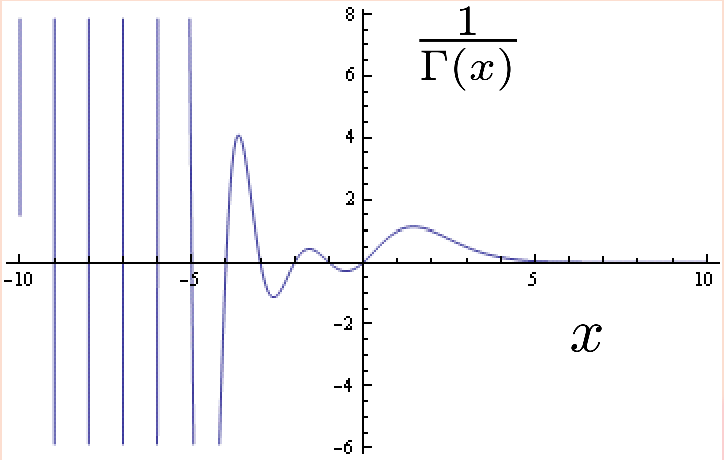
$$m! = \Gamma(m+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

- This is the Riemann-Liouville derivative.
- Can be applied to functions represented by Taylor series
- n can be any real or complex number

Fractional Differentiation (2)

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$



- The result will be zero if m-n+1 is zero or a negative integer;
- Otherwise its non-zero

$$m = 0: \quad \frac{d^n 1}{dx^n} = \frac{x^{-n}}{\Gamma(1-n)}$$

$$\text{e.g.} \quad \frac{d^{1/2} 1}{dx^{1/2}} = \frac{1}{\sqrt{\pi x}}$$

Factorial Function

Key equations for the gamma function are:

$\Gamma(z+1) = z\Gamma(z) = z! = z(z-1)!, \quad (1)$

$\Gamma(1) = 1, \ (2)$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0, \quad (3)$

and

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \quad (4)$

From (4) one can prove other useful things-e.g.,

$\Gamma(1/2) = \sqrt{\pi} \quad (5)$

and

$\Gamma(-n+\delta) = \frac{(-1)^n}{n!\delta}. \quad (6)$

Fractional Differentiation (3)

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

$$\frac{d^p}{dx^p}\left(\frac{d^n x^m}{dx^n}\right) = \frac{\Gamma(m-n+1)}{\Gamma(m-n-p+1)} \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n-p}$$

i.e. $\frac{d^p}{dx^p}\left(\frac{d^n x^m}{dx^n}\right) = \frac{\Gamma(m+1)}{\Gamma(m-n-p+1)} x^{m-n-p}$

Example:

$$\frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}1}{dx^{1/2}}\right) = \frac{\Gamma(1)}{\Gamma(1-1/2-1/2)} x^{-1} = 0$$

We know for any integer n:

$$D^n(e^{ax}) = a^n e^{ax}$$

so we want for any α

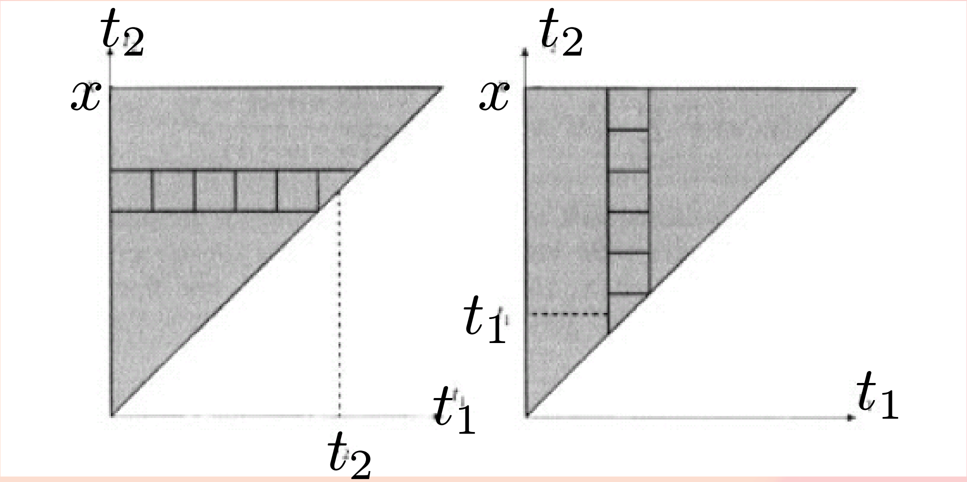
$$D^\alpha(e^{ax}) = a^\alpha e^{ax}.$$

Yet:

$$D^\alpha e^x = D^\alpha(\sum_{n=0}^{\infty} \frac{x^n}{n!}) = \sum_{n=0}^{\infty} \frac{x^{n-\alpha}}{\Gamma(n-\alpha+1)}$$

These don't match unless α is an integer!

Integration as Negative Differentiation



We consider the definite integrals:

$D^{-1}f(x) = \int_0^x f(t)dt, \quad D^{-2}f(x) = \int_0^x \int_0^{t_2} f(t_1)dt_1dt_2.$

In the second integral, we invert the order of integrations, going from left to right diagrams above.

$D^{-2}f(x) = \int_0^x \int_{t_1}^x f(t_1)dt_2dt_1 = \int_0^x f(t_1)(x-t_1)dt_1$

$D^{-3}f(x) = \frac{1}{2}\int_0^x f(t_1)(x-t_1)^2dt_1$

$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t_1)(x-t_1)^{n-1}dt_1$

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Integration as NegativeDifferentiation(2)

We generalize

$D^{-n}f(x) = \frac{1}{(n-1)!}\int_0^x f(t_1)(x-t_1)^{n-1}dt_1$

to give

$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(t_1)(x-t_1)^{\alpha-1}dt_1$

For the singularity at $t_1 \to x$ to be integrable, we require $\alpha > 0$, confirming we are dealing with a negative order of differentiation.

So we write our generalized differential operator with a curly D, putting its order as the superscript, and the lower limit and variable being differentiated as subscripts. The two usual choices for lower limits are 0 and $-\infty$.

$$\mathcal{D}^{\alpha}_{a,x}$$

CUDOS Integration as NegativeDifferentiation(3)

Being clear about implicit limits enables us to clear up the previous difficulty:

$\mathcal{D}_{b,x}^{-1} e^{ax} = \int_b^x e^{ax} dx = \frac{e^{ax}}{a}$ if $b = -\infty$

$\mathcal{D}_{b,x}^{-1} x^p = \int_b^x x^p dx = \frac{x^{p+1}}{p+1}$ if $b = 0$

For any given physical problem, there will be a choice to make about the best value of lower limits.
This choice will control the results of differentiations to fractional powers.

$\mathcal{D}_{0,x}^{\alpha}(x^p) = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}$

and

$\mathcal{D}_{-\infty,x}^{\alpha}(e^{ax}) = a^{\alpha} e^{ax}$

Differentiation as Negative Integration

We can define fractional differentiation on the basis of fractional integration

$\mathcal{D}^{s}_{a,x} = (\frac{d^n}{dx^n})\mathcal{D}^{-(n-s)}_{a,x} f(x)$

with n a positive integer, $\Re(s) > 0$, $\Re(n-s) > 0$.

We have then some familiar properties- linearity:

$\mathcal{D}^{s}_{a,x}(\alpha f(x) + \beta g(x)) = \alpha\mathcal{D}^{s}_{a,x} f(x) + \beta\mathcal{D}^{s}_{a,x} g(x)$

and composition

$\mathcal{D}^{s}_{a,x}\mathcal{D}^{p}_{a,x} f(x) = \mathcal{D}^{s+p}_{a,x} f(x)$,

with $p < 0$ and $f(x)$ finite at $x = a$.

For $p > 0$, see Baumann.

For Leibnitz's rule, we get an infinite series:

$\mathcal{D}^{q}_{a,x}(f(x)g(x)) = \sum_{j=0}^{\infty} \left(\begin{array}{c} q \\ j \end{array} \right) \mathcal{D}^{q-j}_{a,x} f(x) \mathcal{D}^{j}_{a,x} g(x)$,

with the symbol in brackets being

$\Gamma(q+1)/(\Gamma(j+1)\Gamma(q-j+1))$.

- Reprise from last lecture:

$$\frac{d^p x^q}{dx^p} = D^p_{0,x} = \frac{\Gamma(q+1)}{\Gamma(q-p+1)} x^{q-p}$$

the Riemann-Liouville derivative.

Differentiation to a negative power:

$$D^{-\alpha}_{a,x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t_1)(x-t_1)^{\alpha-1} dt_1$$

for $\alpha > 0$.

Most discussions use fractional calculus in one variable. Let's see how we can treat two dimensions, using Fourier transform ideas.

Adding up Harmonics

- Fourier series- sines and cosines adding up to give arbitrary waveforms: harmonics
- Fourier transforms- not just harmonics, but an integral over all frequencies
- In more than one dimension, add up plane waves
- In two dimensions, a plane wave is

$$\exp i(k_x x + k_y y)$$

Wave vector: $(k_x, k_y) = \mathbf{k}, \quad \text{momentum } \hbar\mathbf{k}$

The Fourier transform

- Represent a function in space as an integral over plane waves: inverse transform

$$A(\mathbf{x}) = A(x, y) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{i(k_x x + k_y y)} \tilde{A}(k_x, k_y)$$

Function in space

Function in wave vector space; reciprocal space; momentum space

Direct transform:

$$\tilde{A}(k_x, k_y) = \int dx dy e^{-i(k_x x + k_y y)} A(x, y)$$

Momentum space

- A lot of physics is based on momentum or wavevector space
- Conservation of momentum: $\mathbf{p} = \hbar\mathbf{k}$
- A mathematical reason: derivatives are replaced by algebraic operations

$$A(\mathbf{x}) = A(x,y) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{i(k_x x + k_y y)} \tilde{A}(k_x, k_y)$$

$$\frac{\partial}{\partial x} A(\mathbf{x}) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{i(k_x x + k_y y)} i k_x \tilde{A}(k_x, k_y)$$

$\frac{\partial}{\partial x}$ Partial derivative with respect to x

The Laplacian

- A particularly important operator in physics is the Laplacian
- Take two derivatives with respect to x, two with respect to y and add
- Crops up in electrostatics, magnetostatics, complex variable theory
- Symbol: $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2}{\partial x^2} \to -k_x^2 \qquad \frac{\partial^2}{\partial y^2} \to -k_y^2$$

$$\nabla^2 \to -(k_x^2 + k_y^2)$$

The Laplacian (2)

Once: $$\nabla^2 \rightarrow -(k_x^2 + k_y^2)$$

n times: $$\nabla^{2n} \rightarrow (-1)^n (k_x^2 + k_y^2)^n$$

p times $$\nabla^{2p} \rightarrow e^{i\pi p} (k_x^2 + k_y^2)^p$$

since $$(-1)^n = \cos(n\pi) + i \sin(n\pi) = e^{i\pi n}$$

$$\nabla^{2p} A(x,y) \rightarrow e^{i\pi p} (k_x^2 + k_y^2)^p \tilde{A}(k_x, k_y)$$

The Dirac delta function

- Fourier transforms integrate over extended objects: plane waves
- Need a way of going from extended objects to point-like objects
- This is provided by the Dirac delta function: the Fourier transform of a constant

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}$$

Delta function of angular frequency

$$(2\pi)^2\delta^2(k_x, k_y) = \int_{-\infty}^{\infty} dx dy e^{i(k_x x + k_y y)}$$

Delta function of wave vector (2D)

The Dirac delta function (2)

- The more spread out a function is, the tighter its Fourier transform concentrates around the origin
- A constant is spread out uniformly in space: its Fourier transform concentrates around the origin in reciprocal space
- Another way of thinking about the delta function is that it is a function concentrated around the origin, but having unit area under its curve

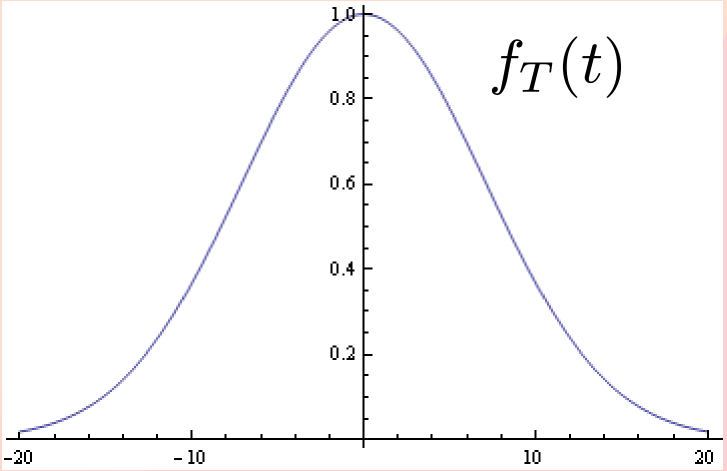
The Dirac delta function (3)

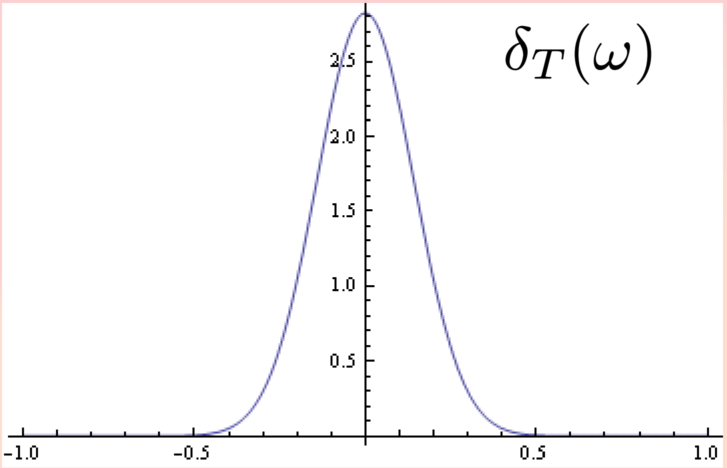
- One representation is based on Gaussian functions

$$f_T(t) = e^{-t^2/T^2} \rightarrow$$
$$\delta(\omega) = \lim_{T\rightarrow\infty} \frac{T}{2\sqrt{\pi}} e^{-\omega^2 T^2/4}$$

Function $\rightarrow$ constant

Transform $\rightarrow$ delta





$$T = 10$$

Green's functions

- A Green's function for a problem in physics is a solution of the governing equation corresponding to a point source
- The point source is just a delta function
- So for example in electrostatics if we look for the Green's function for a point source at the origin, we want to solve

$$\nabla^2 G(x,y) = -\delta^2(x,y)$$

The minus sign is just a convention: other authors have a plus sign

Green's functions (2)

- Once you have the Green's function for a point source, you can get the solution for an arbitrary set of sources by summing the Green's function multiplied by the strength of the source
- You know well the potential for a point electrostatic charge in 3D:

$$G(x,y,z) = \frac{1}{4\pi r}, \ r = \sqrt{x^2 + y^2 + z^2}$$

The Green's function in 2D

We start with

$$(2\pi)^2\delta^2(k_x, k_y) = \int_{-\infty}^{\infty} dxdye^{i(k_x x+k_y y)}$$

and

$$\nabla^2 \int_{-\infty}^{\infty} dxdye^{i(k_x x+k_y y)} =$$

$$-\int_{-\infty}^{\infty} dxdy(k_x^2 + k_y^2)e^{i(k_x x+k_y y)}$$

Hence

$$G(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(k_x x+k_y y)}}{(k_x^2+k_y^2)}$$

The integral is done in polar coordinates:

$$k_x = r\cos(\theta),\ \ k_y = r\sin(\theta).$$

The Green's function in 2D (2)

$$G(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(k_x x + k_y y)}}{(k_x^2 + k_y^2)}$$

In polar coordinates, the angular integral is:

$\int_{-\pi}^{\pi} \exp(ikr\cos\theta) = 2\pi J_0(kr)$.

Here $J_0(z)$ is theBessel function of order zero (of the first kind).

This gives us

$G(\mathbf{x}) = G(r) = \frac{1}{2\pi}\int_0^{\infty} dk \frac{J_0(kr)}{k}$.

If we require that $G(r)$ vanish at $r = a$, we get

$G(r) = \frac{1}{2\pi}\int_0^{\infty} dk \frac{J_0(kr) - J_0(ka)}{k}$.

The Green's function in 2D (3)

$$G(r) = \frac{1}{2\pi}\int_0^\infty dk\,\frac{J_0(kr) - J_0(ka)}{k}.$$

We evaluate this using Frullani's integral

$I(a,b) = \int_0^\infty dx\,\frac{[f(ax) - f(bx)]}{x}$, $a > 0, b > 0$

and $f(x)$ is continuous at $x = 0$. Then

$I(a,b) = f(0)\ln(b/a)$.

Hence

$G(r) = \frac{1}{2\pi}\ln\left(\frac{a}{r}\right)$.

This is the 2D Green's function. It satisfies

$\nabla^2 G(r) = -\delta^2(\mathbf{x})$, $|\mathbf{x}| = r$.

$$G(r) = \frac{1}{2\pi} \ln\left(\frac{a}{r}\right).$$

This is the 2D Green's function. It satisfies $\nabla^2 G(r) = -\delta^2(\mathbf{x}),\ |\mathbf{x}| = r.$

The question we answer here is:

What does this Green's function become if we want to have the operator ∇^{2s}, s arbitrary real or complex?

A Neat Trick

We know the Green's function for the Laplacian:

$G_2(r) = -\frac{1}{2\pi}\ln(r)$

gives

$\nabla^2 G_2(r) = -\delta^2(x,y)$

We want to know what G_{2s} is for which

$\nabla^{2s} G_{2s}(r) = -\delta^2(x,y)$

We write

$\nabla^2 G_2(r) = \nabla^{2s}(\nabla^{2-2s} G_2(r))$

Then quite simply:

$G_{2s}(r) = \nabla^{2-2s} G_2(r)$

So all we need is to apply the Laplacian to an arbitrary power to the log function!

Technical Details (1)

To carry out this calculation, we first need to know $\nabla^{2s} r^{\beta}$. It is obvious that each second derivative reduces the power of r by 2, so

$$\nabla^{2s} r^{\beta} = K(r,\beta) r^{\beta-2s}$$

for some $K(s,\beta)$ which depends on β and s, but not r.

To evaluate $K(s,\beta)$ we need Weber's integral:

$$\int_0^{\infty} r^s J_0(\alpha r) dr = \frac{\Gamma(\frac{1+s}{2})2^s}{\Gamma(\frac{1-s}{2})\alpha^{1+s}}$$

Technical details (2)

We write down the Fourier transform of r^β:

$$r^\beta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Gamma(1+\beta/2)}{\Gamma(-\beta/2)\pi^{\beta+1}k^{\beta+2}} e^{2\pi i(k_x x + k_y y)} dk_x dk_y .$$

To check this expression, convert the double integral to an integral over angle multiplied by an integral over kdk. The integral over angle gives the Bessel function $2\pi J_0(2\pi kr)$. You then get

$$\frac{2\Gamma(1+\beta/2)}{\Gamma(-\beta/2)\pi^\beta}\int_0^\infty k^{-1-\beta} dk J_0(2\pi kr).$$

You then use Weber's integral to verify the result.

Technical details (3)

We next apply ∇^{2s} to the Fourier transform of r^β:

$r^\beta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Gamma(1+\beta/2)}{\Gamma(-\beta/2)\pi^{\beta+1}k^{\beta+2}} e^{2\pi i(k_x x+k_y y)} dk_x dk_y.$

The operator just gives $(2\pi i k)^{2s}$ times the same integral.

This means that in essence the differential operator makes the replacement

$k^{-\beta-2} \to k^{2s-\beta-2}$

We then do the integral in the same way:

convert to polar coordinates, integrate over angle,

and finally use Weber's integral. We obtain:

$\nabla^{2s}(r^\beta) = i^{2s}2^{2s}\frac{\Gamma(1+\beta/2)}{\Gamma(-\beta/2)}\frac{\Gamma(s-\beta/2)}{\Gamma(1-s+\beta/2}r^{\beta-2s}$

Differentiation of the Logarithm

$$\nabla^{2s}(r^\beta) = i^{2s}2^{2s}\frac{\Gamma(1+\beta/2)}{\Gamma(-\beta/2)}\frac{\Gamma(s-\beta/2)}{\Gamma(1-s+\beta/2)}r^{\beta-2s}$$

Suppose we expand this taking β small:

$r^\beta = e^{\beta\ln r} \simeq 1 + \beta\ln r$.

Then this will tell us how ∇^{2s} operates on a constant and $\ln r$.

The only term which causes any problem is

$\frac{1}{\Gamma(z)} \simeq z$, for $|z| << 1$. So for β small,

$\nabla^{2s}(r^\beta) \simeq i^{2s}2^{2s}(\frac{-\beta}{2})\frac{\Gamma(s)}{\Gamma(1-s)}r^{-2s}$

This tells us that

$\nabla^{2s}(1) = 0$: cf differentiation!

$\nabla^{2s}(\ln r) \simeq -i^{2s}2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}r^{-2s}$

The Final Answer

$$\nabla^{2s}(\ln r) \simeq -i^{2s}2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}r^{-2s},$$

so that

$$\nabla^{2-2s}(\ln r) \simeq -i^{2-2s}2^{2-2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}r^{-2+2s}$$

- So we have deduced:

$$G_{2s}(r) = \frac{-r^{2s-2}}{\pi(2i)^{2s}}\frac{\Gamma(1-s)}{\Gamma(s)}$$

Only for s$\rightarrow$ 1 will we get a logarithm!

We have also learned that Fourier transforms can be used as a way of evaluating fractional derivatives.