Murray Klamkin, Amer. Math. Monthly

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This is the third of a number of files listing problems, solutions and other writings of Murray Klamkin.

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The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

Amer. Math. Monthly, 56(1949) 347–348.

Summation, Binomial coefficients

4276 [1947,601]. Proposed by P. A. Pizá, San Juan, P.R.

Let the integers $_{n}K_{c}$ be defined by the relations

 $_{n}K_{1} = 1;$ $_{n}K_{m} = 0, m > n;$ $_{n+1}K_{c} = c(_{n}K_{c} + _{n}K_{c-1}), c > 1$

Prove the following summations

(A)
$$x^{n} = \sum_{j=1}^{n} {}_{n}K_{j} \begin{pmatrix} x \\ j \end{pmatrix}$$

(B)
$$\sum_{a=1}^{x-1} = \sum_{j=1}^{n} {}_{n}K_{j} \begin{pmatrix} x \\ j+1 \end{pmatrix}$$

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute, Brooklyn, N.Y. On the assumption that (A) is true, we have

$$x^{n+1} = \sum_{j=1}^{n} {}_{n}K_{j}\binom{x}{j}x = \sum_{j=1}^{n} {}_{n}K_{j}\left[(j+1)\binom{x}{j+1} + j\binom{x}{j}\right]$$
$$= {}_{n}K_{1}x + \sum_{j=2}^{n}\binom{x}{j}[j \cdot {}_{n}K_{j} + j \cdot {}_{n}K_{j-1}] + (n+1) {}_{n}K_{n}\binom{x}{n+1}$$
$$= {}_{n+1}K_{1}x + \sum_{j=2}^{n} {}_{n+1}K_{j}\binom{x}{j} + {}_{n+1}K_{n+1}\binom{x}{n+1} = \sum_{j=1}^{n+1} {}_{n+1}K_{j}\binom{x}{j}$$

Since (A) is evidently true for n = 1, it is true for all n by induction.

(B) follows immediately from (A) by use of the familiar relation

$$\sum_{a=j}^{x-1} \binom{a}{j} = \binom{x}{j+1}$$

Thus

$$\sum_{a=1}^{x-1} a^n = \sum_{a=1}^{x-1} \sum_{j=1}^n {}_n K_j \binom{a}{j} = \sum_{j=1}^n {}_n K_j \sum_{a=1}^{x-1} \binom{a}{j} = \sum_{j=1}^n {}_n K_j \binom{x}{j+1}$$

Solved also by [6 others, including] E. T. Frankel and Yu-shu Luan.

Frankel points out that (A) and (B) are special cases of general formulas in the calculus of finite differences which express the general term and the sum of a given number of terms of a rational integral function by meansof its leading differences and binomial coefficients. (See Whittaker and Robinson, *The Calculus of Observations*, London, 1924, p.7.) The integers ${}_{n}K_{1}$, ${}_{n}K_{2}$, ..., ${}_{n}K_{c}$ are the leading differences of the *n* th powers of the natural nombers 0^{n} , 1^{n} , 2^{n} , ..., c^{n} .

In consequence, as noted by Yu-shu Luan, we have the following explicit expression for ${}_n K_c$

$$_{n}K_{c} = \sum_{j=1}^{n} (-1)^{j} {\binom{c}{j}} (c-j)^{n}$$

Amer. Math. Monthly, 56(1949) 474–475.

The Sum of a Series

E 844 [1949, 31]. Proposed by Orrin Frink, Pennsylvania State College Sum the series

$$1 + 1/5! + 1/10! + 1/15! + \dots + 1/(5n - 5)! + \dots$$

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. Since the sum of the m th powers of the k k th roots of unity is zero unless m is a multiple of k, in which case the sum is unity, we have, from the Maclaurin expansion of e^z

$$\sum_{j=1}^{k} \exp(w_k^j x) = k \sum_{n=0}^{\infty} x^{kn} / (kn)!$$

where w_k is a primitive k the root of unity. The required sum, obtained by taking x = 1, k = 5, is then

$$S = (1/5) \sum_{j=1}^{5} \exp w_5^j$$

Also solved by [21 others]

Several solvers easily reduced the above sum to

$$S = (1/5)[e + 2e^{\cos 72^{\circ}}\cos(\sin 72^{\circ}) + 2e^{-\cos 36^{\circ}}\cos(\sin 36^{\circ})]$$

If we are merely interested in obtaining a numerical result, however, there is little point of transforming the original series, which converges very rapidly. Thus four terms of the series gives S = 1.00833360890

[F. C.] Smith picked up, as a by-product, the pretty summations

$$\sum_{n=0}^{\infty} (x^n \cos n\theta)/n! = e^{x \cos \theta} \cos(\sin \theta)$$
$$\sum_{n=0}^{\infty} (x^n \sin n\theta)/n! = e^{x \cos \theta} \sin(\sin \theta)$$

[[RKG notes that the above numerical value is given by *three* terms. Four terms give 1.008333608907290289 and five terms give 1.0083336089072902899764536]]

Amer. Math. Monthly, 56(1949) 634.

A series for π

E 854 [1949, 104]. Proposed by Jerome C. R. Li, Oregon State College Show that $\pi = \sum_{n=0}^{\infty} (n!)^2 2^{n+1} / (2n+1)!$

[[Five solutions were published, one by Ragnar Dybvik using the beta function, the second by Paul Carnahan using Wallis's formula, the third by Murray below, the fourth by N. J. Fine using a differential equation, and the fifth by D. H. Browne using Euler's transfomation of series.]]

III. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. (Using the Legendre polynomials). We have

$$a_n = (n!)^2 2^{n+1} / (2n+1)! = \int_{-1}^1 z^n P_n(z) \, dz$$

where $P_n(z)$ is the Legendre polynomial of degree *n*. Therefore

$$\sum_{n=0}^{\infty} a_n = \int_{-1}^{1} \sum_{n=0}^{\infty} z^n P_n(z) \, dz = \int_{-1}^{1} (1-z^2)^{-1/2} \, dz = \pi$$

Amer. Math. Monthly, 57(1950) 39.

Equivalent Concurrent Sections of a Tetrahedron

E 865 [1949, 263]. Proposed by Victor Thébault, Tennie, Sarthe, France

Find a point such that the planes drawn through this point parallel to the faces of a tetrahedrn cut the opposite trihedrals in equivalent triangles. Express the common area of these triangles in terms of the areas of the faces of the tetrahedron.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. Denote the areas of the faces of the tetrahedron by A_i (i = 1, 2, 3, 4), the corresponding altitudes by H_i and the perpendicular distances from the corresponding vertices to the corresponding plane sections by h_i . Let V be the volume of the tetrahedron and A the common area of the parallel sections. Then

$$\sum (H_i - h_i)A_i = 3V + H_iA_i \qquad h_i^2/H_i^2 = A/A_i$$

Therefore

$$3V = \sum \left\{ H_i A_i - H_i (AA_i)^{1/2} \right\} = 12V - \sum H_i (AA_i)^{1/2}$$

or

$$A^{1/2} = 9V / \sum H_i A_i^{1/2} = 3 / \sum A_i^{-1/2}$$

Also

$$h_i = 3H_i A_i^{-1/2} \left/ \sum A_i^{-1/2} \right.$$

The analogous problem for the plane was proposed by J. Neuberg as Question 30, *Mathesis*, 1881, p.148. For this case we have

$$2/k = 1/a + 1/b + 1/c$$

where a, b, c are the sides of the triangle and k is the common length of the concurrent lines which are drawn parallel to the sides of the triangle. This, and the result of the given problem, suggested to [N. D.]Lane for the corresponding problem of an n-dimensional simplex the formula

$$n/k = \sum 1/a$$

where k^{n-1} is the common content of the (n-1)-dimensional cells parallel to the (n-1)-dimensional cell "faces" of the simplex, and a^{n-1} is the content of a "face".

[[I pause to note that Murray does not appear in the Author Index in Amer. Math. Monthly, **57** No.7, part 2: index of Vols. 1 to 56, Aug-Sep 1950.]] Amer. Math. Monthly, 58(1951) 195.

4431. Proposed by M. S. Klamkin, Brooklyn Polytechnic Institute, New York If $S_n = \sum_{1}^{n} 1/r$, prove

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}$$

Amer. Math. Monthly, **59**(1952) 471–472.

A Summation Problem

Solution by Robert Steinberg, University of California, Los Angeles. It suffices to prove the first and third expressions equal since the obvious identity

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2}$$

implies the second. Now we have

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{r=1}^n \frac{1}{r} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{r=1}^\infty \left(\frac{1}{r} - \frac{1}{n+r}\right)$$
$$= \sum_{n=1}^{\infty} \sum_{r=1}^\infty \left(\frac{1}{r(n+r)^2} + \frac{1}{n(n+r)^2}\right) = 2\sum_{n=1}^\infty \sum_{r=1}^\infty \frac{1}{r(n+r)^2}$$

by symmetry. Upon setting r = R and r + n = N + 1, with appropriate changes in the limits of summation, this last form becomes $2\sum_{1}^{\infty} S_N/(N+1)^2$ and the proof is complete.

An interesting corollary may be obtained as follows. We have

$$\sum_{n=1}^{\infty} \frac{S_n}{n^2} = \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{1}{n^2 r}$$

If the indicated summation in the (n, r)-plane is carried out along rational rays through the origin, the first integral point on such a ray is a point (R, N) such that R and Nare relatively prime and all other points on this ray are the integral multiples of (R, N). Thus the contribution of this ray to the sum is

$$\frac{1}{N^2 R} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

The contribution of all rational rays with $0 < R \leq N$ is thus

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{n=1}^{\prime} \frac{1}{N^2 R}$$
(1)

where the inner summation is made over all pairs of integers R and N such that $0 < R \le N$ and R and N are relatively prime. Since the expression (1) is equal to $2\sum_{1}^{\infty} 1/n_3$ we get the desired result:

$$\sum' \frac{1}{N^2 R} = 2.$$

[[Also solved by [5 others and] the proposer.]]

Editorial Note. [J.]Vales and [A.]Petracca obtain the following generalizations. Put

$$S(r,n) = \sum_{i=1}^{n} \frac{1}{i^r} \qquad K(r,n) = \frac{1}{n+r-1} \sum_{i=i}^{n} K(r-1,i)$$

with K(1,n) = 1/n. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^{h+2}} = \sum_{n=1}^{\infty} \sum_{i=1}^{h} \frac{S(i,n)}{(n+1)^{h+2-i}} \qquad \sum_{n=1}^{\infty} \frac{1}{n^{h+1}} = \sum_{n=1}^{\infty} \frac{K(h,n)}{h+n-1}$$

[D. H.]Browne finds

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{S_n}{(n+1)^2} = 5\sum_{n=1}^{\infty} \frac{S_n}{(n+1)^3}$$

but further extensions do not assume so neat a form.

Amer. Math. Monthly, 58(1951) 260.

E 963. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

It had started snowing before noon and three snow plows set out at noon, 1 o'clock and 2 o'clock, respectively, along the same path. If at some later time they all came together simultaneously, find the time of meeting and also the time it started snowing.

Amer. Math. Monthly, **59**(1952) 42.

A Variation of the Snow Plow Problem

Solution by L. A. Ringenberg, Eastern Illinois State College. Let n denote the number of hours before noon that it started to snow and let x, y, z denote the distances that the first, second and third snow plow travelled by t hours past noon. Assume that it snows at a constant rate. Then, if the length units are properly chosen, we have

$$dt/dx = t + n,$$
 $x = 0$ when $t = 0$

with solution

$$t = e^x n - n \tag{1}$$

)

Also

$$dt/dy = t - (e^y n - n), \qquad y = 0 \quad \text{when} \quad t = 1$$

with solution

$$t = e^{y}(n + 1 - ny) - n (2)$$

Also

$$dt/dz = t - [e^{z}(n+1-nz) - n]$$
 $z = 0$ when $t = 2$

with solution

$$t = e^{z}(n+2-nz-z+nz^{2}/2) - n$$
(3)

Let d denote the common value of x, y and z when the plows meet and let T denote the value of t when the meeting occurs. It follows from (1), (2) and (3) that

$$(T+n)/e^d = n = n + 1 - nd = n + 2 - nd - d + nd^2/2$$

Solving we get n = 1/2 and T = 3.195. Therefore it began to snow at 11:30 and the plows met at about 3:12.

[[Solved by [5 others and] the proposer.]]

Amer. Math. Monthly, 58(1951) 261.

Tetrahedron and Concurrent Cevians

E 928 [1950, 483]. Proposed by Victor Thébault, Tennie, Sarthe, France

Given a tetrahedron ABCD and a point O. Denote by A', B', C', D' the intersections of AO, BO, CO, DO with the corresponding faces of the tetrahedron, and set x = AO/A'O, y = BO/B'O, z = CO/C'O, t = DO/D'O. Show that

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. The altitude of OBCD from O is easily shown to be $h_A/(1-x)$, where h_A is the altitude of ABCD from A. Therefore, if V is the volume of ABCD and b_A is the area of the face opposite A,

$$V = \sum (h_A b_A) / 3(1 - x) = \sum V / (1 - x)$$

That is

$$1 = 1/(1-x) + 1/(1-y) + 1/(1-z) + 1/(1-t)$$

or

$$xyzt = 3 - 2(x + y + z + t) + (xy + xz + xt + yz + yt + zt).$$

Amer. Math. Monthly, 58(1951) 268-269.

Two summations

4356 [1949, 479]. Proposed by P. A. Piza, San Juan, Peurto Rico Prove the relations

(a)
$$x^{2n+1} - (x-1)^{2n+1} = \sum_{a=0}^{n} \left[\binom{n+a}{2a+1} + \binom{n+1+a}{2a+1} \right] (x^2 - x)^{n-a}$$

(b) $x^{2n+2} - (x-1)^{2n+2} = (2n-1) \sum_{a=0}^{n} \binom{n+a}{2a+1} (x^2 - x)^{n-a}$

[[The last binomial is misprinted $\binom{n+1}{2a+1}$ in the original. — R.]]

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York. Assume (b) is true for n = k. Then by integrating between 1 and x we obtain

$$x^{2n+1} - (x-1)^{2n+1} = 1 + \sum_{a=0}^{k} \frac{2k+3}{k+1-a} \binom{k+1+a}{2a+1} (x^2 - x)^{k+1-a}$$

which is easily shown to be equivalent to (a) for n = k + 1. Further, upon multiplying (a) for n = k + 1 by 2x + 1, multiplying (b) for n = k by $x^2 - x$, and subtracting, we get exactly (b) for n = k + 1. Since (a) and (b) are obviously true for n = 0, it follows by induction that they are true for all integral n. Amer. Math. Monthly, 58(1951) 569.

4455. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)}{g'(x)}$$

solve for f(x) in terms of g(x) and for g(x) in terms of f(x). (This is an extension of one of the problems proposed for th 1951 William Lowell Putnam Prize Competition.)

Amer. Math. Monthly, **59**(1952) 707.

Functions such that the Derivative of the Quotient is the Quotient of the Derivatives Solution by E. J. Scott, University of Illinois. Using the abbreviations f, g, f', g' for f(x), g(x), etc., we have

$$(gf' - fg'/g^2 = f'/g')$$
(1)

from which

$$f'/f - g'/g = [1 - g/g']^{-1}$$

Integrating, we have

$$\ln f/g = \int [1 - g/g']^{-1} \, dx$$

whence

$$f = g \exp\left\{\int [1 - g/g']^{-1} dx\right\}$$

Similarly, from (1) we obtain

$$(g'/g)^2 - (f'/f)(g'/g) + f'/f = 0$$

whence

$$(g'/g)^2 = \frac{1}{2} \left[f'/f \pm \sqrt{(f'/f)^2 - 4(f'/f)} \right]$$

Integrating and solving for g, we get

$$g = \sqrt{f} \exp\left[\pm\frac{1}{2}\int\sqrt{(f'/f)^2 - 4(f'/f)}\,dx\right]$$

Also solved by [7 others] and the Proposer.

Amer. Math. Monthly, **59**(1952) 408.

A Slow Ship Intercepting a Fast Ship

E 991 [1951, 699]. Proposed by C. S. Ogilvy, Syracuse University

Two ships leave different points at the same time and steam on straight courses at constant but unequal speeds. Find the condition under which the slower ship can intercept (catch) the faster.

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute. If the slower ship intercepts the faster ship then the ratio of the distances traversed is V_S/V_F , where V_S is the velocity of the slow ship and V_F is the velocity of the fast ship. But the locus of a point such that the ratios of its distances from two fixed points is constant, is a circle. Thus if the path of the faster ship intersects this circle it can be intercepted by the slower one.

Editorial Note. The circle is a circle of Apollonius of the two starting points. If θ is the angle between the faster ship's course and the line joining the two starting points, then it is easily shown that the slower ship can set a course to intercept the faster ship if and only if

$$\theta \leq \arcsin(V_S/V_F).$$

If we have inequality here, then there are two courses that the slower ship may set. The problem become more interesting if the sphericity of the earth is taken into account.

Amer. Math. Monthly, **59**(1952) 636.

An Approximation for $\sqrt[n]{a}$

E 1004 [1952, 105]. Proposed by L. R. White, Washington, D.C.

(1) The approximation $\sqrt[n]{a} = 1 + (a-1)/n$ for large n and a close to 1 is well known and frequently used. Show that for large n and all a

$$\sqrt[n]{a} = 1 + (\ln a)/n$$

is a good approximation.

(2) Find

$$\lim_{n\to\infty}\left\{(k-1+\sqrt[n]{a})/k\right\}^n$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. (1) For all a > 0

$$\sqrt[n]{a} = e^{(\ln a)/n} = 1 + (\ln a)/n + O(1/n^2)$$

(2) Using the preseding result we find

$$\lim_{n \to \infty} \left\{ (k - 1 + \sqrt[n]{a})/k \right\}^n = \lim_{n \to \infty} \left\{ 1 + (\ln a)/kn + O(1/n^2) \right\}^n \\ = e^{(\ln a)/k} = a^{1/k}$$

Amer. Math. Monthly, **59**(1952) 637.

An Application of the Newton-Raphson Method

E 1005 [1952, 105]. Proposed by D. W. Dubois, University of Oklahoma Let $x_0 \neq 0$ and a > 0 be two real numbers. Define

 $x_{n+1} = x_n/2 + a/2x_n$ $n = 0, 1, 2, \dots$

Find all values of x_0 and a for which the sequence $\{x_n\}$ converges, and find the limits.

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If we plot y = x/2 + a/2x and y = x we get a geometric insight into the convergence process. The curve is a hyperbola with x = 0 and y = x/s as the asymptotes. The intersections are (\sqrt{a}, \sqrt{a}) and $(-\sqrt{a}, -\sqrt{a})$. Also, the points of intersection are the maximum and minimum points of the hyperbola. From the plot it follows that the sequence converges for all values of a > 0 and $x_0 \neq 0$. If $x_0 > 0$, then $x_n \to \sqrt{a}$; if $x_0 < 0$, then $x_n \to -\sqrt{a}$. [[[R. E.]Greenwood called attention to Whittaker and Robinson, The Calculus of Observations, pp.79–81.]]

Amer. Math. Monthly, **59**(1952) 640.

4508. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York

Find a polynomial F(x) of lowest degree such that $F(x) + \omega^s$ is divisible by $(x - \omega^s)^r$ for $s = 0, 1, 2, \ldots, p - 1$, where ω is a primitive p th root of unity and r is a given positive integer. (This generalizes a problem in Goursat-Hedrick, A Course in Mathematical Analysis, v.1, p.32.)

Amer. Math. Monthly, **61**(1954) 127–128.

Solution by Chih-Yi Wang, University of Minnesota. Since F'(x) is divisible by $(x^p - 1)^{r-1}$, and the binomials $x - \omega^s$ are relatively prime to one another, it is evident that the degree of the polynomial F'(x) cannot be lower than p(r-1) + 1. Let

$$F'(x) = a(x^p - 1)^{r-1}$$

where $a \neq 0$ is some constant which has to be determined. By integrating from 0 to x we get

$$F(x) - F(0) = a \int_0^x (t^p - 1)^{r-1} dt$$

But $F(\omega^s) + \omega^s = 0$ for $s = 0.1, \ldots, p - 1$, so that

$$\begin{aligned} -\omega^s - F(0) &= \int_0^{\omega^s} a(t^p - 1)^{r-1} dt \\ &= \omega^s \left\{ a \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{1}{p[(r-1)-k]+1} \right\} \\ &= \omega^s \left\{ a \int_0^1 (t^p - 1)^{r-1} dt \right\} \qquad s = 0, 1, \dots, p-1. \end{aligned}$$

These p equations are satisfied if and only if

$$F(0) = 0, \qquad a = -1 \left/ \int_0^1 (t^p - 1)^{r-1} dt \right.$$

Hence the required polynomial is

$$F(x) = -\int_0^x (t^p - 1)^{r-1} dt \left/ \int_0^1 (t^p - 1)^{r-1} dt \right|$$

[[Also solved by X, Y, M. S. Klamkin, Z, and the Proposer — who was Murray Klamkin!!]]

Amer. Math. Monthly, **59**(1952) 643

Polynomials with Positive Coefficients

4411 [1951, 343]. Proposed by R. M. Cohn, Rutgers University

If a polynomial equation f(x) = 0, with integral coefficients has no positive roots, there exists a polynomial g(x) with integral coefficients such that all coefficients of $f(x) \cdot g(x)$ are positive.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. It will suffice to prove the theorem for $q(x) = x^2 - ax + b$, $(a, b > o, a^2 < 4b)$. Let $p(x) = (1 + x)^k$. Then all the coefficients of $q(x) \cdot p(x)$ will be positive if

$$\binom{k}{r} - a\binom{k}{r-1} + b\binom{k}{r-2} > 0 \tag{1}$$

for $r = 1, 2, \ldots, k + 1$. This is always possible for k sufficiently large.

f(x) can be factored into linear factors like $x + c, c \ge 0$, and quadratic factors like q(x). We may then take g(x) to be $(1+x)^K$ where K is the sum of the k s determined as above for the individual quadratic factors of f(x). Since the coefficients of f(x) are integers, the coefficients of $f(x) \cdot g(x)$ will be positive integers.

Also solved by [5 others and] the Proposer

Editorial Note. The existence of suitable k may be seen as follows. (1) reduces to

$$(k-r+2)(k-r+1) - ar(k-r+2) + br(r-1) > 0$$

which is true if

$$2k > (2+a)r - 3 + \sqrt{1 + (2a+4b)r - (4b-a^2)r^2}$$
(2)

and the radical is real; if the radical is imaginary (1) is true for all k. Evidently the right member of (2) has a maximum value and we need only choose k greater than this maximum.

[L. J.]Paige and George]Piranian refer to a proof by E. Meissner (Math. Annalen, v.70(1911) p.223) and to the following generalization by D. R. Curtiss (Math. Annalen, v.73(1913) p.424): If f(x) is any polynomial with all its coefficients real, there exists polynomials $f_1(x)$ such that when the product $f_2(x) = f_1(x) \cdot f(x)$ is arranged according to ascending or descending powers of x, the number of variations of sign presented by its coefficients is exactly equal to the number of positive roots of f(x). See also Pólya and Szegö, Aufgaben und Lehrsatze, v.II, problem 190, section 5.

Amer. Math. Monthly, **59**(1952) 648–649.

A Diophantine Equation

4448 [1951, 343]. Proposed by Jekuthiel Ginsburg, Yeshiva College, New York City Solve in positive integers

$$z^4 = \frac{ax^2 + by^2}{a+b}$$

where a and b are given integers.

I. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York. The given equation will be satisfied if

$$an(z^{2} + x) = m(y - z^{2})$$
 $m(z^{2} - x) = bn(y + z^{2})$

m and n being arbitrary integers. Therefore

$$x : y : z^{2} = |m^{2} - 2bmn - abn^{2}| : |m^{2} + 2amn - abn^{2}| : (m^{2} + abn^{2})$$

is a solution provided the constant of proportionality, k, is so chosen that $k(m^2 + abn^2)$ is a perfect square.

Other solutions may be obtained by solving $z^2 = m^2 + abn^2$. See L. E. Dickson, *History* of the Theory of Numbers, v.2, p.425. One form of the solution is

$$m: n: z = |p^2 - abq^2|: 2pq: (p^2 + abq^2)$$

Amer. Math. Monthly, **59**(1952) 702.

4514. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

 $\psi(t) = \text{constant}$ and $\psi(t) = |t|$ are solutions of the integral equation

$$2\psi(t) = \int_{-1}^{1} \psi(2tx) \, dx$$

as is easily verified. Are there any other solutions?

Amer. Math. Monthly, **61**(1954) 203.

Solution by J. E. Wilkins, Jr., Nuclear Development Associates, Inc., White Plains, N.Y. There are others. To find additional solutions, substitute $|t|^a$ for $\psi(t)$. This function will satisfy the integral equation if and only if $2^a = a + 1$. This transcendental equation has exactly two real roots, namely 0 and 1, and has infinitely many complex roots

$$\alpha_n \pm i\beta_n (\log 2)^{-1}$$

such that

$$\alpha_n = -1 + \beta_n (\log 2)^{-1} \cot \beta_n$$
$$2\beta_n (\log 2)^{-1} \exp(-\beta_n \cot \beta_n) = \sin \beta_n$$
$$2n\pi < \beta_n < (4n+1)\pi/2$$

(n = 1, 2, ...). For example, $\beta_1 = 7.454087$, $\alpha_1 = 3.545368 \pm 10.75397i$.

Also solved by [7 others and] the Proposer.

Editorial Note. As shown also by the other solvers, there exist no real solutions of the integral equation, under the assumption that $\psi(x)$ is of class C^1 on [0,1], other than linear combinations of the solutions cited by the Proposer.

Amer. Math. Monthly, **60**(1953) 40.

E 1049. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

If a uniform thin rod of length L is perpendicular to and touching the earth's surface at one end, find the distance between the centre of gravity and the centre of mass of the rod.

Centre of Gravity of a Vertical Rod

Solution by M. Morduchow, Polytechnic Institute of Brooklyn

Let x be distance measured along the rod from the earth's surface. Also, let R be the radius and M the mass of the earth, while ρ is the mass per unit length of the rod. Then, by Newton's inverse-square law, the total force (directed towrds the earth's centre) acting on the rod will be

$$F = \int_0^L GM\rho(R+x)^{-2} \, dx = GMm/R(R+L)$$

where G is the gravitational constant, and $m = \rho L$ is the mass of the rod. If the centre of gravity is at $x = x_g$, then by definition we require that $F = GMm/(R+x_g)^2$. Therefore $(R + x_g)^2 = R(R + L)$, whence

$$x_g = \sqrt{R(R+L)} - R$$

Since the centre of mass is at $x = x_m = L/2$, the centre of gravity will be below the centre of mass, abd the distance between the two centres will be

$$x_m - x_g = L/2 - R[(1 + L/R)^{1/2} - 1]$$

If $L/R \ll 1$, then expansion by the binomial theorem to second powers of L/R yields the approximate result

$$x_m - x_g \approx (1/8)(L/R)L$$

Also solved by [13 others and] the Proposer.

All of these solutions were not the same. [Julian]Braun, [A. R.]Hyde, [L. V.]Mead, [J. V.]Whittaker and the proposer took the distance x_g to be

$$x_g = \frac{\int_0^L x(R+x)^{-2} \, dx}{\int_0^L (R+x)^{-2} \, dx} = (R/L)(R+L)\ln(1+L/R) - R$$

[Azriel]Rosenfeld took the centre of gravity of the rod as that point for which the weight of the portion of the rod above the point is equal to the weight of the portion of the rod below the point. He found

$$x_g = 2RL/(4R+L).$$

All other solutions werelike Murduchow's. Here, for a rod a mile long, the distance between the centres is about 2 inches.

[H. W.]Smith called attention to Kellogg, Foundations of Potential Theory, p.4.

Amer. Math. Monthly, **60**(1953) 186–187.

[[There's a Classroom Note, D. E. WHITFORD and M. S. KLAMKIN, On an elementary derivation of Cramer's rule.]]

Amer. Math. Monthly, **60**(1953) 188.

E 1057. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Find the sum of the first n terms of the series

 $\sec \theta + (\sec \theta \sec 2\theta)/2 + (\sec \theta \sec 2\theta \sec 4\theta)/4 + \cdots$

Amer. Math. Monthly, **60**(1953) 628.

Solution by J. R. Hatcher, Fisk University. Using the identities $\sec \alpha = 2 \sin \alpha / \sin 2\alpha$ and $\csc \alpha = \cot \alpha / 2 - \cot \alpha$, we conclude that the sum $S_n(\theta)$ of the first n terms is

$$S_n(\theta) = 2\sin\theta(\csc 2\theta + \csc 4\theta + \cdots \csc 2^n\theta)$$

= $s\sin\theta(\cot \theta - \cot 2^n\theta)$
= $[2\sin(2^n - 1)\theta]/[\sin 2^n\theta].$

[F.]Underwood pointed out that the problem is essentially Ex.10, p.125 of *Plane Trigonometry, Part II*, by S. L. Loney (1908).

Amer. Math. Monthly, **60**(1953) 423.

4546. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York Evaluate

$$I = \int_0^\infty \frac{\sin\frac{mx}{2}\sin\frac{nx}{2}\sin\frac{m+1}{2}x\sin n + 12x}{x^2\sin^2\frac{x}{2}} dx$$

where m and n are integers and $m \ge n$.

Amer. Math. Monthly, **61**(1954) 649.

Solution by J. V. Whittaker, University of California, Los Angeles. This integral is merely

$$\int_0^\infty \frac{1}{x^2} \sum_{r=1}^m \sum_{s=1}^n \sin rx \sin sx \, dx$$

Integrating by parts and applying the sum and difference formulas, we obtain a known form:

$$\int_0^\infty \frac{\sin rx \sin sx}{x^2} dx = \int_0^\infty \frac{r \cos rx \sin sx + s \sin rx \cos sx}{x} dx$$
$$= \pi s/2 \qquad (r \ge s)$$

Summing over r and s, we find the value of I to be $(\pi/12)n(n+1)(3m-n+1)$. Also solved by [6 others and] the Proposer. Amer. Math. Monthly, **60**(1953) 632.

4561. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York Prove

$$S_{2,p} + \sum_{n=1}^{r} \left[n^2 \binom{n+p}{p} \right]^{-1} = S_{2,r} + \sum_{n=1}^{p} \left[n^2 \binom{n+r}{r} \right]^{-1}$$

where $S_{2,p} = \sum_{n=1}^{p} 1/n^2$.

Amer. Math. Monthly, **62**(1955) 49.

Solution by Leonard Carlitz, Duke University. Let p be an arbitrary, fixed integer. For r = 1, the proposed relation reduces to

$$S_{2,p} + \frac{1}{p+1} = 1 + \sum_{n=1}^{p} \frac{1}{n^2(n+1)}$$

which is evidently true since

$$\sum_{n=1}^{p} \frac{1}{n^2} - \sum_{n=1}^{p} \frac{1}{n^2(n+1)} = \sum_{n=1}^{p} \frac{1}{n(n+1)} = \sum_{n=1}^{p} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{p+1}$$

Assuming now that the stated result holds for r-1, we have

$$S_{2,r} + \sum_{n=1}^{p} \left[n^2 \binom{n+r}{r} \right]^{-1} - S_{2,r-1} - \sum_{n=1}^{p} \left[n^2 \binom{n+r-1}{r-1} \right]^{-1}$$
$$= \frac{1}{r^2} - \sum_{n=1}^{p} \frac{(r-1)!n!}{n^2} \frac{n}{(n+r)!} = \frac{1}{r^2} - \sum_{n=1}^{p} \frac{(r-1)!}{n(n+1) \cdot (n+r)}$$
$$= \frac{1}{r^2} - \frac{(r-1)!}{r} \left\{ \frac{1}{r!} - \frac{1}{(p+1) \cdots (p+r)} \right\}$$
$$= \frac{(r-1)!}{r(p+1) \cdots (p+r)} = \left[r^2 \binom{r+p}{p} \right]^{-1}$$

This evidently completes the induction proof.

Also solved by [5 others and] the Proposer.

Amer. Math. Monthly, **60**(1953) 843.

4552. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York

What derangement of terms of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

will produce a sum which is rational?

Amer. Math. Monthly, **61**(1954) 719.

Solution by V. C. Harris, San Diego State College, California. If neither the order in which the positive terms occur nor the order in which the negative terms occur is changed, but if the terms are rearranged so that k is the limit of the ratio of the number of positive terms to the number of negative terms in the first n terms, then the alteration L in the sum of the series is $L = \frac{1}{2} \log k$. (This result is due to Pringsheim; see Bromwich, An Introduction to the Theory of Infinite Series, Second Edition, Revised, p.76.) Since the sum of the given series is $\log 2$, the sum after alteration is $\log 2 + \frac{1}{2} \log k$. Setting $k = K^2/4$, the sum becomes $\log K$. If this is rational, say m/n, then $k = K^2/4 = (1/4) \exp(2m/n)$. To effect the derangement, set up any sequence of

 $k = K^2/4 = (1/4) \exp(2m/n)$. To effect the derangement, set up any sequence of rationals which has limit $(1/4) \exp(2m/n)$, e.g., convergents of its continued fraction expansion, and take a number of positive terms equal to the denominator, of the successive terms of the sequence. In particular, for k = 1/4, we have

$$0 = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots$$

Also solved by [5 others and] the Proposer.

Amer. Math. Monthly, 60(1953) 484–485.3

A Summation

4489 [1952, 332]. Proposed by D. J. Newman, Harvard University Sum the series

$$\sum_{n=1}^{\infty} \frac{(n/e)^{n-1}}{n!}$$

III. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If we expand e^{az} in powers of w where $z = we^{bz}$ we obtain

$$\frac{e^{az} - 1}{aw} = \sum_{n=1}^{\infty} \frac{\left[(a+nb)w\right]^{n-1}}{n!}$$

(See Bromwich, *Infinite Series*, p.160, ex.4.) Letting z = 1, and then $w = e^{-b}$, there results

$$\frac{e^a - 1}{ae^{-b}} = \sum_{n=1}^{\infty} \left[\frac{a + nb}{e^b} \right]^{n-1} \frac{1}{n!}$$

The proposed problem is a special case of this where we let b = 1 and $a \to 0$.

Amer. Math. Monthly, **60**(1953) 551–552.

N Objects in B Boxes

E 1051 [1953, 114]. Proposed by S. W. Golomb, Harvard University

Given N objects in B boxes, what is a necessary and sufficient condition for at least two boxes to contain the same number of objects?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If no two of the boxes contain the same number of objects, then the total number of objects is

$$0 + 1 + 2 + \dots + (B - 1) = B(B - 1)/2$$

or any greater integer. Therefore a necessary and sufficient condition for at least two boxes to contain the same number of objects is that N < B(B-1)/2.

Amer. Math. Monthly, **60**(1953) 716.

4564. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn Prove

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \frac{\pi^4}{72} \quad \text{where} \quad S_n = \sum_{r=1}^n \frac{1}{r}$$

Amer. Math. Monthly, **62**(1955) 129–130.

Solution by J. V. Whittaker, University of California, Los Angeles. We have, after rearrangement of the terms of the series,

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^3} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(m+n)^3}$$

We first notice that

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2(m+n)^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2} = \frac{\pi^4}{120}$$

Then

$$\begin{split} \sum_{n=1}^{\infty} \frac{1}{n(m+n)^3} &= \sum_{n=1}^{\infty} \left\{ \frac{1}{m^3 n} - \frac{1}{m^3(m+n)} - \frac{1}{m^2(m+n)^2} - \frac{1}{m(m+n)^3} \right\} \\ &= \sum_{n=1}^{\infty} \frac{1}{m^3 n} - \sum_{n=1}^{\infty} \left\{ \frac{1}{m^2(m+n)^2} + \frac{1}{m(m+n)^3} \right\} \end{split}$$

Finally, summing over m from 1 to ∞ , we find

$$\left(S - \frac{\pi^4}{90}\right) = S - \frac{\pi^4}{120} - \left(S - \frac{\pi^4}{90}\right)$$

or $S = \pi^4/72$.

II. Solution by the Proposer. This result is a special case of a theorem due to G. T. Williams (this MONTHLY, 1953, p.25) which may be put in the form

$$2\sum_{n=1}^{\infty} \frac{S_n}{n^p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(j+1)\zeta(p-j)$$

For p = 3 we have

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \frac{1}{2} \left\{ 5\zeta(4) - \zeta^2(2) \right\} = \frac{\pi^4}{72}$$

It may be obtained as easily from a result due to D. H. Browne in connection with Problem no.4431 [1952, 472].

Amer. Math. Monthly, **61**(1954) 199.

4582. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn Show that

(a)
$$\sum_{n=1}^{p} \frac{S_{1,n}^{2}}{n} \sim \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^{2}} + \frac{1}{3} (\gamma + \log p)^{2}$$

(b)
$$\sum_{n=1}^{\infty} \frac{\beta_{2,n}}{n} \sim \frac{\pi}{6} (\gamma + \log p) - \sum_{n=1}^{\infty} \frac{1}{n^3}$$

where γ is Euler's constant and

$$S_{m,n} = \sum_{r=1}^{n} \frac{1}{r^m}.$$

Amer. Math. Monthly, **62**(1955) 447–450. Solution by Leonard Carlitz, Duke University. We shall show that

$$A = \sum_{n=1}^{k} \frac{1}{n} S_{2,n} = \frac{\pi^2}{6} (\gamma + \log k) - \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{\log k}{k}\right)$$
(1)

$$B = \sum_{n=1}^{k} \frac{1}{n} S_{1,n}^2 = \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3} (\gamma + \log k)^3 + O\left(\frac{\log k}{k}\right)$$
(2)

where $S_{m,n} = \sum_{n=1}^{k} n^{-m}$. We recall that

$$S_{1,k} = \log k + \gamma + O\left(\frac{1}{k}\right) \qquad S_{2,k} = \frac{\pi^2}{6} + O\left(\frac{1}{k}\right) \tag{3}$$

We have

$$S_{1,k}S_{2,k} = A + C (4)$$

where

$$C = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=n+1}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{n} = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{r-n}$$

Thus

$$2C = \sum_{r=2}^{k} \frac{1}{r^2 \sum_{n=1}^{r-1}} \left(\frac{1}{n} + \frac{1}{r-n} \right) = \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n(r-n)}$$

$$2C + \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n^2} = \sum_{r=2}^{k} \sum_{n=1}^{r-1} \frac{1}{n^2(r-n)} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=n+1}^{k} \frac{1}{r-n} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=1}^{k-n} \frac{1}{r}$$
$$= \sum_{r=1}^{k-1} \frac{1}{r} \sum_{n=1}^{k-r} \frac{1}{n^2} = \sum_{r=1}^{k-1} \frac{1}{r} \left\{ \frac{\pi^2}{6} + O\left(\frac{1}{k-r}\right) \right\}$$
$$= \frac{\pi^2}{6} S_{1,k-1} + \left(\sum_{r=1}^{k-1} \frac{1}{r(k-r)} \right)$$

But

$$\sum_{r=1}^{k-1} \frac{1}{r(k-r)} = \frac{1}{k} \sum_{r=1}^{k-1} \left(\frac{1}{r} + \frac{1}{k-r}\right) = \frac{2}{k} S_{1,k-1} = O\left(\frac{\log k}{k}\right)$$

so that

$$2C + A = \frac{\pi^2}{6} S_{1,k} + S_{2,k} + O\left(\frac{\log k}{k}\right)$$

Thus (4) becomes

$$2S_{1,k}S_{2,k} = 2A + 2C = A + \frac{\pi^2}{6}S_{1,k} + S_{2,k} + O\left(\frac{\log k}{k}\right)$$

Using (3) and

$$S_{2,k} = \sum_{n=1}^{\infty} \frac{1}{n^2} + O\left(\frac{1}{k^2}\right)$$

we get

$$2\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\}\left\{\frac{\pi^2}{6} + O\left(\frac{1}{k}\right)\right\}$$
$$= A + \frac{\pi^2}{6}\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\} + \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{\log k}{k}\right)$$

which reduces to (1).

In the next place

$$S_{1,k}^3 = B + DS_{1,k} + E (5)$$

where

$$D = \sum_{n=1}^{k} \frac{1}{n} \sum_{s=n+1}^{k} \frac{1}{s} \qquad E = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} \sum_{s=n+1}^{k} \frac{1}{s}$$

Now

$$S_{1,k}^2 = 2D + S_{2,k} \tag{6}$$

As for E, we have

$$E = \sum_{s=2}^{k} \frac{1}{s} \sum_{n=1}^{s-1} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{n=1}^{s} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r} \sum_{n=r}^{s} \frac{1}{n} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r}$$

so that

$$2E = \sum_{s=1}^{k} \frac{1}{s} \left(\sum_{r=1}^{s} \frac{1}{r} \right)^{2} + \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r^{2}} - 2 \sum_{r=1}^{k} \frac{1}{r} \sum_{s=r}^{k} \frac{1}{s^{2}} = B - 2S_{1,k}S_{2,k} + 3A - 2S_{3,k}$$
(7)

Thus using (5), (6), (7) we get

$$S_{1,k}^3 = \frac{3}{2}B + \frac{1}{2}(S_{1,k}^2 - S_{2,k})S_{1,k} - S_{1,k}S_{2,k} + \frac{3}{2}A - S_{3,k}$$

which reduces to

$$S_{1,k}^3 = 3B - 3S_{2,k}S_{1,k} + 3A - 2S_{3,k}$$

= $3B - 3(S_{2,k}S_{1,k} - A - S_{3,k}) + 5S_{3,k}$

Now using (1) and (3) we immediately get (2).

Also solved by the Proposer.

Amer. Math. Monthly, **61**(1954) 263.

4583. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

For $a_n = 1, n = 1, 2, \ldots,$

$$S = \sum_{r=1}^{\infty} [r!]^{-a_r}$$

- -

is transcendental. Find other non-decreasing squences $\{a_n\}$ such that S is transcendental.

Amer. Math. Monthly, **62**(1955) 450–451.

Solution by the Proposer. Let

$$\sum_{r=1}^{n} [r!]^{-a_r}$$

be an approximation to S. If S is algebraic, then by the theorem of Liouville on algebraic numbers

$$\sum_{r=n+1}^{\infty} [r!]^{-a_r} \ge [n!]^{-ma_n} \tag{1}$$

must hold for m > 2 and n sufficiently large. If we take $a_{j+1} > 1 + a_j$, (1) implies

$$\frac{[(n+1)!]^{-a_{n+1}}}{1-\frac{1}{(n+1)!}} \ge [n!]^{-ma_n}$$

which, in turn, implies that

$$2[n!]^{ma_n} \ge [(n+1)!]^{a_{n+1}} \tag{2}$$

But (2) will nothold if we choose a_n of sufficiently high order in n. In fact, since $2[n!]^m < [(n+1)!]^{n+1}$ for n > m, we have

$$2^{n!}[n!]^{mn!} < [(n+1)!]^{(n+1)!}$$

which contradicts (2) with $a_n = n!$ and hence also contradicts (1). Thus S is transcendental for $a_n = n!$ and also for any a_n of equivalent or higher order, such as $(n!)^p$, n^n , etc.

Amer. Math. Monthly, **61**(1954) 350.

4592. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N.Y. Find the sum

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} \log r}{r}$$

Amer. Math. Monthly, **62**(1955) 558–560.

Dirichlet Series

I. Solution by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland. Write

$$f(n) = \sum_{r=1}^{n} \frac{\log r}{r} - \frac{1}{2} \log^2 n$$

then from Cauchy's integral test it follows that f(n) tends to a limit as $n \to \infty$; hence

$$f(2n) - f(n) \to 0 \qquad n \to \infty$$

Now, since $\log 2r = \log 2 + \log r$,

$$\sum_{r=1}^{n} \frac{\log r}{r} = 2\sum_{r=1}^{n} \frac{\log 2r}{2r} - \log 2\sum_{r=1}^{n} \frac{1}{r}$$

hence

$$f(2n) - f(n) = \sum_{r=1}^{2n} \frac{(-1)^{r+1} \log r}{r} + \log 2 \left(\sum_{r=1}^{n} \frac{1}{r} - \log n \right) - \frac{1}{2} \log^2 2$$

Thus, letting n tend to infinity, we have

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} \log r}{r} = \frac{1}{2} \log^2 2 - \gamma \log 2$$

where γ denotes Euler's constant.

II. Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. Choose a > 0 and consider

$$\int_0^\infty \frac{\log x}{e^{ax} + 1} dx = \sum_{r=1}^\infty (-1)^{r+1} \int_0^\infty e^{-rax} \log x \, dx$$
$$= \sum_{r=1}^\infty \frac{(-1)^{r+1}}{ra} \int_0^\infty e^{-u} \log(u/ra) \, du$$
$$= \left\{ \int_0^\infty e^{-u} \log u \, du \right\} \left\{ \sum_{r=1}^\infty \frac{(-1)^{r+1}}{ra} \right\} - \sum_{r=1}^\infty \frac{(-1)^{r+1} \log ra}{ra}$$

Since

$$\int_0^\infty e^{-u} \log u \, du = \Gamma'(1) = -\gamma$$
$$\sum_{r=1}^\infty \frac{(-1)^{r+1}}{ra} = \frac{\log 2}{a}$$
$$\int_0^\infty \frac{\log x}{e^{ax} + 1} \, dx = -\frac{1}{2a} \log 2 \log 2a^2$$

(see solution to problem 4394 [1951, 705]) we obtain

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} \log ra}{ra} = \frac{1}{2a} \log 2 \log 2a^2 - \frac{\gamma \log 2}{a}$$

The present problem is the special case a = 1.

Editorial Note. In a note on The Power Series Coefficients of $\zeta(s)$, Briggs and Chowla obtain, among other results, formulas for the coefficients A_n in the expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n (s-1)^n$$

See this MONTHLY, May 1955, pp.323–325.

Amer. Math. Monthly, **61**(1954) 423.

E 1123. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Given a square $N \times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of 2N - 2 segments. Can it be done with less than 2N - 2 segments?

Amer. Math. Monthly, **62**(1955) 124.

Solution by the Proposer. Assume that it can be done for N = K in such a fashion that we end up in position E as shown in Figure 1. Then, as shown in Figure 2, by



drawing two more lines it can be done in the same fashion for N = K + 1. Figure 3 shows that it can be done in this fashion for N = 3. Thus, by induction, it can be done for all $N \ge 3$.

No one successfully answered the question at the end of the problem.

Amer. Math. Monthly, **62**(1955) 443.

II. Addendum by John Selfridge, U.C.L.A. We answer the question at the end of the problem in the negative.

Let there be R rows and S columns which have none of the given segments lying on them. The $R \times S$ lattice formed by these has 2R + 2S - 4 boundary points if R and Sare each greater than 1. Each oblique line covers at most 2 of these boundary points. Thus in the polygonal covering there are at least R + S - 2 oblique segments, N - Rhorizontal segments and N - S vertical segments, or at least 2N - 2 segments in all.

If R or S is 0 or 1 there are at least 2N - 1 segments.

Amer. Math. Monthly, **61**(1954) 427–428.

4595. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York If

$$[xD]^{n} = \sum_{r=1}^{n} A_{rn} x^{n-r+1} D^{n-r+1}$$

where D is the differential operator, determine A_{rn} .

Amer. Math. Monthly, **62**(1955) 660–661.

Stirling Numbers of the Second Kind

Solution by A. S.Hendler, Rensselaer Polytechnic Institute, Troy N.Y. On writing $[xD]^n = x \cdot D[xD]^{n-1}$, we see that the $A_{r,n}$ must satisfy the equation of partial differences

$$A_{r,n} = A_{r,n-1} + (n-r+1)A_{r-1,n-1}$$

and the initial conditions are $A_{1,n} = A_{n,n} = 1$. Thus the $A_{r,n}$ are completely determined.

However, if we take $A_{n-k+1,n} = B_{k,n}$ (k = 1, 2, ..., n) the equation of partial differences may be written

$$B_{k,n} = B_{k-1,n-1} + kB_{k,n-1}$$

with the initial conditions $B_{n,n} = B_{1,n} = 1$, But $\mathfrak{S}_n^1 = \mathfrak{S}_n^n = 1$, where \mathfrak{S}_n^k are the Stirling numbers of the second kind defined by

$$\mathfrak{S}_{n}^{k} = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} {}_{k}C_{i}i^{n}$$

Also, an easy calculation will verify that

$$\mathfrak{S}_{n-1}^{k-1} + k\mathfrak{S}_{n-1}^k = \mathfrak{S}_n^k$$

Hence $B_{k,n} = \mathfrak{S}_n^k$ and

$$A_{r,n} = \mathfrak{S}_n^{n-2+1} = \frac{1}{(n-r+1)!} \sum_{i=1}^{n-r+1} \sum_{i=1}^{n-r+1} (-1)^{n-r+1-i} \sum_{n-r+1}^{n-r+1} C_i i^n$$

Editorial Note. The problem has been treated in a number of places and the following references were cited by our correspondents: Jordan, *Calculus of Finite Differences*, pp.195–196, (See also pp.168–170 for the Stirling numbers.), Schwatt, *Introduction of Operations with Series* (1924) pp.86ff., and an article by L. Carlitz, *On a class of finite sums*, this MONTHLY **37**(1930) 473–479.

Amer. Math. Monthly, **61**(1954) 470.

E 1129. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Find an integral arithmetic progression with an arbitrarily large number of terms such that no term is a perfect r th power for r = 2, 3, ..., N. Is this still possible if $N = \infty$? bigskip

Amer. Math. Monthly, **62**(1955) 185.

Solution by Azriel Rosenfeld, Columbia University. The progression 2, 6, 10, ..., 4k+2, ... can contain no perfect powers whatsoever. For, a power of an odd integer is odd, and a power of an even integer must be divisible by 4..

An obvious solution, pointed out by [A. R.]Hyde, is any arithmetic progression with common difference d = 0 and with the (invariant) term chosen so as not to be an integral power of an integer, for example the progression 3, 3, 3, Hyde, Leo Moser and [R. E.]Shafer offered deeper solutions to the problem.

Amer. Math. Monthly, **61**(1954) 705–707

Classroom Note: M. S. Klamkin, On the vector triple product.

Amer. Math. Monthly, **61**(1954) 711.

E 1142. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N.Y.

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α , 2β , 2γ with each other.

Amer. Math. Monthly, **62**(1955) 444–445.

Solution by Leon Bankoff, Los Angeles, Calif. The sides of the base of the triangular pyramid determined by the three generating lines are $a = 2y \sin \alpha$, $b = 2y \sin \beta$, $c = 2y \sin \gamma$, where y is the slant height of the cone. The radius of the base of the cone is given by

$$R = abc/4\sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter of the base of the pyramid. Since the semi-vertical angle ϕ is equal to $\arcsin R/y$, we obtain

$$\phi = \arcsin\frac{2\sin\alpha\sin\beta\sin\gamma}{\sqrt{(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)^2} - 2(\sin^4\alpha + \sin^4\beta + \sin^4\gamma)}$$

[Hüseyin]Demir gave the equivalent answer

$$\sin^2 \phi = -\frac{16\sin^2 \alpha \sin^2 \beta \sin^2 \gamma}{\begin{vmatrix} 0 & \sin \alpha & \sin \beta & \sin \gamma \\ \sin \alpha & 0 & \sin \gamma & \sin \beta \\ \sin \beta & \sin \gamma & 0 & \sin \alpha \\ \sin \gamma & \sin \beta & \sin \alpha & 0 \end{vmatrix}}$$

[R. L.]Helmbold considered the analogous problem in an *n*-dimensional vector space.

Amer. Math. Monthly, **62**(1955) 122.

E 1153. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

For any angle θ , show that arbitarily small constructible angles ϕ exist such that $(\theta - \phi)$ can be trisected.

Amer. Math. Monthly, **62**(1955) 584.

Trisection of Angles Arbitrarily Close to a Given Angle

Solution by C. S. Ogilvy, Hamilton College. Let $\phi_n = \theta/4^n$, n = 1, 2, ... For all n, ϕ_n is certainly constructible. Obviously n can be selected so that $\phi_n < \epsilon$ for any ϵ . But

 $(\theta - \phi_n)/3 = \phi_1 + \phi_2 + \dots + \phi_n$

Also solved by [5 others and] the proposer.

Amer. Math. Monthly, **62**(1955) 123.(127?)

4627. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York

In Chrystal, *Textbook of Algebra*, vol.2, p.225, there is the following theorem on the representation of an irrational number:

The number represented by the series

$$\sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

is irrational provided that

(1) r_n and p_n are integers such that $0 < p_n < r_n$

(2) $r_{n+1} > r_n > 1$

(3) the sequence $\{r_1r_2\cdots r_n\}$ includes all powers of the primes.

(A) Construct a counter-example, i.e., a number of the stated form which is rational even though conditions (1), (2) and (3) are satisfied.

(B) Complete the list of conditions (1), (2) and (3) so that the theorem is indeed valid.

Amer. Math. Monthly, **63**(1956) 263.

Solution by Ivan Niven, University of Oregon. (A) The following number is of the required form and is rayional:

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = 1$$

(B) The theorem is valid if the following condition is also satisfied:

(4) $p_n < r_n - 1$ for infinitely many values of n.

For if the series converged to a rational number, a/b, then by condition (3) we could choose an integer m so that b is a divisor of $r_1r_2\cdots r_m$. When we multiply the equation

$$\frac{a}{b} = \sum_{n=1}^{\infty} \frac{p_n}{r_1 r_2 \cdots r_n}$$

by $r_1 r_2 \cdots r_m$, and rearrange the terms to get

$$a\frac{r_1r_2\cdots r_m}{b} - \sum_{n=1}^m \frac{p_nr_1r_2\cdots r_m}{r_1r_2\cdots r_n} = \sum_{i=1}^\infty \frac{p_{m+i}}{r_{m+1}r_{m+2}\cdots r_{m+i}}$$

we note that the left side is an integer. We get a contradiction by establishing that the series on the right converges to a value between 0 and 1. Indeed the series on the right is positive because of condition (1), and by condition (4) the series is less than

$$\sum_{i=1}^{\infty} \frac{r_{m+i} - 1}{r_{m+1}r_{m+2} \cdots r_{m+i}}$$

but this series converges to the value 1 because the sum of the first k terms has the value

$$1 - \frac{1}{r_{m+1}r_{m+2}\cdots r_{m+k}}$$

Thus the proof is complete and we make the following comment. Condition (2) may be replaced by the weaker condition $r_n > 1$, and the theorem remains valid by the same proof.

Also solved by [3 others and] the Proposer.
Amer. Math. Monthly, **62**(1955) 493.

E 1180. Proposed by M. S. Klamkin and Alex Kraus, Polytechnic Institute of Brooklyn

Determine the 2319th digit in the expansion of 1000!

Amer. Math. Monthly, **63**(1956) 190.

Solution by J. B. Muskat, Cambridge, Mass. By Stirling's formula we find that the expansion of 1000! has 2568 digits. Since 5 divides 1000! 249 times, and 2 divides 1000! more than 500 times, the expansion of 1000! terminates in a string of 249 zeros. Further, since 2568 - 249 = 2319, the digit we seek is the last non-zero digit in the expansion. Clearly this digit is even, and depends upon the last digits of the factors 1, 2, 3, ..., 1000 of 1000! after removal of allfactors 5. Taking the factors successively in sets of ten and removing the multiples of 5 we have, for the product of the remaining factors in each set,

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \equiv 6 \pmod{10} \tag{1}$$

There are 100 of these products, leaving 200 numbers which are divisible by 5, and if 5 is factored out of each,200! remains. We then have 20 more products like (1), leaving 40 numbers which are divisible by 5, and if 5 is factored out of each of these, 40! remains. We get 4 more products like (1), leaving 8 numbers which are divisible by 5, and if 5 is factored out of each of these, 8! remains. Dividing out 5 once more, we have left

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \equiv 4 \pmod{10}$$

Now $4 \cdot 6^{124} \equiv 4 \pmod{10}$, whence 4 is the last digit in $(1000!)/5^{249}$. But 2^{249} must also be divided out to complement the 5^{249} . Since $2^{249} \equiv 2 \pmod{10}$, it now follows that the last digit in $(1000!)/10^{249}$, which is the digit we are seeking, is 2.

Amer. Math. Monthly, **63**(1956) 492.

Editorial Note. H. S.Uhler has calculated the exact value of 1000!. See his article, Exact values of 996! and 1000! with skeleton tables of antecedent constants, *Scripta Math.*, **XXI** 261–268. The 2319th digit in the expansion of 1000! appears as 2, as was shown in [1956. 189].

Amer. Math. Monthly, **62**(1955) 494.

E 1147. Proposed by E. P. Starke, Rutgers University

If $\cos \alpha$ is rational $(0 < \alpha < \pi)$, prove there are infinitely many triangles with integer sides having α as one angle. In particular, given $\cos \alpha = r/s$, find a three-parameter solution for the sides a, b, c.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. By the law of cosines

$$s(c+a)(c-a) = b(2cr - bs)$$

This will be satisfied if

$$ms(c+a) = n(2cr - bs)$$
 and $n(c-a) = mb$

or if

$$na + mb - nc = 0$$
 and $msa + nsb + (ms - 2nr)c = 0$

It follows that

$$a = ts(m^{2} + n^{2}) - 2tmnr$$
 $b = 2tn(nr - ms)$ $c = ts(n^{2} - m^{2})$

This problem has been solved previously by Züge, Archiv Math. Phys.(2) **17**(1900) 354. See Dickson, History of the Theory of Numbers, Vol.II, p.215.

Amer. Math. Monthly, 62(1955) 734.

4664. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Are there any other laws of attraction beside the inverse square law such that the time of descent (from rest) through any striaght tunnel through a uniform spherical planet is independent of the path?

Amer. Math. Monthly, **75**(1968) 415.

Editorial Note. The solution of this problem appears as a Scientific Laboratory of the Ford Motor Company publication, preprint of a paper by M. S. Klamkin and D. J. Newman, *On some inverse problems in dynamics*, to appear in the Quarterly of Applied Mathematics. The abstract of the paper follows.

It is known that the time of traverse of a freely falling body through a straight tunnel connecting any two points of the surface of a uniform spherical planet is isochronous. We show here that if the isochronous property is to hold for any planet with a spherical symmetric density, then the density must be constant. Also it is shown that if the isochronous property is to hold for any uniform spherical planet subject to a central force law, then the force law must be inverse square. However, the isochronous property can hold for one uniform spherical planet with a different force law of attraction. Amer. Math. Monthly, **63**(1956) 39.

E 1199. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

A flexible chain of length L is suspended from its endpoints. Determine themaximum area between the chord joining the endpoints and the hanging curve.

Amer. Math. Monthly, **63**(1956) 495–496.

Maximum Area Between a Hanging Chain and its Chord

Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pa. Let -x + w and x + w be the abscissas of two points on the curve having the equation $y = \cosh x$, such that the figure formed by the arc and the chord joining the two points is similar to the chain of length L and the chord joining its points of suspension. Let A be the plane area bounded by the chain and its chord, and let z be the length of the chord. Then

$$z = L(x^{2} + \sinh^{2} w \sinh^{2} x)^{1/2} / \cosh w \sinh x$$
$$A = L^{2}(x \cosh x - \sinh x) / 2 \cosh w \sinh^{2} x$$

If w is held constant, dA/dz = 0 when

$$x = 2(\sinh x \cosh x)/(1 + \cosh^2 x) = 1.6062$$
, approximately.

For any x, A is a maximum when w = 0, which occurs when and only when the chord is horizontal. Then

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x$$
 and $z = Lx/\sinh x$

so the maximum area is approximately $0.1549L^2$, the points of suspension being joined by a horizontal chord whose length is approximately 0.6716L.

Editorial Note. Eliminating x from

$$A = L^2(x \cosh x - \sinh x)/2 \sinh^2 x$$
 and $L = 2 \sinh x$

we find

$$A_{\rm max} = (L^2 + 4)^{1/2} \sinh^{-1}(L/2) - L$$

Amer. Math. Monthly, **63**(1956) 126.

4677. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.

For what values of θ does the following series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n \theta} ?$$

Amer. Math. Monthly, **64**(1957) 124-125.

Solution by Leonard Carlitz, Duke University. Put $\theta = \alpha \pi$, where we may assume that $0 < \alpha < 2$. Also put $\alpha = 2^{-n_1} + 2^{-n_2} + 2^{-n_3} + \cdots$ $(0 \le n_1 < n_2 < \cdots)$ and $k_r = n_{r+1} - n_r$ $(r = 1, 2, \ldots)$.

Clearly we may assume that $\alpha \neq m/2^t$ where *m* is an integer. We shall prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n \sin 2^n \alpha \pi} \tag{1}$$

converges if and only if, as $r \to \infty$,

(a)
$$2^{k_r}/n_r \to 0$$

and the following series converges

(b)
$$\sum_{r=1}^{\infty} 2^{k_r} / n_r^2$$

Proof. The necessity of (a) is obvious, for otherwise the n_r th term of (1) does not approach zero. In the next place it follows from the identity

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \dots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

that

$$\sum_{n=1}^{N} \frac{1}{n \sin 2^{n} \alpha \pi} = \sum_{n=1}^{N} \frac{1}{n} (\cot 2^{n-1} \alpha \pi - \cot 2^{n} \alpha \pi)$$
$$= \cot \alpha \pi - \sum_{n=1}^{N-1} \frac{\cot 2^{n} \alpha \pi}{n(n+1)} - \frac{1}{N} \cot 2^{N} \alpha \pi \qquad (2)$$

It follows from (a) that as $N \to \infty$, $(\cot 2^N \alpha \pi)/N \to 0$. As for

$$\sum_{n=1}^{N-1} \frac{\cot 2^n \alpha \pi}{n(n+1)}$$
(3)

note first that the only negative terms are those for which $n = n_r - 1$; since the series

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n_r - 1} \alpha \pi}{n_r (n_r - 1)}$$

is evidently convergent, we may ignore such terms in (3). In other words, if

$$\sum_{n=1}^{\infty} \frac{\cot 2^n \alpha \pi}{n(n+1)} \tag{4}$$

converges, it converges absolutely. Consequently the convergence of (4) implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\cot 2^{n_r} \alpha \pi}{n_r (n_r + 1)} \tag{5}$$

Since the fractional part of $2^{n_r}\alpha$ is equal to $\alpha = 2^{n_r - n_{r+1}} + 2^{n_r - n_{r+2}} + \cdots$, it is clear that (5) converges if and only if

$$\sum_{r=1}^{\infty} \frac{2^{k_r}}{n_r(n_r+1)}$$

converges; this is equivalent to (b).

Conversely when (a) and (b) hold, it is clear from (2) that it is only necessary to prove the convergence of (4). But

$$\left|\sum_{n=1}^{N} \frac{\cot 2^{N} \alpha \pi}{n(n+1)}\right| \le \sum_{n=1}^{N} \frac{|\csc 2^{n} \alpha \pi|}{n(n+1)}$$

and $\sin 2^n \alpha \pi$ is negative only for $n = n_r$. Then the convergence of

$$\sum_{n=1}^{\infty} \frac{\csc 2^n \alpha \pi}{n(n+1)}$$

is a consequence of (b), while the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)\csc 2^n \alpha \pi}$$

is easily proved by sumation by parts as in (2), This completes the proof.

Remark. The series (1) is certainly convergent if the differences k_r are bounded; in particular, (1) converges for rational α (except $m/2^t$). An example of divergence is furnished by $\sum 2^{-r^2}$; the condition (a) is not satisfied.

For $n_r = [r \log_2 r]$ both (a) and (b) are satisfied; for

$$n_r = [r(\log_2 r + \log_2 \log_2 r)]$$

neither (a) nor (b) holds; while (a) is satified but (b) is not if

$$n_r = [r(\log_2 r + \log_2 \log_2 r - \log_2 \log_2 \log_2 r)].$$

Amer. Math. Monthly, 63(1956) 191.

4680. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass. Solve the following generalization of Clairaut's differential equation

$$y - xy' + \frac{x^2y''}{2!} + \dots + (-1)^{n-1}\frac{x^{n-1}y^{(n-1)}}{(n-1)!} + (-1)^n\frac{x^nF(y^{(n)})}{n!} = G(y^{(n)})$$

Amer. Math. Monthly, 64(1957) 204.

Solution by the proposer. After differentiating the given equation and replacing $y^{(n)}$ by r, we can rewrite it as

$$dx^{n}/dr - x^{n}F'(r)H(r) = (-1)^{n-1}G'(r)n!H(r)$$
 where $H(r) = (r - F(r))^{-1}$

The standard solution of this first order linear equation is

$$x^{n} = \exp\left\{\int F'(r)H(r)\,dr\right\}\int (-1)^{n-1}G'(r)n!H(r)\exp\left\{-\int F'(r)H(r)\,dr\right\}\,dr$$

or $x = \phi(r)$. Now $(d^n y)/(dx^n) = r$, whence

$$y = \int \cdots \int r \, (dx)^n = \int \phi'(r) \int \phi'(r) \cdots \int r \phi'(r) \, (dr)^n$$

This last equation and the equation $x = \phi(r)$ constitute the parametric form of the solution.

It is to be noted that there is no singular solution unless r = F(r) in which case the equation reduces to that treated by Witty, this MONTHLY, 1952, pp.100–102. See also the proposer's note, this MONTHLY, 1953, pp.97–99.

Amer. Math. Monthly, **63**(1956) 425–426.

A Conditional Inequality

E 1195 1955, 728. Proposed by G. E. Bardwell, University of Denver

If n = 2, 3, 4, ..., and m is fixed and positive, for what values of p less than 1 is $\ln n < mn^p$?

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. If $mn^p > \ln n$, then

 $p > \ln (\ln n)/m / \ln n = (\ln x - \ln m)/x$

where $x = \ln n$. Now the maximum value of $(\ln x - \ln m)/x$ occurs when x = em and is equal to 1/em. Thus

$$1 > p > 1/em$$
 provided $m > 1/e$.

Amer. Math. Monthly, **63**(1956) 665.

E 1238. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Determine integral values of n > 0 such that 3^n , 3^{n+1} , 3^{n+2} all have the same number of digits in their denary expansions.

Amer. Math. Monthly, **64**(1957) 367.

Solution by Joe Lipman, University of Toronto. If n is an integer such that 3^n , 3^{n+1} , 3^{n+2} all have the same number of digits in their denary expansions. then

$$\underbrace{10000\ldots}_{k \text{ digits}} < 3^n < \underbrace{11111\ldots}_{k \text{ digits}}$$

Now the mantissa of log 11111... is 0.04575749056... If the inequality is satisfied, $n \log 3 =$ an integer + a decimal fraction between zero and 0.04575749056. But $\log 3 = 0.47712125472$, which is just slightly greater than 10/21. [[misprinted as 1/21]] Therefore we can expect the *n*s to recur at intervals of 21 or 23. Thus we have

n	$n\log 3$
21	10.019546349
42	20.039092698
65	31.012881557
86	41.032427906
109	52.006216764
130	62.025763114
151	72.045309463
174	83.019098321

[[We interrupt the program to observe that we've corrected the arithmetic in the above table and in the following paragraph (and the preceding one!) The sequence of values of n is A001682 in Sloane's OEIS, where the following values are given:

 $\begin{array}{l} 21,42,65,86,109,130,151,174,195,218,239,262,283,\ 304,327,348,371,392,415,436,457,\\ 480,501,524,545,\ 568,589,610,633,654,677,698,721,742,763\] \end{array}$

A comparison of 174 and 21 shows that the corresponding mantissae differ by only 0.00044802789. This is because $153 \log 3 = 72.999551972108$. Thus any number of the form 21 + 153k, where k < 0.019546349/0.00044802789 = 43.6275... will be one of the required n. For k = 44 the resulting mantissa is 0.99983312188. Subtracting $0.954242509 = 2 \log 3$ we get the mantissa 0.04559061244. So instead of 21 + 153(44), use 21 + 153(44) - 2 = 6751. Now using the sequence $6741 + 153k_1$, where $k_1 < 0.04559061244/0.00044802789 = 101.758$, repeat the above process and get more n s. Then derive the new sequence $22194 + 153k_2$, and so on. In this way arbitrarily large n s can be determined as long as tables of sufficient accuracy are available.

Also solved by [15 others and] the proposer.

Amer. Math. Monthly, 63(1956) 667.

E 1209 [1956, 186]. Proposed by Hüseyin Demir, Zonguldak, Turkey

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D. E, F and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio D(E, F, E', F') is the same for all triangles ABC.

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. By a central projection, triangle ABC and its incircle (I) can be transformed into an equilateral triangle and its incircle. The anharmonic ratio D(E, F, E', F') is invariant under this transformation and consequently is constant for all triangles. It is easy to show that D(E, F, E', F') = 4. Amer. Math. Monthly, 63(1956) 724.

E 1245. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn If

$$b_{n+1} = \int_0^1 \min(x, a_n) \, dx$$
 $a_{n+1} = \int_0^1 \max(x, b_n) \, dx$

prove that the sequences $\{a_n\}$ and $\{b_n\}$ both converge and find their limits.

Amer. Math. Monthly, **64**(1957) 436.

Soution by N. J. Fine, University of Pennsylvania. For any a_0 , b_0 , it is easy to see that a_n and b_n both lie between 0 and 1 for all $n \ge 2$. The recurrence formulas then become (for $n \ge 2$)

$$a_{n+1} = (1+b_n^2)/2$$
 $b_{n+1} = a_n - a_n^2/2$

If we assume that $\lim a_n = a$, $\lim b_n = b$, they must satisfy

$$a = (1 + b^2)/2$$
 $b = a - a^2/2$

from which we get

$$a+b-1 = (a+b-1)(b-a+1)/2$$

Since the factor $(b - a + 1)/2 \neq 1$, we have a + b = 1, and this yields $a = 2 - \sqrt{2}$, $b = \sqrt{2} - 1$. To show that $a_n \to a$, $b_n \to b$, we write $a_n = a + \delta_n$, $b_n = b + \epsilon_n$. The recurrence formulas become, after an easy reduction,

$$\delta_{n+1} = (b + \epsilon_n/2)\epsilon_n$$
 $\epsilon_{n+1} = (b - \delta_n/2)\delta_n$

Now $|\epsilon_n| = |b_n - b| \le \max(b, 1 - b) = a$ and $|\delta_n| = |a_n - a| \le \max(a, 1 - a) = a$. Hence $|b + \epsilon_n/2| \le b + a/2 = 1/\sqrt{2}$, $|b - \delta_n/2| \le b + a/2 = 1/\sqrt{2}$. Therefore $|\delta_{n+1}| \le |\epsilon_n|/\sqrt{2}$ and $|\epsilon_{n+1}| \le |\delta_n|/\sqrt{2}$. This shows that $\delta_n \to 0$, $\epsilon_n \to 0$ and the proof is complete.

Also solved by [9 others and] the proposer.

Amer. Math. Monthly, **63**(1956) 725–726.

A Trigonometric Inequality

E 1212. Proposed by H. A. Osborn, University of California, Berkeley

Show that t > 0 implies $(2 + \cos t)t > 3 \sin t$.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. This problem and a generalization both appear in Durell and Robson, Advanced Trigonometry, p.83 and p.100. We will prove the generalization: If $a \ge 2b > 0$ and $\pi \ge t > 0$, then

$$(a+b\cos t)t > (a+b)\sin t$$

If the inequality is true for the range $\pi \ge t > 0$, it follows immediately that it will be true for for all t > 0. Since the two sides of the inequality are equal for t = 0, the inequality will follow if we can show that the derivative of the left hand side is greater than the derivative of the right hand side, that is, if

$$a + b\cos t - bt\sin t > (a + b)\cos t$$

or

$$\tan t/2 > bt/a$$

which is true if $a \ge 2b$.

Setting a = 2, b = 1, we get $(2 + \cos t)t > 3\sin t$.

The similar inequality, $(2 + \cosh t)t > 3 \sinh t$, t > 0, is given on p.115.

Amer. Math. Monthly, 63(1956) 729.

4716. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Determine the equation of motion if $\bar{V}_s = \lambda \bar{V}_t$ where \bar{V}_s and \bar{V}_t are the averages of velocity with respect to distance and time, respectively, in any time interval starting at t = 0. What is the minimum eigenvalue λ ?

Solution by N. J. Fine, University of Pennsylvania. Assuming that s(0) = 0, we see that $\bar{V}_t = s/t$, and so

$$\frac{1}{s} \int_0^s v \, ds = \lambda \frac{s}{t}$$

Multiply by s and differentiate, to get

$$v = \lambda \left(2\frac{s}{t} - \frac{1}{v}\frac{s^2}{t^2} \right)$$
 and $v = \mu \frac{s}{t}$

where $\mu = \lambda \pm \sqrt{\lambda^2 - \lambda}$. Hence $s = ct^{\mu}$. Except for the trivial case $\mu = \lambda = 0$, we must have $\mu > 1/2$ to ensure finiteness of \bar{V}_s and $\lambda = \mu^2/(2\mu - 1) \ge 1$. Therefore every $\lambda \ge 1$ is an eigenvalue, with the solution $s = ct^{\mu}$, $\mu = \lambda \pm \sqrt{\lambda^2 - \lambda}$.

Also solved by [2 others and] the proposer

Amer. Math. Monthly, 64(1957) 204–205.

A Summation Problem

4682 [1956, 191]. Proposed by R. C. Lyness, Preston, England

(a) Prove that when the series

$$1 + \sum_{r=1}^{\infty} \binom{r\alpha}{r-1} \frac{x^r}{r}$$

is convergent, its sum, y, satisfies $y = 1 + xy^{\alpha}$.

(b) Prove also that

$$\sum_{r=1}^{\infty} \binom{r\alpha+\beta-1}{r-1} \frac{x^r}{r} = \frac{y^\beta-1}{\beta}$$

I. Solution by M. S. Klamkin, AVCO Research Division, Lawrence, Mass. Case (a) is a special case of (b) which, in turn, is an application of Lagrange's reversion formula (See Bromwich, Infinite Series, p.158): If y = xf(y), then $g(y) = \sum_{0}^{\infty} p_n y^n$ where np_n is the coefficient of y^{-1} in the expansion of $g'(y)/x^n$.

Here $g(y) = (y^{\beta} - 1)/\beta$, and $f(y) = y^{\alpha+1}/(y-1)$. It follows that the coefficient of y^{-1} in the expansion of $y^{\beta-1}y^{\alpha n}(y-1)^{-n}$ is

$$(-1)^n \binom{n-1-\alpha n-\beta}{n-1} = \binom{n\alpha+\beta-1}{n-1}$$

by the binomial theorem. Thus

$$\frac{y^{\beta} - 1}{\beta} = \sum_{n=1}^{\infty} \binom{n\alpha + \beta - 1}{n-1} \frac{x^n}{n}$$

Amer. Math. Monthly, **64**(1957) 428–431.

[[There's a Murray Klamkin Classroom Note:

On a Graphical Solution of Linear Differential Equations]]

Amer. Math. Monthly, **64**(1957) 504.

E 1275. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass. Solve for x:

$$\int_0^x s^{8/3} (1-s)^{4/3} \, ds = \int_0^1 t^{8/3} / (1+t)^{-6} \, dt$$

Amer. Math. Monthly, **65**(1958) 124.

Solution by Calvin Foreman, Baker University. Set t = s/(1-s) to obtain

$$\int_0^x s^{8/3} (1-s)^{4/3} \, ds = \int_0^{1/2} s^{8/3} (1-s)^{4/3} \, ds$$

Since the integral on the left is a monotonically increasing function of x, the only solution is x = 1/2.

Also solved by [19 others and] the proposer.

Amer. Math. Monthly, 64(1957) 437.

4744. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.

Three congruent ellipses are mutually tangent. Determine the maximum of the area bounded by the three ellipses.

Amer. Math. Monthly, **73**(1966) 89–90.

Solution by Michael Goldberg, Washington, D.C. The extremal positions can be found as equilibrium states of the following hydromechanical analogy. The minimum enclosed area is obtained by uniform external pressure on the ellipses while the maximum enclosed area is produced by a pressure within the enclosed area. In either case, the resultant force due to the pressure on an ellipse is directed along the perpendicular bisector of the chord joining the points of contact. To obtain equilibrium, the ellipse will turn until these forces are concurrent. But this can occur only when the chord is parallel to an axis of the ellipse, and this applies to each of the ellipses. Hence, the maximum enclosed area is attained when the arcs of least curvature bound the area: that is, when the chords are parallel to the major axes and form an equilateral triangle. The least area is attained when the chords are parallel to the minor axes and form an equilateral triangle.

The enclosed areas can be derived as follows. Begin with a circle of radius a and two tangents while make an included angle of 2A. The the area between the circle

and the tangents is $a^2(\cot A + A - \pi/2)$. If the figure is rotated about the bisector of the angle 2A so that the circle projects into an ellipse of minor semi-axis b, then the included area becomes $ab(\cot A + A - \pi/2)$, but the angle between the tangents becomes 2B, where $a \tan B = b \tan A$. Thus the minimum area for the given problem is obtained when $B = \pi/3$, $\tan B = \sqrt{3}$, $\tan A = 3^{1/2}a/b$, and the total enclosed area is $3ab(b/a\sqrt{3} - \tan^{-1}(b/a\sqrt{3}))$.

If the figure is rotated about the normal to the bisector of angle 2A so that the circle projects into an ellipse of minor semi-axis b, then the new included area again becomes $ab(\cot A + A - \pi/2)$ but the new angle between the tangents becomes 2B, where $b \tan B = a \tan A$. For the maximum area of the given problem, $B = \pi/3$, $\tan B = \sqrt{3}$, $\tan A = b\sqrt{3}/a$, and the total enclosed area is $3ab(a/b\sqrt{3} - \tan^{-1}(a/b\sqrt{3}))$.

The foregoing procedures can be used when the three ellipses are not congruent, even when other curves are used. The following theorem may be stated: A necessary condition for an extremum for the area enclosed by three (or more) mutually tangent curves is the equilibrium condition that the normals at the points of contact of each curve make equal angles with the respective chords joining these points of contact.

The adjoining figure is an example of the maximum area enclosed by three unequal ellipses.

[[The figure taxes my present capabilities, but is well worth including. Will see what I can do. Note how long it took for a solution to Murray's problem to appear. Later: in endeavoring to simplify the situation, one of the ellipses has turned into a circle! Perhaps someone can do better?]]

Amer. Math. Monthly, 64(1957) 510.

4752. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.

Determine a set of n distinct, nonzero terms such that their geometric mean is the geometric mean of their arithmetic and harmonic means.

Amer. Math. Monthly, 65(1958) 455–456.

Solution by Emil Grosswald, University of Pennsylvania. Let S_j be the *j* th fundamental symmetric function of the *n* terms a_1, \ldots, a_n . Then their harmonic, geometric and arithmetic means are

$$H = nS_n/S_{n-1}$$
 $G = S_n^{1/n}$ $A = S_1/n$

respectively, and the condition $G^2 = HA$ of the problem becomes

$$_{n-1} = S_1 S_n^{1-2/n} \tag{1}$$

It is therefore sufficient to take as a_1, \ldots, a_n the roots of

$$x^{n} - S_{1}x^{n-1} + \dots + (-1)^{n-1}S_{n-1}x + (-1)^{n}S_{n} = 0$$
(2)

with arbitrary $S_2, S_3, \ldots, S_{n-2}$ and any S_1, S_{n-1}, S_n which satisfy (1). In order to have all terms different from zero it is sufficient to take $S_n \neq 0$; and $a_i \neq a_j$ for $i \neq j$ is assured if the coefficients of (2) are chosen so that its discriminant does not vanish.

Editorial Note. Several explicit sets were proposed. The simplest are: (i) n successive terms of any geometric progression, (ii) [n/2] distinct pairs of reciprocals, with the addition of the element 1 in case n is odd.

Amer. Math. Monthly, 64(1957) 661–663.

[[There's a Murray Klamkin Mathematical Note:

An Application of the Gauss Multiplication Theorem]]

Amer. Math. Monthly, **64**(1957) 710–715.

[[There's a Murray Klamkin Article:

A Moving Boundary Filtration Problem or "The Cigarette Problem"]]

Amer. Math. Monthly, **64**(1957) 742.

E 1295. Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.

Show that all the roots of $\tan z = z/(1 + m^2 z^2)$, where m is real, are real.

Amer. Math. Monthly, **65**(1958) 450.

I. Solution by J. W. Haake. Armour Research Foundation, Tucson, Arizona. We note that if z is a root of the given equation, then so are \bar{z} and -z. Hence, in discussing imaginary roots z = a + ib, $b \neq 0$, we may, for convenience, assume $a \geq 0$, b > 0. Now in the given equation, set z = a + ib ($b \neq 0$, $\tan z = (\sin z)/(\cos z)$, $sinib = i \sinh b$, $\cos ib = \cosh b$, and then multiply each side of the reulting equation by the conjugate of the denominator of that side. We obtain

$$\frac{(\sin 2a)/2 + (i\sinh 2b)/2}{\cos^2 a\cosh^2 b + \sin^2 a\sinh^2 b} = \frac{a[1+m^2(a^2+b^2)] + ib[1-m^2(a^2+b^2)]}{[1+m^2(a^2+b^2)]^2 + 4m^4a^2b^2}$$

Since real and imaginary parts must be equal, we obtain, by taking the ratio of real part to imaginary part on each side,

$$\frac{\sin 2a}{\sinh 2b} = \frac{a}{b} \left[\frac{1 + m^2(a^2 + b^2)}{1 - m^2(a^2 + b^2)} \right]$$

For real m, this implies the impossible inequality

$$(\sin r)/(\sinh s) \ge r/s$$
 $r \ge 0, \ s > 0$

It follows that we cannot have $b \neq 0$, and z must be real.

Also solved by [6 others and] the proposers.

Amer. Math. Monthly, 64(1957) 742.

The Steensholt Inequality for a Tetrahedron

E 1264 [1957, 272]. Proposed by Victor Thébault, Tennie, Sarthe, France

If an interior point P of a tetrahedron ABCD is projected orthogonally into A', B', C', D' on the planes of the faces BCD, CDA, DAB, ABC and if the areas of the faces are denoted by A, B, C, D, show that

$$A(PA) + B(PB) + C(PC) + D(PD) \ge 3[A(PA') + B(PB') + C(PC') + D(PD')]$$

I. Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass. Represent the volume of the volume of the tetrahedron by V. Then $V = (1/3) \sum A(PA')$. Also, $V = (1/3)h_AA$ and $h_A \leq PA + PA'$. Hence $(1/3) \sum A(PA') = (1/12) \sum h_AA \leq (1/12) \sum (PA + PA')A$ or $\sum A(PA) \geq 3 \sum A(PA')$.

Editorial Note. This problem extends to the tetrahedron a property of the triangle given by Gunnar Steensholt, this MONTHLY [1956, 571]. If the tetrahedron is isosceles (thatis, equifacial), the inequality reduces to

$$PA + PB + PC + PD \ge 3(PA' + PB' + PC' + PD')$$

which establishes the Erdős-Mordell inequality for the tetrahedron $(2\sqrt{2} \text{ in place of } 3)$ for this special type of tetrahedron.

Amer. Math. Monthly, 65(1958) 47.

4772. Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.

It is easy to show that there exist consecutive prime pairs such that their difference is arbitrarily large. Do there exist prime triplets P_1 , P_2 , P_3 such that $\min(P_2 - P_1, P_3 - P_2)$ is arbitrarily large?

Amer. Math. Monthly, 65(1958) 781.

Solution by P. T. Bateman, University of Illinois. The question of the problem was answered affirmatively by Sierpiński [Colloq. Math., 1(1948) 193–194]. The following stronger results have since been obtained. Erdős [Pub. Math. Debrecen, 1(1949) 33–37] proved that for any positive number C there exist consecutive prime triples such that $\min(P_2 - P_1, P_3 - P_2) > C \log P_3$. Walfisz [Doklady Akad. Nauk SSSR (N.S.) 90(1953) 711–713] proved that for almost all primes p the distance of the closet prime on either side is greater thaan $(\log p)/(\log \log \log p)^2$. Prachar [Monats. Math., 58(1954) 114– 116] showed that Walfisz's result is still true if $(\log \log \log p)^2$ is replaced by any function of p which tends to infinity with p.

The following proof is similar to Sierpiński's but differs somewhat in detail. Let q be any prime greater than 2. Then (q-1)! - 1 and q! are relatively prime. In fact, the prime factors of q! are the primes not exceeding q. Clearly (q-1)! - 1 is not divisible by any of the primes less than q, while $(q-1)! - 1 \equiv -2 \pmod{q}$ by Wilson's theorem. Since (q-1)! - 1 and q! are relatively prime, Dirichlet's theorem guarantees the existence of a prime such that

$$p \equiv (q-1)! - 1 \pmod{q!}$$

Now the q integers following p and the q-2 integers preceding p are composite since

$$p \pm k + 1 \equiv p + 1 \equiv (q - 1)! \equiv 0 \pmod{k}$$

if $2 \le k \le q-1$, and since $p+2 \equiv (q-1)!+1 \equiv 0 \pmod{q}$. But q may be taken as large as desired and so the assertion of Sierpiński is established.

Amer. Math. Monthly, 65(1958) 122.

E 1302. Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.

A square is divided into two parts by an arbitray diameter through its centre. Determine the locus of the centroid of one of the equal areas.

Amer. Math. Monthly, 65(1958) 629.

Solution by B. H. Bissinger, Lebanon Valley College. Consider the square with vertices at $(\pm 1, \pm 1)$. Taking moments about the coordinate axes we find the centroid of the

region below the diameter of slope m, $|m| \leq 1$. has coordinates x = m/3, $y = -1/2 + m^2/6$. Considerations of symmetry prove the closed path of the centroid for one complete revolution of the diameter consists of four parabolic sections whose equations are

$$\begin{array}{ll} 4y^2 = (3x^2 - 1)^2 & x^2 \leq 1/9 \\ 4x^2 = (3y^2 - 1)^2 & 1/9 \leq x^2 \leq 1/4 \end{array}$$

It is interesting to note that the direction of the path of the centroid is parallel to the parametric diameter and that therefore that the derivative exists at the four points $(\pm 1/3, \pm 1/3)$ where the parabolas are pieced together.

[M. J.]Pascual considered the analogous problem in three space, where the square is replaced by a cube and the diametral line by a diametral plane. [C. S.]Ogilvy proposed the allied problem: What is the locus of the centroid if the arbitrary diameter is fixed and the square is rotated?

Amer. Math. Monthly, 65(1958) 124.

Divergent Integrals

4728 [1957, 201]. Proposed by R. P. Boas, Jr., Northwestern University

A. M. Rudov has propounded a proof that if f(x) is continuous and the first of the following integrals converges, then the second diverges

$$\int_{1}^{\infty} f(x) \, dx \qquad \int_{1}^{\infty} x^{-2} \{f(x)\}^{-1} \, dx$$

(a) Construct a counter-example. (b) More generally, show that if g(x) and $\phi(x)$ are positive and $\int_{1}^{\infty} \phi(x) dx$ diverges, then at least one of

$$\int_{1}^{\infty} \phi(x)g(x) \, dx \quad \text{and} \quad \int_{1}^{\infty} \{\phi(x)/g(x)\} \, dx$$

diverges.

I. Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass. (a) With $f(x) = (-1)^n/x$, $n = [\sqrt{x}]$, both integrals are easily seen to be convergent.

(b) By the Schwartz inequality,

$$\int_{1}^{\infty} \phi(x)g(x) \, dx \cdot \int_{1}^{\infty} \phi(x)/g(x) \, dx \ge \left[\int_{1}^{\infty} \phi(x) \, dx\right]^{2}$$

When $\int_1^{\infty} \phi(x) dx$ diverges, then at least one of the two integrals on the left must diverge.

Amer. Math. Monthly, 65(1958) 212.

E 1295. Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.

An equiproduct point of a curve is defined to be a point such that the product of the two segments of any chord through the point is constant. (1) Show that if every point inside a curve is equiproduct, the curve must be a circle. (2) What is the maximum number of equiproduct points a noncircular oval can have?

Amer. Math. Monthly, 66(1959) 69

Equiproduct Points

Editorial Note. The problem is not new. For solution and discussion see problem E 705 [1946, 395] and [1947, 164]. In the latter reference will be found a discussion of a paper by K. Yanagihara in $T\hat{o}h\hat{o}ku$ Math. J.(1917) touching on the same problem, together with analogous theorems for three dimensional space. The principal result is the theorem: A convex closed curve having two distinct interior equiproduct points is a circle.

Amer. Math. Monthly, 65(1958) 285.

An Insolvable Diophantine Equation

E 1281 [1957, 592]. Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.

Prove that no perfect square is 7 more than a perfect cube.

Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass. This is a problem due to V. A. Lebesgue in 1869. A reference to this and the following proof are given in H. Davenport, *The Higher Arithmetic*, (1952) p.160.

If $y^2 = x^3 + 7$, then x must be odd, since a number of the form 8k + 7 cannot be a square. Now

 $y^{2} + 1 = (x + 2)(x^{2} - 2x + 4) = (x + 2)[(x - 1)^{2}]$

and the final factor is of the form 4n + 3, and hence must have a prime factor of this same form. But it kis well known that $y^2 + 1$ cannot have a prime factor of this form.

L. E. Dickson, History of the Theory of Numbers, vol.2, p.534, gives the Lebesgue reference.

Amer. Math. Monthly, 65(1958) 289.

4787. Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.

Express as a single definite integral

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(m+n+r)!m^n n^r}{m!n!r!(r+1)^{m+n+r+1}}$$

Amer. Math. Monthly, **66**(1959) 66.

An Incorrect Proposal

Editorial Note. As several readers were quick to point out, the stated series is divergent. Attempts to rework the original idea into a well-posed problem have not been satisfactory. The editor and the proposer apologize for a careless oversight.

Amer. Math. Monthly, 65(1958) 370.

4792. Proposed by M. S. Klamkin and D. J. Newman, AVCO Research Division, Lawrence, Mass.

Show that the following operators are reducible:

(1) $x^n D^{2n}$ (2) $x^{2n} D^n$

and thus solve the differential equations

(1a) $[x^n D^{2n} - \lambda]y = 0$ (2a) $[x^{2n} D^n - \lambda]y = 0.$

[[Was a solution published? Have I missed it?]]

Amer. Math. Monthly, 65(1958) 779.

4816. Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.

Solve the integral equation

$$\int_0^\infty t^3 \phi(x-t) \, dt = a \left\{ \int_0^\infty t^2 \phi(x-t) \, dt \right\}^b$$

where a and b are independent of x.

Amer. Math. Monthly, 66(1959) 731–732.

Solution by Robert Weinstock, University of Notre Dame. Introducing the change of variable u = t - x and defining F by

$$F(x) = \int_0^\infty t^2 \phi(x-t) \, dt = \int_{-x}^\infty (u+x)^2 \phi(-u) \, du \tag{1}$$

we may differentiate the integral equation and obtain

$$3\int_{-x}^{\infty} (u+x)^2 \phi(-u) \, du = ab \left\{ \int_{-x}^{\infty} (u+x)^2 \phi(-u) \, du \right\}^{b-1} \cdot 2\int_{-x}^{\infty} (u+x)\phi(-u) \, du$$

that is

$$3 = ab[F(x)]^{b-2}F'(x) = ab(d/dx)\{(b-1)^{-1}[F(x)]^{b-1}\}$$

provided $b \neq 1$; the case b = 1 is handled separately below. We introduce m = b/(1-b)and integrate to obtain

$$F(x) = [c - (3/am)x]^{-m-1}$$
(2)

where c is an arbitrary constant.

Differentiating (1) three times and using (2) we obtain

$$\phi(x) = \frac{1}{2}F'''(x) = \frac{1}{2}(3/am)^3(m+1)(m+2)(m+3)[c-(3/am)x]^{-m-4}$$

For the existence of the integrals involved we must have m > 0 (whence 0 < b < 1), a > 0 and x < amc/3. In case b = 1, the differential equation for F reads F'(x) = (3/a)F(x), whence $F(x) = 2(a/3)^3Be^{3x/a}$ where B is an arbitrary constant. We then have $\phi(x) = \frac{1}{2}F'''(x) = Be^{3x/a}$ which satisfies the integral equation for b = 1, provided a > 0, for all x.

If a = 0, there is clearly only the trivial solution $\phi(x) = 0$.

Also solved by [3 others and] the proposer.

Amer. Math. Monthly, 66(1959) 66.

4826. Proposed by M. S. Klamkin and L. A. Shepp, AVCO Research and Development, Wilmington, Mass.

If $\phi(x) = x/1^2 - x^3/3^2 + x^5/5^2 - x^7/7^2 + \cdots$, express $\phi(1)$ in terms of $\phi(2 - \sqrt{3})$, thus obtaining a more rapidly converging expansion.

Amer. Math. Monthly, 66(1959) 819-820.

A Result of Ramanujan

Note by Emil Grosswald, University of Pennsylvania. In Ramanujan's paper, On the integral $\int_0^x t^{-1} \tan^{-1} t \, dt$, he obtains the result

$$2\phi(1) = 3\phi(2 - \sqrt{3}) + \frac{1}{4}\pi\log(2 + \sqrt{3})$$

(See J. Indian Math. Soc., VII(1915) 93–96; also *Collected Papers*, pp.40–43.) Ramanujan's treatment, with some details filled in, follows.

One observes that

$$\phi(x) = \int_0^x \frac{\arctan t}{t} \, dt \tag{1}$$

Next one shows that, for $0 < x < \pi/2$,

$$\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \dots = \phi(\tan x) - x\log(\tan x)$$
(2)

holds. Since both sides of (2) approach zero as $x \to 0$, it is sufficient to show that their derivatives are equal, i.e., using (1), that

$$\frac{2\cos 2x}{1} + \frac{2\cos 6x}{3} + \frac{2\cos 10x}{5} + \dots = -\log(\tan x) \quad 0 < x < \pi/2 \tag{3}$$

Since (3) is a known cosine Fourier expansion, (2) is proved. With $x = \pi/12$, $\tan x = 2 - \sqrt{3}$, (2) becomes:

$$\frac{1}{2}\left\{\frac{1}{1^2} + \frac{2}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} - \frac{2}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{2}{15^2} + \cdots\right\} = \phi(2 - \sqrt{3}) - \frac{\pi}{12}\log(2 - \sqrt{3})$$

The left member is $2\phi(1)/3$ as is shown by rewriting it as

$$\frac{1}{2}\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\} + \frac{3}{2}\left\{\frac{1}{3^2} - \frac{1}{9^2} + \frac{1}{15^2} + \cdots\right\}$$

The desired result now follows immediately.

Amer. Math. Monthly, 66(1959) 62.

E 1348. Proposed by M. S. Klamkin and Raphael Miller, AVCO Research and Development, Wilmington, Mass.

Find the locus of the centroids of all equilateral triangles inscribed in an ellipse.

Solution by Sister Mary Stephanie, Georgian Court College, Lakewood, New Jersey. The problem is not new; it appears on p.170 of C. Smith, Conic Sections, Macmillan (1937). The solution given there is essentially as follows. Let the ellipse be $x^2/a^2 + y^2/b^2 = 1$. If the eccentric angles of the vertices of an inscribed triangle are A, B, C, the centroid of the triangle is given by

$$x = a(\cos A + \cos B + \cos C)/3$$

$$y = b(\sin A + \sin B + \sin C)/3$$

The circumcentre of the triangle is given by

$$x = (a^{2} - b^{2}) \{\cos A + \cos B + \cos C + \cos(A + B + C)\}/4a$$

$$y = (b^{2} - a^{2}) \{\sin A + \sin B + \sin C - \sin(A + B + C)\}/4b$$

Since in an equilateral triangle the centroid coincides with the circumcentre, in this case we have

$$\frac{4ax}{(a^2 - b^2) - 3x/a} = \cos(A + B + C)$$

$$\frac{4by}{(b^2 - a^2) - 3y/b} = -\sin(A + B + C)$$

Squaring and adding we obtain the ellipse

$$(a^{2} + 3b^{2})^{2}x^{2}/a^{2} + b^{2} + 3a^{2}({}^{2}y^{2}/b^{2}) = (a^{2} - b^{2})^{2}$$

as the required locus.

Amer. Math. Monthly, 66(1959) 146–147.

4828. Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.

Do the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ converge, where

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n) dx$$

$$b_{n+1} = \int_0^1 \min(x, c_n, a_n) dx$$

$$c_{n+1} = \int_0^1 \max(x, a_n, b_n) dx$$

and mid (a, b, c) = b if $a \ge b \ge c$.

Amer. Math. Monthly, 66(1959) 922–923.

Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Buffalo, N.Y. The proposed sequences are convergent, with the limits 3/8, 1/2, 5/8 respectively.

Evidently $\min(x, b_n, c_n) \le x \le \max(x, a_n, b_n)$ so that if $\min(x, a_n, c_n) = x$, we have

$$\min(x, b_n, c_n) \le \min(x, a_n, c_n) \le \max(x, a_n, b_n) \tag{1}$$

Now, if mid $(x, a_n, c_n) = a_n$, we have either $x \le a_n$ or $c_n \le a_n$, so that $a_n \le \max(x, a_n, b_n)$ implies (1) in this case also. A similar argument holds for mid $(x, a_n, c_n) = c_n$, so that (1) is true in all cases. By integration there results $a_{n+1} \le b_{n+1} \le c_{n+1}$, $n = 1, 2, \ldots$

Now we have

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n) \, dx \le \int_0^1 d \, dx = \frac{1}{2}$$

and similarly $c_{n+1} \geq \frac{1}{2}$. Using this

$$b_{n+2} = \int_0^1 \operatorname{mid} \left(x, a_{n+1}, c_{n+1} \right) dx = \int_0^{1/2} \max(x, a_{n+1}) dx + \int_{1/2}^1 \min(x, c_{n+1}) dx$$
$$\leq \int_0^{1/2} \frac{1}{2} dx + \int_{1/2}^1 x dx = 5/8$$

Dually, $b_{n+2} \ge 3/8$. Since $3/8 \le b_{n+2} \le c_{n+2}$, $a_{n+3} = \int_0^1 \min(x, b_{n+2}, c_{n+2}) dx > 0$, and similarly $c_{n+3} < 1$.

It is now assumed that n is so large that $0 < a_n \le b_n \le c_n < 1$.

$$a_{n+1} = \int_0^{b_n} x \, dx + \int_{b_n}^1 b_n \, dx = \frac{2b_n - b_n^2}{2}$$
$$b_{n+1} = \int_0^{a_n} a_n \, dx + \int_{a_n}^{c_n} x \, dx + \int_{c_n}^1 c_n \, dx = \frac{a_n^2 - c_n^2 + 2c_n}{2}$$
$$c_{n+1} = \int_0^{b_n} b_n \, dx + \int_{b_n}^1 x \, dx = \frac{b_n^2 + 1}{2}$$

Thus

$$b_{n+2} = \frac{1}{2} \left[\left(\frac{2b_n - b_n^2}{2} \right)^2 - \left(\frac{b_n^2 + 1}{2} \right)^2 + 2 \left(\frac{b_n^2 + 1}{2} \right) \right]$$
$$= \frac{1}{2} + \frac{(2b_n - 1)(-2b_n^2 + 2b_n + 1)}{8} = b_n - \frac{(2b_n - 1)^3 + 5(2b_n - 1)}{16}$$

Since $0 < -2b_n^2 + 2b_n + 1$ whenever $0 < b_n < 1$, either

$$\frac{1}{2} \le b_{n+2} \le b_n$$
 or $\frac{1}{2} > b_{n+2} > b_n$

It follows that $\lim b_{2n} = \lim b_{2n+1} = \lim b_n = \frac{1}{2}$. Then

$$\lim a_n = \lim a_{n+1} = \lim \frac{2b_n - b_n^2}{2} = \frac{3}{8}$$
$$\lim c_n = \lim c_{n+1} = \lim \frac{b_n^2 + 1}{2} = \frac{5}{8}$$

Amer. Math. Monthly, 66(1959) 312.

E 1361. Proposed by M. S. Klamkin, AVCO Research and Development

If A, B, C are angles of a triangle, show that

 $\csc A/2 + \csc B/2 + \csc C/2 \ge 6$

Amer. Math. Monthly, 66(1959) 916.

I. Solution by Leon Bankoff, Los Angeles, Calif. Consider the angle bisectors AD, BE, CF concurrent at the incentre I of the triangle ABC. It is known that the sum of the ratios in which a point within a triangle divides the cevians of this point is never less than 6 (E 1043 [1953, 421]). Since the inradius $r \leq (ID, IE, IF)$, it follows that $AI/r + BI/r + CI/r \geq 6$.

[[This is just the first of no fewer than SEVEN published solutions.]]

Amer. Math. Monthly, 66(1959) 423.

E 1368. Proposed by M. S. Klamkin, AVCO Research and Development

Show that if all roots of $ax^4 - bx^3 + cx^2 - x + 1 = 0$ are positive, then $c \ge 80a + b$. Amer. Math. Monthly, **67**(1960) 84–85.

Solution by A. J. Goldman, National Bureau of Standards, Washington, D.C. The proof will show that 80 is the best possible constant.

(i) If a = b = c = 0, the result is true.

(ii) If a = b = 0, $c \neq 0$, then the roots r_1 , r_2 of $cx^2 - x + 1 = 0$ are positive, so $1/c = r_1r_2 > 0$, hence c > 0 and the result is true.

(iii) If a = 0, $b \neq 0$, then the roots of $bx^3 - cx^2 + x - 1 = 0$ are positive. Since $r_1r_2r_3 = r_2r_3 + r_3r_1 + r_1r_2 = 1/b$, we have $1/r_1 + 1/r_2 + 1/r_3 = 1$, which (since all $r_i > 0$) shows that all $r_i \ge 1$; hence $c/b = \sum r_i \ge 3$, so $c \ge 3b \ge b$ and the relation holds.

(iv) Suppose $a \neq 0$. Then

$$(c-b)/a = 1/2 \sum_{i \neq j} r_i r_j - \sum_k r_k = \left\{ \left(\sum r_i\right)^2 - \sum r_i^2 \right\} / 2 - \sum f_i$$

Call this $f(r_1, r_2, r_3)$. We can use Lagrangian multipliers to determine the minimum of $f(r_1, r_2, r_3)$ in the open subset $r_1 > 0$, $r_2 > 0$, $r_3 > 0$, $r_4 > 0$ of the constraint set

$$\prod_{i=1}^{4} r_i = \sum_{j=1}^{4} \prod_{i \neq j} r_i \qquad (=1/a)$$

i.e., $\sum 1/r_i = 1$, since on this set the function is bounded below. The minimum occurs when all $r_i = 4$, $f_{\min} = 80$; that is, $(c - b)/a \ge 80$ as desired.

Amer. Math. Monthly, 66(1959) 427.

4848. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

Without performing any integration determine the ratio

$$\int_{0}^{1} \frac{dt}{\sqrt{1-t^{4}}} : \int_{0}^{1} \frac{dt}{\sqrt{1+t^{4}}}$$

Amer. Math. Monthly, **67**(1960) 300.

Solution by A. B. Farnell, Convair Research Laboratory, San Diego, California. Since the first integral involved is convergent, and

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \int_0^{\frac{1}{4}\pi} \frac{d(\sqrt{\sin 2\theta})}{\cos 2\theta} = \int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sqrt{\sin 2\theta}}$$
$$\int_0^1 \frac{dt}{\sqrt{1+t^4}} = \int_0^{\frac{1}{4}\pi} \frac{d(\sqrt{\tan \theta})}{\sec \theta} = \frac{1}{\sqrt{2}} \int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

the desired ratio is $\sqrt{2}$.

Editorial Note. Several solvers used contour integration and transformations of the complex plane. In this way [G. E.]Raynor and the proposer obtained the more general result

$$\int_0^1 dt/(1-t^{2n})^{1/n} : \int_0^1 dt/(1+t^{2n})^{1/n} = \sec(\pi/2n), \ n = 2, 3, \dots$$

Amer. Math. Monthly, 66(1959) 515.

4855. Proposed by M. S. Klamkin, AVCO Research and Development

The angle of intersection between two spheres is constant for all points on any curve of intersection. Are there other surfaces with this property?

[[Was a solution to this ever published??]]

Amer. Math. Monthly, 66(1959) 816.

4828. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

A smooth centro-symmetric curve has the property that the centroid of any half-area which is formed by chords through the centre is equidistant from the centre. Show that the curve is a circle.

Amer. Math. Monthly, 67(1960) 809-810.

Solution by Harley Flanders, University of California, Berkeley. Represent the curve in polar coordinates by $r = r(\theta)$, a periodic function with $r(\alpha + \pi) = r(\alpha)$. Assume r is differentiable. After multiplying r by a suitable constant, the condition on the centroid is

$$\left[\int_{\alpha}^{\alpha+\pi} r^3 \cos\theta \,d\theta\right]^2 + \left[\int_{\alpha}^{\alpha+\pi} r^3 \sin\theta \,d\theta\right]^2 = \left[\int_{\alpha}^{\alpha+\pi} r^2 \,d\theta\right]^2$$

Differentiation with respect to α yields

$$\cos\alpha \int_{\alpha}^{\alpha+\pi} r^3 \cos\theta \, d\theta + \sin\alpha \int_{\alpha}^{\alpha+\pi} r^3 \sin\theta \, d\theta = 0$$

and a second differentiation,

$$-2r^3 - \sin\alpha \int_{\alpha}^{\alpha+\pi} r^3 \cos\theta \, d\theta + \cos\alpha \int_{\alpha}^{\alpha+\pi} r^3 \sin\theta \, d\theta = 0$$

Another differentiation gives r' = 0, whence r is a constant and the curve is a circle.

Editorial Note. The proposer remarks that if the locus of the centroiod were an ellipse instead of a circle, the same argument proves that the original curve must be a homothetic ellipse. He also conjectures that an analogous hypothesis regarding a surface in three-space will lead to a sphere.

Amer. Math. Monthly, 67(1960) 87.

4885. Proposed by M. S. Klamkin and D. J. Newman, AVCO Research, Lawrence, Mass.

Determine the unique solution of the integral equation

$$F(x_1, x_2, \dots, x_n) = ! + \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} F(y_1, y_2, \dots, y_n) \, dy_1 dy_2 \cdots dy_n$$

(The uniqueness when n = 2 was one of the problems in the 1958 Putnbam competition.)

Amer. Math. Monthly, 68(1961) 73.

Solution by P. G. Rooney, University of Toronto. Let $u_0 = 1$ and

$$u_{r+1}(x_1,\ldots,x_n) = 1 + \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} u_r(y_1,\ldots,y_n) \, dy_1 \cdots dy_n$$

The, by induction,

$$u_r(x_1...,x_n) = \sum_{m=0}^r (x_1 x_2 \cdots x_n)^m / (m!)^n$$

Clearly $u(x_1, \ldots, x_n) = \lim u_r(x_1, \ldots, x_n)$ exists uniformly in any bounded region of *n*-space, and satisfies the integral equation. Thus a solution is

$$u(x_1,...,x_n) = \sum_{m=0}^{\infty} (x_1 x_2 \cdots x_n)^m / (m!)^n$$

Now, if u and v are two bounded measurable solutions and w = u - v, then

$$w(x_1, \dots, x_n) = \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} w(y_1, \dots, y_n) \, dy_1 \dots dy_n$$

Hence, if M is a bound for |w|, then by induction

$$|w(x_1,\ldots,x_n)| \le M |x_1 x_2 \cdots x_n|^m / (m!)^n \to 0$$

as $m \to \infty$, and u is unique.

Amer. Math. Monthly, 67(1960) 187.

4889. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

If k points are distributed at random at the vertices of a regular n-gon, determine the probability that the centre of gravity of the k masses lies in a circle of radius r about the centre of the n-gon. What does the probability function reduce to when $n \to 0$? [[Was a solution ever published??]]

Amer. Math. Monthly, 67(1960) 693.

E 1430. Proposed by M. S. Klamkin, AVCO Research and Advanced Development What is the highest order of multiplicity a root can have for the equation

$$x(x-1)(x-2)\cdots(x-n+1) = \lambda ?$$

Amer. Math. Monthly, 68(1961) 298.

Solution by D. C. B. Marsh, Colorado School of Mines. Since the polynomial $p(x) = x(x-1)(x-2)\cdots(x-n+1)$ has all of its zeros real and distinct, so does p'(x). For $p(x) - \lambda$ to have a zero of multiplicity m, p'(x) must have this zero with multiplicity m-1. Therefore any root of $p(x) = \lambda$ can have multiplicity no greater than 2. That 2 is possible is shown by the case $\lambda = p(z)$ where z is a zero of p'(x).

Amer. Math. Monthly, 67(1960) 702–703.

The Inverse Tangent Integral

4865 [1959, 728]. Proposed by L. Lewin, Enfield, England Defining the inverse tangent integral of the second order by

$$Ti_2(x) = \int_0^x \frac{\tan^{-1}(x)}{x} \, dx = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \cdots$$

prove that $6Ti_2(1) - 4Ti_2(1/2) - 2Ti_2(1/3) - Ti_2(3/4) = \pi \log 2$.

Solution by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass. The proposer has show that

$$Ti_{2}\left(\frac{y^{2}}{2}\right) + \frac{1}{2}Ti_{2}\left\{\frac{y(2+y)}{2(1+y)}\right\} + \frac{1}{2}Ti_{2}\left\{\frac{2(y-1)}{y(2-y)}\right\} + Ti_{2}\left\{\frac{y}{2+y}\right\}$$
$$-Ti_{2}\left(\frac{y}{2-y}\right) + Ti_{2}\left(\frac{1}{1+y}\right) - Ti_{2}(y-1) = 2Ti_{2}(1) + \frac{\pi}{4}\log\left\{\frac{y(2-y)}{2(1+y)}\right\}$$

(See L. Lewin, *Dilogarithms and Associated Functions*, London, 1958, p.37.) The desired result follows upon letting y = 1.

Editorial Note. The problem and the proposer's solution were received before the publication of the book. A variety of similar formulas may be obtained, e.g.

$$Ti_2(7/24) + 2Ti_2(1/7) + 6Ti_2(1/3) - 8Ti_2(1/2) + Ti_2(3/4) + (\pi/2)\log(3/2) = 0.$$

Amer. Math. Monthly, **67**(1960) 802.

E 1431. Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Advanced Development

If $a_{n+1} = (1 + a_n a_{n-1})/a_{n-2}$ and $a_1 = a_2 = a_3 = 1$, show that a_n is an integer.

Amer. Math. Monthly, 68(1961) 379–380.

I. Solution by J. L. Pietenpol, Columbia University. Define a sequence $\{b_n\}$ of integers by

$$b_1 = b_2 = b + 3 = 1$$
 $b_4 = 2$ $b_n = 4b_{n-2} - b_{n-4}$ $(n > 4)$

Then

$$b_{n+1}b_{n-2} - b_n b_{n-2} = (4b_{n-1} - b_{n-2})b_{n-2} - (4b_{n-2} - b_{n-4})b_{n-1}$$

= $b_{n-1}b_{n-4} - b_{n-2}b_{n-3}$

so that, by induction, $b_{n+1}b_{n-2} - b_nb_{n-2} = 1$, or $b_{n+1} = (1 + b_nb_{n-1})/b_{n-2}$, and hence $\{a_n\} = \{b_n\},\$

II. Solution by H. E. Bray, Rice University. The solution of the problem is implicit in the following

THEOREM. If $a_{n+1} = (k + a_n a_{n-1})/a_{n-2}$ and $a_1 = a_2 = 1$, $a_3 = p$, where k, p are positive integers such that (k, p) = 1, a necessary and sufficient condition that a_n be an integer is that k = rp - 1, where r is an integer.

[[Proof supplied]]

[H. O.]Pollak showed that the recurrence relation gives integers whenever a_1 , a_2 , $(a_3 + a_1)/a_2$, $(a_4 + a_2)/a_3$ are integers. [R. A.]Spinelli showed that, apart from translations, there are only two different positive integer sequences satisfying the recurrence relation, namely $a_1 = a_2 = a_3 = 1$ and $a_1 = a_3 = 1$, $a_2 = 2$.

NUMBER THEORY

Perfect numbers

Amer. Math. Monthly, 67(1960) 1028.

E 1445. Proposed by M. S. Klamkin, AVCO Research and Advanced Development

A number n is defined as **almost perfect** if $\sum_{d|n} d = 2n \pm 1$. Are there any other almost perfect numbers besides numbers of the form 2^m ?

[[This was unsolved, and reproposed, with an asterisk, fifteen year later, at *Amer. Math. Monthly*, 82(1975) 73, together with the reference

R. P.Jerrard & Nicholas Temperley, Almost perfect numbers, *Math. Mag.*, **46**(1973) 84–87.

In 2006, the only thing known beyond what is stated in the problem, is that numbers having sum 2n + 1 must be square.]]

Amer. Math. Monthly, 67(1960) 1034.

4940. Proposed by M. S. Klamkin, AVCO Research and D. J. Newman, Brown University

Problem no.151 in the "Scottish Book" of problems due to Wavre poses the question of the existence of a harmonic function defined in a region containing a cube in its interior such that it vanishes on all its edges. Show that such a function $\neq 0$ exists for any number of dimensions.

Amer. Math. Monthly, **69**(1962) 173.

Solution by Fred Suvorov, Princeton University. Consider

 $h(x_1, x_2, \dots, x_n) = (\sinh(n-1)^{1/2} x_1) \cdot (\sin x_2) \cdots (\sin x_n)$

h is harmonic in *n*-space and vanishes on the *n*-cube of side π . (And, of course, many other places.)

Also solved by the proposer[s].

Amer. Math. Monthly, 68(1961) 67.

4946. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + 1/n$. Sum $\sum_{1}^{\infty} S_n/n!$

Amer. Math. Monthly, 69(1962) 239–240.

II. Solution by J. W. Wrench, Jr., The David Taylor Model Basin, Washington, D.C. Consider the more general sum $y = \sum_{n=1}^{\infty} S_n x^n / n!$ The function y is found to satisfy the differential equation $xy' - xy = e^x - 1$. The solution, satisfying the condition $y \to 0$ when $x \to 0$, is

$$y = e^x [\gamma + \ln x - Ei(-x)]$$

where γ is Euler's constant and $Ei(-x) = -\int_x^\infty e^{-t}t^{-1} dt$, x > 0, is the exponential integral for a negative real argument.

When x = 1, we find for the sum of the proposed series the expression $e \cdot [\gamma - Ei(-1)]$, which is numerically equal to 2.16538221532693635942, to 20 decimal places.

III. Comment by J. H. van Lint, Technical University, Eindhoven, Netherlands. The solution can be found in Erdélyi-Magnus-Oberhettinger-Tricomi: Higher Transcendental Functions, part 2, p.143, formula (5).

Amer. Math. Monthly, 68(1961) 807.

4983. Proposed by M. S. Klamkin, AVCO Research, and L. A. Shepp, University of California

Determine the number of different products, $P_n(r)$, if the factors are to be taken r+1 at a time, in $a_1a_2a_3\cdots a_n$ by inserting parentheses and keeping the order of the elements a_i unchanged. The different products which arise will be due entirely to the nonassociative character of the multiplication. The explicit products for n = 4, r = 1 are given by $((a_1a_2)(a_3a_4))$, $(a_1(a_2(a_3a_4)))$, $(((a_1a_2)a_3)a_4)$, $(a_1((a_2a_3)a_4)))$, $((a_1(a_2a_3))a_4)$. Whence, $P_4(1) = 5$. This problem generalizes the case for r = 1 (Bateman Project, Higher Transcendental Functions, III, 1955, p.230).

Amer. Math. Monthly, **69**(1962) 931.

Solution by John B. Kelly, Michigan State University. We follow the method of generating functions given by N. Jacobson (Lectures in Abstract Algebra, vol.I, pp.18–19) for the case r = 1. Let

$$y = \sum_{n=1}^{\infty} P_n(r) x^n$$

One easily observes the recursion formula

$$P_n(r) = \sum P_{n_1}(r) P_{n_2}(r) \cdots P_{n_{r+1}}(r)$$
(1)

the summation being extended over all solutions of $n_1 + n_2 + \cdots + n_{r+1} = n$ in nonnegative integers. From (1) it follows that $y^{r+1} - y + x = 0$, whence using a method given by J. S. Frame for inverting trinomials (this MONTHLY, April 1957) we find that

$$P_n(r) = 0 \qquad n \not\equiv 1 \pmod{r}$$
$$P_n(r) = \frac{1}{n} \binom{(r+1)k}{k} \quad \text{where} \quad n = kr+1$$

(See also Pólya und Szegö, Aufgaben und Lehrsätze aus der Analysis, Berlin, 1954, Bd.I, Aufgabe 211.)

Editorial Note. The present result is included in Some problems of nonassociative combinations, [1] I. M. H. Etherington, Edinburgh Math. Notes, **32**(1940) 1–6, and [2] I. M. H. Etherington and A. Erdélyi, ibid., **32**(1940) 7–12. It is observed in [1] that $P_n(r)$ is the number of ways in which a convex polygon with n + 1 sides can be divided into (r + 2)-gons by nonintersecting diagonals.

For a related problem which includes the present result as a special case, see G. N. Raney, *Functional composition patterns and power series reversion*, Trans. Amer. Math. Soc., **94**(1960) 441–451.

Amer. Math. Monthly, **69**(1962) 236.

5014. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

It is well known that an equilateral triangle cannot be imbedded in a square lattice. However, it can be done in a cubic lattice. Can this be extended, i.e. can any regular polygon be imbedded in a cubic lattice of high enough dimension?

Amer. Math. Monthly, **70**(1963) 447–448.

Solution by H. E. Chrestenson, Reed College. Suppose that a regular n-gon is imbedded in a lattice. If s and d are the lengths of the sides and the shortest diagonal, respectively, the law of cosines gives

$$d^{2} = 2s^{2} - 2s^{2}\cos(\pi - 2\pi/n) = 2s^{2} + s^{2}(2\cos 2\pi/n)$$

Since s^2 and d^2 are integers, $2\cos 2\pi/n$ must be rational. A theorem of D. H. Lehmer (see I. Niven, *Irrational Numbers*, Carus Monograph No.11, p.37) states that $2\cos 2\pi/n$ is an algebraic ineger of degree $\phi(n)/2$. Thus $\phi(n)$ must be 2, whence n is 3, 4 or 6. To imbed a regular hexagon, let the origin and A and B be lattice point vertices of an equilateral triangle (e.g. A: (4,1,1) and B: (1,4,1).) By expanding the triangle by a factor of 3 and reflecting in the centroid we see that the origin, 2A - B, 3A. 2A + 2B, 3B and 2B - A are vertices of a regular hexagon. Thus a regular n-gon can be imbedded in a lattice if and only if n = 3, 4 or 6, and in these cases a cubic lattice suffices. Amer. Math. Monthly, **69**(1962) 438.

5028. Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.

It is known that (a) any two quadrics which have a common enveloping cone intersect in plane curves, and (b) any two enveloping cones of a quadric intersect in plane curves. Does each of these properties characterize quadrics only?

[[Was a solution ever published? Have I missed it?]]

Amer. Math. Monthly, **69**(1962) 809.

E 1538. Proposed by M. S. Klamkin and Jerry Yos, AVCO Corporation

A simple closed curve has the property that there exist inscribed squares of the same dimension in every direction. Must the curve be a circle?

Amer. Math. Monthly, **70**(1963) 669–670.

Solution by Marlow Sholander, Western Reserve University. No. Each unit square can be inscribed in an "eyepiece" of the spectacle-shaped boundary of the union of circles $(x \pm \sqrt{2})^2 + y^2 \leq 1/2$ with the rectangle $-\sqrt{2} \leq x \leq \sqrt{2}, 1 \leq 2y \leq \sqrt{2}$. (A variation of the spectacles gives the same answer for *n*-gons.)

II. Solution by Michael Goldberg, Washington, D.C. The curve need not be a circle. Circumscribe a square about an oval of constant width. Holding the oval fixed, rotate the square about the oval. Then all four of the vertices of the square describe a new oval. This new oval is not a circle, yet it has the property that the inscribed square within it may be turned through all orientations. See Michael Goldberg, "Rotors tangent to n fixed circles", J. Math. Phys., **37**(1958) 70.

The proposers furnished a counter-example given by

$$x = a\cos\theta + \epsilon\cos 4\theta$$
$$y = a\sin\theta + \epsilon\sin 4\theta$$

where $16\epsilon < a$. Here there is only one square in each direction.

J. J. Schäffer, of the Instituto de Matematica y Estadistica. Monevideo, Uruguay, pointed out that this problem is considered by Günter Lumer in "Polígonos inscriptibles en curvos convexas", *Rev. Un. Mat. Argentina*, **17**(1955) 97–102. Considerable information, such as the fact that the curve may be convex and have area larger or smaller than the circle, is contained in the paper.
Amer. Math. Monthly, **69**(1962) 1011.

A Well-Known Series

E 1509. Proposed by A. G. Konheim, IBM, Yorktown Heights, New York

With -1 < t < 1, sum the series

$$1 + \frac{2}{3}t^2 + \frac{2}{3}\frac{4}{5}t^4 + \frac{2}{3}\frac{4}{5}\frac{6}{7}t^6 + \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{7}t^8 + \cdots$$

II. Solution by M. S. Klamkin, University of Buffalo. The result follows from the known sum

$$(\sin^{-1}t)^2 = t^2 + \frac{2}{3}\frac{t^4}{2} + \frac{2}{3}\frac{4}{5}\frac{t^6}{3} + \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{t^8}{4} + \cdots$$

by differentiatind both sides and dividing by 2t.

The series was located in Smithsonian Mathematical Formulae and Tables of Elliptic Functions, 6.42 No.55, p.122; Chrystal, Algebra, vol.2, 1906, Ex. xx, No.7, p.335 (cites Pfaf as source); Bromwich, An Introduction to the Theory of Infinite Series, Prob.2, p.197 (claims known to Euler); Knopp, Theory and Application of Infinite Series, Ex.123, Chap.VIII, p.271; Schuh Leerboek der Differentiaal en Integraalreckening, vol.2, pp.154–156; Hobson, Treatise on Plane Trigonometry, eqs.20,21,22, pp.279–280; M. R. Spiegel, this MONTHLY, **60**(1953) 243–247; Taylor, Advanced Calculus, p.632; Edwards, Differential Calculus for Beginners (1899) p.78.

Amer. Math. Monthly, **70**(1963) 438.

E 1588. Proposed by M. S. Klamkin, The State University of New York at Buffalo

An ellipse has the property that the sum of the moments of inertia of its area about two orthogonal tangents is constant. Does this property characterize the ellipse?

Amer. Math. Monthly, **71**(1964) 213.

Solution by the proposer. It follows from the parallel-axis transfer theorem that in order for this property to hold for a given curve, its orthoptic curve (locus of intersection of orthogonal tangents) must be a circle whose centre coincides with the centroid of the area enclosed by the given curve. Another curve having this property (see R. C. Yates, *A Handbook of Curves and Their Properties*, J. W. Edwards, Ann Arbor, 1947) is the deltoid

$$x = a(2\cos t + \cos 2t)$$

$$y = a(2\sin t - \sin 2t)$$

which is a 3-cusped hypocycloid whose orthoptic is the inscribed circle.

Amer. Math. Monthly, **70**(1963) 572–573.

5109. Proposed by M. S. Klamkin, State University of New York at Buffalo and A. L. Tritter, Data Processing, Inc.

Given the infinite permutation

where the second row is formed by taking in order from the natural numbers, 1 odd, 2 even, 3 odd, ..., 2n even, 2n + 1 odd, What is the cycle structure of this permutation?

Amer. Math. Monthly, **71**(1964) 570.

Solution by George Bergman, Harvard University. Let I_n designate the set of n integers $\{i \mid \frac{1}{2}n(n-1) < i \leq \frac{1}{2}n(n+1)\}$. Examination of the given permutation shows that it acts on I_n by the law: $i \to 21 - u_n$ where $u_n = \frac{1}{2}n^2$ if n is even, $u_n = \frac{1}{2}(n^2 + 1)$ if n is odd. The "pivot" of this action is u_n ; u_n is fixed, numbers of I_n less than u_n are decreased, numbers of I_n greater than u_n are increased.

But we see that even the greatest integer in I_n is not increased as far as u_{n+1} , and even the least integer in I_{n+1} is not decreased as far as u_n ; hence the interval $J_n = \{i \mid u_n \leq i < u_{n+1}\}$ is sent into itself. This J_n contains 2[n/2] + 1 elements. Let us represent them by the integers 0 through 2[n/2], writing j for $u_n + j$. Then the action of our permutation is: $j \to 2j$ for $j \leq [n/2], j \to 2j - 2[n/2] - 1$ otherwise. In other words, the elements of J_n are permuted exactly as the residue classes (mod 2[n/2]+1) are permuted under multiplication by 2.

The nature of the permutation is as follows: for each divisor d of 2[n/2]+1, the elements $i = u_n + j$ of J_n such that (2[n/2] + 1, j) = d form a cycle of order f((2[n/2] + 1)/d), where f(k) is the least m such that $k \mid 2^m - 1$. This number-theoretic function is described in standardtexts. For example, let n = 15, $J_n = \{i \mid 113 \leq i < 128\}$, represented by $\{j \mid 0 \leq j < 15\}$. The permutation for these integers is

The cycles are given by:

$$\begin{array}{ll} d=1:&(1\ 2\ 4\ 8),\ (7\ 14\ 13\ 11)\sim(114\ 115\ 117\ 121),\ (120\ 127\ 126\ 124)\\ d=3:&(3\ 6\ 12\ 9)\sim(116\ 119\ 125\ 122)\\ d=5:&(5\ 10)\sim(118\ 123)\\ d=15:&(0)\sim(113)\ \text{fixed} \end{array}$$

f takes on every integral value (for $f(2^m - 1) = m$); therefore all cycles are finite, and there are infinitely many cycles of every finite order.

Amer. Math. Monthly, **70**(1963) 671.

Polynomial Multiple of a Polynomial

E 1540 [1962, 809]. Proposed by Azriel Rosenfeld, Yeshiva University

Prove that every polynomial has a nonzero polynomial multiple whose exponents are all divisible by 1000000.

I. Solution by M. S. Klamkin, State University of New York at Buffalo. Let the given polynomial be

$$P(x) = \prod_{i} (x - r_i)$$

Let

$$Q(x) = x^a \prod_i \left[(x^a - r_i^a) / (x - r_i) \right]$$

It follows immediately that P(x)Q(x) is a polynomial whose exponents are all divisible by a. Now let $a = 10^6$.

Amer. Math. Monthly, **70**(1963) 679.

Expansion of a Definite Integral

5035 [1962, 570]. Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan

Let α be a fixed positive number. Prove that

$$\int_{0}^{\frac{1}{2}\pi} t^{\alpha} \cos^{2n} t \, dt = \frac{1}{2} \, \Gamma\left(\frac{\alpha+1}{2}\right) \Big/ n^{(\alpha=1)/2} - \frac{1}{12} \, \Gamma\left(\frac{\alpha+5}{2}\right) \Big/ n^{(\alpha+3)/2} + O(1/n^{(\alpha+5)/2})$$
as $n \to \infty$.

Solution by M. S. Klamkin, State University of New York at Buffalo. First expand t^{α} into the series

$$t^{\alpha} = A_1 \sin^{\alpha} t + A_2 \sin^{\alpha+2} t + A_3 \sin^{\alpha+4} t + \cdots$$

(This is a Lagrange reversion of a power series.) It follows immediately that $A_1 = 1$, $A_2 = \alpha/6$. Since

$$\int_{0}^{\frac{1}{2}\pi} \sin^{m} t \cos^{2n} t \, dt = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2}m+n+1)}$$

it follows that

$$\int_{0}^{\frac{1}{2}\pi} t^{\alpha} \cos^{2n} t \, dt = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2}\alpha+n+1)} + \frac{\alpha}{12} \frac{\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+n+2)} + \cdots$$

Expanding out the Gamma functions for large n by Stirling's approximation, i.e.,

$$\Gamma(n+1) = \left(\frac{n}{e}\right)^n \sqrt{2n\pi} \left[1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right]$$

we obtain the proposed expansion.

Amer. Math. Monthly, **72**(1965) 789–790.

Oops! E 1676 Again

E 1721 [1964, 911]. Proposed by J. C. Van Rhijn, Vollenhove, The Netherlands

Given an ellipse E with foci F_1 and F_2 , a point P outside E, the tangents PR_1 and PR_2 from P to E, and a positive number f (0 < f < 1). Find the locus of P if $PR_1 \cdot PR_2 = f \cdot PF_1 \cdot PF_2$.

Editor's comment: This problem was posed as E 1676 [1964, 317] and a solution published in this MONTHLY, 72(1965) 188–189.

Comment by M. S. Klamkin, University of Minnesota. The result follows immediately from some results on ellipses in C. Zwikker, Advanced Plane Geometry, North-Holland, Amsterdam (1950) pp.98 & 112: If the parametric equation of the ellipse E is $z = a \cos \theta + 1b \sin \theta$ and R_1 and R_2 are given by $\theta = u + q$ and $\theta = u - q$ respectively, then

$$-PR_1 \cdot PR_2 = \left(\frac{a+b}{2}\right)^2 e^{2iu} + \left(\frac{a-b}{2}\right)^2 e^{-2iu} - \frac{c^2}{2}\cos 2q \qquad (c^2 = a^2 - b^2)$$

and $PF_1 \cdot PF_2 = -PR_1 \cdot PR_2/\cos^2 q$. If now $|PR_1 \cdot PR_2| = f|PF_1 \cdot PF_2|$, then $\cos^2 q = f$ and P is given by $(a \cos u + ib \sin u)/\cos q$; in other words, the locus of P is an ellipse confocal with E.

Following Zwikker, we then note the following generalization of the usual reflection property of ellipses: The tangents to an ellipse from a point make equal angles with the lines joining that point to the foci.

Amer. Math. Monthly, 72(1965) 921–922.

Inversion of Convolutal Sequences

5231 [1964, 923]. Proposed by H. W. Gould, West Virginia University

Let x, z be real numbers. Prove that each of the following systems implies the other:

$$B_n = \sum_{k=0}^n \binom{z}{k} x^k A_{n-k} \qquad A_n = \sum_{k=0}^n \binom{-z}{k} x^k B_{n-k}$$

I. Solution by M. S. Klamkin, University of Minnesota. If F(x, z, k) and G(x, z, k) satisfy

$$\sum_{k=0}^{\infty} F(x, z, k) t^k \cdot \sum_{k=0}^{\infty} G(x, z, k) t^k \equiv 1$$

then either of the following systems implies the other:

$$B_n = \sum_{k=0}^n F(x, z, k) A_{n-k} \qquad A_n = \sum_{k=0}^n B_{n-k} G(x, z, k)$$

These transform equations follow by direct substitution and noting that

$$\sum_{k=0}^{s} G(x, z, k) F(x, z, s - k) = \begin{cases} 1 & s = 0 \\ 0 & s > 0 \end{cases}$$

The special case for the present problem requires

$$F(x, z, k) = \begin{pmatrix} z \\ k \end{pmatrix} \qquad G(x, z, k) = \begin{pmatrix} -z \\ k \end{pmatrix}$$
$$\sum F \cdot t^{k} = (1 + xt)^{z} \qquad \sum G \cdot t^{k} = (1 + xt)^{-s}$$

Amer. Math. Monthly, 73(1966) 546-547.

Observations with Zero Dispersion

E 1775 [1965, 316]. Proposed by George Purdy, University of Reading, England

Under what conditions do real x_1, \ldots, x_n satisfy the equation $x_1^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2/n$ for $n \ge 1$?

III. Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Mich.. The problem here is a special case of a well-known result for convex functions, i.e., if $\phi(t)$ is convex in $t \ge 0$, then

$$\phi\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \le \frac{\phi(x_1)+\phi(x_2)+\dots+\phi(x_n)}{n}$$

 $(x_i \ge 0)$, with equality only if the x_i are equal or $\phi(t)$ is linear. For the present case $\phi(t) = t^2$ and we must have $x_1 = x_2 = \cdots = x_n$

Amer. Math. Monthly, **73**(1966) 553–554.

Solving
$$\sum f(d)f(n/d) = 1$$

5293 [1965, 555]. Proposed by Martin J. Cohen, Beverly Hills, California

Find a function f such that $\sum f(d)f(n/d) = 1$ for every positive integer n, where the sum is taken over all d which divide n (including 1 and n).

II. Solution by M. S. Klamkin, Ford Motor Company. The problem may be extended to find a function F such that

$$\sum_{d_1d_2\cdots d_r=n} F(d_1)F(d_2)\cdots F(d_r) = 1$$

for every positive integer n where the sum is taken over all d_r which divide n (including 1 and n).

Consider the formal product of r identical Dirichlet series:

$$\left\{\sum \frac{F(n)}{n^s}\right\} = \sum \frac{G(n)}{n^s}$$

[See Hardy and Wright, Theory of Numbers, p.248.] Then

$$G(n) = \sum_{d_1 d_2 \cdots d_r = n} F(d_1) F(d_2) \cdots F(d_r) = 1$$

whence,

$$\sum \frac{F(n)}{n^s} = \zeta(s)^{1/r} = \prod_{\text{primes}} \{1 - 1/p_n^s\}^{-1/r}$$

multiplied possibly by an r th root of unity. Let

$$\{1 - 1/p_n^s\}^{-1/r} = 1 + \frac{a_1}{p_n^s} + \frac{a_2}{p_n^{2s}} + \frac{a_3}{p_n^{3s}} + \cdots$$

then $a_m = \binom{-1/r}{m}$. It now follows that if

$$n = p_{a_1}^{i_1} \cdot p_{a_2}^{i_2} \cdots p - a_s^{i_s}$$
 then $F(n) = a_{i_1} a_{i_2} \cdots a_{i_s}$

multiplied by a fixed r th root of unity. The original problem corresponds to the special case r = 2.

Amer. Math. Monthly, **73**(1966) 779–780.

A Special Case of a Theorem of Whitney

5293 [1965, 555]. Proposed by R. A. Bell, Kansas City, Mo.

Suppose that g(x) has its first n+1 derivatives defined and continuous in [-1,1]. Define y(x) = g(x)/x for $x \neq 0$ and y(0) = g'(0). If g(0) = 0, prove that $y^{(n)}(0) = d^n y/dx^n|_{x=0}$ exists and equals $g^{(n+1)}(0)/(n+1)$.

II. Solution by M. S. Klamkin, Mathematical and Theoretical Sciences Scientific Laboratory, Ford Motor Company, Dearborn, Michigan. Let

$$D^m[g(x)/x] = F_m(x)/x^{m+1}$$

so that $DF_0(x) = D\{xD^0[g(x)/x]\} = Dg(x) = x^0g(x)$. Assume that

$$DF_m(x) = x^m D^{m+1}g(x) \tag{(\star)}$$

Then

$$\frac{F_{m+1}(x)}{x^{m+2}} = D^{m+1}\frac{g(x)}{x} = D\frac{F_m(x)}{x^{m+1}} = \frac{xDF_m(x) - (m+1)F_m(x)}{x^{m+1}}$$

whence $F_{m+1}(x) = xDF_m(x) - (m+1)F_m(x)$ and

$$DF_{m+1}(x) = xD^2F_m(x) - (m+1)DF_m(x)$$

= $xD[x^mD^{m+1}g(x)] - (m+1)x^mD^{m+1}g(x)$
= $x^{m+1}D^{m+1}g(x)$

It follows by induction that (\star) is true for all nonnegative integers m.

By L'Hospital's Rule, $\lim_{x\to 0} D^n[g(x)/x]$ will exist if $\lim_{x\to 0} [DF_n(x)/Dx^{n+1}]$ exists. By (\star) , this latter limit is $g^{(n)}(0)/(n+1)$.

Amer. Math. Monthly, **73**(1966) 894.

A Consequence of Problem 4964

E 1795 [1965, 665]. Proposed by N. D. Kazarinoff, University of Michigan

Let ABCDEF be a convex hexagon such that the perimeters of the triangle ABF, BCD, DEF and BDF are the same. Show that the hexagon must be a triangle, that is, it must have three 180° angles. Compare 4964 [1962, 672].

Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. Draw $\triangle GHI$, where $GB \parallel AC$, $HD \parallel CE$ and $IF \parallel AE$. Since the hexagon is convex it lies inside or on $\triangle GHI$. By the result of problem 4964 [1962,672], the perimeter of $\triangle BFD$ cannot be less than each of the triangles GFB, FDI and BDH, and if $\triangle BFD$ has the same perimeter as one of these other triangles, then all the four triangles have the same perimeter. Since $\triangle ABF \leq \triangle GBF$ in perimeter and similarly for the other two pairs of triangles, $\angle ABC = \angle CDE = \angle EFA = \pi$, and the hexagon is a triangle. The above result would still be valid if we replaced "perimeter" of the triangles by "area". This follows from a corresponding result to problem 4964 and is also a solved problem in this MONTHLY.

Amer. Math. Monthly, **73**(1966) 894.

An Asymptotic Formula

E 1795 [1965, 665]. Proposed by Louis Comtet, Boulogne, France

Show that, as $n \to \infty$, $s(n) = 1^n + 2^n + 3^n + \dots + n^n$ is asymptotic to $en^n/(e-1)$. Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. Let $\nu(n)|geq0$ for $n = 1, 2, 3, \dots$, and set $S(n) = \sum_{r=1}^{n} r^{\nu(n)}$. By Maclaurin's Integral Test,

$$S(n) \ge \int_0^n x^{\nu(n) \, dx \ge S(n) - n^{\nu(n)}}$$

Consequently,

1. If $\nu(n)/n \to 0$, then $S(n) \sim n^{\nu(n)}/[1 + \nu(n)]$ 2. If $n/\nu(n) \to 0$, then $S(n) \sim n^{\nu(n)}$ 3. If $\nu(n)/n \to a+ > 0$, then $1/a \le S(n) \le 1 + 1/a$

In Case 3 we have

$$\begin{split} S(n)/n^{\nu(n)} &= \sum_{r=0}^{n-1} \exp[\nu(n) \log(1-1/n)] = \sum_{r=0}^{n-1} \exp[-\nu(n)[r/n + O(r^2/n^2)]] \\ &= \sum_{r=0}^{n-1} e^{-ar} e^{O(1)} \to \sum_{r=0}^{\infty} e^{-ar} = \frac{e^a}{e^a - 1} \end{split}$$

by Tannery's theorem [see, for example, T. J. Bromwich, Introduction to the Theory of Infinite Series, 2nd ed.(1925) p.136].

Amer. Math. Monthly, **74**(1967) 81.

A Consequence of the Mean Value Theorem

E 1802 [1965, 666]. Proposed by Dov Avishalom, Tel-Aviv, Israel For functions of class C_n prove that

$$f^{(n)}(a) = \lim_{h \to 0} \left\{ \frac{1}{h^n} \sum_{k=0}^n \left[(-1)^{n-k} \binom{n}{k} f(a+kh) \right] \right\}$$

I. Solution by M. S. Klamkin, Ford Scientific Laboratory. First, it is well known that

$$\sum_{k=0}^{n} (-1)^{k} k^{r} \binom{n}{k} = \begin{cases} 0 & (r=0,1,\dots,n-1) \\ n! & (r=n) \end{cases}$$

(Easily proved by differentiating $(1 - x)^n = \sum_{k=0}^n (-1)^k {n \choose k} x^k$ r times $(r \le n)$ and evaluating at x = 1.) Consequently, it follows from L'Hospital's rule that the desired limit equals

$$\frac{1}{n!} \left\{ D^n \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} F(a+kh) \right\}_{k=0} = F^n(a).$$

Amer. Math. Monthly, **74**(1967) 206.

5461. proposed by M. S. Klamkin, Ford Scientific Laboratory

Show that it is possible in E_n to have n + 1 mutually orthogonal spheres. What is the maximum number of such spheres?

Amer. Math. Monthly, **75**(1968) 202–203.

Solution by Seymour Schuster, University of Minnesota. Consider a hypersphere of unit radius centred at the origin of a rectangular coordinate system in E_n . An inversion with respect to any hypersphere whose centre is two units from the origin transforms the *n* coordinate hyperplanes and the unit hypersphere into n + 1 mutually orthogonal hyperspheres.

Conversely any n mutually orthogonal hyperspheres can be inverted into n mutually orthogonal hyperplanes, and we recall that n is the maximum number of mutually orthogonal hyperplanes in n-space. Any two distinct hyperspheres orthogonal to n mutually orthogonal hyperplanes must be concentric. Thus n + 1 is the maximum number of mutually orthogonal hyperspheres in E_n .

J. D. E. Konhauser states that the number is n + 2 if imaginary hyperspheres are admitted.

Amer. Math. Monthly, **74**(1967) 318.

E 1966. proposed by M. S. Klamkin, Ford Scientific Laboratory

Show how to construct a regular tetrahedron if the vertices lie on four given parallel planes.

Amer. Math. Monthly, **75**(1968) 675.

Solution by the proposer. Let the distances between successive planes be a, b and c. Starting with any regular tetrahedron, locate points D, E on edge OA such that OD : DE : EA = a : b : c. On edge OB locate F such that OF : FB = a : b. Now draw a plane through O, a plane through B and E and a plane through A all parallel to the plane through F, D and C.

This gives us a configuration similar to the one we wish to construct, which can now be done by similar figures.

[V. F.]Ivanoff notes that it is possible to construct a tetrahedron under the given conditions similar to any given tetrahedron.

[[In connexion with the next item, I'm sure that 'Sheila M. Kaye' of McGill is Murray!]]

Amer. Math. Monthly, **74**(1967) 337.

On Commutative Rings

5377 [1966, 312]. proposed by Erwin Just and Norman Schaumberger, Bronx Community College

In a ring R each element x stisfies the equation $x = x^{n+1}$ for some integer n. Prove that $x^n y = yx^n$ for each x and y contained in R.

I. Solution by Shiela M. Kaye, McGill University. We must assume $n \equiv n(x) > 0$.

(a) $x^{n(x)}$ is idempotent, since $(x^{n(x)})^2 = x^{n(x)+1}x^{n(x)-1} = x^{n(x)}$.

(b) Let ab = 0. Then $ba = (ba)^{n(ba)+1} = b(ab)^{n(ba)}a = 0$.

(c) Let $x, y \in R$ and let n = n(x). Then from (a), $yx^n = yx^{2n}$, whence $(y - yx^n)x^n = 0$. Also from (b), $x^n(y - yx^n) = 0$, so that $x^ny = x^nyx^n$.

(d) Similarly, from $x^n y = x^{2n} y$, we deduce $yx^n = x^n yx^n$, which completes the proof.

Amer. Math. Monthly, **74**(1967) 1141–1142.

Convergence of a Series

E 1884 [1966, 411]. proposed by A. F. Beardon, University of Maryland

Prove that the series

$$\sum_{n_1,...,n_k}^{\infty} \frac{1}{(n_1^2 + \dots + N_k^2)^p}$$

converges if and only if p > k/2.

II. Solution by M. S. Klamkin, Ford Scientific Laboratory. By the integral test (for m dimensions) the series will converge or diverge with the integral

$$\int_1^\infty \int_1^\infty \cdots \int_1^\infty \frac{dn_1 \, dn_2 \cdots dn_k}{(n_1^2 + n_2^2 + \cdots + n_k^2)^p}$$

or equivalently (using spherical coordinates) with the integral $\int_c^{\infty} (r^{k-1} dr/r^{2p})$ which converges if and only if 2p - k + 1 > 1, or p > k/2.

We are assuming p is a constant, otherwise the result is not valid, e.g., $\sum_{n=1}^{\infty} n^{1+1/n}$ diverges.

Amer. Math. Monthly, **75**(1968) 298.

Minors of a Bidiagonal Matrix

5377 [1966, 312]. proposed by D. Z. Djoković, University of Belgrade, Yugoslavia

Let $A_n = (a_{ij})$ be an $n \times n$ matrix such that $a_{ii} = a_i$, i = 1, 2, ..., n; $a_{i,i+1} = b_i$, i = 1, 2, ..., n-1; $a_{ij} = 0$ otherwise. Let M be the minor of det A_n obtained by deleting the rows $i_1, i_2, ..., i_k$ $(1 \le i_1 < i_2 < \cdots < i_k \le n)$ and the columns $j_1, j_2, ..., j_k$ $(1 \le j_1 < j_2 < \cdots < j_k \le n)$. Prove that

$$M = (a_1 a_2 \cdots a_{j_1-1})(b^{j_1} b^{j_1+1} \cdots b_{i_1-1})(a_{i_1+1} a_{i_1+2} \cdots a_{j_2-1})$$

$$\cdot (b_{j_2} b_{j_2+1} \cdots b_{i_2-1})(a_{i_2+1} a_{i_2+2} \cdots a_{j_2-1}) \cdots (b_{j_k} b_{j_k+1} \cdots b_{i_k-1})$$

$$\cdot (a_{i_k+1} a_{i_k+2} \cdots a_n)$$

if $(1 \le j_1 \le i_1 < j_2 \le i_2 < j_3 \le i_3 < \dots < j_k \le i_k \le n)$; M = 0 otherwise. We take $(a_r a_{r+1} \cdots a_s) = 1$ whenever s < r.

Solution by M. S. Klamkin, Ford Scientific Laboratory. We employ induction. Assume that the result hold for all matrices A_r , r = 1, 2, ..., n and for all k = 1, 2, ..., r. Now consider A_{n+1} . If $n + 1 = j_k > i_k$ then $A_{n+1} = 0$. If $n + 1 = i_k > j_k$ delete row i_k and column j_k and expand by minors using the last column, giving bA_{n-1} . If i_k and $j_k < n + 1$, then $A_{n+1} = a_{n+1}A_n$. By the inductive hypothesis the result also holds for A_{n+1} . Since it clearly holds for A_2 it is valid for all A_n .

Amer. Math. Monthly, **76**(1969) 946–947.

Sums of Powers of Integers

E 2136 [1968, 1113]. proposed by A. Inselberg and B. Dimsdale, IBM Los Angeles Scientific Center

Let

$$S_r = \sum_{k=1}^n k^r$$

It is well known that $S_3 = S_1^2$. Are there other values of p, q, u, v such that $S_p^u = S_q^v$ for all n?

Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. We will consider p, q, u, v to be any real numbers. By the Euler-MacLaurin expansion, the first few terms of the asymptotic expansion of S_r for r > -1 is given by

$$S_r \sim \frac{n^{r+1}}{r+1} + \frac{1}{2}n^r + \frac{rn^{r-1}}{12} + \cdots$$

Thus

$$S_{p}^{u} \sim \left(\frac{n^{p+1}}{p+1}\right)^{u} \left\{ 1 + \frac{p+1}{2n} + \frac{p(p+1)}{12n^{2}} + \cdots \right\}^{u}$$
$$S_{q}^{v} \sim \left(\frac{n^{q+1}}{q+1}\right)^{v} \left\{ 1 + \frac{q+1}{2n} + \frac{q(q+1)}{12n^{2}} + \cdots \right\}^{v}$$

Since $S_p^u = S_q^v$ for all n, we must have by equating the first few terms of the expansion that

$$(p+1)^u = (q+1)^v \tag{1}$$

$$(p+1)u = (q+1)v$$
 (2)

$$\frac{u(u-1)(p+1)^2}{8} + \frac{up(p+1)}{12} = \frac{v(v-1)(q+1)^2}{8} + \frac{vq(q+1)}{12}$$
(3)

(assuming that there are at least three terms in both expansions). It now follows from (2) and (3) that p = q and then that u = v. If the expansion of S_p has less than three terms, then p = 0 or p = 1. In this case the term up(p+1)/12 does not appear in (3). This then leads to q = 3p. Thus for p = 1, either q = 1 or q = 3, and for p = 0, q = 0.

We now cosider the cases $p, q \leq 1$. The case $p = -1 \neq q$ is ruled out since here $S_p \sim \ln n$ and S_q is not. For p, q < -1 we have

$$S_p^u \sim \left\{ \zeta(-p) - \frac{n^{p+1}}{p+1} + \frac{n^p}{2} - \cdots \right\}^u$$

On comparison of the first two terms of the expansion for S_p^u with that for S_q^u [[can't read the last exponent, neither in JSTOR, nor in the original.]] we must have p = q. Thus the only solution is the known identity $S_3 = S_1^2$.

[John] Ivie and many others note that the solution is well known and has been published. See D. Allison, A note on sums of powers of integers, this MONTHLY, 1961, p.272. A related result is developed in S. Cavior, A theorem on power sums, in the April 1968 Fibonacci Quarterly, pp.157–161. He considers the more general problem of finding polynomials

$$f(x) = \sum_{i=0}^{r} a_1 x^i$$
 $g(x) = \sum_{i=0}^{s} b_i x^i$

over the real field such that

$${f(1) + \dots + f(n)}^p = {g(1) + \dots + g(n)}^q$$

for positive integral r, p, s, q. For this condition to hold, it is shown that the only monic solutions occur when p = 2, q = 1 and

$$f(x) = a + x$$
 $g(x) = x^3 + 3ax^2 + (2a^2 - a)x - a^2$

where a is an arbitrary real constant. (For a = 0 this is the result of the present problem.) Cavior also considers the problem of finding non-monic polynomials f and g for arbitrary p and q, and proves a general theorem.

Amer. Math. Monthly, **76**(1969) 1063.

E 2197. Proposed by M. S. Klamkin, Ford Scientific Laboratory and D. J. Newman, Yeshiva University

Solve the functional equation $F(x^m) = [F(x)]^n$.

[[Was a solution ever published??]]

Amer. Math. Monthly, **77**(1970) 774.

Solution by the proposers. Let $\log F(x) = G(x)(\log x)^{\alpha}$ where $\alpha = (\log n)/(\log m)$. Then $G(x^m) = G(x)$. A general solution for G(x) is $G(x) = H(\log \log x)$. where H is periodic with period $\log m$. Thus

$$F(x) = \exp[H(\log \log x) \cdot (\log x)^{\alpha}]$$

It is to be noted that the problem was deliberately incompletely fomulated in that no class of functions F and no domain of x were specified, nor the constants m and n. If F(x) is to be real, then it is assumed that x > 1. In the above it is also assumed that m, n > 0 and $m \neq 1$. (The case m = 1 is easily handled. If mn = 0, the equation may be solved by inspection. The above solution is also valid when m, n are not bothpositive, provided α can be chosen so that $m^{\alpha} = n$.)

Amer. Math. Monthly, **76**(1969) 1138.

E 2203^{*}. Proposed by M. S. Klamkin, Ford Scientific Laboratory

It is known that if $0 \le x_1 \le x_2 \le \cdots \le x_n$, $(n \ge 3)$, then

$$x_1^{x_2}x_2^{x_3}\cdots x_n^{x_1} \ge x_2^{x_1}x_3^{x_2}\cdots x_1^{x_n}$$

Are there any other nontrivial permutations $\{a_1\}$ and $\{b_i\}$ of the $\{x_i\}$ such that

$$a_1^{a_2}a_2^{a_3}\cdots a_n^{a_1} \ge b_2^{b_1}b_3^{b_2}\cdots b_1^{b_n}$$
?

Amer. Math. Monthly, 77(1970) 1008–1009.

Solution (adapted) by G. L. Watson, University College, London, England. For n = 3 there is no other nontrivial permutation of the x_i of the form required. For n = 4 there are other solutions. For one such solution, note that $x_3/x_1 \ge 1$, $x_4/x_3 \ge 1$, $x_3 - x_2 \ge 0$, $x_3 - x_1 \ge 0$ imply

$$(x_3/x_1)^{x_3-x_2}(x_4/x_3)^{x_3-x_1} \ge 1$$

whence (upon multiplying both sides by $x_2^{x_4}/x_3^{x_2}$)

$$x_1^{x_2} x_2^{x_4} x_4^{x_3} x_3^{x_1} \ge x_1^{x_3} x_3^{x_2} x_2^{x_4} x_4^{x_1}$$

For n > 4, the possibilities increase rapidly. For example, with n = 5,

$$(x_5/x_2)^{x_4-x_3}(x_2/x_1)^{x_5-x_2} \ge 1$$

implies

$$x_1^{x_2} x_2^{x_3} x_3^{x_5} x_5^{x_4} x_4^{x_1} \ge x_1^{x_5} x_5^{x_3} x_3^{x_2} x_2^{x_4} x_4^{x_1}$$

Amer. Math. Monthly, **76**(1969) 1152.

5705. Proposed by M. S. Klamkin, Ford Scientific Laboratory

Find the general solution of the differential equation

$$[xD^{n+1} + 2nD^n - xD - n]y = 0.$$

Amer. Math. Monthly, 77(1970) 1020.

Solution by Robert Heller, Mississippi State University. It can be shown by induction that $xD^n + nD^{n-1} = D^n x$ from which it follows that the given equation may be written $(D^n - 1)(xD + n)y = 0$. Hence

$$(xD+n)y = \sum_{k=1}^{n} c_k e^{a_k x}$$

where a_1, \ldots, a_n are the *n* distinct *n* th roots of unity. Multiplication by x^{n-1} gives

$$D(x^n y) = \sum_{k=1}^n c_k x^{n-1} e^{a_k x}$$

Repeated integration by parts shows that

$$x^{n}y = c_{0} + \sum_{k=1}^{n} \left[c_{k}e^{a_{k}x} \sum_{p=1}^{n} (-1)^{p-1} \frac{(n-1)!}{(n-p)!} (a_{k}x)^{n-p} \right]$$

Taking limits of both sides as $x \to 0$ we see that although solutions y may exist on $(-\infty, \infty)$, such solutions are not expressed with precisely n + 1 arbitrary constants. On $(-\infty, 0)$ or on $(0, \infty)$ we have the general solution

$$y = c_0 x^{-n} + x^{-n} \sum_{k=1}^{n} \left[c_k e^{a_k x} \sum_{p=1}^{n} (-1)^{p-1} \frac{(n-1)!}{(n-p)!} (a_k x)^{n-p} \right]$$

Amer. Math. Monthly, **77**(1970) 79.

E 2209. Proposed by M. S. Klamkin, Ford Scientific Laboratory

Determine the locus of the centroids of all triangles similar to a given triangle and inscribed in another given triangle.

Amer. Math. Monthly, 77(1970) 1012–1013.

Solution by Michael Goldberg, Washington, D.C. If a triangle of given shape grows so that the vertices trace fixed straight lines in the plane, then every point of the triangle will trace a straight line.

If the fixed straight lines are the sides of the triangle ABC, and the variable inscribed triangle is DEF, then its centroid P describes a straight line. However, there are some orientations for which the vertices of DEF cannot be confined to the straight line segments of the triangle ABC. Also, as DEF is turned, the motion of the vertices will change direction as they shift from oneline of ABC to another line of ABC. Hence the complete locus of the centroid consists of three straight line segments, shown in dotted lines in the figure.



If, in addition to direct symmetry, reflected symmetry is acceptable, then three more straight line segments are to be added to the locus of the centroid.

[[The above picture is a special case — in simplifying the calculations I indvertently made the two triangles similar. Here is the more general picture, with the reflected segments shown dashed. — I've gone mad and given dotted only; dashed only (the inner triangle ought to be reflected here); both; and enlarged — take your pic!! R.]]





Several solvers called attention to the well-known underlying theorem which can be found in Peterson's text, also Johnson's, and elsewhere.

Amer. Math. Monthly, 77(1970) 79.

E 2214. Proposed by M. S. Klamkin, Ford Scientific Laboratory and B. Ross Taylor, York High School

It is intuitive that every simple *n*-gon (n > 3) possesses at least one interior diagonal. For a simple *n*-gon what is the leasr number of diagonals which, except for their endpoints, lie wholly in its interior?

Amer. Math. Monthly, 77(1970) 1111–1112.

Solution by Anders Bager, Hjørring, Denmark. The two tangents from a point P outside a circle Γ touch Γ in points A and B. Connect A and B with a broken line consisting of n-2 chords succeeding each other along the smaller arc from A to B. Join P to A and B to obtain a simple *n*-gon with exactly n-3 inner diagonals (all issuing from P).

The number n-3 is minimal. This is trivially so if n = 3. Suppose it tru for some n and consider an arbitrary simple (n + 1)-gon. From this cut off a triangle such that two sides are sides of the (n + 1)-gon, and the third side an inner diagonal. This is always possible and leaves a simple n-gon which, by assumption, has at least n - 3 inner diagonals. Hence the (n + 1)-gon has at least (n - 3) + 1 = (n + 1) - 3 inner diagonals. Thus the assertion of the problem is true by induction.

[Also solved by ten others, including the proposers and ...]]

R. B. Eggleton establishes the result that a simple *n*-gon has precisely n-3 inner diagonals if and only if no two of its diagonals intersect.

Amer. Math. Monthly, 77(1970) 83-84.

A Minimum Partition Problem

E 2171 [1969, 414]. Proposed by Kenneth Jackman, Federal Electric Corporation, Fairbanks, Alaska

Given N, what is the smallest W for which $B_1 + B_2 + \cdots + B_c = W$ and $B_1 B_2 \cdots B_c \ge N$ with all B_k positive integers.

Note. The statement of the problem is ambiguous since it is not clear whether the integer c is fixed. Solutions were submitted for both cases.

I. (c fixed.) Solution by David Zeitlin, Minneapolis, Minnesota. From the arithmeticgeometric inequality, we have

$$\frac{W}{c} = \frac{B_1 + B_2 + \dots + B_c}{c} \ge \sqrt[c]{B_1 B_2 \cdots B_c} \ge \sqrt[c]{n}$$

Thus $W = c\sqrt[c]{N}$ if integral; otherwise $W + [c\sqrt[c]{N}] + 1$.

II. (c not fixed.) Solution by M. S. Klamkin, Ford Scientific Laboratory. The dual of this problem is to find the largest number which can be obtained as the product of positive integers whose sum is $\leq S$. This problem was proposed by Leo Moser and solved by L. Carlitz [Problem 125, Pi Mu Epsilon Journal, Fall, 1961]. If P(S) denotes the maximum product, it was shown that

$$P(S) = \begin{cases} 3^m & \text{if } S=3m \\ 4 \cdot 3^{m-1} & \text{if } S=3m+1 \\ 2 \cdot 3^m & \text{if } S=3m+2 \end{cases}$$

Here S is partitioned into as many 3s as possible.

It now follows immediately that if $P(S) + 1 \le N \le P(S+1)$ then $W_{\min} = S + 1$ (the corresponding partition is not unique in general).

Amer. Math. Monthly, 77(1970) 192.

E 2216. Proposed by M. S. Klamkin, Ford Scientific Laboratory

Which of the two integrals

$$\int_{0}^{1} x^{x} dx \qquad \qquad \int_{0}^{1} \int_{0}^{1} (xy)^{xy} dx dy$$

is larger?

Solution by R. A. Groeneveld, Mount Holyoke College. Making the substitution u = xy, the second integral may be written

$$\int_0^1 \int_0^1 \frac{u_u}{x} \, dx \, du = -\int_0^1 u^u (\log u) \, du$$

Since

$$\int_0^1 u^u (1 + \log u) \, du = u^u |_0^1 = 0$$

the two stated integrals are equal.

J. Gillis proves the following generalization: Define

$$I_r = \int_0^1 \int_0^1 \cdots \int_0^1 (x_1 x_2 \cdots x_r)^{x_1 x_2 \cdots x_r} dx_1 dx_2 \cdots dx_r$$

 $r = 1, 2, \ldots$. Then $I_1 = I_2 < I_3 < I_4 < \cdots$, and $\lim_{r \to \infty} I_r = 1$. [M. M.]Klein reports the computer value of I_1 is 0.78343051.

Amer. Math. Monthly, **78**(1971) 676.

II. Comment by C. D. Olds, San Jose Sate College. For readers who wonder how the computer value reported by Klein might be obtained, the following manipulations (easily justified) may be of interest.

$$I = \int_{0}^{1} x^{x} dx = \int_{0}^{1} e^{x \ln x} dx = \int_{0}^{1} \sum_{n=0}^{\infty} \frac{(x \ln x)^{n}}{n!} dx$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{1} (x \ln x)^{n} dx = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)^{n+1}n!} \int_{0}^{\infty} e^{-t} t^{n} dt$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)^{n+1}n!} \Gamma(n+1) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)^{n+1}} = 0.78343051 \dots$$

The series is particularly attractive because of its rapid convergence

Amer. Math. Monthly, **77**(1970) 308.

E 2226. Proposed by M. S. Klamkin, Ford Scientific Laboratory

If one altitude of a tetrahedron intersects two other altitudes, then all four altitudes are concurrent.

Amer. Math. Monthly, 78(1971) 201.

II. Solution by Simeon Reich, Israel Institute of Technology. Let ABCD be the given tetrahedron, and let h_A intersect h_B and h_C . Then AB is perpenducular to CD and AC is perpendicular to BD (Nathan Altshiller-Court, Modern Pue Solid Geometry, 2nd Ed., §204). This can be expressed by

 $AB \cdot (AD - BC) = 0$ $AC \cdot (AD - AB) = 0$

Hence $AD \cdot (AB - AC) = 0$. That is, AD is perpendicular to BC. It follows that the altitudes are concurrent (*loc. cit.* §208, §212).

Amer. Math. Monthly, **77**(1970) 403.

E 2231^{*}. Proposed by M. S. Klamkin, Ford Scientific Laboratory

It is a known result that if the centroid of the vertices and the centroid of the area (both uniformly weighted) of a quadrilateral coincide, then the figure is a parallelogram. If the centroids of the vertices, of the edges, and of the area (all uniformly weighted) of a pentagon all coincide, must the figure be a regular pentagon?

Amer. Math. Monthly, **78**(1971) 302.

Solution by W. G. Wild, Wisconsin State University. The answer is no.

Consider the pentagon with vertices at $(\pm 7k/20, 0)$, $(\pm k/2, 4/7)$ and (0,1). The centroids of the area and of the vertices coincide at (0,3/7). The centroid of the edges is located at the solution of the equation

$$\frac{3}{7} = \frac{\frac{11}{7}\sqrt{\left(\frac{k}{2}\right)^2 + \left(\frac{3}{7}\right)^2} + \frac{4}{7}\sqrt{\left(\frac{3k}{20}\right)^2 + \left(\frac{4}{7}\right)^2}}{\frac{7}{10}k + 2\sqrt{\left(\frac{k}{2}\right)^2 + \left(\frac{3}{7}\right)^2} + 2\sqrt{\left(\frac{3k}{20}\right)^2 + \left(\frac{4}{7}\right)^2}}$$

(The first moment of the edge masses about the x-axis divided by the total edge mass.) These trun out to be $k = \pm 1.04228$ and ± 2.59575 .

A more general solution is provided by studying the pentagon with vertices at $(\pm ak/2, 0)$, $(\pm k/2, b)$, (0,1). The centroid of the vertices is at (0, (2b + 1)/5) and if a is equal to $(2 - b)/(3b + b^2)$, then the area centroid coincides with that of the vertices, designated by $(0, \bar{y})$, and the relation (analogous to the one in the special case above) which expresses \bar{y} as a function of k assures us that the centroid of the edge can be made to coincide if the equation has solutions. The related existence study is routine but tedious.

[Don]Coppersmith cites the pentagon with vertices $(0, 2\sqrt{19})$, $(\pm 9, \sqrt{19})$ and $(\pm 4, -2\sqrt{19})$, along with a more general form. [Harry]Lass gives $(\pm a, 0)$, $(\pm a, 1)$ and $(0, 1 + \frac{1}{2}\sqrt{6})$, where a is the positive root of a certain quadratic equation.

Amer. Math. Monthly, 77(1970) 522.

E 2197. Proposed by M. S. Klamkin, Ford Scientific Laboratory and D. J. Newman, Yeshiva University

Show that if the integral of the reciprocal of a nonconstant polynomial is a rational function, then the polynomial must be of the form $(ax + b)^n$.

Amer. Math. Monthly, **78**(1971) 408.

Solution by G. A. Heuer and C. V. Heuer, Concordia College. If the rational function, in its lowest terms, is f(x)/g(x), f and g polynomials, then

$$[g(x)f'(x) - f(x)g'(x)]/[g(x)]^2 = 1/p(x)$$

where p is a polynomial. If x - r is a (possibly complex) factor of the numerator on the left, it divides $[g(x)]^2$, so divides g(x), so divides f(x)g'(x), and therefore g'(x); thus $(x - r)^2 | g(x)$. By induction one finds that if $(x - r)^m$ divides the numerator then $(x - r)^m | g'(x)$ and $(x - r)^{m+1} | g(x)$. Thus the two terms in the numerator separately divide $[g(x)]^2$. Since f(x) and g(x) are relatively prime, f(x) is a constant. Thus $g'(x) | [g(x)]^2$ and every linear factor of g'(x) divides g(x). It follows that if $(x - r)^m | g'(x)$ then $(x - r)^{m+1} | g(x)$. Since the degree of g(x) is only one more than that of g'(x), g(x) cannot have two different linear factors. The desired result follows.

Amer. Math. Monthly, **78**(1971) 905.

Comment and solution by L. R. Abramson, Riverside Research Institute, New York. The published solution I is in error: if f, g and p are polynomials such that f/g is in its lowest terms and (f/g)' = 1/p, then f need not be constant, for it is not necessarily true that each of fg', gf' divides g^2 . For example, let f(x) = x - 1, g(x) = x and $p(x) = 1/x^2$.

The solution may be corrected as follows. Evidently deg $f \leq \deg g$. If deg $f = \deg g$, then we can write $f/g = c + f_1/g$ where deg $f_1 < \deg g$. Since f_1/g is another antiderivative for p, there is no loss of generality in assuming that deg $f < \deg g$. Let the leading terms of f and g be respectively ax^s and bx^t . Then the leading term of gf' - fg' is $ab(s-t)x^{s+t-1}$, since $s - t \neq 0$. Inspection rules out the cases s = 0, t = 1 and s = 1, t = 0; hence $s + t \geq 2$ and so $s + t_1 \geq 1$. As in the published solution every m-fold root of gf' - fg' is an (m+1)-fold root of g. Thus $t = \deg g \geq (s+t-1) + d$, where d is the number of distinct roots of gf' - fg'. But $d \geq 1$, whence s = 0 and d = 1. In other words, f is constant and g' has exactly one distinct linear factor; i.e., $g(x) = (ax + b)^n$ for some $n \geq 2$.

Amer. Math. Monthly, **77**(1970) 532.

5736. Proposed by M. S. Klamkin, Ford Scientific Laboratory

Solve the nonlinear difference equation of r th order

$$D_n = a_1 D_{n-1}^{m+1} + a_2 D_{n-1}^m D_{n-2}^{m+1} + \dots + a_r D_{n-1}^m D_{n-2}^m \cdots D_{n-r+1}^m D_{n-r}^{m+1}$$

(m, r, a - i constants).

[[I can't make sense of the superscripts, but that doesn't necessarily mean anything. Later: there's a correction on p.774, taking out + signs from round the second \cdots !]]

Amer. Math. Monthly, **78**(1971) 554.

Solution by the proposer. By considering the case r = 2, one is led, after some trial and error, to rewrite the given equation in the form

$$1 = a_1\phi_n + a_2\phi_n\phi_{n-1} + \dots + a_r\phi_n\phi_{n-1} \cdots \phi_{n-r+1}$$

in which we have replaced D_{n-1}^{m+1}/D_n by ϕ_n . By letting $\phi_n = \psi_n/\psi_{n+1}$ we obtain the linear difference equation

$$\psi_{n+1} = a_1 \psi_n + a_2 \psi_{n-1} + \dots + a_r \psi_{n-r+1}$$

which has the general solution

$$\psi_n = \sum_{i=1}^r k_i R_i^n$$

where R_i are the roots of $x^r = a_1 x^{r-1} + a_2 x^{r-2} + \dots + a_r$

Retracing our substitutions, we get in turn

$$\phi_n = \psi_n / \psi_{n+1} \qquad D_n \phi_n = D_{n-1}^{m+1}$$

or equivalently, $\log D_n = (m+1) \log D_{n-1} - \log \phi_n$. Let $\log D_n = (m+1)^n A_n$; then $A_n - A_{n-1} = -\log \phi_n^{(m+1)^{-n}}$. Thus

$$A_n = -\log\left\{e^{-A_0}\prod_{j=1}^n \phi_j^{(m+1)^{-j}}\right\}$$

and finally

$$D_n = e^{A_0(m+1)^n} \prod_{j=1}^n \phi_j^{-(m+1)^{n-j}}$$

The equation arose as a generalization in a study of the frequency spectrum of a mass-spring system which forms a rooted Cayley tree.

Amer. Math. Monthly, 77(1970) 890.

5754. Proposed by M. S. Klamkin, Ford Scientific Laboratory

Let L(a, c) equal the perimeter of an ellipse with semi-axes a and c $(a \ge c)$. Show that if $a \ge b$, then

$$L^{2}(a,c) - 16a^{2} \ge L^{2}(b,c) - 16b^{2}$$

Amer. Math. Monthly, **78**(1971) 202.

[[There was a correction, the above \geq replacing an earlier = .]]

Amer. Math. Monthly, 78(1971) 919–920.

Solution by K. F. Andersen, Royal Roads Military College, Victoria, B.C. Parameterize the ellipse in the usual trigonometric manner and then let

$$n = n(t) = (a^2 \sin^2 t + c^2 \cos^2 t)^{1/2} \qquad v = v(t) = (b^2 \sin^2 t + c^2 \cos^2 t)^{1/2}$$

Then $u \ge v$ since $a \ge b$, and by Schwartz's inequality we obtain:

$$\begin{split} L^{2}(a,c) - L^{2}(b,c) &= (L(a,c) - L(b,c))(L(a,c) + L(b,c)) \\ &= \left(\int_{0}^{2\pi} (u-v) \, dt\right) \left(\int_{0}^{2\pi} (u+v) \, dt\right) \\ &= \left(\int_{0}^{2\pi} \frac{(u^{2} - v^{2})}{(u+v)} \, dt\right) \left(\int_{0}^{2\pi} \frac{(u^{2} - v^{2})}{(u-v)} \, dt\right) \\ &\geq \left(\int_{0}^{2\pi} \frac{(u^{2} - v^{2})^{1/2}}{(u+v)^{1/2}} \cdot (u^{2} - v^{2})^{1/2} (u-v)^{1/2} \, dt\right)^{2} \\ &= \left(\int_{0}^{2\pi} (u^{2} - v^{2})^{1/2} \, dt\right)^{2} \\ &= (a^{2} - b^{2}) \left(2 \int_{0}^{\pi} \sin t \, dt\right)^{2} \\ &= 16(a^{2} - b^{2}). \end{split}$$

Since equality holds in Schwartz's inequality if and only if there are constants m, n, not both zero, such that m(u - v) = n(u + v) almost everywhere, we have equality above if and only if a = b.

Editorial Note. The error in the first printing of the problem was noted by several solvers who derived interesting consequences from the false proposal. In particular, [E. D.]Bolker shows that the original statement implies $\pi^2 = 8$.

[In connexion with the next item, it's first of all interesting that Oppy's address was, in 1965 and 1967, University of Malaya, Kuala Lumpur.

Here is what was published at 1967, 441:

Solution by C. S. Venkataraman, Trichur, India. Let A' be the midpoint of BC; D the foot of the perpendicular from A upon BC; O the overline; and R the circumradius of triangle ABC. We use the following three well-known results:

(i)
$$\prod \cos A \le \frac{1}{8}$$

- (ii) $\sum \cos^2 A = 1 2 \prod \cos A$
- (iii) $2R\cos B\cos C = OD$

The angles A, B, C being acute implies that O lies inside the triangle ABC. Now from the triangle ODA', right-angled at D, we have OD|leqOA'. It follows that the maximum value of OD is OA'. But when OD = OA', the triangle OBC is isosceles with OB = OC, whence AB = AC. Hence from (iii) $2R \cos B \cos C$ is maximum when AB = AC. Similarly $2R \cos C \cos A$ is a maximum when BC = AB. Then AC = BCand $2R \cos A \cos B$ is also maximum, and triangle ABC is equilateral. Thus it follows that

$$\sum (2R\cos B\cos C)^2 \le \sum (2R \cdot \frac{1}{2} \cdot \frac{1}{2})^2 = 3R^2/4$$

Therefore $4\sum \cos^2 B \cos^2 C \leq \frac{3}{4}$. Using (i) this gives

$$16\prod \cos^2 A + 4\sum \cos^2 B \cos^2 C \le 16(1/64) + 3/4 = 1$$

which is the required result (1).

Further (i) and (ii) imply

$$\sum \cos^2 A \ge 1 - 2(1/8) = 3/4 \ge 4 \sum \cos^2 B \cos^2 C$$

which is result (2). Clearly, equality arises in both cases if and only if ABC is equilateral.]]

Amer. Math. Monthly, **79**(1972) 394–395.

Two Triangle Inequalities

E 1838 [1965, 1129; 1967, 440]. Proposed by A. Oppenheim, University of Ghana Suppose that ABC is an acute-angle triangle; then

$$16 \prod \cos^2 A + 4 \sum \cos^2 B \cos^2 C \le 1$$
(1)
$$4 \sum \cos^2 B \cos^2 C \le \sum \cos^2 A$$
(2)

Equality occurs when ABC is equilateral or right-angles isosceles and in no other case.

II. Comment and solution by Murray Klamkin, Ford Scientific Laboratory. By virtue of the weak inequality conditions, ABC can be restricted to non-obtuse triangles rather than acute triangles.

In a personal communication, A. W. Walker has pointed out that there is a flaw in the published solution [1967, 441]. He notes that the solution "derives" and uses the inequality $16 \sum \cos^2 B \cos^2 C \leq 3$; however, this is invalid—just consider an isosceles right triangle. (By continuity, there exist acute non-isosceles triangles which violate the inequality.)

We prove (2) of the problem and show how (1) follows from it. By using $2\cos^2 A = 1 + \cos 2A$ and then making the transformations $A' = \pi - 2A$, etc., we see that (2) becomes equivalent (after dropping primes) to the following:

$$3\sum \cos A \ge 3 + 2\sum \cos B \cos C \tag{3}$$

where now ABC is an arbitrary triangle. Inequality 6.12 of O. Bottema et al., *Geometric Inequalities*, Nordhoff, Groningen, 1969, states $2R + 5r \ge h_a + h_b + h_c$. Since $h_a = AH + HD = 2R \cos A + 2R \cos B \cos C$, etc., it follows that

$$2R + 5r \ge 2R \sum \cos A + 2R \sum \cos B \cos C$$

and hence $5(1 + r/R) \ge 3 + 2\sum \cos A + 2\sum \cos B \cos C$ which reduces to (3) since $1 + r/R = \sum \cos A$.

Now, using (2) we establish a stronger inequality than (1), viz.

$$16\prod\cos^2 A + \sum\cos^2 A \le 1 \tag{4}$$

Since $1 - \sum \cos^2 A = 2 \prod \cos A$, (4) is equivalent to

$$(\prod \cos A)(1 - 8 \prod \cos A) \ge 0 \tag{5}$$

But $\prod \cos A \ge 0$ since the triangle is non-obtuse and $8 \prod \cos A \le 1$ by 2.24 of Bottema et al. Thus (5) is established. We note that there is equality in (5) if and only if the triangle is equilateral or a right triangle. This implies that there is equality in (1) if and only if the triangle is equilateral or right isosceles.

Amer. Math. Monthly, 80(1973) 75.

E 2393. Proposed by M. S. Klamkin, Ford Motor Company

Parallel lines are drawn through the vertices A_0, A_1, \ldots, A_n of a given simplex of volume V, terminating in the opposite faces (extended if necessary) in the points B_0 , B_1, \ldots, B_n , respectively.

(1) Show that the volume of the simplex determined by B_0, B_1, \ldots, B_n is nV

(2) Show that the volume of the simplex determined by the vertices A_0, A_1, \ldots, A_r , $B_{r+1}, B_{r+2}, \ldots, B_n$ is given by $V'_r = |n - r - 1|V$.

Amer. Math. Monthly, 80(1973) 1145–1146.

Solution by Leon Gerber, St. John's University. Parallel lines are drawn through the vertices A_0, A_1, \ldots, A_n , etc. Let the weights of the point P with respect to the given simplex be (p_i) where $\sum_{i=0}^{n} p_i = s$, with s = 1 if P is a proper point and s = 0 if P is improper. The the cevians A_iP (which are parallel if P is improper) meet the face opposite A_i in $B_i = (b_{ij})$ where $b_{ii} = 0$ and $b_{ij} = p_j/(s - p_i)$. Thus the ratio of the content of $A_0A_1 \ldots A_{r-1}B_r \ldots B_n$ to that of the given simplex is

$$\det \begin{vmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ b_{r,0} & \cdots & b_{r,r-1} & b_{r,r} & \cdots & b_{n,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n,0} & \cdots & b_{n,r-1} & b_{n,r} & \cdots & b_{n,n} \end{vmatrix}$$
$$= \det \begin{vmatrix} 0 & \cdots & p_n/(s-p_r) \\ \cdots & \cdots & \cdots \\ p_r/(s-p_n) & \cdots & 0 \end{vmatrix}$$
$$= (r-n) \prod_{i=r}^n p_i/(p_i-s)$$

Amer. Math. Monthly, 80(1973) 75.

E 2394. Proposed by S. L. Greitzer, Rutgers University, and M. S. Klamkin, Ford Motor Company

A line is drawn through the centroid G of a simplex A_0, A_1, \ldots, A_n intersecting the faces (extended if necessary) in points B_0, B_1, \ldots, B_n , respectively. Show that

$$\sum_{i=1}^{n} \frac{1}{GB_i} = 0$$

where GB_i denotes the directed distance from G to B_i . Show also that the above property characterizes the point G as the centroid; i.e., if the above sum vanishes for all arbitrary lines, then G is the centroid.

This generalizes known results for triangles and tetrahedrons.

Amer. Math. Monthly, 81(1974) 283–285

Solution by Mildred L. Stancl, Nichols College, Dudley, Massachusetts. Let G be an arbitrary point which does not lie on a face of a simplex $A_0A_1 \ldots A_n$. (The word face throughout means extended face, i.e., the (n-1)-dimensional affine subspace spanned by n of the points A_0, A_1, \ldots, A_n .) Let L be an arbitrary line through G and let B_0, B_1, \ldots, B_n be defined as follows: If L intersects the face opposite A_i , let B_i be the point of intersection and call B_i finite. If L does not intersect the face opposite A_i , let B_i is infinite.

Since the fact that $\sum 1/GB_i = 0$ is immediate if all B_i are infinite, assume that L intersects at least one face in the point B, where B is one of B_0, B_1, \ldots, B_n . (Unless otherwise noted, all summations run from 0 to n.) Then B is finite and for each B_j which is finite the following statements hold:

- (i) The *j* th barycentric coordinate of B_j is zero.
- (ii) $B_j = (1 t_j)G + t_jB$ where t_j is a unique nonzero real number.
- (iii) $GB_i = t_i GB$ where GB is nonzero.

Now let G be the barycentre (centroid) of the simplex, so that $G = (n + 1)^{-1} \sum A_k$ and let b_0, b_1, \ldots, b_n be the barycentric coordinates of B so that $B = \sum b_k A_k$, If B_r is an infinite point, then the r th barycentric coordinate of B is $(n + 1)^{-1}$. This remark is verified by noting the existence of points P and Q lying in the face opposite A_r such that B is a point of the line segment with endpoints A_r and P, and G is a point of the line segment with endpoints A_r and Q. The fact that L does not intersect the face opposite A_r means that the line segment with endpoints P and Q lying in that face is parallel to the line segment with endpoints B and G lying in L. Thus if $s = (n + 1)^{-1}$, then

$$B = (1 - s)P + sA_r, \qquad G = (1 - s)Q + sA_r$$

The equality $b_r = (n + 1)^{-1}$ follows since P and Q have r th barycentric coordinate zero. Since $\sum b_k = 1$, the following statement holds:

(iv) If G is the barycentre, if exactly N $(1 \le N \le n + 1)$ of the points B_0, B_1, \ldots, B_n are finite, and if B with barycentric coordinates b_0, b_1, \ldots, b_n is one of the finite points, then $\sum_{i}^{\prime} b_i = N/(n + 1)$.

(The notation \sum_{j}^{\prime} throughout this solution means the summation over those j for which B_j is finite.)

If B_j is a finite point, statement (ii) implies that

$$B_j = \sum \left[\frac{1}{n+1}(1-t_j) + t_j b_k\right] A_k$$

and (i) implies that

$$t_j = \frac{1}{1 - (n+1)b_j}$$

Since $\sum 1/GB_i = \sum_j' 1/GB_j$, statement (iii) implies that

$$\sum \frac{1}{GB_i} = \frac{1}{GB} \sum_{j=1}^{\prime} [1 - (n+1)b_j]$$

It now follows from (iv) that $\sum 1/GB_i = 0$.

Conversely, if G is any point such that $\sum 1/GB_i = 0$ for all lines through G, then G does not lie on a face of $A_0A_1 \dots A_n$ for otherwise the denominator of at least one of the summands would vanish. Let A_k be any one of A_0, A_i, \dots, A_n and consider the line through G and A_k . The point A_k lies in all but one of the faces; hence, n of B_0 , B_1, \dots, B_n are equal to A_k and are finite. The remaining one, B_k , is also finite, for otherwise the sum of the reciprocals of the directed distances would be n/GA_k which is nonzero. Let g_0, g_1, \dots, g_n be the barycentric coordinates of G and let a_0, a_1, \dots, a_n be the barycentric coordinates of A_k . The statements (ii) and (iii) imply

$$t_i = \frac{g_i}{g_i - a_i}$$

Statement (iii) implies that

$$0 = \sum \frac{1}{GB_i} = \frac{1}{GA_k} \left(n + \frac{1}{t_k} \right) = \frac{1}{GA_k} \left(n + 1 - \frac{a_k}{g_k} \right)$$

Since $a_k = 1$ it follows that $g_k = (n + 1)^{-1}$ and G is the barycentre. Also solved by $[\dots]$ the proposers.

Amer. Math. Monthly, 80(1973) 323

Non-Negative Forms

E 2348. Proposed by Leonard Carlitz, Duke University

Let P be a point in the interior of the triangle ABC. Let R_1 , R_2 , R_3 denote the distances from P to the vertices of ABC and let r_1 , r_2 , r_3 denote the perpendicular distances from P to the sides of ABC. Show that

$$\sum R_1(r_1 + r_3) \ge \sum (r_1 + r_2)(r_1 + r_3)$$
(1)
$$\sum (R_1 + R_2)(R_1 + R_3) \ge 4 \sum (r_1 + r_2)(r_1 + r_3)$$
(2)

with equality if and only if ABC is equilateral and P is its centre.

Solution by M. S. Klamkin, Ford Scientific Laboratory. To satisfy (1) we prove a stronger inequality. For the triangle ABC let a, b, c be the lengths of the sides BC, CA, AB respectively. From [1, p.107] we have

$$R_1 \ge \frac{r_2 c + r_3 b}{a}$$
 $R_2 \ge \frac{r_1 c + r_3 a}{b}$ $R_3 \ge \frac{r_1 b + r_2 a}{c}$ (3)

with equality if and only if ABC is equilateral and P is its centre. We now prove that

$$\sum a^{-1}(r_2c + r_3b)(r_2 + r_3) \ge \sum (r_1 + r_2)(r_1 + r_3)$$
(4)

This inequality implies (1). This inequality is actually valid for all real r_1 , r_2 , r_3 since it will be shown to be a non-negative quadratic form with equality if and only if a = b = c. The matrix associated with (4) is given by

$$M = \begin{bmatrix} \frac{b^2 + c^2 - bc}{bc} & \frac{a + b - 3c}{2c} & \frac{a + c - 3b}{2b} \\ \frac{b + a - 3c}{2c} & \frac{c^2 + a^2 - ca}{ca} & \frac{b + c - 3a}{2a} \\ \frac{c + a - 3b}{2b} & \frac{c + b - 3a}{2a} & \frac{a^2 + b^2 - ab}{ab} \end{bmatrix}$$

As is well known, (4) is a non-negative form if three principal minors M_1 , M_2 , M_3 of M are non-negative. After some algebraic manipulation we find that

$$bcM_1 = (b-c)^2 + bc > 0$$

$$4abc^{2}M_{2} = 4c^{2}(\sum a^{2} - \sum ab) + ab(2\sum ab - \sum a^{2}) > 0 \quad \text{and} \\ (x+y)^{2}(y+z)^{2}(z+x)^{2}M_{3} = (\sum xy)(\sum x^{2} - \sum xy)(\sum x^{2} + 3\sum xy) \ge 0$$

with equality if and only if x = y = z, or equivalently a = b = c. Here we simplify the calculation of M_3 by using the duality transformation [2]

$$a = y + z$$
 $b = z + x$ $c = x + y$

where x. y, z are arbutrary non-negative numbers, not all zero.

Inequality (2) follows from adding the following two inequalities found in [1, p.110]:

$$3\sum R_2R_3 \ge 12\sum r_2r_3$$
 $\sum R_1^2 \ge 4\sum r_1^2$

These inequalities are inequalities if and only if ABC is equilateral and P is its centre.

1. O. Bottema et al., Geometric Inequalities, Noordhoff, Groningen, 1969.

2. M. S. Klamkin, *Duality in triangular inequalities*, Ford Motor Company preprint, July 1971.

Amer. Math. Monthly, 80(1973) 807.

E 2428. Proposed by M. S. Klamkin, Ford Motor Company

If a - i (1 = 1, 2, ..., n) denote real numbers, show that

$$n\min(a_i) \le \sum a_i - S \le \sum a_i + S \le n\max(a_i)$$

where

$$(n-1)S^{2} = \sum_{1 \le i < j \le n} (a_{i} - a_{j})^{2} \qquad (S \ge 0)$$

and with equality if and only if $a_i = \text{constant}$.

Amer. Math. Monthly, 81(1974) 782–783.

I. Solution by Ellen Hertz, Bronx Community College; Carolyn MacDonald, University of Missouri; Wolfe Snow, Brooklyn College; and Melvin Tews, University of California, Berkeley (independently). We can assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then

$$S^{2} = \frac{1}{n-1} \sum_{i=2}^{n} \sum_{j=1}^{i-1} (a_{i} - a_{j})^{2} \leq \frac{1}{n-1} \sum_{i=2}^{n} (i-1)(a_{i} - a_{1})^{2}$$
$$\leq \sum_{i=2}^{n} (a_{i} - a_{1})^{2} \leq \left\{ \sum_{i=1}^{n} (a_{i} - a_{1})^{2} < \left\{ \sum_{i=1}^{n} (a_{i} - a_{1})^{2} < \left\{ \sum_{i=1}^{n} (a_{i} - a_{1})^{2} < \left\{ \sum_{i$$

Taking square roots we obtain

$$na_1 \le \sum_{i=1}^n a_i - S$$

Similarly,

$$S^{2} \leq \frac{1}{n-1} \sum_{j=1}^{n-1} (n-j)(a_{n}-a_{j})^{2} \leq \left\{ \sum_{j=1}^{n} (a_{n}-a_{j}) \right\}^{2}$$

from which it follows that

$$\sum_{j=1}^{n} a_j + S \le na_n$$

It is clear the equality holds anywhere if and only if it holds throughout and this is true if and only if $a_i = \text{constant}$.

[[the misprint – see below – in the last displayed inequality has been corrected]]

II. Comment by O. P. Lossers, Technological University, Eindhoven, the Netherlands. A statistical interpretation is possible. Let a_1, a_2, \ldots, a_n $(n \ge 2)$ be a random sample f a random variable A with mean μ and variance σ^2 . As estimates for μ and σ^2 one usually takes

$$\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i$$
 $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (a_i - \bar{a})^2$

The inequalities in the problem then take the form

$$\min a_i \le \bar{a} - \frac{s}{\sqrt{n}} \le \bar{a} + \frac{s}{\sqrt{n}} \le \max a_i \tag{*}$$

Note that if A_n is the random variable defined by averaging samples of size n from A, then the mean of A_n is also μ , but its variance is σ^2/n , so that quantities in (*) are related to the parameters of A_n .

The proposer notes that the case n = 3 is due to D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970, p.215.

Amer. Math. Monthly, 82(1975) 401.

III. Comment by C. L. Mallows, Bell Laboratories, Murray Hill, New Jersey. The left-hand inequality, $na_1 \leq \sum a_i - S$, will be an equality if and only if

 $a_1 = a_2 = \cdots = a_{n-1}$ (assuming as in the published solution that $a_1 \leq a_2 \leq \cdots \leq a_n$), contrary to the statement on lines 10–11 of p.783. This follows by taking r = 1 in Corollary 6.1 of my paper (jointly with Donald Richter), Inequalities of Chebyshev type involving conditional expectations, Ann. Math. Statist., **40**(1969) 1922–1932. Similarly the dual inequality $\sum a_j + S \leq na_n$ (note the misprint on line 9 of p.783) is an equality if and only if $a_2 = a_3 = \cdots = a_n$. This follows, too, from my paper by using the inequality dual to that in Corollary 6.1 (i.e., using u_r instead of v_r). Thus equality holds in both if and only if n = 2 or all a_i are equal. Certainly $\sum a_i - S = \sum a_1 + S$ if and only if S = 0, i.e., if and only if all a_i are equal. Amer. Math. Monthly, 81(1974) 1143–1145.

Six equal regions? Yes, Seven? No.

E 2391. Proposed by V. R. R. Uppuluri, Oak Ridge National Laboratory

It is well known that three chord can divide a circular disk into at most seven pieces. Can these seven pieces all have the same area?

IV. Comment by M. S. Klamkin, Ford Motor Company (similar comment by R. C.Buck, University of Wisconsin at Madison). The answer is negative even if the circular region is replaced by a convex region; see R. C. Buck and E. F. Buck, Equipartition of convex sets, Math. Mag., 22(1949) 195–198, where it is shown that at most six of the regions can have the same area, and that these equal regions must be the six outer ones.

Amer. Math. Monthly, 81(1974) 291–292

Alpha-max, Beta-min, and a limit for e

E 2406 [1973, 316]. Proposed by Erwin Just and Norman Schaumberger, Bronx Community College

What is the maximum value of α and the minimum value of β for which

$$\left(1+\frac{1}{n}\right)^{n+\alpha} \le e \le \left(1+\frac{1}{n}\right)^{n+\beta}$$

for all positive integers n?

Solution by M. S. Klamkin, Ford Motor Company. On taking logarithms we obtain

$$\alpha_{\max} = \inf_{n} \left\{ \frac{1}{\log(1+1/n)} - n \right\} \qquad \beta_{\min} = \sup_{n} \left\{ \frac{1}{\log(1+1/n)} - n \right\}$$

We now show that the function

$$F(x) = \frac{1}{\log(1 + 1/x)} - x$$

is monotonically decreasing for x > 0 by showing that its derivative is positive:

$$F'(x) = \frac{1}{x(x+1)[\log(1+1/x)]^2} - 1 = \frac{\sinh^2 u}{u^2} - 1 > 0$$

where $e^{2u=1+1/x}$. Thus, $\alpha_{\max} = 1/\log 2 - 1 = 0.4426950$ and $\beta_{\min} = \lim_{n \to \infty} F(n)$. By expanding $\log(1+x)$ in a Maclaurin series, we have

$$F(n) = \left[\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n_3}\right)\right]^{-1} - n$$

from which it follows that $\beta_{\min} = \lim_{n \to \infty} F(n) = \frac{1}{2}$.

Amer. Math. Monthly, 81(1974) 525.

5974. Proposed by M. S. Klamkin, Ford Motor Company

Prove that aside from a polynomial of integration of degree 2n - 1,

$$x^{2n-1} \int \frac{dx}{x^2} \int \frac{dx}{x^2} \cdots \int \frac{dx}{x^2} \int x^{2n-1} F(x) \, dx = \int \int \cdots \int F(x) \, (dx)^{2n}$$

where there are 2n integrals on each side.

[[I don't understand this sentence!]]

Amer. Math. Monthly, 83(1976) 143–144.

Solution by A. B. Farnell, Colorado State University. Consider

$$y_n = x^{n-1} \int \frac{dx}{x^2} \int \frac{dx}{x^2} \int \cdots \int \frac{dx}{x^2} \int G(x) dx$$

where there are n integrals involved. We propose to show by induction that

$$y_n^{(n)} = x^{1-n} G(x)$$

This expression is readily verified for n = 1, 2, 3. Thus we assume it valid for $y_{n-1}^{(n-1)}$. Then

$$y'_{n} = (n-1)x^{n-2} \int \cdots \int G(x) \, dx + x^{n-3} \int \cdots \int G(x) \, dx$$
$$xy'_{n} = (n-1)y_{n} + y_{n-1}$$

Differentiating (n-1) times, we obtain

$$xy_n^{(n)} + (n-1)y_n^{(n-1)} = (n-1)y_n^{(n-1)} + x^{2-n}G(x)$$

or

$$y_n^{(n)} = x^{1-n} G(x)$$

This shows that

$$x^{n-1} \int \frac{dx}{x^2} \cdots \int x^{n-1} F(x) \, dx = \int \cdots \int F(x) \, (dx)^n$$

modulo a polynomial of degree (n-1) for all n.
ALGEBRA

Inequalities: exponentials

Amer. Math. Monthly, 81(1974) 660.

E 2483. Proposed by M. S. Klamkin, Ford Motor Company

Let x be nonnegative and let m, n be integers with $m \ge n \ge 1$. Prove that

$$(m+n)(1+x^m) \ge 2n\frac{1-x^{m+n}}{1-x^n}$$

Amer. Math. Monthly, 82(1975) 758–760.

IV. Solution by the proposer. As above, we can assume that 0 < x < 1 and m > n, and we shall show that the inequality is strict. Also, we shall not restrict m and n to be integers, but shall allow them to be any positive real numbers m > n > 0. Now let m + n = r and m - n = s so that r > s > 0 and let $t = x^{1/2}$ so that 0 < t < 1. Rearranging the desired inequality we get

$$\frac{1-t^{2r}}{rt^r} > \frac{1-t^{2s}}{st^s}$$

On letting $t = e^{-y}$ we get

$$\frac{\sinh ry}{ry} > \frac{\sinh sy}{sy}$$

where now $0 < y < \infty$. But $x^{-1} \sinh x$ has a power series expansion with only positive coefficients, so that it is a strictly increasing function on $(0, \infty)$.

Editor's comment. Assume that m > n are positive real numbers and that $x \neq 1$ is nonnegative. We have seen that

$$(m+n)(1+x^m) > \frac{2n(1-x^{m+n})}{1-x^n}$$
 (A)

It is not hard to see that the following inequality is actually equivalent to (A); let us call it the **dual** of (A):

$$\frac{2m(1-x^{m+n})}{1-x^m} > (m+n)(1+x^n)$$
(B)

(Both (A) and (B) can be rearranged to assert that

$$(m-n) - (m+n)x^n + (m+n)x^m - (m-n)x^{m+n}$$

is positive if $0 \le x < 1$ and negative if x > 1.)

Lepson shows that

$$\frac{2m(1-x^{m+n})}{1-x^n} > (m+n)(1+x^m)$$
(C)

In the same way, the dual of this inequality is

$$(m+n)(1+x^n) > \frac{2n(1-x^{m+n})}{1-x^m}$$
 (D)

(Both (C) and (D) are equivalent to the statement that

$$(m-n) + (m+n)x^n - (m+n)x^m - (m-n)x^{m+n}$$

is positive if $0 \le x < 1$ and negative if x > 1.) These inequalities can be displayed conveniently by the following diagram, where the arrows run from the larger quantities to the smaller:

$$\begin{array}{c} \frac{2m(1-x^{m+n})}{1-x^n} \longrightarrow (m+n)(1+x^m) \longrightarrow \frac{2n(1-x^{m+n})}{1-x^n} \\ & \downarrow & 0 \le x < 1 \\ & \downarrow & x > 1 \\ \frac{2m(1-x^{m+n})}{1-x^m} \longrightarrow (m+n)(1+x^n) \longrightarrow \frac{2n(1-x^{m+n})}{1-x^m} \end{array}$$

(The inequality in the centre of the diagram can run either way, according as x > 1 or $0 \le x < 1$.) Note that in the case $0 \le x < 1$, there is actually a chain of five inequalities, and that in the limit as $x \to 1^-$, the middle three equalities of the chain approach equality.

The proposer comments that the special case of his inequality corresponding to n = 1 and m = 2p+1 appears as Problem 4.8 in D. S. Mitrinović, *Elementary Inequalities*, Noordhoff, Groningen, 1964, p.95, and as Equation 3.2.4 in D. S. Mitrinović and P. M. Vasić, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970 p.198. He observes that similar results by V. I. Levin (3.2.12), by Beck (3.2.11), and by J. M. Wilson (3.2.27), which appear in the latter reference can be extended by analogous methods.

Amer. Math. Monthly, **81**(1974) 659.

E 2480. Proposed by M. S. Klamkin, Ford Motor Company

If $a_i \ge 0$, $\sum a_i = 1$ and $0 \le x_i \le 1$ for i = 1, 2, ..., n, prove that

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} + \dots + \frac{a_n}{1+x_n} \le \frac{1}{1+x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}}$$

When does equality hold?

Amer. Math. Monthly, 82(1975) 670–671.

Solution by Miriam Beesing, Junior, Hamline University. Assume without loss of generality that $a_i > 0$ for all *i*. The proposed inequality follows from a straightforward application of Jensen's inequality for concave functions: If f is strictly concave (downwards) on an interval I, then for all $y_i \in I$,

$$\sum a_i f(y_i) \le f\left(\sum a_i y_i\right)$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$. (See Roberts and Varberg, *Convex Functions*, Academic Press, 1973, pp.189, 192.) To apply this to the proposed inequality, let $y_i = \log x_i$ and $f(y) = (1+e^y)^{-1}$, assuming for the moment that $x_i > 0$ for all *i*. Since $0 < x_i \le 1$, it follows that $-\infty < y_i \le 0$ and since $f''(y) = e^y(e^y-1)(1+e^y)^{-3} < 0$ if y < 0, we see that f is strictly concave downwards on $(-\infty, 0]$. Then

$$\sum_{i=1}^{n} \frac{a_i}{1+x_i} = \sum_{i=1}^{n} a_i (1+\exp y_i)^{-1} \le \left\{ 1+\exp\left(\sum_{i=1}^{n} a_i y_i\right) \right\}^{-1}$$
$$= \left\{ 1+\prod_{i=1}^{n} \exp(a_i y_i) \right\}^{-1} = \left\{ 1+\prod_{i=1}^{n} x_i^{a_i} \right\}^{-1}$$

with equality if and only if $y_1 = \cdots = y_n$, i.e., if and only if $x_1 = \cdots = x_n$.

If some $x_i = 0$, the above proof breaks down, but this case is easily handled on its own merits. Again, equality holds if and only if $x_1 = \cdots = x_n$, which in this case means that they are all 0.

If we allow $a_i = 0$ (and assume $0^0 = 1$), then the condition for equality becomes $x_i =$ constant for all *i* for which $a_i > 0$.

Finally we remark that the inequality is reversed if $x_i \ge 1$ for all *i*. This is because f''(y) > 0 on $(0,\infty)$ and thus *f* is concave upwards on $[0,\infty)$.

Editor's comment. The proposer notes that the special case n = 2 of our problem is a lemma of D. Borwein, used in his solution to Problem 5333 [1965,1030; 1966, 1022]. Borwein's lemma was used to prove the special case $a_1 = \cdots = a_n = n^{-1}$ of our problem; once this was shown, 5333 follows from a trivial application of the arithmetic-geometric mean inequality. [Hans]Kappus uses Borwein's

lemma as a starting point and obtains our result by an easy induction. Kappus also notes that this problem generalizes Problem 254 in *Elem. Math.*, 11(1956) p.112.

[R. J.]Evans shows that the problem generalizes to infinite sequences $\{a_i\}$ and $\{x_i\}$ with $a_i \ge 0$, $\sum a_i = 1$ and $0 \le x_i \le 1$. [E. B.]Rockower shows, using Riemann sums, that the following continuous analog holds: Suppose that a(t) and x(t) are continuous functions on the interval [0,1] (the actual interval is not important) and that $a(t) \ge 0$, $\int a(t) dt = 1$ and $0 \le x(t) \le 1$. Then

$$\int_0^1 a(t) [1+x(t)]^{-1} \, dt \le \left\{ 1 + \exp \int_0^1 a(t) \log x(t) \, dt \right\}^{-1}$$

with equality if and only if x(t) is constant on the set where a(t) > 0. The above results can be subsumed by the following generalization, which can be shown by first considering simple functions and then passing to the limit: Let (X, A, μ) be a measure space with $\mu(X) = 1$ and let f be a measurable real-valued function on X which satisfies $0 \le f(x) \le 1$ almost everywhere. Then

$$\int (1+f)^{-1} \, d\mu \le \left\{ 1 + \exp \int \log f \, d\mu \right\}^{-1}$$

with equality if and only if f(x) is almost everywhere constant.

Amer. Math. Monthly, 81(1974) 660

E 2483. Proposed by M. S. Klamkin, University of Waterloo

Let x be non-negative and let m, n be integers with $m \ge n \ge 1$. Prove that

$$(m+n)(1+x^m) \ge 2n\frac{1-x^{m+n}}{1-x*n}$$

Amer. Math. Monthly, 82(1975) 759

IV. Solution by the proposer. As above [i.e., in earlier solutions] we can assume that 0 < x < 1 and m > n, and we shall show that [in these cases] the inequality is strict. Also, we shall not restrict m and n to be integers, but shall allow them to be any positive real numbers m > n > 0. Nowlet m + n = r and m - n = s so that r > s > 0 and let $t = x^{1/2}$ so that 0 < t < 1. Rearranging the desired inequality get

$$\frac{1-t^{2r}}{rt^r} > \frac{1-t^{2s}}{st^s}$$

On letting $t = e^{-y}$ we get

$$\frac{\sinh ry}{ry} > \frac{\sinh sy}{sy}$$

where now $0 < y < \infty$. But $x^{-1} \sinh x$ has a power series expansion with only positive coefficients, so that it is a strictly increasing function on $(0, \infty)$.

Amer. Math. Monthly, 81(1974) 666–668

Area Summations in Partitioned Convex Quadrilaterals

E 2423 [1973, 691]. Proposed by Lyles Hoshek, Monterey Park, California, and B. M. Stewart, Michigan State University

Let there be given a plane convex quadrilateral of area A. Divide each of its four sides into n equal segments and join the corresponding points of division of opposite sides, forming n^2 smaller quadrilaterals. Prove: (a) the n smaller quadrilaterals in any diagonal (ordinary or broken) have a composite area equal to A/n; (b) The composite area of any row of smaller quadrilaterals and its complementary row (row i and row n+1-i) is equal to 2A/n. (In particular, if n is odd this implies that the composite area of the middle row is A/n.)

Solution by Donald Batman, M.I.T. Lincoln Laboratory, and M. S. Klamkin, Ford Motor Company. We obtain more general results by dividing one pair of opposite sides into n equal segments and the other pair of sides into m equal segments, as shown in the figure B



Denote the given quadrilateral by OBDC, where O is the origin. If X is a point in the plane, then we make the usual identification of X with the vector **X** from the origin to the point X. Define p, q by

$$\mathbf{D} = (p+1)\mathbf{B} + (q+1)\mathbf{C}.$$

Note that p, q > -1 and also p + q > -1 since the quadrilateral is convex. The points of division will be denoted by P(r, s), with $r = 0, 1, \ldots, m$ and $s = 0, 1, \ldots, n$; e.g., P(0, 0) = O and P(m, n) = D. Let Q(r, s) denote the small quadrilateral whose

upper left-hand vertex is P(r, s) and partition Q(r, s) into the two triangles $\Delta(r, s)$ and $\Delta'(r, s)$ as shown in the figure.

One can show that for suitable scalars x and y

$$\mathbf{P}(\mathbf{r}, \mathbf{s}) = \frac{r}{m}\mathbf{B} + x\left\{\mathbf{C} + \frac{r}{m}(\mathbf{D} - \mathbf{C} - \mathbf{B})\right\} = \frac{s}{n}\mathbf{C} + y\left\{\mathbf{B} + \frac{s}{n}(\mathbf{D} - \mathbf{B} - \mathbf{C})\right\}$$

Since **B** and **C** are linearly independent, we find that x = s/n and y = r/m. Thus

$$\mathbf{P}(\mathbf{r}, \mathbf{s}) = \frac{r}{m} \left\{ 1 + \frac{sp}{n} \right\} \mathbf{B} + \frac{s}{n} \left\{ 1 + \frac{rq}{m} \right\} \mathbf{C}$$
(1)

Since $\mathbf{P}(r+1, s) - \mathbf{P}(r, s)$ and $\mathbf{P}(r, s+1) - \mathbf{P}(r, s)$ are independent of r and s respectively, each segment of the figure is divided into equal parts—m for the "horizontal" segments and n for the "vertical" segments (as shown in the figure).

For the area $|\Delta(r,s)|$ of $\Delta(r,s)$ we have

$$2|\Delta(r,s)| = |\{\mathbf{P}(r+1,s) - \mathbf{P}(r,s)\} \times \{\mathbf{P}(r,s+1) - \mathbf{P}(r,s)\}|$$
$$= \frac{1}{mn} \left\{1 + \frac{sp}{n} + \frac{rq}{m}\right\} |\mathbf{B} \times \mathbf{C}|$$
(2)

and similarly

$$2|\Delta'(r,s)| = \frac{1}{mn} \left\{ 1 + \frac{(s+1)p}{n} + \frac{(r+1)q}{m} \right\} |\mathbf{B} \times \mathbf{C}|$$
(3)

Note also that if A is the area of OBDC, then

$$2A = (p+q+2)|\mathbf{B} \times \mathbf{C}| \tag{4}$$

Look now at any $\triangle(r, s)$ and its centro-symmetric $\triangle'(m-1-r, n-1-s)$. From (2), (3) and (4) we have

$$|\Delta(r,s)| + |\Delta'(m-1-r,n-1-s)| = \frac{A}{mn}$$
(5)

For m, n odd it follows from this that the central small quadrilateral has area A/mn. (The special case m = n = 3 was established using along synthetic proof by B. Greenberg, *That area problem*, Math. Teacher **64**(1971) 79–80.)

If we take any small quadrilateral Q(r, s) and its centro-symmetric quadrilateral Q(m-1-r, n-1-s) we see from (5) that

$$\begin{aligned} |Q(r,s)| &+ |Q(m-1-r,n-1-s)| = |\Delta(r,s)| + |\Delta'(r,s)| \\ &+ |\Delta(m-1-r,n-1-s)| + |\Delta'(m-1-r,n-1-s)| \\ &= \frac{A}{mn} + \frac{A}{mn} = \frac{2A}{mn} \end{aligned}$$

which proves part (b).

From (2), (3) and (4) we have

$$\begin{aligned} |Q(r,s)| &= |\Delta(r,s)| + |\Delta'(r,s)| \\ &= \frac{A}{mn(p+q+2)} \left\{ 2 + \frac{(2s+1)p}{n} + \frac{(2r+1)q}{m} \right\} \end{aligned}$$

Let m = n; we can now show that part (a) now follows from this formula. In fact we can show that the result hold not only for broken diagonals, but for "generalized diagonals", i.e., for selections of n smaller quadrilaterals with one from each row and each column, as in the individual terms of a matrix expansion. More precisely, let σ be a permutation of $(0, 1, \ldots, n-1)$; an easy computation shows that

$$\sum_{r=0}^{n-1} |Q(r, r\sigma)| = \frac{A}{n}$$

giving the result.

We note that Problem E 1548 [1963, 892] and its generalizations follow from the above results,

Amer. Math. Monthly, 81(1974) 902.

E 2495. Proposed by M. S. Klamkin, University of Waterloo, Ontario, and L. A. Shepp, Bell Telephone Laboratories

[[Shepp's name added as a proposer at Amer. Math. Monthly, 82(1975) 168.]]

Let n be a natural number. Evaluate the following limit:

$$I_n = \lim_{x \to \infty} \left\{ \frac{(\log x)^{2n}}{2n} - \int_0^x \frac{(\log t)^{2n-1}}{1+t} \, dt \right\}$$

Amer. Math. Monthly, 82(1975) 938.

Solution by Watson Fulks, University of Colorado. Consider the slightly modified problem where 2n is replaced by an arbitrary positive integer k. Then the substitutions $y = \log x$ and $u = \log t$ reduce the problem to the determination of $f(\infty) = \lim_{y\to\infty} f(y)$ where f is given by

$$f(y) = \frac{y^k}{k} - \int_{-\infty}^y \frac{u^{k-1}}{1 + e^{-u}} \, du$$

We note that

$$f(0) = -\int_{-\infty}^{0} \frac{u^{k-1}}{1+e^{-u}} \, du = (-1)^k \int_{0}^{\infty} \frac{u^{k-1}}{1+e^u} \, du$$

so that

$$f(y) = f(0) + \int_0^y f'(t) dt = f(0) + \int_0^y \frac{u^{k-1}}{1 + e^u} du$$

from which

$$f(\infty) = [1 + (-1)^k] \int_0^\infty \frac{u^{k-1}}{1 + e^u} \, du$$

By formula (6), p.312 of *Tables of Integral Transforms*, Vol.1, Erdélyi et al., McGraw-Hill, 1954, or by expanding $(1 + e^u)^{-1}$ in powers of e^{-u} and integrating termwise, this is

$$f(\infty) = [1 + (-1)^k]\zeta(k)\Gamma(k)[1 - 2^{1-k}]$$

Further, f can be written in the form

$$f(y) = f(\infty) - \int_y^\infty \frac{u^{k-1}}{1+e^u} \, du$$

from which the asymptotic behavior of f as $y \to \infty$ is easily deduced, again by expanding $(1 + e^u)^{-1}$. In particular

$$f(y) = f(\infty) - e^{-y} \sum_{j=0}^{k-1} {j \choose k-1} y^j \Gamma(k-j) + O(e^{-2y} y^{k-1})$$

= $f(\infty) - e^{-y} y^{k-1} [1 + O(1/y)].$

Editor's comment. As noted by a number of solvers, the solution may be expressed in terms of Bernoulli numbers using the formula $(2n)!\zeta(2n) = 2^{2n-1}\pi^{2n}|B_{2n}|$.

Amer. Math. Monthly, 81(1974) 903-904

Minimal Curve for Fixed Area

E 2185 [1969, 825; 1970, 531]. Proposed by Michael Goldberg, Washington, D.C.

Given a convex quadrilateral. Find the shortest curve which divides it into two equal areas.

Comment by M. S. Klamkin, Ford Motor Company. Although the properties of the shortest bisecting arc as given by both Ogilvy and Goldberg are correct, neither solver has really supplied a full mathematical proof. At a geometry seminar held at Michigan State University several years ago, Branko Grünbaum raised again the more general problem of determining the shortest arc which divides a given simply connected area in a fixed ratio, and which lies wholly within the area. Grünbaum noted that Norbert Wiener [*The shortest line dividing an area in a given ratio*, Proc. Cambridge Philos. Soc., **18**(1914) 56–58] proved that (if such an arc exists) it must consist of an arc of a finite or infinite circle or a chain of such arcs having the propertythat two successive arcs meet only on the boundary of the given area. At the end of this paper is the footnote, "It is almost self-evident that the shortest line to divide a convex area in a given ratio is a single arc of a circle, but this I have not been able to prove." This conjecture includes E 2185 as a special case.

Wiener's shortest-line conjecture went unproved for almost sixty years, but in 1973, Richard Joss, a student of Grünbaum, announced that he had proved it [Notices A.M.S., June 1973, Abstract 705-D1, p.A-461].

Amer. Math. Monthly, 81(1974) 904–905

Symmedian Point of a Triangle

E 2347 [1972, 303; 1973, 321]. Proposed by Leonard Carlitz, Duke University

Let P denote a point in the interior of the triangle ABC. Let α , β , γ denote the angles of ABC. Let R_1 , R_2 , R_3 denote the distances from P to the vertices of ABC and let r_1 , r_2 , r_3 denote the distances from the sides of ABC. Show that

$$R_1^2 \sin^2 \alpha + R_2^2 \sin^2 \beta + R_3^2 \sin^2 \gamma \le 3(r_1^2 + r_2^2 + r_3^2)$$

with equality if and only if P is the symmetry point of ABC.

IV. Solution by M. S. Klamkin, Ford Motor Company. The published solution, which is rather long, involves Lagrange multipliers, which should always be avoided whenever possible in proving elementary triangle inequalities. Furthermore, the solution is incomplete since sufficiency was not established.

We give a generalization by starting with the known inequality [1, p.7]

$$xR_1^2 + yR_2^2 + zR_3^2 \ge \frac{a^2yz + b^2zx + c^2xy}{x + y + z}$$
(1)

where x, y, z are arbitrary real numbers such that x + y + z > 0 and where there is equality if and only if $x/F_1 = y/F_2 = z/F_3$ (F_1 denotes the area of BPC, etc.). (A physical interpretation of (1) is that the polar moment of inertia of three masses x, y, z located at A, B, C respectively, is minimized by taking the axis through the centroid of the masses.)

For any inequality of the form $\phi(R_1, R_2, R_3, a, b, c) \geq 0$ there is a dual equality $\phi(r_1, r_2, r_3, R_1 \sin \alpha, R_2 \sin \beta, R_3 \sin \gamma) \geq 0$, obtained by considering the pedal triangle of P. Here the distances from P to the vertices of the pedal triangle are r_1, r_2, r_3 and the sides of the pedal triangle are $R_1 \sin \alpha, R_2 \sin \beta, R_3 \sin \gamma$, respectively. Thus the dual of (1) is

$$xr_1^2 + yr_2^2 + zr_3^2 \ge \frac{yzR_1^2\sin^2\alpha + zxR_2^2\sin^2\beta + xyR_3^2\sin^2\gamma}{x + y + z}$$
(2)

Then the stated inequality corresponds to the special case x = y = z of (2). There is equality if and only if the point P is the centroid of the pedal triangle and consequently if and only if P is the symmedian point of ABC [2, Theorem 350]. Coincidentally, the stated inequality appears in the same form in [1, p.10]. By applying (1) to the right hand side of (2) we obtain

$$\frac{\sum y z R_1^2 \sin^2 \alpha}{x + y + z} \ge \frac{4F^2}{\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}}$$
(3)

Inequalities (2) and (3) also provide a strengthening and a generalization of the following known inequality [2, Theorem 349], [3, Item 12.54, p.118]:

$$r_1^2 + r_2^2 + r_3^2 \ge \frac{4F^2}{a^2 + b^2 + c^2}$$

1. M. S. Klamkin, Nonnegative Quadratic Forms and Triangle Inequalities, Ford Motor Company Preprint, June 1971. (Also see Notices A.M.S., Oct.1971, p.966.)

- 2. R. A. Johnson, Advanced Euclidean Geometry, Dover, New York, 1960.
- 3. O. Bottema etal., Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.

Amer. Math. Monthly, 81(1974) 1034.

5997. Proposed by M. S. Klamkin, Ford Motor Company

Prove that

$$|x^{p-1}(x-1)^p(x-2)^p\cdots(x-n)^p| \le \Gamma((1+n)^p)$$

where $0 \le x \le n$ and p, n are real and ≥ 1 . (This inequality has been given by A. Ostrowski for integral p, n. See Mitrinović and Vasić, Analytic Inequalities, Springer-Verlag, 1970, p.198.)

Amer. Math. Monthly, 83(1976) 490–491.

Solution by Thomas Foregger, Bell Laboratories, Murray Hill, New Jersey. Let

$$f_n(x) = |x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p|$$

We show that

$$f_n(x) \leq \left(\frac{p-1}{2p-1}\right)^{p-1} \left(\frac{p}{2p-1}\right)^p \{\Gamma(1+n)\}^p \quad \text{if } 0 \leq x < 1$$

$$f_n(x) \leq \frac{1}{n} \{\Gamma(1+n)\}^p \quad \text{if } 1 \leq x \leq n$$

First, suppose that $0 \le x \le 1$. Then, using $\Gamma(n+1) = n!$ and the relation $|(x-k)^p| = (k-x)^p \le k^p$ for $0 \le x < 1$ and k = 1, 2, ..., n, we have that

$$\frac{f_n(x)}{(n!)^p} = x^{p-1} \frac{(1-x)^p}{1^p} \frac{(2-x)^p}{2^p} \cdots \frac{(n-x)^p}{n^p} \le x^{p-1} (1-x)^p$$

Elementary calculus shows that the right hand side has a maximum value of

$$\left(\frac{p-1}{2p-1}\right)^{p-1} \left(\frac{p}{2p-1}\right)^p$$

Next, suppose that $1 \le x \le n$ and let m = [x]. Then

$$|(x-k)^{p}| = \begin{cases} (x-k)^{p} & \text{if } 1 \le k \le m \\ (k-x)^{p} & \text{if } m+1 \le k \le n \end{cases}$$

so that

$$\frac{f_n(x)}{(n!)^p} = \frac{x^{p-1}}{n^{p-1}} \prod_{k=1}^{m-1} \frac{(x-k)^p}{(n-k)^p} \cdot \frac{(x-m)^p}{n} \cdot \prod_{k=m+1}^n \frac{(k-x)^p}{(k-m)^p}$$

Clearly, $0 \le x - k \le n - k$ if $1 \le k \le m - 1$ and $0 \le k - x \le k - m$ if $m + 1 \le k \le n$. Also $x \le n$. Thus

$$\frac{f_n(x)}{(n!)^p} \le \frac{(x-m)^p}{n} \le \frac{1}{n}$$

Editor's note. [Emil]Grosswald observes that the inequality written as $|x(x-1)\cdots(x-n)|x^{-1/p} \le n!$ is an easy consequence of Ostrowski's theorem.

APPLIED MATHEMATICS

Physics: projectiles

Amer. Math. Monthly, 82(1975) 520–521.

E 2535. Proposed by M. S. Klamkin, University of Waterloo

A body is projected in a uniform gravitational field and is subject to a resistance which is a function of its speed $|\mathbf{v}|$. If the acceleration **a** of the body always has a constant direction, no matter what the initial velocity \mathbf{v}_0 , show that $\mathbf{a} = \mathbf{a}_0 e^{-kt}$ for some constant k.

Amer. Math. Monthly, 83(1976) 657.

Solution by Harry Lass and Robert M. Georgevic (jointly), Jet Propulsion Laboratory, Pasadena, California. We have $\mathbf{a} = \mathbf{g} - f(v)\mathbf{v} = \lambda(t)\mathbf{a}_0$ where \mathbf{g} is the constant gravitational acceleration, \mathbf{a} and \mathbf{v} the acceleration and velocity of the body, $v = |\mathbf{v}|$, t the time and $\mathbf{a}_0 = \mathbf{a}(0)$, $\lambda(0) = 1$. Differentiating with respect to time yields

$$(\lambda'(t) + f(v)\lambda(t))\mathbf{a}_0 = -f'(v)\frac{dv}{dt}\mathbf{v}$$

For **v** not parallel to \mathbf{a}_0 it follows that f'(v) = 0 and $\lambda'(t) = -f(v)\lambda(t)$. Hence f(v) = k (a constant), $\lambda(t) = e^{-kt}$ and $\mathbf{a} = e^{-kt}\mathbf{a}_0$.

SOLID GEOMETRY

Tetrahedra: planes

Amer. Math. Monthly, 82(1975) 661.

Largest Cross-Section of a Tetrahedron

E 1298^{*} [1958, 43]. Proposed by H. D. Grossman

It is not difficult to show that the longest linear section of a triangle is the longest side of the triangle. Is the greatest planar section of a tetrahedron the largest face of the tetrahedron?

Solution by Murray S. Klamkin, University of Waterloo. The answer is yes, as was shown in the solution to the identical Advanced Problem 5006 [1962, 63; 1963, 338; 1963, 1108]. Curiously enough, the corresponding result for a 5-simplex is false: See D. W. Walkup, A simplex with a large cross section, this MONTHLY, **75**(1968) 34–36.

GEOMETRY

N-dimensional geometry: simplexes

Amer. Math. Monthly, 82(1975) 756.

E 2548. Proposed by Murray S. Klamkin, University of Waterloo

Let A_0, A_1, \ldots, A_n be distinct points which lie on a hyperplane. Suppose that these points are parallel projected into another hyperplane and that their images are B_0, B_1, \ldots, B_n respectively. Prove that for any $r = 0, 1, \ldots, n$ the volumes of the simplexes spanned by $A_0, A_1, \ldots, A_r, B_{r+1}, B_{r+2}, \ldots, B_n$ and by $B_0, B_1, \ldots, B_r, A_{r+1}, A_{r+2}, \ldots, A_n$ are equal.

Amer. Math. Monthly, 83(1976) 815.

Solution by Aage Bondesen, Espergaerde, Denmark. Call the two given hyperplanes α and β , and let the given parallel projection p be done from α to β . The set of midpoints of the segments Ap(A), for A in α , is a hyperplane γ . Let r be the affine reflection in γ such that r(A) = p(A) for A in α . Since r is volume-preserving, the desired result is immediate.

ALGEBRA

Maxima and minima

Amer. Math. Monthly, 83(1976) 54.

E 2573. Proposed by Murray S. Klamkin, University of Waterloo

If n positive real numbers vary such that the sum of their reciprocals is fixed and equal to A, find the maximum value of the sums of the reciprocals of the n numbers taken j at a time.

Amer. Math. Monthly, 84(1977) 298–299.

IV. Solution and a generalization by the proposer. Let a_1, \ldots, a_n be positive and define

$$S_r = \sum_{\text{sym}} \frac{1}{x_1 + \dots + x_r}$$
 and $T_r = \frac{r^2 S_r}{\binom{n-1}{r-1}}$

We shall prove that $T_1 \ge T_2 \ge \cdots \ge T_n$

Proof. The inequality

$$T_{n-1} = (n-1)S_{n-1} \ge n^2 S_n = T_n \tag{1}$$

follows easily from the Cauchy-Schwartz inequality because

$$\frac{n-1}{S_n} = (n-1)(x_1 + \dots + x_n) = \sum_{\text{sym}} (x_1 + \dots + x_{n-1})$$

Using (1), we have (r = 2, 3, ..., n - 1)

$$(r-1)\sum_{\text{sym}} \frac{1}{y_1 + \dots + y_{r-1}} \ge \frac{r^2}{y_1 + \dots + y_r}$$

for any positive y_1, y_2, \ldots, y_n . Replacing (y_1, \ldots, y_r) by $(x_{k_1}, \ldots, x_{k_r})$ and summing over all r-tuples (k_1, \ldots, k_r) such that $1 \leq k_1 < \cdots < k_r \leq n$ we obtain $(r-1)(n-r+1)S_{r-1} \geq r^2S_r$, i.e., $T_{r-1} \geq T_r$

Note that the proposed inequality is identical with $T_1 \ge T_j$

GEOMETRY

Triangles: medians

Amer. Math. Monthly, 83(1976) 59–60.

Extended Medians of a Triangle

E 2505 [1974, 1111]. Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York

Let a, b, c be the sides of a triangle ABC, and let m_a , m_b , m_c be the medians to sides a, b, c respectively. Extend the medians so as to meet the circumcircle again, and let these chords be M_a , M_b , M_c respectively. Show that

$$M_{a} + M_{b} + M_{c} \ge \frac{4}{3}(m_{a} + m_{b} + m_{c})$$
(1)
$$M_{a} + M_{b} + M_{c} \ge \frac{2}{3}\sqrt{3}(a + b + c)$$
(2)

When does equality occur?

II. Solution to (1) by Paul Erdős and M. S. Klamkin, University of Waterloo, Ontario. Since $m_a(M_a - m_a) = a^2/4$, etc., (1) can be rewritten as

$$3\sum a * 2/m_a \ge 4\sum m_a \qquad (1')$$

It is known that one can form a triangle having sides m_a , m_b , m_c with its respective medians being 3a/4. 3b/4, 3c/4. Thus (1') is equivalent to

$$4\sum m_a^2/a \ge 3\sum a \qquad 1''$$

Since $4m_a^2 = 2b^2 + 2c^2 - a^2$, etc., (1") reduces to $\sum (b^2 + c^2)/a \ge 2\sum a$ or

$$\sum (b+a)(b-a)^2/ab \ge 0$$

GEOMETRY

Polygons: convex polygons

Amer. Math. Monthly, 83(1976) 200.

Area of a Convex Polygon

E 2514. Proposed by G. A. Tsintsifas, Thessaloniki, Greece

Let P be a convex polygonand let K be the polygon whose vertices are the midpoints of the sides of P. A polygon M is formed by dividing the sides of P (cyclically directed) in a fixed ratio p:q where p+q=1. Show that

$$|M| = (p-q)^2 |P| + 4pq|K|$$

where |M| denotes the area of M, etc.

II. Comment by M. S. Klamkin, University of Waterloo. It follows as an easy consequence that $\min |M| = |K|$ occurs for $p = q = \frac{1}{2}$.

ALGEBRA

Inequalities: fractions

Amer. Math. Monthly, 83(1976) 483.

E 2603. Proposed by Murray S. Klamkin, University of Waterloo, Ontario Let $x_i > 0$ $(1 \le i \le n)$. Prove that

$$r \cdot \sum \frac{x_1 x_2 \cdots x_r}{x_1 + x_2 + \cdots + x_r} \le \binom{n}{r} \left(\frac{x_1 + \cdots + x_n}{n}\right)^{r-1}$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

(The "symmetric" sum above consists of $\binom{n}{r}$ terms.)

Amer. Math. Monthly, 84(1977) 743.

Solution by Lawrence A. Shepp, Bell Laboratories, Murray Hill, New Jersey. If $x_i \neq x_j$ then replacing x_i and x_j by $\frac{1}{2}(x_i + x_j)$ increases the left hand side and leaves the right hand side constant as is easy to see. Thus the maximum of the left side under fixed $x_1 + \cdots + x_n$ occurs only for $x_1 = \cdots = x_n$ in which case a direct calculation shows that the equality holds.

SOLID GEOMETRY

Analytic geometry

Amer. Math. Monthly, 84(1977) 218–219.

Volume and Surface Area of a Solid

E 2563 [1975, 937]. Proposed by J. Th. Korowine, Athens, Greece

Let f_1 and f_2 be non-negative periodic functions of period 2π and let h > 0. Let $P_1(\theta)$ and $P_2(\theta)$ be the points whose cylindrical coordinates are $(f_1(\theta), \theta, 0)$ and $(f_2(\theta), \theta, 0)$ respectively. Find integrals for the volume and surface area of the solid bounded by the planes z = 0, z = h and the lines $P_1(\theta)P_2(\theta)$.

Solution by M. S. Klamkin, University of Waterloo, Ontario, Canada. If a point (r, θ, z) lies on the surface generated by the motion of the segment $P_1(\theta)P_2(\theta)$ then

$$r = r(z,\theta) = f_1(\theta) + \frac{z}{h}(f_2(\theta) - f_1(\theta)) \qquad 0 \le z \le h$$

Then the volume we want to find is given by

$$V = \frac{1}{2} \int_0^{2\pi} \left(\int_0^h r(z,\theta)^2 \, dz \right) \, d\theta = \frac{h}{6} \int_0^{2\pi} (f_1^2 + f_1 f_2 + f_2^2) \, d\theta$$

This can also be written in the form

$$V = \frac{h}{6}(B_1 + 4M + B_2)$$

where B_i (i = 1, 2) are the areas of the bases and M is the area of the mid-cross-section. Explicitly

$$B_i = \frac{1}{2} \int_0^{2\pi} f_i^2 d\theta \qquad (i = 1, 2) \qquad \qquad M = \frac{1}{2} \int_0^{2\pi} \left(\frac{f_1 + f_2}{2}\right)^2 d\theta$$

The lateral surface area S is given by the well-known area integral in cylindrical coordinates $\int_{a}^{2\pi} \int_{a}^{b} \int_{a}^{b}$

$$S = \int_0^{2\pi} \int_0^{\pi} \sqrt{r^2 + (rr_z)^2 + r_\theta^2} \, dz \, d\theta$$

where r_z and r_{θ} are the partial derivatives of $r = r(z, \theta)$. In our case we have

$$r_z = \frac{1}{h}(f_2 - f_1) \qquad \qquad r_\theta = f_1' + \frac{z}{h}(f_2' - f_1')$$

Then the entire area of the solid is $B_1 + B - 2 + S$.

Editor's Comment. The expression $r^2 + (rr_z)^2 + r_{\theta}^2$ is a quadratic in z. The formula for S which emerges after this integration is given by [L.]Kuipers but it is too complicated to state it here.

Amer. Math. Monthly, 85(1978) 386–387.

An Old Result

E 2637 [1977, 134]. Proposed by Armand E. Spencer, State University College, Potsdam, N.Y.

If $a_0, a_1, \ldots, a_{n-1}$ are integers show that

$$\prod_{0 \le i < j \le n-1} \frac{a_i - a_j}{i - j}$$

is also an integer.

Comments. M. S. Klamkin informs us that this is the same as problem 132 in G. Pólya and G. Szegö, *Problems and Theorems in Analysis* II, Springer 1976, p.134. The solution appears as a special case of Problem 96 on pp.96 and 229:

By row manipulations we have

$$\begin{vmatrix} 1 & \dots & 1 \\ \binom{x_1}{1} & & \binom{x_n}{1} \\ \vdots & & \vdots \\ \binom{x_1}{n-1} & \dots & \binom{x_n}{n-1} \end{vmatrix} = \begin{pmatrix} n-1 \\ i=1 \end{pmatrix} \cdot \begin{vmatrix} 1 & \dots & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ \binom{x_1}{n-1} & \dots & \binom{x_n}{n-1} \end{vmatrix} = \prod_{0 \le i < j \le n-1} \frac{x_i - x_j}{i - j}$$

Klamkin, Henry Ricardo and the proposer refer to H. W. Segar, *Messenger of Math.*, **22**(1892) 57–67 where this result is stated. The proofs appear in the same journal by Cayley, **22**(1892) 186–190 and by Segar, **23**(1893) 31–37. Ricardo has also located this problem in D. K. Faddeev and I. S. Sominskii, *Problems in Higher Algebra*, Freeman 1965, Problem 269 and in Shklarsky, Chentzov and Yaglom, *The USSR Olympiad Problem Book*, Freeman 1962, Problem 62.

Ronald Evans refers to his joint paper with I. M. Isaacs, *Proc. Amer. Math. Soc.*, **58**(1976) 51–54 where the result is attributed to O. H. Mitchell, *Amer. J. Math.*, **4**(1881) 341–344.

Weyl's formula for the dimension of simple SU(n)-modules shows that the number in the problem is equal to the dimension of such a module and hence is an integer.

ALGEBRA

Inequalities: exponentials

Amer. Math. Monthly, 86(1979) 222.

S 6. Proposed by M. S. Klamkin and A. Meir, University of Alberta Let $x_i > 0$ for i = 1, 2, ..., n with $n \ge 2$. Prove that

$$(x_1)^{x_2} + (x_2)^{x_3} + \dots + (x_{n-1})^{x_n} + (x_n)^{x_1} \ge 1$$

Amer. Math. Monthly, 87(1980) 302–303.

Solution by David Hammer. University of California, Davis. For n = 2, the solution is in E 1342 [1959, 513]. Hence we assume that n > 2. We also may assume that $0 < x_i < 1$ for all *i*. Let *S* be the given sum. Since *S* is invariant under cyclic permutation of the x_i we may assume that x_3 is minimal among the x_i and hence that $x_3 \le x_1$. Then

$$S \ge (x_1)^{x_2} + (x_2)^{x_3} \ge (x_3)^{x_2} + (x_2)^{x_3} \ge 1$$

as desired.

The sharpness of the inequality is shown by the example with n = 4 and

$$a = r^{-r^r}$$
 $b = r^{-r}$ $c = r^{-1}$ $d = 1$

Then $a^b + b^c + c^d + d^a = 3r^{-1} + 1$ and one can let $r \to \infty$.

A slight generalization is the following: Let $x_i > 0$ for i = 1, 2, ..., n and let s be a permuation of 1, 2, ..., n which is the product of m disjoint cycles of length at reast 2 and has k fixed points. Then

$$\sum_{i=1}^{n} (x_i)^{x_{s(i)}} \ge m + k \left(\frac{1}{e}\right)^{1/e}$$

This follows immediately from S6 and the well-known fact that the minimum of x^x occurs when x = 1/e.

Editorial note. F. S. Cater gave the generalization that

$$(x_1)^{x_2} + (x_2)^{x_3} + \dots + (x_{n-1})^{x-n} + (x_n)^{x_1} > 1 + (n-2)\min(x_1^{x_2}, x_2^{x_3}, \dots, x_{n-1}^{x_n}, x_n^{x_1})$$

NUMBER THEORY

Modular arithmetic: coprime integers

Amer. Math. Monthly, 86(1979) 306.

S 9. Proposed by M. S. Klamkin and A. Liu, University of Alberta

(a) Determine all positive integers n such that gcd(x,n) = 1 omplies that $x^2 \equiv 1 \pmod{n}$.

(b) Determine all positive integers n such that $xy+1 \equiv 0 \pmod{n}$ implies that $x+y \equiv 0 \pmod{n}$.

Amer. Math. Monthly, 87(1980) 488.

Solution by Arnold Adelberg, Grinnell College, and Jeffery M. Cohen, graduate student, University of Pittsburgh (independently). (a) We show that n satisfies the condition if and only if $n \mid 24$. First, if n has a prime divisor p > 3 and q is the product of all the prime divisors of n different from p, then the Chinese Remainder Theorem implies the existence of an x with $x \equiv 1 \pmod{q}$ and $x \equiv 2 \pmod{p}$. [Let q = 1 if n is a power of p.] Then gcd(x, n) = 1 and $x^2 \equiv 4 \neq 1 \pmod{p}$ so that $x^2 \not\equiv 1 \pmod{n}$. Next, if $n = 2^r 3^s$, then gcd(5, n) = 1 and $5^2 \equiv 1 \pmod{n}$ if and only if $n \mid 24$.

Conversely, if $n \mid 24$ then gcd(x, 2) = 1 implies $x^2 \equiv 1 \pmod{8}$ and gcd(x, 3) = 1 implies $x^2 \equiv 1 \pmod{3}$ so that n satisfies the condition.

(b) Let A and B be the sets of integers n satisfying the conditions in (a) and (b) respectively. We show that A = B.

Let $n \in A$. The $xy + 1 \equiv 0 \pmod{n}$ implies that $gcd(x, n) = 1, x^2 \equiv 1 \pmod{n}$ and

$$x + y \equiv x + x^2 y \equiv x(1 + xy) \equiv x \cdot 0 \equiv 0 \pmod{n}$$

Thus $A \subseteq B$.

Now let $n \in B$. Then gcd(x, n) = 1 implies that there is an integer y with $xy \equiv -1 \pmod{n}$ which implies $x + y \equiv 0 \pmod{n}$ and so $x^2 \equiv x(-y) \equiv 1 \pmod{n}$, i.e., $B \subseteq A$. Hence A = B.

Editor's Note. M. J. DeLeon established several generalizations of S 9 dealing with the property of a pair (m, n) of positive integers such that gcd(a, m) = 1 implies $a^n \equiv 1 \pmod{m}$. He also referred to problem B-1 of the December 6, 1969 William Lowell Putnam Mathematical Competition, which asked for a proof that $n \equiv -1 \pmod{24}$ implies $24 \mid \sigma(n)$.

SOLID GEOMETRY

Tetrahedra: opposite edges

Amer. Math. Monthly, 86(1979) 392.

S 12. Proposed by M. S. Klamkin, University of Alberta

If $a, a_1; b, b_1; c, c_1$ denote the lengths of the three pairs of opposite edges of an arbitrary tetrahedron, prove that $a + a_1, b + b_1, c + c_1$ satisfy the triangle inequality.

[['sides' changed to 'edges' in the above – R.]]

Amer. Math. Monthly, 87(1980) 576.

[[Spelling 'Nelson' corrected to 'Nelsen' in the following — R.]]

Solution by Roger B. Nelsen, Lewis and Clark College. Without loss of generality, we may assume that the edges of of one face are a, b and c; while the edges emanating from the vertex opposite that face are a_1, b_1 and c_1 . Hence the edges of the four triangular faces of the tetrahedron are (a, b, c); (a_1, b_1, c) ; (a_1, b, c_1) and (a, b_1, c_1) . Each triangle satisfies the triangle inequality, so that

$$a < b + c$$

 $a_1 < b_1 + c$
 $a_1 < b + c_1$
 $a < b_1 + c_1$

Adding and dividing by 2 gives $a + a_1 < b + b_1 + c + c_1$. The other two forms of the inequality are similarly obtained. Note that this result also holds for the degenerate tetrahedron consisting of four distinct coplanar points.

Amer. Math. Monthly, 87(1980) 309.

6294. Proposed by M. S. Klamkin, University of Alberta

If $n = n_1 + n_2 + \dots + n_r$ where $n_i \ge 0$,

[[I've changed n_r to n_i — R.]]

prove that

$$\frac{n^n}{\prod n_i^{n_i}} \ge \frac{\Gamma(1+n)}{\prod \Gamma(1=n_i)} \ge \frac{(n+1)^{n+1}}{\prod (n_i+1)^{n_i+1}}$$

Amer. Math. Monthly, 89(1982) 216–217.

Solution by the proposer. The two inequalities will follow easily if we can show that the two functions

$$F(x) = \log(x^x/\Gamma(1+x))$$
 and $G(x) = \log(\Gamma(1+x)/(1+x)^{1+x})$

are superadditive, e.g., $F(x + y) \ge F(x) + F(y)$ for $x \ge 0$. Since F(0) = G(0) = 0, it suffices to show that both F(x) and G(x) are convex or equivalently that $F''(x) \ge 0$, $G''(x) \ge 0$ (See D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970, pp.22–23).

$$F'(x) = 1 + \log x - \Gamma'(1+x) / \Gamma(1+x) = 1 + \log x - \psi(1+x)$$

$$F''(x) = 1/x - \psi'(1+x) = \frac{1}{x} - \left\{ \frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \cdots \right\}$$

By the integral test,

$$\psi'(1+x) \le \frac{1}{(1+x)^2} + \int_{1+x}^{\infty} \frac{dt}{t^2} = \frac{1}{(1+x)^2} + \frac{1}{(1+x)} = \frac{2+x}{(1+x)^2} < \frac{1}{x}$$

Thus F(x) is superadditive and

$$\sum F(n_k) \le F\left(\sum n_i\right) = F(n)$$

which establishes the left-hand half of (1).

$$G'(x) = \psi(1+x) - 1 - \log(1+x)$$

$$G''(x) = \psi, (1+x) - 1/(1+x)$$

To show $G''(x) \ge 0$, first note that

$$G''(x) = \int_0^\infty \frac{t e^{-(x+1)t}}{1 - e^{-t}} dt - \int_0^\infty e^{-(x+1)t} dt = \int_0^\infty \frac{t - 1 + e^{-t}}{1 - e^{-t}} e^{-(x+1)t} dt$$

so $G''(x) \ge 0$, since $e^{-t} \ge 1 - t$ for $t \ge 0$. Thus, G(x) is also superadditive which gives the right-hand half of (2).

It is to be noted that the case of the left-hand inequality, when the n_i are non-negative integers, reduces to a problem of Leo Moser (*Math. Mag.*, **31**(1957) 113). The neat solution by Chi-yi Wong was to first write $(n_1 + n_2 + \cdots + n_r)^n = n^n$ and then to note that each term of the multinomial expansion of the left-hand expression is less then n^n .

Editorial Note. The solution by [Otto G.]Ruehr suggests a further problem: Determine

$$\sup \left\{ \alpha : \sum f_{\alpha}(t_i) \ge f_{\alpha}(t) \quad \text{whenever} \quad t = t_1 + t_2 + \dots + t_r \quad t_i \ge 0 \quad r = 1, 2, \dots \right\}$$

where

$$f_{\alpha}(x) = \log\{\gamma(1+x)/(x+\alpha)^{x+\alpha}\}$$

Clearly from the problem, $0 \le \alpha \le 1$.

Amer. Math. Monthly, 87(1980) 675.

6312^{*}. Proposed by M. S. Klamkin, University of Alberta

Prove or disprove that the set of n equations in n unknowns

$$x_1^{l_i} + x_2^{l_i} + \dots + x_n^{l_i} = 0$$
 $(i = 1, 2, \dots, n)$

where the l_i are relatively prime positive integers, has only the trivial solution $x_i = 0$ (i = 1, 2, ..., n) if and only if each m = 2, 3, ..., n divides at least one l_i .

Amer. Math. Monthly, 89(1982) 505.

Solution by Constantine Nakassis, Gaithersburg, Maryland. Let n > 2 be an even number (n = 2k); suppose that the only even number in $l_1, l_2 \ldots l_n$ is l_1 (for example take $l_1 = n!$ and let $l_2, \ldots l_n$ be the first n-1 primes that follow n). Consider any kcomple numbers which satisfy

$$y_1^{l_i} + y_2^{l_i} + \dots + y_n^{l_i} = 0$$

Let $x_{2i-1} = y_i$, $x_{2i} = -y_i$ for i = 1, 2, ..., k. It is clear than that the proposed system has nontrivial solutions. (The starred assertion is true if n = 2, but false if n = 2k+1 > 3.)

The case n = 3 remains open; the starred assertion has been established by the proposer for many triples.

Amer. Math. Monthly, 89(1982) 498.

E 2958. Proposed by M. S. Klamkin, University of Alberta

Let x, y, z be positive, and let A, B, C be angles of a triangle. Prove that

$$x^{2} + y^{2} + z^{2} \ge 2yz\sin(A - \pi/6) + 2zx\sin(B - \pi/6) + 2xy\sin(C - \pi/6).$$

Amer. Math. Monthly, **92**(1985) 289.

Solution by C. S. Karuppan Chetty, Regional Engineering College, Tiruchirapalli, India.

Solution I. Let

$$f(x, y, z) = x^2 + y^2 + z^2 - 2yz\sin\alpha - 2zx\sin\beta - 2xy\sin\gamma$$

where $\alpha = A - \pi/6$, $\beta = B - \pi/6$ and $\gamma = C - \pi/6$. Since

$$\sin \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma$$

we can express f(x, y, z) as

$$(x - y\sin\gamma - z\sin\beta)^2 + (y\cos\gamma - z\cos\beta)^2$$

and the result follows.

In fact, the given inequality holds for any real x, y, z and for all A, B, C with $A + B + C = \pi$.

Solution II. Defining f(x, y, z) as above, the matrix of the quadratic form f is

$$U = \begin{bmatrix} 1 & -\sin\gamma & -\sin\beta \\ -\sin\gamma & 1 & -\sin\alpha \\ -\sin\beta & -\sin\alpha & 1 \end{bmatrix}$$

It is easily seen that det U = 0, and hence the leading principal minors of U are 1, $\cos^2 \gamma$ and 0. Therefore f is positive semidefinite and the result follows.

Amer. Math. Monthly, 89(1982) 593–594.

E 2962. Proposed by M. S. Klamkin, University of Alberta, Canada

It is known that if the circumradii R of the four faces of a tetrahedron are congruent, then the four faces of the tetrahedron are mutually congruent (i.e., the tetrahedron is isosceles) [1]. It is also known that if the inradii of the four faces of tetrahedron are congruent, then the tetrahedron need not be isosceles [2]. Show that if Rr is the same for each face of a tetrahedron, the tetrahedron is isosceles.

1. Crux Mathematicorum, **6**(1980) 219.

2. Crux Mathematicorum, **4**(1978) 263.

Amer. Math. Monthly, **92**(1985) 290.

Solution by O. P. Lossers, Eindhoven University of Technology, The Netherlands. Let ABCD be tetrahedron. We put $\overline{BC} = a$, $\overline{CA} = b$, $\overline{AB} = c$, $\overline{DA} = a_1$, $\overline{DB} = b_1$, $\overline{DC} = c_1$. The fact that the product of the circumradius and the inradius is the same for each face is then expressed by

$$\frac{abc}{a+b+c} = \frac{ab_1c_1}{a+b_1+c_1} = \frac{a_1bc_1}{a_1+b+c_1} = \frac{a_1b_1c}{a_1+b_1+c} = 2Rr$$

The first equality is equivalent to

$$bb_1(c-c_1) + cc_1(b-b_1) = a(b_1c_1 - bc)$$

or

$$c_1(a+c)(b-b_1) + b(b_1+a)(c-c_1) = 0$$
(1)

We have two more equations obtained from (1) by cyclic permutation, namely

$$c(c_1 + b)(a - a_1) + a_1(b + a)(c - c_1) = 0$$

$$b_1(b + c)(a - a_1) + a(a_1 + c)(b - b_1) = 0$$
(2)
(3)

Considering (1), (2) and (3) as a system of linear equations for the unknowns $a - a_1$, $b - b_1$, $c - c_1$ we observe that the determinant of the system has the form

$$\begin{vmatrix} 0 & p & q \\ r & 0 & s \\ t & u & 0 \end{vmatrix} = pst + qru$$

and does not vanish because all its elements are obviously positive. Hence $a = a_1$, $b = b_1$, $c = c_1$, which proves the statement.

Amer. Math. Monthly, 89(1982) 601.

Unequal Trigonometric Sums

E 2874. Proposed by Naoki Kimura and Tetsundo Sekiguchi, University of Arkansas Let $n \ge 3, 0 < A_i \le \pi/2, i = 1, 2, ..., n$. Assume $\sum_{i=1}^n \cos^2 A_i = 1$. Prove

$$\sum \tan A_i \ge (n-1) \sum \cot A_i$$

Solution by M. S. Klamkin, University of Alberta, and V. Pambuccian, student (Rumania) (independently). The inequality can be rewritten as

$$\sum \frac{1}{\sin A_i \cos A_i} \ge n \sum \frac{\cos A_i}{\sin A_i} \tag{1}$$

Since by Cauchy's inequality,

$$\sum \sin A_i \cos A_i \sum \frac{1}{\sin A_i \cos A_i} \ge n^2$$

(1) will follow from the stronger inequality

$$n \ge \sum \sin A_i \cos A_i \sum \frac{\cos A_i}{\sin A_i} \tag{2}$$

By letting $x_i = \cos^2 A_i$ and $S = \sum x_i$ (2) can be expressed in the homogeneous form

$$\frac{1}{n}\sum x_i \ge \left\{\frac{1}{n}\sum \sqrt{x_i(S-x_i)}\right\} \left\{\sum \sqrt{x_i/(S-x_i)}\right\}$$
(3)

Finally, we can assume $x_1 \leq x_2 \leq \cdots \leq x_n$ so that $\{x_i(S - x_i)\}$ and $x_i/(S - x_i)$ are monotonic in the same sense and (3) follows by Chebyshev's inequality with equality if and only if x_i are all equal. (See Hardy, Littlewood and Pólya, *Inequalities*, pp.43–44.)

REMARKS: By Cauchy's inequality again, (3) interpolates the power mean inequality

$$\frac{1}{n}\sum x_i \ge \{\frac{1}{n}\sum \sqrt{x_i}\}^2$$

Inequality (3) can be extended to

$$\frac{1}{n}\sum x_i \ge \{\frac{1}{n}\sum x_i^r (S-x_i)^s\}\{\frac{1}{n}\sum x_i^{1-r} (S-x_i)^{-s}\}$$

where $1 \ge r \ge s \ge 0$. Actually it even suffices to take $r/s \ge \max x_i/(S - x_i)$ (this ensures that $x^r(S - x)^s$ is an increasing function).

[L.]Kuipers showed that $n\sqrt{n-1}$ is at the same time a lower bound for the left member and an upper bound for the right member of the inequality in the proposal.

Amer. Math. Monthly, **90**(1983) 54.

E 2981. Proposed by M. S. Klamkin, University of Alberta, Canada

If the three medians of a spherical triangle are equal, must the triangle equilateral? Note that the sides of a (proper) spherical triangle are **minor** arcs of great circles and thus its perimeter is $< 2\pi$.

Amer. Math. Monthly, **93**(1986) 300–302.

Composite solution. Several incorrect solutions were submitted. The following is a composite solution, portions of which were contributed by C. Gorsch, W. Meyer, the proposer and the editors.

No, surprisingly the triangle need not be equilateral; however, it must be isosceles, and is otherwise severely limited.

Let a, b, c denote the angles subtended at the centre of the sphere by the sides of the triangle; let m_a , m_b , m_c likewise denote the angles subtended by the medians from a, b, c respectively.

Using dot products, or the spherical law of cosines, or other means, the following may easily be shown:

$$\cos m_a = \frac{\cos b + \cos c}{2\cos a/2} \qquad \cos m_b = \frac{\cos c + \cos a}{2\cos b/2} \qquad \cos m_c = \frac{\cos a + \cos b}{2\cos c/2}$$

If the medians are equal—i.e., $m_a = m_b = m_c$ — then

$$\frac{\cos b + \cos c}{2\cos a/2} = \frac{\cos c + \cos a}{2\cos b/2} = \frac{\cos a + \cos b}{2\cos c/2}$$

and conversely.

It is clear that these equations hold if a = b = c. Moreover, they cannot hold if a, b, c are all different. However, we will show that they may hold if two are equal but the third is different, i.e., the triangle is isosceles but not equilateral.

Suppose, then, that b = c. The condition for equality of the medians becomes

$$\frac{\cos a + \cos b}{2\cos c/2} = \frac{\cos b + \cos c}{2\cos a/2}$$

Let $x = \cos a$ and $y = \cos b = \cos c$. Then, using the half-angle formula, and cancelling common twos, we obtain

$$\frac{2y}{\sqrt{1+x}} = \frac{x+y}{\sqrt{1+y}}$$

The graph of this equation may be shown by standard methods of analytic geometry to consist of the line y = x together with a portion of the ellipse $x^2 + 3xy + 4y^2 + x + 3y = 0$.

The line may be disregarded completely, as it corresponds to the case of equilateral triangles. The major axis of the ellipse is inclined at 1/4 of a right angle clockwise from the *x*-axis. The ellipse is otherwise difficult to describe nicely; but it is easily verified that it contains the following points:

$$\begin{array}{rcl} A(0,0) & B(1,-1/2) & C(9/7,-6/7) & D(1,-1) \\ E(0,-3/4) & F(-1/2,-1/2) & G(-1,0) & H(-5/7,1/7) \end{array}$$

Points C and G are extreme in the x direction; points D and H are extreme in the y direction.

[[The ellipse is not too difficult to describe. Its centre is at (1/7, -3/7), the major and minor semi-axes are $2(3 \pm \sqrt{2})/7$ and have slopes $-\pi/8, 3\pi/8$. — R.]]

The relevant portion of the ellipse is below the line 2y = -(x+1)—the rest is introduced as an "extraneous root". This line passes through G and D.



Not even all points (x, y) on this portion of the ellipse will do, however. Some lead to improper spherical triangles. To be proper, the perimeter a + 2b cannot exceed 2π . On the other hand, $a \ge 2b$ would lead to a flat or impossible triangle. Hence $a/2 < b < \pi - a/s$. Since the cosine is decreasing over $[0, \pi]$,

$$\cos a/2 > \cos b > \cos(\pi - a/2) = -\cos a/2$$

Thus $\cos^2 b < \cos^2 a/2 = (1 + \cos a)/2$ or $2y^2 < x + 1$.

This corresponds to the region inside a parabola opening along the positive x axis and passing through (-1, 0) and $(1, \pm 1)$.

The parabola intersects the ellipse at G(-1,0), F(-1/2,-1/2) and D(1,-1). Point G is a double root and, though the parabola touches the ellipse here, it remains outside it. The parabola is inside the ellipse from F to D. Only the portion of the ellipse within the parabola corresponds to viable cases—this is the section from G to F. With this restriction, $a = \cos^{-1} x$ is between 120° and 180°, whereas $b = \cos^{-1} y$ is between 90° and 120°. Any such values correspond to triangles meeting the required condition.

C. Gorsch has noted that though these values lead to proper triangles with the required property, they are barely proper in that their perimeters are all very close to the limiting value 2π . In fact he alleges the existence of a single extremum at approximately $b = 102^{\circ}$ for which the perimeter is least; but even so the perimeter there is within 4° of 360° .

W. Meyer has noted that the triangle determined by the midpoints of the sides of the original triangle has the property that its two sides are 120° each and the remaining side is between 90° and 120° . He asserts that, in fact, an arbitrary triangle meeting these conditions may be given, and then a triangle with the desired medians-equal property may easily be circumscribed around it.

The proposer raises two related questions: whether the triangle must be equilateral if the three altitudes are equal; and likewise if the angle bisectors are equal. He alleges that in the former case, $\sin a = \sin b = \sin c$, and thus the triangle again need not be equilateral due to the rise-fall behavior of sine in $[0, \pi]$. He leaves open the apparently more difficult second question. Amer. Math. Monthly, **90**(1983) 569.

6440. Proposed by M. S. Klamkin, J. McGregor and A. Meir, University of Alberta Let F(x), G(x) be two functions in $L_1(-\infty, \infty)$ which satisfy

$$\int_{-\infty}^{\infty} F(x) \, dx = \int_{-\infty}^{\infty} G(x) \, dx = 1$$

Show that for any λ in (0,1) there is a set $E \subseteq (-\infty, \infty)$ such that

$$\int_{E} F(x) \, dx = \int_{E} G(x) \, dx = \lambda.$$

Amer. Math. Monthly, **92**(1985) 289–290.

Solution 1 by Gerald A. Edgar, The Ohio State University. By Liapunoff's theorem [see W. Rudin, Functional Analysis, second edition, McGraw-Hill, Theorem 5.5], the set S in \mathbb{R}^2 defined by

$$S = \left\{ \left(\int_E F(x) \, dx, \int_E G(x) \, dx \right) : E \text{ measurable} \right\}$$

is convex. Since $(0,0) \in S$ and $(1,1) \in S$, it follows that $(\lambda, \lambda) \in S$ for any $\lambda \in (0.1)$.

Solution 2 by the proposers. (This solution is more elementary than the above.) Let $A = \{x : F(x) > G(x)\}, B = \mathbb{R} \setminus A$, let $A_i = A \cap (-\infty, t), \quad B_t = B \cap (-\infty, t)$ and let

$$\alpha(t) = \int_{A_t} (F(x) - G(x)) \, dx \beta(t) = \int_{B_t} (G(x) - F(x)) \, dx$$

Then $\alpha(t)$ and $\beta(t)$ are monotone non-decreasing continuous functions with $\alpha(-\infty) = \beta(-\infty)$ and $\alpha(\infty) = \beta(\infty)$. Therefore, for every t there exists $\gamma = \gamma(t)$ such that $\alpha(t) = \beta(\gamma)$. Now set $C_t = A_t \cup B_{\gamma}$. The clearly $C_t \subset C_s$ when $t \leq s$ and $\mu(C_s) \to \mu(C_t)$ when $s \to t^+$. Hence the function

$$\int_{C_t} F(x) \, dx = \int_{C_t} G(x) \, dx =: H(t)$$

is continuous. Since $C_{-\infty} = \emptyset$ and $C_{\infty} = \mathbb{R}$, it follows that H(t) attains all its values between 0 and 1. Thus, for $\lambda \in (0, 1)$, there is a number τ such that $H(\tau) = \lambda$ and $E = C_{\tau}$ has the required property. Amer. Math. Monthly, **93**(1986) 812.

E 3180. Proposed by M. S. Klamkin, University of Alberta, Canada

If A, B, C are angles of a triangle, prove that

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \ge 1 + \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$$

Amer. Math. Monthly, **95**(1988) 771–773.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. We consider the function V defined by

$$V(A, B, C) = \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} - \left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right)$$

on the compact set S defined by $A \ge 0$, $B \ge 0$, $C \ge 0$ and $A + B + C = \pi$. V attains an absolute maximum and minimum; we will show that the minimum is 1 and the maximum is $3(\sqrt{3}-1)/2$.

Suppose C = 0, so that $A + B = \pi$. The identity $\cos(\pi/2 - x) = \sin x$ implies V(A, B, 0) = 1. Similarly V(0, B, C) = V(A, 0, C) = 1 so V is identically 1 on the boundary of S, and we need only consider interior points.

For fixed $C \neq 0$ we have

$$V(A, B, C) = \left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) - \left(\sin\frac{A}{2} + \sin\frac{B}{2}\right) + \left(\cos\frac{C}{2} - \sin\frac{C}{2}\right)$$
$$= 2\cos\frac{A-B}{4} \cdot \left(\cos\frac{A+B}{4} - \sin\frac{A+B}{4}\right) + \cos\frac{C}{2} - \sim\frac{C}{2}$$
$$= 2\cos\frac{A-B}{4} \cdot \left(\cos\frac{\pi-C}{4} - \sin\frac{\pi-C}{4}\right) + \cos\frac{C}{2} - \sin\frac{C}{2}$$

Since $|x| < \pi/4$ implies $\cos x > \sin x$, the last expression is minimized only when A = 0 or B = 0, and it is maximized for a fixed $A \neq 0$ when B = C. Hence V attains ite maximum only at $A = B = C = \pi/3$, where $V = 3/2(\sqrt{3} - 1)$. Thus we have $1 < V(A, B, C) \le 3/2(\sqrt{3} - 1)$, with lower equality only at a degenerate triangle and upper inequality only at an equilateral triangle.

Editorial Comment. Severa; readers used Lagrange multipliers to find the extrema of V. Lagrange's method seems particularly suitable for studying the extrema of such functions of the angles of a triangle. For example, it shows that the cosine sum and the sine sum separately satisfy the inequalities

$$2 < \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \le \frac{3}{2}\sqrt{3}$$
$$1 < \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$$

with lower equality only for a degenerate triangle and upper equality only for an equilateral triangle. H. Guggenheimer remarks that the difference between the maximum and minimum values of V is surprisingly small when compared to the corresponding differences in the inequalities just quoted. Amer. Math. Monthly, **94**(1987) 71.

E 3183*. Proposed by M. S. Klamkin, University of Alberta, Canada

Let P' denote the convex *n*-gon whose vertices are the midpoints of the sides of a given convex *n*-gon *P*. Determine the extreme values of

(i) Area P' / Area P.

(ii) Perimeter P' / Perimeter P.

Amer. Math. Monthly, 96(1989) 157-161.

Solution by David B. Secrest (student), University of Illinois, Urbana. The extreme values are summarized in the following table:

n	3	4	5	≥ 6
$\max\left(\frac{\operatorname{area} P'}{\operatorname{area} P}\right)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\min\left(\frac{\operatorname{area} P'}{\operatorname{area} P}\right)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\max\left(\frac{\operatorname{perim} P'}{\operatorname{perim} P}\right)$	$\frac{1}{2}$	1	1	1
$\min\left(\frac{\operatorname{perim} P'}{\operatorname{perim} P}\right)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

We divide the proof into seven cases.

Case 1. n = 3. The triangle P' is similar to P with sides exactly half as long; the conclusion is immediate for all four extrema.

Case 2. n = 4, Area Ratio. Given a quadrilateral $E_1E_2E_3E_4$, let E'_1 , E'_2 , E'_3 , E'_4 be the midpoints of the sides $\overline{E_1E_2}$, $\overline{E_2E_3}$, $\overline{E_3E_4}$, $\overline{E_4E_1}$ respectively. By similar triangles, area $(E'_1E_2E'_2) = \frac{1}{4}$ area $(E_1E_2E_3)$ and

$$\operatorname{area}(E'_{3}E_{4}E'_{4}) = \frac{1}{4}\operatorname{area}(E_{3}E_{4}E_{1})$$

 \mathbf{SO}

$$\operatorname{area}(E_1'E_2E_2') + \operatorname{area}(E_3'E_4E_4') = \frac{1}{4}\operatorname{area}(E_1E_2E_3E_4)$$

Likewise

$$\operatorname{area}(E_4'E_1E_1') + \operatorname{area}(E_2'E_3E_3') = \frac{1}{4}\operatorname{area}(E_1E_2E_3E_4)$$

 \mathbf{SO}

$$\frac{\operatorname{area}(E_1'E_2'E_3'E_4')}{\operatorname{area}(E_1E_2E_3E_4)} = \frac{1}{2}$$

Case 3. $n \ge 6$, Maximum Area Ratio. A degenerate n-gon, $P = E_1 E_2 \dots E_n$, with $E_1 = E_2$, $E_3 = E_4$ and $E_5 = E_6 = \dots = E_n$ will have $\operatorname{area} P'/\operatorname{area} P = 1$, which is the largest the ratio can be if P is convex.

Case 4. $n \geq 5$, Minimum Area Ratio. Given an n-gon $P = E_1E_2...E_n$ draw all the "corner triangles", i.e. the ones made by connecting three consecutive vertices of the n-gon. Let X be some point inside corner triangle $E_2E_3E_4$. X cannot be inside any of the other corner triangles except possibly $E_1E_2E_3$ and $E_3E_4E_5$ because all the others are disjoint from $E_2E_3E_4$. However, $E_1E_2E_3$ and $E_3E_4E_5$ are disjoint, so X can be in at most one of them. In other words, no point inside the N-gon P can be inside more than two of the corner triangles. Thus

$$\sum \operatorname{area}(\operatorname{corner triangles}) \le 2 \operatorname{area} P.$$

But the triangles made by connecting midpoints of adjacent sides of P (which will be referred to hereafter as "midpoint corner triangles") each has 1/4 the area of the corresponding corner triangle, so that

$$\sum \operatorname{area(midpoint\ corner\ triangles)} \le \frac{1}{2}\operatorname{area} P$$

Adding area P' to both sides of this inequality and noticing that P' and the midpoint corner triangles together make up P, we get

$$\operatorname{area} P \leq \frac{1}{2} \operatorname{area} P + \operatorname{area} P'$$

or

$$\frac{\operatorname{area} P'}{\operatorname{area} P} \ge \frac{1}{2}$$

It is possible to attain this minimum ratio of $\frac{1}{2}$ by setting $E_3 = E_4 = \cdots = E_n$

Case 5. n = 5, Maximum Area Ratio. Given a convex pentagon $P = E_1 E_2 E_3 E_4 E_5$, if we could prove that

$$\operatorname{area}(E_1 E_3 E_5) \le \operatorname{area}(E_5 E_1 E_2) + \operatorname{area}(E_2 E_3 E_4) + \operatorname{area}(E_4 E_5 E_1)$$

then, by adding $\operatorname{area}(E_1E_3E_5) + \operatorname{area}(E_3E_4E_5)$ to both sides, we could get

area
$$P \leq \sum \operatorname{area}(\operatorname{corner triangles})$$

or

$$\frac{1}{4}\operatorname{area} P + \operatorname{area} P' \leq \operatorname{area} P \quad \text{or} \quad \operatorname{area} P'/\operatorname{area} P \leq \frac{3}{4}$$

We will show that by choosing a suitable labelling of the vertices of P, we can prove the stronger result, that

$$\operatorname{area}(E_1 E_3 E_5) \le \operatorname{area}(E_5 E_1 E_2) + \operatorname{area}(E_2 E_3 E_4)$$
 (1)

In the convex pentagon P there must be a pair of adjacent angles that add to more than π since the average sum of pairs of adjacent angles in a pentagon is $6\pi/5$. Assume $\angle E_4 + \angle E_5 > \pi$. (see FIGURE 1).

[[Warning: The label ' E_5 ' is misplaced in the original figure. It is corrected in the figure below. — R.]]

This implies that the extension of $\overline{E_1E_5}$ beyond E_5 and the extension of $\overline{E_3E_4}$ beyond E_4 intersect. Draw the lines ℓ_1 and ℓ_2 through E_2 and parallel to $\overline{E_1E_5}$ and $\overline{E_3E_4}$ respectively. Draw $\overline{E_1E_3}$. Pick a point D on $\overline{E_1E_3}$ so that it inside the parallelogram formed by ℓ_1 , ℓ_2 and the extensions of $\overline{E_1E_5}$ and $\overline{E_3E_4}$. Draw lines through E_4 and through E_5 parallel to $\overline{E_1E_3}$. By performing a reflection of the picture if necessary, we can assume that E_5 is closer to $\overline{E_1E_3}$ than E_4 is.

We establish some inequalities:

$$\operatorname{area}(DE_3E_5) \le \operatorname{area}(DE_3E_4)$$
 (2)

because they share a common base $\overline{DE_3}$ but E_4 is farther away from that base. Likewise, for the base $\overline{E_5E_1}$,

$$\operatorname{area}(E_5 E_1 D) \le \operatorname{area}(E_5 E_1 E_2) \tag{3}$$

and, for the base $\overline{E_3E_4}$

$$\operatorname{area}(DE_3E_4) \le \operatorname{area}(E_2E_3E_4) \tag{4}$$

Now (2) and (4) yield area $(DE_3E_5) \leq \operatorname{area}(E_2E_3E_4)$. By adding (3) to the last inequality we get (1) and thus area $P'/\operatorname{area} P \leq \frac{3}{4}$.

It is possible to attain this maximum ratio by setting $E_1 = E_2$ and $E_3 = E_4$.

Case 6. $m \ge 4$, Maximum Perimeter Ratio. An n-gon $P = E_1 E_2 \dots E_n$ which has $E_1 = E_2$ and $E_3 = E_4 = \dots = E_n$ will have perim P = perim P'.

It is impossible to get a larger ratio than 1 since, by the triangle inequality, the length of one of the sides of P' is at most that of the two half sides of P that it replaces. (This argument works even if P is not convex.)

Case 7. $n \ge 4$, Minimum Perimeter Ratio. Given an n-gon $P = E_1 E_2 \dots E_n$ let D_1 be the point of intersection of $\overline{E_n E_2}$ and $\overline{E_1 E_3}$ and in general D_k the point of intersection of $\overline{E_{k-1} E_{k+1}}$ and $\overline{E_k E_{k+2}}$ where the subscripts are taken mod n. (See FIGURE 2.) By the triangle inequality,

$$E_1E_2 \leq D_1E_2 + E_1D_1$$

$$E_2E_3 \leq D_2E_3 + E_2D_2$$

$$\vdots$$

$$E_nE_1 \leq D_nE_1 + E_nD_n$$

 E_4

Also, $0 \leq D_1 D_2 + D_2 D_3 + \dots + D_n D_1$. Adding we obtain

 E_4

perim
$$P \leq (D_1E_2 + E_1D_1) + D_1D_2 + (D_2E_3 + E_2D_2) + D_2D_3 + \cdots$$

$$= D_1E_2 + (E_1D_1 + D_1D_2 + D_2E_3) + (E_2D_2 + D_2D_3 + D_3E_4) + \cdots$$

$$= E_1E_3 + E_2E_4 + \cdots + E_nE_2$$

$$= 2 \text{ perim } P'$$

 \mathbf{SO}

$$\operatorname{perim} P'/\operatorname{perim} P \ge \frac{1}{2}$$

If $E_1 E_2 \dots E_n$ is a polygon with $E_2 = E_3 = \dots = E_n$ then perim P' = 1/2 perim P.
Amer. Math. Monthly, **94**(1987) 382.

One Cubic Factorization Implies Another

E 3083 [1985, 286]. Proposed by Gunnar Blom, University of Lund and Lund Institute of Technology, Sweden

If the relation

$$x^{3} + y^{3} + z^{3} = (x + y)(x + z)(y + z)$$

is satisfied by x_0, y_0, z_0 where

$$(x_0^2 + y_0^2 - z_0^2)(x_o^2 + z_0^2 - y_0^2)(y_0^2 + z_0^2 - x_0^2) \neq 0$$

then it is also satisfied by

$$x_1 = \frac{1}{y_0^2 + z_0^2 - x_0^2}$$
 $y_1 = \frac{1}{x_0^2 + z_0^2 - y_0^2}$ $z_1 = \frac{1}{x_0^2 + y_0^2 - z_0^2}$

Observation by M. S. Klamkin, University of Alberta, Canada. This problem is given as problems 194 and 196 in Wolstenholme, Mathematical Problems, Cambridge University Press, London (1878 or 1891).

194. Having given the equations

$$\begin{aligned} x^3 + y^3 + z^3 &= (y+z)(z+x)(x+y) \\ a(y^2 + z^2 - x^2) &= b(z^2 + x^2 - y^2) = c(x^2 + y^2 - z^2) \end{aligned}$$

prove that

$$a^{3} + b^{3} + c^{3} = (b + c)(c + a)(a + b)$$

196. If $x = b^2 + c^2 - a^2$, $y = c^2 + a^2 - b^2$ and $z = a^2 + b^2 - c^2$ prove that

$$y^{3}z^{3} + z^{3}x^{3} + x^{3}y^{3} - xyz(y+z)(z+x)(x+y)$$

is the product of four factors, one of which is

$$4abc + (b + c - a)(c + a - b)(a + b - c)$$
$$(= (b + c)(c + a)(a + b) - a^{3} - b^{3} - c^{3})$$

and the other three are formed from this by changing the signs of a, b, c respectively. All of these assertions can be proved by direct verification.

Amer. Math. Monthly, **94**(1987) 384–385.

Maximizing a Cyclic Sum of Powers of Differences

E 3087 [1985, 287]. Proposed by Weixuan Li, University of Waterloo, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada

Let $a_i, i = 1, 2, ..., n$ be real numbers such that $0 \le a_i \le 1$, where $n \ge 2$. Find a best upper bound for

$$S_n = (a_1 - a_2)^2 + (a_2 - a_3)^2 + \dots + (a_n - a_1)^2$$

and determine all cases in which this bound is attained.

Solution by M. S. Klamkin and A. Meir, University of Alberta, Edmonton, Alberta, Canada. More generally we find the maximum of the cyclic sum

$$S(n,p) = |a_1 - a_2|^p + |a_2 - a_3|^p + \dots + |a_n - a_1|^p$$

Clearly the case n = 2m is trivial for which max S(n, p) = 2m and achieved by either

$$a_1 = a_3 = \dots = a_{2m-1} = 1$$
 $a_2 = a_4 = \dots = a_{2m} = 0$

or vice-versa. For the odd case, we show that

$$\max S(2m+1,p) = \begin{cases} 2m-1+2^{1-p} & \text{for } 0 1 \end{cases}$$
(1)

Proof. Suppose that the above maximum is achieved for the values $a_1, a_2, \ldots, a_{2m+1}$. Then either (i) all a_i are 0 or 1, or (ii) there exists an a_i with $0 < a_i < 1$. In case (i), $a_v = a_{v+1}$ for some v and thus $S(2m+1,p) \leq 2m$ for any p > 0. In case (ii), let $0 < a_v < 1$. Then $a_{v-1} < a_v$ and $a_{v+1} < a_v$ is impossible since $a_v = 1$ would yield a larger sum. Similarly $a_{v-1} > a_v$ and $a_{v+1} > a_v$ is impossible. Thus $a_{v-1} > a_v > a_{v+1}$ or vice-versa. In both cases,

$$|a_{v-1} - a_v|^p + |a_v - a_{v+1}|^p \le \begin{cases} |a_{v-1} - a_{v+1}| \le 1 & \text{if } p > 1\\ 2\left\{\frac{1}{2}|a_{v-1} - a_{v+1}|\right\}^p \le 2^{1-p} & \text{if } 0$$

Since clearly $\sum_{i \neq v-1,v} |a_i - a_{i+1}|^p \leq 2m - 1$ for all p > 0, inequality (1) follows, the bound being attained in the case $0 by taking, for example, <math>a_1 = 0$, $a_2 = \frac{1}{2}$, $a_{2k} = 0$, $a_{2k+1} = 1$, $2 \leq k \leq m$.

Amer. Math. Monthly, **94**(1987) 996.

E 3239. Proposed by M. S. Klamkin, University of Alberta

Show that if A is any three-dimensional vector and B, C are unit vectors, then

$$[(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} + \mathbf{C})] \times (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{B} + \mathbf{C}) = 0$$

Interpret the result as a property of spherical triangles.

Amer. Math. Monthly, **97**(1990) 531–532.

Solution by Walter Janous, Ursulinengymnasium, Innsbruck, Austria. Starting from the formula

$$(\mathbf{D} \times \mathbf{E}) \times \mathbf{F} = (\mathbf{F} \cdot \mathbf{D})\mathbf{E} - (\mathbf{F} \cdot \mathbf{E})\mathbf{D}$$

we get

$$\begin{aligned} [(\mathbf{A} + \mathbf{B}) &\times & (\mathbf{A} + \mathbf{C})] \times [\mathbf{B} \times \mathbf{C}] \\ &= & [(\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{A} + \mathbf{B})][\mathbf{A} + \mathbf{C}] - [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} + \mathbf{C})][\mathbf{A} + \mathbf{B}] \\ &= & [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}][\mathbf{A} + \mathbf{C}] - [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}][\mathbf{A} + \mathbf{B}] \\ &= & [(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}][\mathbf{C} - \mathbf{B}] \end{aligned}$$

whence the result follows, since

$$(\mathbf{C} - \mathbf{B}) \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{C} \cdot \mathbf{C} - \mathbf{B} \cdot \mathbf{B} = 1 - 1 = 0$$

Editorial comment. No two solvers gave the same "interpretation of the result as a property of spherical triangles". The proposer showed that the result is equivalent to the assertion that a great circle which bisects two sides of a spherical triangle intersects the third side 90° from its midpoint. His argument follows.

First, if we have two great circles on a unit sphere determined by the two pairs of points B', C' and B, C, then the two antipodal points of intersection of these circles lie on the line ℓ of intersection of the planes of B', C', O and B, C, O where O is the centre of the sphere. Then a vector \mathbf{V} on ℓ is given by $\mathbf{V} = (\mathbf{B}' \times \mathbf{C}') \times (\mathbf{B} \times \mathbf{C})$. (Here \mathbf{B} denotes the vector from O to the point B, etc.)

Second, let the respective midpoints of the sides of the spherical triangle ABC be A', B', C'. Then

$$A' = (B + C)/|B + C|$$
 $B' = (A + C)/|A + C|$ $C' = (C + B)/|A + B|$

and $\mathbf{V} \cdot \mathbf{A}' = 0$. The equivalence now follows.

Amer. Math. Monthly, **95**(1988) 358.

An Exponential Inequality

E 3151 [1986, 401]. Proposed by Peter Ivady, Institute for Economy and Organization, Budapest, Hungary

Let $x \ge 0, x \ne 1, \lambda \ge 1$ and $0 \le \beta \le 2$ be real numbers. Prove that

$$\left(\frac{x^{\lambda}-1}{x-1}\right)^{\beta} \le \lambda \left(\frac{x^{\lambda\beta}-1}{x^{\beta}-1}\right)$$

Solution by M. S. Klamkin, University of Alberta. Replacing x by 1/x leaves the inequality unchanged, so it suffices to consider only x > 1 (it is trivial for x = 0). Because $(e^{2\lambda at}-1)/(e^{2at}-1) = (e^{\lambda at}/e^{at}) \cdot (\sinh \lambda at/\sinh at)$, the hyperbolic substitution $x = e^{2t}$ converts the inequality to

$$\lambda \frac{\sinh \lambda \beta t}{\sinh^{\beta} \lambda t} \ge \frac{\sinh \beta t}{\sinh^{\beta} t} \tag{1}$$

for t > 0, $\lambda \ge 1$ and $2 \ge \beta \ge 0$. Equation (1) holds with equality for $\lambda = 1$, so it suffices to show that the left side is a nondecreasing function of λ , or equivalently that its logarithmic derivative with respect to λ is non-negative, i.e.,

 $(1/\lambda) + \beta t \coth \lambda \beta t - \beta t \coth \lambda t \ge 0$. By multiplying through by $\lambda \sinh \lambda t \cdot \sinh \lambda \beta t$ and using the addition formula for sinh, we transform this inequality into

$$\sinh \lambda t \sinh \lambda \beta t \ge \lambda \beta t \sinh \lambda t (\beta - 1) \tag{2}$$

Since sinh is negative for negative arguments, (2) holds for $1 \ge \beta \ge 0$ and we need only consider $2 \ge \beta \ge 1$. At $\beta = 2$, (2) reduces to $(\sinh \lambda t)(\sinh 2\lambda t - 2\lambda t) \ge 0$ which follows from $\sinh y \ge y$ for $y \ge 0$. To establish (2) for $2 > \beta \ge 1$ it suffices to show that the logarithmic derivative of the left side with respect to β is less than that of the right side. This reduces to showing

$$\frac{\lambda t}{\tanh \lambda \beta t} \le \frac{1}{\beta} + \frac{\lambda t}{\tanh \lambda t(\beta - 1)}$$

This follows immediately from the fact that tanh is an increasing function.

Amer. Math. Monthly, **95**(1988) 658–659.

Weizenböck Generalized

E 3150 [1986, 400]. Proposed by George A. Tsintsifas, Thessaloniki, Greece

Let ABC be a triangle with sides a, b, c and area F. It is well known that $a^2 + b^2 + c^2 \ge 4F\sqrt{3}$. If p, q, r are arbitrary positive real numbers, prove that

$$\frac{p}{q+r}a^2 + \frac{q}{r+p}b^2 + \frac{r}{p+q} \ge 2F\sqrt{3}$$

Solution III and generalization by M. S. Klamkin, University of Alberta, Canada. Replace a, b, c by a_1, a_2, a_3 and p, q, r by p_1, p_2, p_3 . We derive more generally an inequality for $S + \sum p_i a_i^{4n}/(k-p_i)$ where $1 \ge n \ge 0$ and $a_i \ge 0$. By Cauchy's inequality, we have $2(S + \sum a_i^{4n}) = \sum (k-p_i) \sum a_i^{4n}/(k-p_i) \ge (\sum a_i^{2n})^2$. Letting $R = \sum a_i^n$ this is equivalent to

$$2S|geqR\prod(R-2a_i^n)\tag{1}$$

with equality if and only if $a_1^{2n}/(p_2 + p_3) = a_2^{2n}/(p_3 + p_1) = a_3^{2n}/(p_1 + p_2)$. If the $\{a_i^n\}$ are not the sides of a triangle, then the right side of (1) is negative. Suppose that $\{a_i\}$ form a triangle and that $1 \ge n \ge 0$. Then $\{a_i^n\}$ also form a triangle whose area we denote by F_n . The right side of (1) is 16 times the square of the side-length formula for area, yielding $2S \ge 16F_n^2$.

We now use Oppenheim's generalization [1] of the Finsler-Hadwiger inequality, i.e., $4F_n/\sqrt{3} \ge (4F/\sqrt{3})^n$ for $1 \ge n \ge 0$, with equality for n < 1 if and only if $a_1 = a_2 = a_3$. With (1) this yields the generalization

$$2S \ge 3(4F/\sqrt{3})^{2n} \qquad (0 \le n \le 1) \tag{2}$$

The proposed inequality corresponds to the special case n = 1/2. The special case n = 1 corresponds to the proposer's problem #1051 in *Crux Mathematicorum*, **11**(1985) 187.

[1.] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., No.461-497(1974) 257– 263; MR 51 #6552.

The inequality $a^2 + b^2 + c^2 \ge 4F\sqrt{3}$ goes back to Weitzenböck, Math. Z., 5(1919) 137–146.

Amer. Math. Monthly, **96**(1989) 55.

E 3305. Proposed by M. S. Klamkin, University of Alberta, Edmonton

If a, b, c are the sides of a triangle with given semiperimeter s, determine the maximum values of

(i)
$$(b-c)^2 + (c-a)^2 + (a-b)^2$$

(ii) $|(b-c)(c-a)| + |(c-a)(a-b)| + |(a-b)(b-c)|$
(iii) $(b-c)^2(c-a)^2(a-b)^2$

Amer. Math. Monthly, 97(1990) 349.

Solution by David Callan, University of Bridgeport, Bridgeport, CT. The maximum vakues are $2s^2$, s^2 , $s^6/108$ respectively. Let a, b, c denote the sides in decreasing order. Note that $s/2 \leq b \leq s$. Form a new (degenerate) triangle with the same perimeter and middle side, the new sides in decreasing order being s, b, s - b. In switching to the new triangle, no difference between side-lengths decreases; in fact, each difference increases if the original triangle is nondegenerate. Hence each part of the problem is the maximization of a polynomial in b over [s/2, s]. The polynomials are:

(i)
$$(2b-s)^2 + b^2 + (s-b)^{@}$$

(ii) $b(2b-s) + b(s-b) + (s-b)(2b-s)$

(iii)
$$(2b-s)^2b^2(s-b)^2$$

These maximizations can be done by elementary calculus, giving maxima as listed above, occurring at b = s, s, $(3 + 3^{1/2})s/6$ respectively.

Editorial comment. Most solvers treated the problem as a constrained maximization problem in several variables.

Amer. Math. Monthly, **96**(1989) 154.

E 3307. Proposed by Peter Andrews, Wilfrid Laurier University, Murray Klamkin, University of Alberta, Edmonton, Alberta, and Edward T. H. Wang, Wilfrid Laurier University

The celebrated Morley triangle of a given triangle ABC is the equilateral triangle whose vertices are the intersections of adjacent pairs of internal angle trisectors of ABC. If s, R, r, F and s_M, R_M, r_M, F_M are the semiperimeter, the circumradius, the inradius, and the area, respectively, of ABC and its Morley triangle, determine the maximum of (i) s_M/s , (ii) R_M/R , (iii) r_M/r and (iv) F_M/F .

Amer. Math. Monthly, **98**(1991) 755–760.

Editorial remark. The first, second and fourth ratios considered in the problem achieve their maxima when $A = B = C = \pi/3$ and have greatest lower bound zero. The third ratio, r_M/r , achieves a positive minimum when $A = B = C = \pi/3$ and has least upper bound equal to 2/9. The calculation of these extremaare given in their approximate order of difficulty.

Solution of (ii) by the proposers. We prove that

$$R_M/R \le (8/\sqrt{3})\sin^3(\pi/9)$$

with equality only when the original triangle is equilateral. We require the known fact that the Morley triangle has side length $8R\sin(A/3)\sin(B/3)\sin(C/3)$. (Cf. [1], [2] or [3] in the list of references.) Since the circumradius of an equilateral triangle is $1/\sqrt{3}$ times its side-length, we have

$$R_M/R = (8/\sqrt{3})\sin(A/3)\sin(B/3)\sin(C/3)$$

Now if $g_2(x) = \log \sin(x/3)$, then $g''_2(x) = -\{3\sin(x/3)\}^{-2}$. Since $g''_2(x) < 0$ on $(0, \pi)$, we have by concavity

$$g_2(A) + g_2(B) + g_2(C) \le 3g_2(\{A + B + C\}/3) = 3g_2(\pi/3)$$

or

 $\sin(A/3)\sin(B/3)\sin(C/3) \le \sin^3(\pi/9)$

with equality only if $A = B = C = \pi/3$. Thus the claimed result follows.

Solution of (iii) by the proposers. We prove that

$$(8/\sqrt{3})\sin^3(\pi/9) \le r_M/r < 2/9$$

with equality on the left only if the original triangle is equilateral. We begin by using the formulas $r_M = R_M/2$ and $r = 4R \sin(A/2) \sin(B/2) \sin(C/2)$ to obtain

$$r_M/r = (1/8)(R_M/R)/\{\sin(A/2)\sin(B/2)\sin(C/2)\}$$

or, by the solution of part (ii)

$$\frac{r_m}{r} = \frac{1}{\sqrt{3}} \frac{\sin(A/3)\sin(B/3)\sin(C/3)}{\sin(A/2)\sin(B/2)\sin(C/2)}$$

Now if $g_3(x) = \log\{\sin(x/3)/\sin(x/2)\}$, then

$$g_3''(x) = -\{3\sin(x/3)\}^{-2} + \{2\sin(x/2)\}^{-2} \\ = \frac{\{\sin(x/3)/(x/3)\}^2 - \{\sin(x/2)/(x/2)\}^2}{\{6x^{-1}\sin(x/3)\sin(x/2)\}^2}$$

Since $(\sin t)/t$ is strictly decreasing on $(0, \pi/2)$, it follows that $g''_3(x) > 0$ for $0 < x < \pi$. Hence by convexity

$$g_3(A) + g_3(B) + g_3(C) \ge 3g_3(\{A + B + C\}/3) = 3g_3(\pi/3)$$

or

$$\frac{\sin(A/3)\sin(B/3)\sin(C/3)}{\sin(A/2)\sin(B/2)\sin(C/2)} \ge \left\{\frac{\sin(\pi/9)}{\sin(\pi/6)}\right\}^3 = 8\sin^3(\pi/9)$$

Thus we have the lower bound $r_M/r \ge (8/\sqrt{3}) \sin^3(\pi/9)$ with equality only if $A = B = C = \pi/3$.

To get an upper bound we note that if $0 < A \leq C \leq B < \pi$ and if ϵ is a small positive number, then

$$g_3(A - \epsilon) + g_3(B + \epsilon) - g_3(A) - g_3(B)$$

= $\int_0^{\epsilon} \{g'_3(B + t) - g'_3(A - t)\} dt > 0$

since g'_3 is strictly increasing on $(0, \pi)$. Thus r_M/r increases if we increase the largest angle of the triangle and decrease another angle by the same amount. Hence r_M/r is less than the value we get for the degenerate triangle with angles $0, 0, \pi$. Thus

$$\frac{r_M}{r} < \frac{1}{\sqrt{3}} \frac{2}{3} \frac{2}{3} \frac{\sqrt{3}}{2} = \frac{2}{9}$$

Combining our two results we obtain the result claimed. Not that r_M/r is confined to the relatively narrow interval (0.18479,0.22223).

Solution of (i) by the editors. We prove that

$$s_M/s \le (8/\sqrt{3})\sin^3(\pi/9)$$

with equality if and only if $A = B = C = \pi/3$. We have

$$s_M/s = \{24R\sin(A/3)\sin(B/3)\sin(C/3)\}/(a+b+c) \\ = \{12\sin(A/3)\sin(B/3)\sin(C/3)\}/\{\sin A + \sin B + \sin C\} \\ = \{3\sin(A/3)\sin(B/3)\sin(C/3)\}/\{\cos(A/2)\cos(B/2)\cos(C/2)\}$$

If
$$g_1(x) = \log\{\sin(x/3)/\cos(x/2)\}$$
, then $g'_1(x) = (1/2)\tan(x/2) + (1/3)\cot(x/3)$ and
 $g''_1(x) = \{2\cos(x/2)\}^{-2} - \{3\sin(x/3)\}^{-2}$

which does not have constant sign on $(0.\pi)$; specifically $g_1''(x) < 0$ for $0 < x < \theta_1 = 6 \arcsin(1/4) = 1.516...$ and $g_1''(x) > 0$ for $\theta_1 < x < \pi$. Thus simple convexity-concavity arguments alone do not suffice for (i) and we must make a more detailed analysis by treating three cases:

- (a) each of A, B, C is at most θ_1
- (b) two of A, B, C exceed θ_1
- (c) exactly one of A, B, C exceeds θ_1

Suppose (a) holds. Since $g''_1(x) < 0$ for $0 < x < \theta_1$ we have by concavity

$$g_1(A) + g_1(B) + g_1(C) \le 3g_1(\{A + B + C\}/3) = 3g_1(\pi/3)$$

[[The penult '3' is missing from the original. - R.]]

and it follows that in this case

$$s_m/s \le 3 \frac{\sin^3(\pi/9)}{\cos^3(\pi/6)} = \frac{8}{\sqrt{3}} \sin^3(\pi/9) = 0.18479\dots$$

with equality holding if and only if $A = B = C = \pi/3$.

Next suppose that (b) holds, say $A \ge B > \theta_1$. It follows that

$$\begin{array}{rcl} A &=& \pi-B-C<\pi-\theta_1\\ &B &<& (A+B+C)/2=\pi/2\\ &C=\pi-A-B<\pi-2\theta_1 \end{array}$$

Since $g'_1(x) > 0$ for $0 < x < \pi$ we have

$$\frac{\sin(A/3)}{\cos(A/2)} < \frac{\sin((\pi - \theta_1)/3)}{\cos((\pi - \theta_1)/2)} < 0.76$$
$$\frac{\sin(B/3)}{\cos(B/2)} < \frac{\sin(\pi/6)}{\cos(\pi/4)} = \sqrt{2}/2$$
$$\frac{\sin(C/3)}{\cos(c/2)} < \frac{\sin((\pi - 2\theta_1)/3)}{\cos((\pi - 2\theta_1)/2)} < 0.037$$

It follows that in case (b) we have $s_M/s < 0.06$, a bound smaller than that found in case (a).

Finally suppose that (c) holds. Say $A > \theta_1$ and $B, C < \theta_1$. Since $g''_1(x) < 0$ on $(0, \theta_1)$, we have by concavity

$$g_1(B) + g_1(C) \le 2g_1((B+C)/2)$$

with strict inequality for $B \neq C$. Thus we may assume B = C. Since $A > \theta_1$

$$B = \frac{1}{2}(\pi - A) < \frac{1}{2}(\pi - \theta_1) = 0.81275\dots$$

If we set $f_1(x) = 2g_1(x) + g_1(\pi - 2x)$, we have $s_M/s = 3 \exp f_1(B)$. Now $f'_1(x) = 2g'_1(x) - 2g'_1(\pi - 2x)$ and we claim that $f'_1(x) > 0$ for $0 < x \le (\pi - \theta_1)/2$. Indeed

$$g_1'(x) = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{3}\cot\frac{x}{3} > \frac{x}{4} + \frac{1 - \frac{x^2}{18}}{\frac{x}{3}} > \frac{1}{x}$$

and

$$g_1'(\pi - 2x) = \frac{1}{2}\cot x + |frac_1| \tan\left(\frac{\pi}{6} + \frac{2x}{3}\right) < \frac{1}{2x} + \frac{1}{3}\tan\left(\frac{\pi}{6} + \frac{2x}{3}\right)$$

For $0 < x \le 0.82$ we have

$$f_1'(x) > \frac{1}{x} - \frac{2}{3} \tan\left(\frac{\pi}{6} + \frac{2x}{3}\right) > \frac{1}{0.82} - \frac{2}{3} \tan\left(\frac{\pi}{6} + \frac{1.64}{3}\right) > 0.007$$

Thus. in this case

$$s_M/s < 3 \exp f\left(\frac{\pi}{2} - \frac{\theta_1}{2}\right) = 0.16976 \dots < 0.18479 \dots$$

and hence the maximum of s_M/s can occur only in case (a) with A = B = C. Thus the claimed result $s_M/s \leq (8/\sqrt{3}) \sin^3(\pi/9)$ is established.

Solution of (iv) by the editors. We prove that

$$F_M/F \le (64/3)\sin^6(\pi/9) = 0.034148\dots$$

with equality if and only if ABC is equilateral. Our argument is similar to that for (i), but is a little more complicated in case (c).

We have

$$F = abc/(4R) = 2R^2 \sin A \sin B \sin C$$

and

$$F_M = (\sqrt{3}/4) \{8R\sin(A/3)\sin(B/3)\sin(C/3)\}^2$$

so that

$$\frac{F_M}{F} = 8\sqrt{3} \frac{\{\sin(A/3)\sin(B/3)\sin(C/3)\}^2}{\sin A \sin B \sin C}$$

If $g_4(x) = \log\{\sin^2(x/3)/\sin x\}$, then $g'_4(x) = (2/3)\cot(x/3) - \cot x$ and

$$g_4''(x) = {\sin x}^{-2} - 2{3\sin(x/3)}^{-2}$$

Now $g_4''(x)$ is negative for

$$0 < x < 3 \arcsin\left(\{3(2-\sqrt{2})/8\}^{1/2}\right) = 1.46342\ldots = \theta_4$$

and is positive for $\theta_4 < x < \pi$. We treat three cases.

(a) If each of A, B, $C \leq \theta_4$, then, since $g''_4(x) < 0$ for $0 < x < \theta_4$ we have by concavity

$$g_4(A) + g_4(B) + g_4(C) \le 3g_4(\{A + B + C\}/3) = 3g_4(\pi/3)$$

with equality if and only if A = B = C. Thus in this case

$$F_M/F \le 8\sqrt{3} \exp\{3g_4(\pi/3)\} = (64/3) \sin^6(\pi/9)$$

(b) If $A \ge B > \theta_4$ then $A < \pi - \theta_4$, $B < \pi/2$ and $C < \pi - 2\theta_4$. Now $g'_4(x) > 0$ for $0 < x < \pi/2$ since $\tan(x/3) < \tan x$ there, and $g'_4 > 0$ for $\pi/2 < x < \pi$, since $\cot(x/3) > 0$ and $-\cot x \ge 0$ there. It follows that in this case

$$F_M/F \leq 8\sqrt{3} \frac{\{\sin((\pi - \theta_4)/3)\sin(\pi/6)\sin((\pi - 2\theta_4)/3)\}^2}{\sin(\pi - \theta_4)\sin(\pi/2)\sin(\pi - 2\theta_4)}$$

= 0.023552...

(c) If $A > \theta_4 \ge B$, C, then, since $g''_4(x) < 0$ for $0 < x < \theta_4$ we have by concavity

$$g_4(B) + g_4(C) \le 2g_4((B+C)/2)$$

with equality if and only if B = C. Thus we may assume that B = C and $B = (\pi - A)/2 < (\pi - \theta_4)/2 = 0.83908...$ Let $f_4(x) = 2g_4(x) + g_4(\pi - 2x)$. We shall show that

$$g'_4(x) > g'_4(\pi - 2x) \qquad 0 < x \le (\pi - \theta_4)/2$$
 (*)

and conclude that f_4 is increasing on this range.

Suppose first that $x \in I + (0.7, (\pi - \theta_4)/2]$. Since g'_4 is decreasing for $0 < x < \theta_4$ and increasing for $\theta_4 < x < \pi$, we have

$$\begin{array}{rcl} g_4'(x) & \geq & g_4'((\pi-\theta_4)/2) = 1.42306\dots \\ & > & 1.18884\dots = g_4'(\pi-1.4) \geq g'(\pi-2x) \end{array}$$

so (*) holds for $x \in I$.

Next, suppose that $x \in J = (0, o.7]$. We begin by showing that

$$3\sqrt{2}\sin(x/3) < \sin(2x)$$
 $x \in J$

Indeed

$$\frac{\sin(2x)}{\sin(x/3)} = 2\cos\left(\frac{5x}{3}\right) + 2\cos x + 2\cos\left(\frac{x}{3}\right)$$

is clearly decreasing on J and we have

$$\sin(2x)/\sin(x/3) > \sin 1.4/\sin(0.7/3) = 4.26192... > 3\sqrt{2}$$

Now $g'_4(x) > (1/3) \cot(x/3)$, since $\tan(x/3) < (\tan x)/3$ for $x \in J$, and also we have

$$g'_4(\pi - 2x) = \frac{2}{3} \tan\left(\frac{\pi}{6} + \frac{2x}{3}\right) + \cot(2x)$$

Let $h(x) = (1/3) \cot(x/3) - \cot(2x)$. We prove (*) for $x \in J$ by showing that

$$h(x) > (2/3) \tan(\pi/6 + 2x/3)$$
 (**)

holds on this range. We have seen above that

$$h'(x) = 2\csc^2(2x) - \frac{1}{9}\csc^2\left(\frac{x}{3}\right) = \left(\frac{\sqrt{2}}{\sin 2x}\right)^2 - \left(\frac{1}{3\sin(x/3)}\right)^2 < 0$$

for $x \in J$. Thus $h(x) \ge h(0.7) = 1.23007...$ On the other hand, if $x \in J$, we have

$$(2/3)\tan(\pi/6 + 2x/3) \le (2/3)\tan(\pi/6 + 1.4/3) = 1.01637\dots$$

so that (**) and hence (*) holds for $x \in J$.

It follows that $f_4(x)$ is increasing for $0 < x \le (\pi - \theta_4)/2$ so that in case (c) we have

$$F_M/F \le 8\sqrt{3} \exp f_4((\pi - \theta_4)/2) = 0.032120\dots$$

which is smaller than the estimate found in (a). Thus the claimed result $F_M/F \leq (64/3) \sin^6(\pi/9)$ is established.

1. H. S. M. Coxeter & S. L. Greitzer, *Geometry Revisited*, New Mathematical Library **19**(1967) pp.47–50.

2. Clarence Lubin, A proof of Morley's theorem, this MONTHLY, 62(1955) 110-112.

3. K. Venkatachaliengar, An elementary proof of Morley's theorem, this MONTHLY, 65(1958) 612–613.
 No other correct solutions were received.

Amer. Math. Monthly, **96**(1989) 356.

E 3320. Proposed by M. S. Klamkin, University of Alberta, Edmonton

Determine positive constants a and b such that the inequality

$$yz + zx + xy \ge a(y^2z^2 + z^2x^2 + x^2y^2) + bxyz$$

holds for all nonnegative x, y, z with x + y + z = 1 and is the best possible inequality of this form (in the sense that the inequality need not hold if a or b is increased).

Amer. Math. Monthly, 97(1990) 619–620.

Solution by Mark Ashbaugh, University of Missouri, Columbia, MO. The family (a, b) = (a, 9 - a) for $0 < a \leq 4$ gives all solution pairs to this problem. Using the values (x, y, z) = (1/3, 1/3, 1/3) and (x, y, z) = (1/2, 1/2, 0), we obtain the two inequalities $a + b \leq 9$ and $a \leq 4$. Thus we need only prove that the desired inequality is valid for the pairs in the family described, i.e., that the inequality

$$yz + zx + xy \ge a(y^2z^2 + z^2x^2 + x^2y^2) + (9 - a)xyz$$
 for $0 < a \le 4$

holds for nonnegative x, y, z with x + y + z = 1. If we let $P = (x + y + z)^2(yz + zx + xy)$. $Q = y^2 z^2 + z^2 x^2 + x^2 y^2$ and R = (x + y + z)xyz, then this is equivalent to proving

$$P(x, y, z) \ge aQ(x, y, z) + (9 - a)R()x, y, z \text{ for } 0 < a \le 4$$

for all nonnegative x, y, z, since these terms are homogeneous.

The proof is now completed by showing that $P \ge 4Q + 5R$ and that $Q \ge R$, because together these imply $P \ge aQ + (4-a)Q + 5R \ge aQ + (9-a)R$ for $0 < a \le 4$. To obtain the first inequality $P \ge 4Q + 5R$ we note that

$$P = 2Q + 5R + x^{3}y + xy^{3} + y^{3}z + yz^{3} + x^{3}z + xz^{3}$$

and then

$$P - 4Q - 5R = xy(x - y)^{2} + yz(y - z)^{2} + xz(x - z)^{2} \ge 0$$
 (*)

To obtain the second inequality, $Q \ge R$, we apply the Cauchy-Schwarz inequality to the vectors V = (xy, yz, zx) and W = (xz, xy, yz) and conclude that

$$Q = \|V\| \cdot \|W\| \ge V \cdot W = x^2 yz + xy^2 z + xyz^2 = R$$
(**)

The points x, y, z for which equality holds are easy to determine. If a < 4, equality occurs in (**) exactly when x = y = z or at least two of x, y, z are 0, and (*) also holds with equality in those cases, so for a < 4 the only instances of equality in the triangle given by x + y + z = 1 and x, y, $z \ge 0$ are the centre (1/3, 1/3, 1/3) and the

corners (1,0,0), (0,1,0) and (0,0,1). When a = 4 the situation changes because (**) is not needed, and (*) also gives equality if one of x, y, z is 0 and the other two are equal. Thus if a = 4 we also have equality at (1/2,1/2,0), (1/2,0,1/2) and (0,1/2,1/2), the midpoints of the sides of the above triangle.

Finally, we note that the best inequality of the form $P \ge bR$ is $P \ge 9R$, which follows from the fact that the entire discussion above applies also when a = 0; equality holds in $P \ge 9R$ only when x = y = z or when two of x, y, z are zero. The best inequality of the form $P \ge aQ$ is $P \ge 4Q$, also from the discussion above; equality holds in $P \ge 4Q$ only when two of x, y, z are zero or when one of them is zero and the other two are equal.

Amer. Math. Monthly, **96**(1989) 641.

E 3337. Proposed by M. S. Klamkin, University of Alberta, Edmonton

Suppose the two longest edges of a tetrahedron are a pair of opposite edges. Prove that the three edges incident to some vertex of the tetrahedron are congruent to the sides of an acute triangle.

Amer. Math. Monthly, **97**(1990) 927–928.

Solution by Jesús Ferrer, Universidad Complutense, Madrid, Spain. Let O, A, B, C be the vertices of the tetrahedron, and suppose that OA and BC are the longest edges. If the edges incident to each vertex fail to be the side lengths of an acute triangle, then the law of cosines implies the following inequalities:

$$OA^{2} \ge OB^{2} + OC^{2} \qquad BC^{2} \ge OB^{2} + AB^{2}$$
$$OA^{2} \ge AB^{2} + AC^{2} \qquad BC^{2} \ge OC^{2} + AC^{2} \qquad (*)$$

(The inequalities (*) are valid *a fortiori* if some triples do not satisfy the triangle inequality.) Summing these inequalities we obtain

$$OA^2 + BC^2 \ge OB^2 + OC^2 + AB^2 + AC^2$$

Expressing the vector \overrightarrow{BC} as $\overrightarrow{OC} - \overrightarrow{OB}$ (and similarly for \overrightarrow{AB} and \overrightarrow{AC}), we can use the fact that squared length equals vector inner product to rewrite this as

$$0 \geq OA^{2} + OB^{2} + OC^{2} - 2\overrightarrow{OA} \cdot \overrightarrow{OB} - 2\overrightarrow{OA} \cdot \overrightarrow{OC} + 2\overrightarrow{OB} \cdot \overrightarrow{OC} \\ = \|\overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}\|^{2}$$

that is, $\overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC} = 0$. Thus $\overrightarrow{OC} = \overrightarrow{BA}$ and $\overrightarrow{OB} = \overrightarrow{CA}$, which requires OBAC to be a plane parallelogram. Since equality must hold in all four parts of (*), in fact OBAC is a rectangle.

In all other cases at least one of the inequalities (*) fails, and the lengths of segments incident to the corresponding vertex are the side lengths of an acute triangle.

Amer. Math. Monthly, **99**(1992) 169–170.

The Area of a Pedal of a Pedal Triangle

E 3392 [1990, 528]. Proposed by Antal Bege, Miercurea-Ciuc, Romania

Given an acute-angle triangle ABC with overline H, let A_1 , B_1 , C_1 be the feet of the altitudes from A, B, C respectively, and let A_2 , B_2 , C_2 be the feet of the perpendiculars from H onto B_1C_1 , C_1A_1 , A_1B_1 respectively. Prove that

area
$$(\Delta ABC) \ge 16 \operatorname{area} (\Delta A_2 B_2 C_2)$$

and determine when equality holds.

Solution II and generalization by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. More generally, the pedal triangle of a triangle ABC with respect to a point P is the triangle whose vertices A_1 , B_1 , C_1 are the feet of the perpendiculars from P onto the sides of ABC. For P lying within or on ABC it is known [2, p.139] that

$$[A_1B_1C_1] = [ABC](1 - OP^2/R^2)/4 \le [ABC]/4$$

where [] denotes area and O, R are, respectively, the circumcentre and circumradius of ABC. There is equality if and only if P coincides with O (and this requires that ABC be non-obtuse). Then, if $A_2B_2C_2$ is the pedal triangle of $A_1B_1C_1$ with respect to P,

$$[A_2 B_2 C_2] \le [A_1 B_1 C_1]/4 \le [ABC]/16$$

For equality in both places here, P must be the circumcentre of both ABC and $A_1B_1C_1$. This requires that ABC is equilateral.

Editorial comment. Many solvers used analytic and other means to establish the relation $[A_2B_2C_2] = 4(\cos A \cos B \cos C)^2[ABC]$. This is a consequence of $R_2 = 2R \cos A \cos B \cos C$ (where R_2 is the circumradius of $A_2B_2C_2$ **2**, p.191) and the similarity of $A_2B_2C_2$ and ABC. The required inequality then follows from the easy inequality $\cos A \cos B \cos C \leq 1/8$.

Walther Janous suggest \$1.9 of [1] as a good reference for properties of iterated pedal triangles. He also points out the related inequality $[ABC]^5 \ge R^8(27/4)^2[A_2B_2C_2]$ that can be obtained from inequalities found in [3, p.271].

1. H. S. M. Coxeter & S. L. Greitzer, *Geometry Revisited*, New Mathematical Library 19(1967).

2. R. A. Johnson, Modern Geometry, Houghton Mifflin, 1929.

3. D. S. Mitrinović, J. E.Pečarić & V. Volonec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

Amer. Math. Monthly, **99**(1992) 872.

10256. Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Let A_i , A'_i (i = 1, 2, 3, 4) be the vertices of a rectangular parallelepiped \mathcal{P} , with A'_i diametrically opposite to A_i . Let P be any interior point of \mathcal{P} . Prove that

$$S \le 2(PA_1 \cdot PA_1' + PA_2 \cdot PA_2' + PA_3 \cdot PA_3' + PA_4 \cdot PA_4')$$

where S denotes the surface area of \mathcal{P} .

Amer. Math. Monthly, **101**(1994) 478–479.

Solution by Robin J. Chapman, University of Exeter, U.K. Choose Cartesian coordinates with P as origin, and suppose that the faces of \mathcal{P} lie in the planes x = -a, x = A, y = -b, y = B, z = -c and z = C. The inequality becomes

$$\begin{split} \sqrt{a^2 + b^2 + c^2} \sqrt{A^2 + B^2 + C^2} + \sqrt{a^2 + b^2 + C^2} \sqrt{A^2 + B^2 + c^2} \\ + \sqrt{a^2 + B^2 + c^2} \sqrt{A^2 + b^2 + C^2} + \sqrt{A^2 + b^2 + c^2} \sqrt{a^2 + B^2 + C^2} \\ \ge AB + Ab + aB + ab + CA + Ca + cA + ca + BC + Bc + bC + bc \end{split}$$

Now by the Cauchy-Schwarz inequality

$$\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2} \ge aB + bC + cA$$

and also

$$\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2} \ge Ab + Bc + Ca$$

Hence

$$\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2} \ge (aB + bC + cA + Ab + Bc + Ca)/2$$

and by adding the similar inequalities obtained from the other terms on the left hand side of the main inequality, we get the main inequality.

Editorial comment. No other solver used coordinates based at P to simplify the formulas. Also, note that it is not necessary to require P to be an interior point. The interpretation as surface area is possible whenever A + a, B + b and C + c are all positive, and this choice can be made for any P. One reader noted that the result is false if one reads the terms $PA_i \cdot PA'_i$ as inner products of vectors. In this interpretation, the sum of inner products is negative whenever P is an interior point. We apologize for not noticing that this confusion was possible.

Amer. Math. Monthly, **100**(1993) 75.

10275. Proposed by Murray S. Klamkin and A. Liu, University of Alberta, Edmonton, Alberta, Canada

Let \mathcal{A} be a regular *n*-gon with edge length 2. Denote the consective vertices by A_0, \ldots, A_{n-1} and introduce A_n as a synonym for A_0 . Let \mathcal{B} be a regular *n*-gon inscribed in \mathcal{A} with vertices B_0, \ldots, B_{n-1} where B_i lies on A_iA_{i+1} and $|A_iB_i| = \lambda < 1$ for $0 \leq i < n$. Also let C_i be the point on A_iA_{i+1} with $|A_iC_i| = \alpha_i \leq \lambda$ for $0 \leq i < n$ and let \mathcal{C} denote the *n*-gon, also inscribed in \mathcal{A} , with vertices C_0, \ldots, C_{n-1} .

With $P(\mathcal{F})$ denoting the perimeter of the figure \mathcal{F} , prove that $P(\mathcal{C}) \geq P(\mathcal{B})$.

Amer. Math. Monthly, **102**(1995) 76–77.

Solution ii by Roy Barbara, Lebanese University, Fanar, Lebanon. First we formulate a method for comparing lengths.

Lemma. Let ABCD be a convex broken line. Assume AB = CD and that the angles at B and C are equal. Denote by I, J and K the midpoints of AB, BC and CD respectively. Let R be between A and I, T between B and J, and U between C and K. Let S also lie on BT. Then $RS + SU \ge RT + TU$.

Proof.

Α



Let V be the reflection of U across BC. Denote by O the intersection of RV and BC. Using similar triangles, it is clear that O is between J and C. Thus T is inside the triangle RSV. Therefore $RS + SU = RS + SV \ge RT + TV = RT + TU$.

D

Now we apply the lemma to solve the problem. Consider the *n*-gon \mathcal{C} ; a first application of the lemma to $A_{n-1}A_0A_1A_2$ (R, S, T, U being C_{n-1}, C_0, B_0, C_1 respectively) means that replacing the vertex C_0 by B_0 will decrease the perimeter of \mathcal{C} . More generally, if we denote by \mathcal{F}_i the *n*-gon with vertices $B_0, \ldots, B_i, C_{i+1}, \ldots, C_{n-1}$ ($0 \le i \le n-1$), by repeated use of the lemma, we obtain $P(\mathcal{C}) \ge P(\mathcal{F}_0) \ge P(\mathcal{F}_1) \ge \cdots \ge P(\mathcal{F}_{n-1})$. Since \mathcal{F}_{n-1} is \mathcal{B} , the proof is complete.

Note that we have proved a more general result: the *n*-gon \mathcal{B} need not be regular; it is only necessary that $|A_iB_i| < \frac{1}{2}|A_iA_{i+1}|$.

Amer. Math. Monthly, **101**(1994) 575.

10256. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the extreme values of

$$\frac{1}{1+x+u} + \frac{1}{1+y+v} + \frac{1}{1+z+u}$$

where $xyz = a^3$, $uvw = b^3$ and x, y, z, u, v, w > 0.

Amer. Math. Monthly, **103**(1996) 817–818.

Solution by Raphael Robinson, University of California, Berkeley, CA. We must assume that a and b are positive. Let S denote the sum of the three fractions. We show that: (1) the greatest lower bound for S is 0; (2) if $a + b \le 1/2$, then S attains a maximum value of 3/(1 + a + b); and (3) if a + b > 1/2, then the lest upper bound for S is 2 and this value is not attained.

In our analysis, we shall allow boundary value of 0 and ∞ for the variables if they can be approached by finite positive values satisfying the hypotheses. If (x, y, z) is a boundary triple, then one of the variables must be 0 and one must be ∞ , the third being arbitrary. The minimum S on the boundary is 0, attained for $x = y = w = \infty$, z = u = 0, and this is the overall minimum. The maximum S on the boundary is 2, attained for x = y = u = v = 0, $z = w = \infty$. It remains to be seen whether larger values of S can occur in the interior.

Suppose that the maximum S occurs at an interior point (x, y, z, u, v, w). Hold z, u, v, w fixed and vary x and y. Since xy is then constant, we see that dy/dx = -y/x. Hence

$$\frac{dS}{dx} = -\frac{1}{(1+x+u)^2} + \frac{1}{(1+y+v)^2} \cdot \frac{y}{x}$$

This must vanish, so that

$$x(1+y+v)^2 = y(1+x+u)^2$$
 (*)

A similar argument shows that

$$u(1+y+v)^2 = v(1+x+u)^2$$

Thus u/x = v/y and these must also equal w/z. So we may put

$$u = hx$$
 $v = hy$ $w = hz$

where h = b/a. Condition (*) becomes

$$x (1 + (1 + h)y)^2 = y (1 + (1 + h)x)^2$$

which reduces to $x - y = (1 + h)^2 x y (x - y)$. Hence

$$x = y$$
 or $xy = (1+h)^{-2}$

Similarly

$$x = z$$
 or $xz = (1+h)^{-2}$

and

$$y = z$$
 or $yz = (1+h)^{-2}$

If we do not have x = y = z, then, by permuting the variables if necessary, we may assume that x = y, $xz = yz = (1 + h)^{-2}$, hence

$$x = y = (1+h)^2 a^3$$
 $z = (1+h)^{-4} a^{-3}$

so that

$$x + u = y + v = (a + b)^3$$
 $z + w = (a + b)^{-3}$

These yield

$$S = \frac{2 + (a+b)^3}{1 + (a+b)^3}$$

Since this value is less than 2, it cannot be the maximum for S. The only remaining possibility is that x = y = z = a, u = v = w = b, hence S = 3/(1 + a + b). This furnishes the maximum for S if it is at least 2 (i.e., if $a + b \le 1/2$).

Amer. Math. Monthly, **102**(1995) 363–364.

Area of a Roulette

10254 [1992, 782]. Proposed by E. Ehrhart, Université de Strasbourg, Strasbourg, France The curve traced out by a fixed point of a closed convex curve as that curve rolls without slipping along a second curve will be called a "roulette". Let S be the area

of one arch of a roulette traced out by an ellipse of area s rolling on a straight line. Prove or disprove that $S \ge 3s$, with equality only if the ellipse is a circle.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. We will show that the inequality is equivalent to $(a - b)(a - 2b) \ge 0$ where a and b are the major and minor semi-axes of the ellipse, respectively. Consequently there will be equality if a = b or a = 2b. There will be strict inequality only if a > 2b.

Recall that the **pedal** of a given curve with respect to a point P is the locus of the foot of the perpendicular from P to a variable tangent line to the curve. The desired result follows from the following results of Steiner that can be found in B. Williamson, *The Integral Calculus*, Longmans Green and Co., London, 1941, 201–203.

(A) When a closed curve rolls on a straight line, the area between the line and the roulette generated in a complete revolution by any point on the rolling curve is double the area of the pedal of the rolling curve, this pedal being taken with respect to the generating point.

(B) The area of the pedal of an ellipse of semiaxes a and b with respect to any point P is given by $\pi(a^2 + b^2 + |OP|^2)/2$, where O is the centre of the ellipse. In the interest of simplicity, these theorems have been stated only when P lies on the curve. This is not an essential restriction.

Clearly, the minimum of S for P on the ellipse occurs for |OP| = b. Hence $S \ge 3s$ is equivalent to

$$\pi(a^2 + 2b^2) \ge 3\pi ab$$
 or $(a-b)(a-2b) \ge 0$

Editorial comment. The other solvers were able to work through the Calculus without references, but as one of them said: "... I hope some are more elegant in the way they prove the result; I just ground out the integral ...".

The work leading to the formulation of this problem can be found in E. Ehrhart, Les roulettes d'ellipses. L'Ouvert, **62**(1991) 43–45.

Other references to Steiner's theorem found by the editors are E. Goursat, A Course in Mathematical Analysis, Vol.1, Dover, 1959, where it is problem 23 (with hints) on p.207, and J. Edwards, A Treatise on the Integral Calculus, Chelsea, 1955, article 673, pp.696–697, which refers back to W. H. Besant, Tract on Roulettes and Glisettes, 1870 (and not to Steiner).

[[Surely Steiner was earlier than 1870 ?? — R.]]

Richard Holzsager suggested that it would be interesting to find the convex curve C and point P on C which gives the minimum ratio of area of the roulette to the area of the curve. He conjectured that C is given by the arc of the epicycloid $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$ with its end points $(\pm 2, 0)$ connected by a line segment. The point P is (0,0). For this curve, the ratio can be calculated to be 8/3 will the above results show that when C is an ellipse, the smallest value is $2\sqrt{2}$.

Amer. Math. Monthly, **102**(1995) 463.

10453. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Prove that the following two properties of the altitudes of an n-dimensional simplex are equivalent:

i) the altitudes are concurrent.

ii) the feet of the altitudes are the orthocentres of their respective faces.

Amer. Math. Monthly, 105(1998) 563-565.

Solution II by Mark D. Myerson, U.S. Naval Academy, Annapolis, MD. We first show (i)Longrightarrow(ii). Assume that the altitudes are concurrent. Let AA' be the altitude rom a fixed vertex A to the opposite face, an (n-1)-simplex a, and let BB'be the altitude from any other fixed vertex B to its opposite face b. Since AA' and BB' meet, they determine a (2-dimensional) plane π . Since π contains AA', it is perpendicular to a; similarly, it is perpendicular to b. Thus π is perpendicular to the (n-2)-simplex $a \cap b$. Hence the perpendicular projection to a (parallel to AA') carries BB' into the altitude from B to $a \cap b$ in a. Applying this for all choices of B shows that this projection sends all altitudes of the original simplex other than AA' to altitudes of a and all of these altitudes contain A'. Thus, A' is the orthocentre of a.

We now show that (ii) \Longrightarrow (i). Assume that the altitudes of an *n*-simplex Σ in \mathbb{R}^n pass through the orthocentres of the faces. Let AA' be the altitude from a fixed vertex Ato face a. Then A' is the orthocentre of a. Also, for a vertex B different from A, BA' is perpendicular to $a \cap b$, where b is the face of Σ opposite B. Let π be the (2-dimensional) plane determined by BA' and AA'. Since AA' is perpendicular to a, AA' is perpendicular to $a \cap b$, and so π must be perpendicular to $a \cap b$. Since π is also 2-dimensional and $a \cap b$ is (n-2)-dimensional, they must span \mathbb{R}^n , and so π must contain a vector perpendicular to b. Thus π is perpendicular to b, and the altitude BB' from B to b lies in π . Since BB' and AA' lie in the same plane and are not parallel, they must meet. Since A and B were chosen arbutrarily, it follows that any two altitudes of Σ intersect. Consider a third altitude CC' of Σ . By a similar argument, we know that, when projected onto a in a perpendicular fashion, the images of both BB' and CC' pass through A'. Thus the intersection point of BB' and CC' must lie on AA'. It follows that each altitude passes through the point of intersection of AA' and BB'.

Editorial comment. A simplex meeting the conditions of the problem is called **orthocentric**. Another characterization is: A simplex is orthocentric if and only if any two disjoint edges are orthogonal.

This topic has a long history, and expositions can be found in N. A. Court, Notes on the orthocentric tetrahedron, this MONTHLY, 41(1934) 499–502, N. A. Court, The tetrahedron and its altitudes, *Scripta Math.*, 14(1948) 85–97, and H. Lob, The orthocentric simplex in space of three and higher dimensions, *Math. Gaz.*, 19(1935) 102–108. A related article is by L. Gerber, The orthocentric simplex as an extremeal simplex, *Pacific J. Math.*, 56(1975) 97–111. The latter deals primarily with extremal problems in *n* dimensions whose solution is an orthocentric simplex, but includes a brief discussion of the properties of such simplices.

Amer. Math. Monthly, **102**(1995) 841.

10482. Proposed by Emre Alkan, Bosphorus University, Istanbul, Turkey, and Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Given a regular *n*-gonal pyramid with apex P and base $A_1A_2...A_n$, denote $\angle A_iPA_{i+1}$ by α with $0 < \alpha \leq 2\pi/n$. If points B_i are chosen on the rays PA_i (i = 1, 2, ..., n), determine the maximum and minimum values of

$$\frac{|PB_1| + |PB_2| + \dots + |PB_n|}{|B_1B_2| + |B_2B_3| + \dots + |B_nB_1|}$$

Amer. Math. Monthly, **107**(1998) 673–674.

Solution by O. P. Lossers, University of Technology, Eindhoven, The Netherlands. The expression lies between 1/2 and $1/(\sin(\alpha/2))$. This follows by summing over *i* from 1 to *n* (with $B_{n+1} = B_1$) the inequalities

$$|B_iP| + |PB_{i+1}| \ge |B_iB_{i+1}| \ge \sin\frac{\alpha}{2} \cdot (|B_iP| + |PB_{i+1}|) \tag{(*)}$$

The first inequality in (*) is just the triangle inequality. Since $a^2 = b^2 + c^2 - 2bc \cos \alpha$ is equivalent to $a^2 = (b+c)^2 \sin^2 \frac{\alpha}{2} + (b-c)^2 \cos^2 \frac{\alpha}{2}$, the second inequality in (*) follows from the law of cosines applied to $\Delta B_i P B_{i+1}$ with $a = |B_i B_{i+1}|$.

The second inequality in (*) also follows in a more direct way by observing that in any convex quadrilateral the sum of the lengths of the diagonals is more than the sum of the lengths of either pair of opposite sides. Now apply this to the trapezoid $B_i B'_i B_{i+1} B'_{i+1}$ where B'_i is chosen on the ray PB_{i+1} such that $|PB_i| = |PB'_i|$ and B'_{i+1} the corresponding point on the ray PB_i .

Equality in the lower bound occurs only if of any two consecutive B_i at least one coincides with P (but not all B_i coincide with P). The upper bound is attained only if all B_i are the same (positive) distance from P.

Editorial comment. The Anchorage Math Solutions Group pointed out a physical interpretation of the maximum. Namely, with P positioned above the centroid of the base in a uniform gravitational field, consider a spatial polygon with vertices on the rays PA_i . Think of this polygon as a loop of rope passing through heavy rings that can slide freeely on the rays PA_i while the rope can pass freely through the rings. If released, the rope and rings slide into a position of static equilibrium in which the rope is taught and forms a polygon $B_1B_2...B_n$, $B_i \in PA_i$, with the B_i equidistant fro P. This position of the B_i will be such as to minimize the potential energy of the system, which is equivalent to maximizing $\sum |PB_i|$ or maximizing $\sum |PB_i| / \sum |B_iB_{i+1}|$ with $\sum |B_iB_{i+1}|$ fixed. Amer. Math. Monthly, **102**(1995) 929.

10488. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the extreme values of the sum of the lengths of three concurrent and mutually orthogonal chords of a given sphere of radius R if the point of concurrency is at a distance d from the centre.

Amer. Math. Monthly, **105**(1998) 371–372.

Solution by John H. Lindsey II, Fort Myers, FL. We generalize the problem to the case of n concurrent and mutually orthogonal chords of an (n-1)-dimensional sphere of radius R in \mathbb{R}^n . We may assume the chords lie along the coordinate axes with the origin as the point of concurrency. Let the sphere centre be $P = (d_1, d_2, \ldots, d_n)$. Then $d^2 = d_1^2 + d_2^2 + \cdots + d_n^2$ and the square of the distance of the sphere centre to the chord along the *i*th coordinate axis is $d^2 - d_i^2$. Thus the length of this chord is $s\sqrt{R^2 - d^2 + d_i^2}$.

As the (n-1)-sphere is compact, we may locate P to maximize S the sum of the lengths of the chords. We claim that P must be along the diagonal of some orthant. Otherwise, suppose $||d_i < |d_j|$ for some i and j. Note that $\sqrt{a+x} + \sqrt{a-x}$ is decreasing in xfor 0 < x < a. Letting $a = R^2 - d^2 + (d_i^2 + d_j^2)/2$ and $x = (d_j^2 - d_i^2)/2$ we see that decreasing d_j^2 and increasing d_i^2 by the same small amount would increase S, yielding a contradiction. It follows that $d_i^2 = d^2/n$ for all i. Hence the maximum value is

$$2n\sqrt{R^2 - \frac{n-1}{n}d^2} \qquad \qquad \mathbf{U}$$

Similarly, we may locate P to minimize S. Suppose two of the d_i are nonzero, say $0 < |d_i| \le |d_j|$. Again, increasing d_j^2 and decreasing d_i^2 by the same small amount decreases S, a contradiction. It follows that there is an index j such that $d_i = 0$ for all $i \ne j$ and $d_i^2 = d^2$. Hence the minimum value is $2R + 2(n-1)\sqrt{R^2 - d^2}$.

Editorial comment. This result may be compared with problem E 3460 [1991, 755; 1993, 87] part of which dealt with the sum of the squares of the lengths of such chords.

Nelson M. Blachman and L. Scibani also considered the case in which the *n* chords could be extended to have a point of intersection outside the sphere. The first part of the selected solution shows that there is a chord along the *i*th coordinate axis if $d_i^2 > d^2 - R^2$. To assure existence of extrema, one should also allow the chord to degenerate to a tangent when $d_i^2 = d^2 - R^2$. The analysis in the selected solution now shows that the minimum occurs when there is an index *j* such that $d_i^2 = d^2 - R^2$ for all $i \neq j$ and $d_j^2 = (n-1)R^2 - (n-2)d^2$. This leads to a lower bound of $2\sqrt{nR^2 - (n-1)d^2}$ for $R \leq d \leq \sqrt{n/(n-1)}R$.

Amer. Math. Monthly, **103**(1996) 599.

10482. Proposed by Fred Galvin and Saul Stahl, University of Kansas, Lawrence, KS, Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Alberta, Canada, and Harry Tamvakis and Marcin Mazur, The University of Chicago, Chicago, IL

Prove that for any $2^n - 1$ lattice points $A_1, A_2, \ldots, A_{2^n-1}$ in the *n*-dimensional lattice of points with integer coordinates, there is a lattice point P, distinct from the A_i , such that none of the open segments $PA_1, PA_2, \ldots, PA_{2^n-1}$ contains any lattice points.

Solution by Nasha Komanda, Central Michigan University, Mt. Pleasant, MI. If a lattice point $C = (c_1, \ldots, c_n)$ is between lattice points $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ then there exist relatively prime positive integers u and v such that $(u+v)c_i = ua_i + vb_i$ for $1 \le i \le n$, which implies that u+v divides $a_i - b_i$.

For the given lattice points A_1, \ldots, A_{2^n-1} let $A_i = (a_{i1}, \ldots, a_{in})$. It suffices to find a lattice point $P = (x_1, \ldots, x_n)$ distinct fro each A_i , such that for each i the n differences $x_1 - ai1, \ldots, x_n - a_{in}$ have no common factor.

Let p be a prime. Since there are at least 2^n distinct values for (y_1, \ldots, y_n) modulo p, where y_1, \ldots, y_n are integers, there exist integers $y_1(p), \ldots, y_n(p)$ such that for each ithe vector $(y_1(p), \ldots, y_n(p))$ does not agree with (a_{i1}, \ldots, a_{in}) modulo p. Let Y be the set of primes less than 2^n . For $1 \le j \le n$ we apply the Chinese Remainder Theorem to find an integer y_j such that $y_j \equiv y_j(p)$ modulo p for each $p \in Y$ and $y_j \ne a_{ij}$ for $1 \le i \le 2^n - 1$.

Let X be the set of all primes that are at least 2^n and that divide $y_j - a_{ij}$ for some i, j. For each $p \in X$ there is an integer z(p) such that $z(p) \not\equiv a_{i1}$ for $1 \leq i \leq 2^n - 1$. Let $x_j = y_j$ for $2 \leq j \leq n$. Apply the Chinese Remainder Theorem again to find x_1 such that $x_1 \equiv y_1 \pmod{p}$ for $p \in Y$ and $x_1 \equiv z(p) \pmod{p}$ for $p \in X$. For each i, the differences $x_1 - ai1, \ldots, x_n - a_{in}$ have no common factor, as desired. Amer. Math. Monthly, 106(1999) 587.

Using the Walls to Find the Centre

10386 [1994, 477]. Proposed by Jordan Tabov, Bulgarian Academy of Sciences, Sofia, Bulgaria

Let a tetrahedron with vertices A_1 , A_2 , A_3 , A_4 have altitudes that meet in a point H. For any point P, let P_1 , P_2 , P_3 and P_4 be the feet of the perpendiculars from P to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$ and $A_1A_2A_3$ respectively. Prove that there exist constants a_1 , a_2 , a_3 and a_4 such that one has

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}$$

for every point P.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. More generally, let H and P be any two points in the space of the given tetrahedron and let P_1 , P_2 , P_3 , P_4 be the feet of the lines through P parallel to HA_1 , HA_2 , HA_3 , HA_4 in the faces of the tetrahedron opposite A_1 , A_2 , A_3 , A_4 respectively. Then there exist constants a_1 , a_2 , a_3 , a_4 , independent of P such that

$$a_1\overrightarrow{PP_1} + a_2\overrightarrow{PP_2} + a_3\overrightarrow{PP_3} + a_4\overrightarrow{PP_4} = \overrightarrow{PH}$$

Let **V** denote the vector from an origin outside the space of the given tetrahedron to any point V in the space of the tetrahedron. Then H and P have the representations (barycentric coordinates)

$$\mathbf{H} = x_1 \mathbf{A}_1 + x_2 \mathbf{A}_2 + x_3 \mathbf{A}_3 + x_4 \mathbf{A}_4 \qquad (x_1 + x_2 + x_3 + x_4 = 1)$$
$$\mathbf{P} = u_1 \mathbf{A}_1 + u_2 \mathbf{A}_2 + u_3 \mathbf{A}_3 + u_4 \mathbf{A}_4 \qquad (u_1 + u_2 + u_3 + u_4 = 1)$$

Since P_1 has the representation $\mathbf{P}_1 = r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4$ where $r_2 + r_3 + r_4 = 1$ we must have

$$r_2\mathbf{A}_2 + r_3\mathbf{A}_3 + r_4\mathbf{A}_4 - \mathbf{P} = \lambda_1(\mathbf{H} - \mathbf{A}_1)$$

Since \mathbf{A}_1 , \mathbf{A}_2 , \mathbf{A}_3 , \mathbf{A}_4 are independent vectors, we get $\lambda_1 = u_1/(1-x_1)$, so that $\overrightarrow{PP_1} = (\mathbf{P}_1 - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_i)u_i/(1-x_i)$. Similarly,

$$(\mathbf{P}_1 - \mathbf{P}) = (\mathbf{H} - \mathbf{A}_i) \frac{u_i}{1 - x_i}$$
 for $i = 1, 2, 3, 4$

Choosing $a_i = 1 - x_i$ we obtain

$$\sum a_i(\mathbf{P}_i - \mathbf{P}) = \sum u_i(\mathbf{H} - \mathbf{A}_i) = \mathbf{H} - \mathbf{P} = \overrightarrow{PH}$$

This proof generalizes to give an analogous result for n-dimensional simplices.

Murray Klamkin, (Two Year) Coll. Math. Journal

Richard K. Guy

June 22, 2006

File last updated 2006-05-22.

This is the (lost count!) of a number of files listing problems, solutions and other writings of Murray Klamkin.

The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

The Two-Year Coll. Math. J. didn't have a problems section at the outset. The first occurrence is at

Two-Year Coll. Math. J., 4 No.1 (Winter 1973) 73.

The first publication of Murray that I found was:

PROBABILITY

Selection problems: points

Two-Year Coll. Math. J., 10 No.1 (Jan., 1979) 53–54.

[[He is also listed as a solver for each of the other problems (97–100) in this issue!]]

Random Points on a Line Segment

96. (Sept. 1977) Proposed by Milton H. Hoehn, Santa Rosa Jr. College, Santa Rosa, CA

On a line segment of length 1, n points are selected at random. What is the expected value of the sum of the distances between all pairs of these points?

Solution by M. S. Klamkin, University of Alberta, Edmonton, Alberta. Let X_1, X_2, \ldots, X_n be independent random variables each uniformly distributed on the unit interval. Let F'' be any function defined and absolutely integrable on the unit interval such that F'' is equal to the second derivative of some function F. (It is sufficient to choose F''(x) = x and $F(x) = x^3/6$.) Let $S = \sum_{1 \le i < j \le n} F''(|X_i - X_j|)$. Then the expected value

$$E(S) = \sum_{1 \le i < j \le n} \int_0^1 \cdots \int_0^1 \int_0^1 F''(|x_i - x_j|) \, dx_1 \, dx_2 \cdots \, dx_n$$

= $\binom{n}{2} \int_0^1 \int_0^1 F''(|x_1 - x_2|) \, dx_1 \, dx_2 = 2\binom{n}{2} \int_{x_2 = 0}^1 \int_{x_1 = 0}^{x_2} F''(x_2 - x_1) \, dx_2 \, dx_1$
= $2\binom{n}{2} (F(1) - F(0) - F'(0))$

For the given problem, let $F(x) = x^3/6$, giving E(S) = n(n-1)/6.

ANALYSIS

Differential equations: functional equations

Two-Year Coll. Math. J., 10 No.2 (Mar., 1979) 128–129. Quadratic Mean Value Theorem

101. (Sept.1977) Proposed by Louis Alpert, Bronx Community College, NY, and Jerry Brantley, Macomb County Community College, Mt. Clemens, MI (independently)

Determine all functions f defined on $(-\infty, \infty)$ such that for all $a \neq b$,

$$f'((a+b)/2) = (f(a) - f(b))/(b-a).$$

[Solution by John Oman and U. V. Satyanarayan (independently)]]

Comment by M. S. Klamkin, University of Alberta, Edmonton, Canada. The following theorem is stated and proved in M. S. Klamkin & D. J. Newman, On some inverse theorems in potential theory, *Quart. Appl. Math.*, **26**(1968) 277–280:

Theorem. If f(x + h) - f(x - h) = 2hf'(x) holds identically in x for two distinct positive values of h, then f(x) is a quadratic polynomial.

GEOMETRY

Regular polygons: inscribed polygons

Two-Year Coll. Math. J., 10 No.4 (Sept., 1979) 128–129.

146. Proposed by M. S. Klamkin, University of Alberta, Canada

Prove that the smallest regular n-gon which can be inscribed in a given regular n-gon will have its vertices at the midpoints of the sides of the given n-gon.

Two-Year Coll. Math. J., 12 No.1 (Jan., 1980) 64.

Solution by Howard Eves, University of Maine, Lubec, ME. Let $A_1A_2A_3...A_n$ be the given regular *n*-gon and let $P_1P_2P_3...P_n$ be an inscribed *n*-gon, where P_1 lies on A_1A_2 , P_2 on A_2A_3 , ..., P_n on A_nA_1 . Then $A_1P_1/P_1A_2 = A_2P_2/P_2A_3 = \cdots =$ A_nP_n/P_nA_1 . For the inscribed *n*-gon to have minimum area, each of the *n* congruent triangles cut off from $A_1A_2...A_n$ by $P_1P_2...P_n$ must have maximum area. Now, by the familiar formula for the area of a triangle in terms of two sides and the included angle, triangle $P_nA_1O_1$ will be maximum when the product $(A_1P_n)(A_1P_1) = (A_1P_1)(P_1A_2)$ is a maximum. But when a line segment A_1A_2 is partitioned into two parts by a point P_1 such that the product of the two parts is a maximum, P_1 is the midpoint of A_1A_2 .

GEOMETRY

Parallelograms

Two-Year Coll. Math. J., 10 No.5 (Dec., 1979) 361–362. Least Squares Property of the Centroid

117. (June 1978) Proposed by Norman Schaumberger, Bronx Community College, NY

Let *E* be the intersection of the diagonals of parallelogram ABCD and let *P* and *Q* be points on a circle with center *E*. Prove that $PA^2 + PB^2 + PC^2 + PD^2 = QA^2 + QB^2 + QC^2 + QD^2$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Canada. Let a_1 and a_2 be vectors from the origin to the vertices at the respective ends of one of the diagonals of the parallelogram. Similarly, let a_3 and a_4 be the vectors to the endpoints of the other diagonal, and let g be the vector to the point of intersection of the diagonals. Since the diagonals of the parallelogram bisect one another, $g = \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a_3 + a_4)$. We call the endpoint of g the centroid of the vertices because, clearly, $g = \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$. The solution is completed by the following theorem, which, among other things, allows us to generalize the problem, for example, by substituting a skew quadrilateral ABCD and the centroid E of its vertices for the parallelogram and the intersection of the diagonals. Without loss of generality, we will take the center of the circle to be at the origin.

Theorem. Let a_1, a_2, \ldots, a_n be vectors belonging to a Euclidean space, let $g = (1/n) \sum_{k=1}^{n} a_k$ and let c be a positive real number. Then $\sum_{k=1}^{n} ||a_k - p||^2$ is constant for all vectors p belonging to the sphere $||p||^2 = c$ having center at 0 if, and only if, g = 0.

Proof. From the vanishing of the inner product $(\sum_{k=1}^{n} (a_k - g), g - p) = (0, g - p) = 0$ it follows that

$$\sum_{k=1}^{n} \|a_k - p\|^2 = \sum_{k=1}^{n} \|a_k - g + g - p\|^2$$
$$= \sum_{k=1}^{n} \|a_k - g\|^2 + n\|g - p\|^2 \qquad (*)$$

Sufficiency of the condition g = 0 is now obvious. The proof is completed by observing that, if $g \neq 0$, then the values of $||g - p||^2$ in the last term in (*) at p = cg/||g|| and at p = -cg/||g||, respectively, are different.

Corollary. For given vectors a_1, a_2, \ldots, a_n and all vectors p, a unique minimum value of $\sum_{k=1}^{n} ||a_k - p||$ is achieved at $p = (1/n) \sum_{k=1}^{n} a_k$

Proof. The factor $||g-p||^2$ in the last term in (*), with p now representing an arbitrary vector, is greater when p is not g than when p is g.

Two-Year Coll. Math. J., **11** No.2 (Mar., 1980) 137–138. Lattice Point Principle

129. (Nov. 1978) Proposed by Warren Page, New York City Community College, Brooklyn, NY

For any $n^m + 1$ $(n \ge 2)$ lattice points in *m*-space, prove that there is at least one pair of points $\{P, Q\}$ such that (P - Q)/n is a lattice point.

[[Solution by Warren Ruud, Santa Rosa Junior College. CA]]

Editor's note: Klamkin noted that a problem equivalent to the case m = 3, n = 2 was given in the 1971 William Lowell Putnam Competition.

GEOMETRY

Triangle inequalities: sides

Two-Year Coll. Math. J., 11 No.2 (Mar., 1980) 138–141. Sharpening of Heron's Inequality

130. (Nov. 1978. Corrected June, 1979) Proposed by Aron Pinker, Frostburg State College, Frostburg, MD

Let a, b and c be the lengths of the sides of a triangle, P its perimeter and K its area. Prove that

(1)
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{9}{P}$$
 (4) $a^2 + b^2 + c^2 \ge 4\sqrt{3} K$
(2) $a^2 + b^2 + c^2 \ge \frac{P^2}{3}$ (5) $a^3 + b^3 + c^3 \ge \frac{p^3}{9}$
(3) $P^2 \ge 12\sqrt{3} K$

Composite of solutions by Donald C. Fuller, Gainesville J.C., GA; M. S. Klamkin, University of Alberta, Edmonton, Canada; Jack McCown, Central Oregon C.C., Bend, OR; and Thomas C. Wales, St. Mark's School, Southboro, MA

With no loss of generality, let $a \ge b \ge c$. Assume that a = b = c is false. (If a = b = c, then all the inequality signs below, except those of the form $k > k_0$ or $k < k_0$ where k_0 is either 0 or 1, should be changed to equals signs.) Both (1) and (2) are examples of Cauchy's inequality:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - (1+1+1)^2 = \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \left(\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}}\right)^2 + \left(\sqrt{\frac{b}{c}} - \sqrt{\frac{c}{b}}\right)^2 > 0$$

proves (1) and

$$(a^{2} + b^{2} + c^{2})(1 + 1 + 1) - (a + b + c)^{2} = (a - b)^{2} + (b - c)^{2} + (c - a)^{2} > 0$$

proves (2). Inequality (2) is also an example of Chebychev's inequality, as are (5) and

$$\frac{a^3 + b^3 + c^3}{3} > \frac{a^2 + b^2 + c^2}{3} \cdot \frac{a + b + c}{3} \tag{6}$$

Clearly (5) follows from (6) and (2), and

$$3(a^3 + b^3 + c^3) - (a^2 + b^2 + c^2)(a + b + c) = (a^2 - b^2)(a - b) + (a^2 - c^2)(a - c) + (b^2 - c^2)(b - c) > 0$$

proves (6). We call (3) Heron's inequality and we prove it by using Heron's formula and the inequality of arithmetic and geometric means: If s = P/2, then

$$K^{2}/s = (s-a)(s-b)(s-c) < \left(\frac{1}{3}((s-a)+(s-b)+(s-c))\right)^{3} = (s/3)^{3}$$

from $K^2 < s^4/27$, we obtain $K < s^2/(3\sqrt{3}) = P^2/(12\sqrt{3})$. Finally, (4) follows from (2) and (3). This completes the solution. Note that we have used

$$3(a^{k+1} + b^{k+1} + c^{k+1}) - (a^k + b^k + c^k)(a + b + c) = (a^k - b^k)(a - b) + (a^k - c^k)(a - c) + (b^k - c^k)(b - c) > 0$$

for k = 1 and k = 2, when, clearly, it holds for any positive number. It follows by induction that

$$a^k + b^k + c^k > P^k/3^{k-1} \tag{7}$$

for k equal to any positive integer > 1. Combining (7) with (3) proves

$$a^{k} + b^{k} + c^{k} > (12\sqrt{3})^{k/2}/3^{k-1}$$
(8)

which is the extension of (3) and (4) to all positive integer values of k.

Substituting 1/a, 1/b and 1/c for a, b and c, respectively, in (7), and then using (1) yields

$$\left(\frac{1}{a}\right)^k + \left(\frac{1}{b}\right)^k + \left(\frac{1}{c}\right)^k > \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^k / 3^{k-1} > 3^{k+1}/P^k$$

which shows that (7) holds also for all integer values of k that are less than 0. The fact is that (7) holds for k equal to any real number such that either k > 1 or k < 0, and that, if 0 < k < 1, then (7) holds, providing [[provided that?]] the direction of the inequality in (7) is reversed. This can be deduced from Hölder's inequality. We have only, in the case k > 1, to raise both sides of the following inequality to the k th power:

$$a + b + c < (a^k + b^k + c^k)^{1/k}(1 + 1 + 1)^{1-1/k}$$

In case $0 \neq k < 1$ we do the same to the inequality

$$a + b + c > (a^k + b^k + c^k)^{1/k} (1 + 1 + 1)^{1 - 1/k}$$

taking care, in case k < 0, to again reverse the direction of the inequality. Since(7) holds for all real k > 1, so does (8).

Comment by Allen Kaufman, Peat Marwick Mitchell & Co., New York, NY. It is easily shown by elementary calculus that the area $\sin \theta (1 + \cos \theta)$ of the triangle $T = T(\theta)$ $(0 < \theta < \pi)$ whose vertices are the points (-1, 0), $(\cos \theta, \sin \theta)$ and $(\cos \theta, -\sin \theta)$ is greatest when $\theta = \pi/3$ (i.e., when T is equilateral), and that a smaller area is obtained when $\theta \neq \pi/3$. That is, the area of an equilateral triangle inscribed in a circle is greater than the area of an inscribed triangle which is not equilateral, providing [[provided that?]] the latter triangle is isosceles. This conclusion remains correct if we delete the words "providing [[provided that?]] the latter triangle is isosceles" for the area of an inscribed triangle having, say, chord \overline{AB} as one of its sides is certainly not greater than the area of one of the two inscribed triangles that have \overline{AB} for their common base. We refer to the geometric mean of the lengths of the sides of a triangle as the geometric mean of the triangle, and similarly for other means.

Theorem. Given a triangle which is not equilateral, if its geometric mean is,say, d, then its area is less than the area of the equilateral triangle of perimeter 3d.

Proof. Let U be a triangle with area u, circumradius x, and whose sides a, b and c are not all equal. We know that abc = 4ux. Let V be an equilateral triangle of perimeter 3d, which has area v and circumradius y, so that $d^3 = 4vy$. Assume that $abc = d^3$. It follows from ux = vy that u < v if and only if x > y. Consider the equilateral triangle of perimeter 3h which, like U, has circumradius x. It was proved above that the area, known to be $h_3/(4x)$, of this equilateral triangle is greater than u. From $h^3 > 4ux = 4vy = d^3$ we obtain h > d. Since h > d, x > y and the proof is complete.

Clearly, the theorem is equivalent to the inequalities

$$(\sqrt{3}/4)(abc)^{2/3} > K$$
 or $(abc)^{1/3} > (4K/\sqrt{3})^{1/2}$

in which a, b and c represent the sides of any triangle which is not equilateral, and K is the area of the triangle. By the inequality of arithmetic and geometric means, if r is a real number > 0, then

$$((a^r + b^r + c^r)/3)^3 > a^r b^r c^r$$
 or $((a^r + b^r + c^r)/3)^{1/r} > (abc)^{1/3}$

It follows that

$$((a^r + b^r + c^r)/3)^{1/r} > (4K/\sqrt{3})^{1/2} \qquad (r > 0)$$
(*)

Clearly, this result can be expressed in the following way.

Corollary. If r is a positive number, and a triangle which is not equilateral is given whose exponential mean of order r is, say, d, then the area of this triangle is less than the area of the equilateral triangle of perimeter 3d.

To see that (*) does not hold for r < 0, consider the isosceles triangle whose base has length c, whose altitude has length 2/c, and which has legs of equal lengths, a = b, which depend on c. As $c \to 0$, a and b clearly increase beyond all bounds, but Kremains always equal to 1. If r < 0, then, as $c \to 0$, the left side of (*) tends to 0, while the right side remains equal to a positive constant. This contradiction shows that the real number r in (*) must be positive.

Klamkin noted that it is not possible, simply by reversing the inequality signs, to make (*) valid for negative values of r. Consider the triangle determined by a = b = 1, $c = 2 \sin \theta$, where θ is less than $\pi/2$ and closeto $\pi/2$. As $\theta \to \pi/2$, the area $K = \sin \theta \cos \theta \to 0$, and

$$((a^r + b^r + c^r)/3)^{1/r} = ((2 + (2\sin\theta)^r)/3)^{1/r} \to ((2+2^r)/3)^{1/r} > 0$$

This shows that the inequality

$$((a^r + b^r + c^r)/3)^{1/r} < (4K/\sqrt{3})^{1/r}$$

is not valid.

Editor's Note: Most solvers mailed solutions correcting and solving the problem prior tothe printing of a corrected version. Many of them noted that the problem is closely related to TYCMJ problem 98 (Sept. 1977 and Jan. 1979). Klamkin, Kocher, Prielipp and Rabinowitz gave the following references: O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969; H. S. Hall & S. R. Knight, *Higher Algebra*, Macmillan, London, 1891; and D. S. Mitrinovic, *Elementary Inequalities*, Stechert-Hafner, New York, 1964.

GEOMETRY

Maxima and minima: triangles

Two-Year Coll. Math. J., **11** No.4 (Sept., 1980) 279–280. Generalization of a Property of the Symmedian Point

140. (Mar. 1979) Proposed by Norman Schaumberger, Bronx Community College, NY

Locate a point P in the interior of a triangle such that the sum of the squares of the distances from P to the sides of the triangle is a minimum.

Solution by M. S. Klamkin, University of Alberta, Canada. It is known result that the point in the plane for which the sum of the squares of the distances to the sides of a triangle is a minimum is the symmedian point (the isogonal conjugate to the centroid). [R. A. Johnson, Advanced Euclidean Geometry, Dover, NY, 1960, p.216.]

More generally, it is not much harder to locate an interior point P which minimizes

$$S = x_1 r_1^n + x_2 r_2^n + x_3 r_3^n$$

where x_1, x_2, x_3 are given arbitrary nonnegative numbers, n > 1, and r_1, r_2, r_3 denote the distances from P to the sides of the triangle. By Hölder's inequality

$$(x_1 r_1^n + x_2 r_2^n + x_3 r_3^n)^{1/n} \left\{ \left(\frac{a_1^n}{x_1}\right)^{1/(n-1)} + \left(\frac{a_2^n}{x_2}\right)^{1/(n-1)} + \left(\frac{a_3^n}{x_3}\right)^{1/(n-1)} \right\} \\ \ge a_1 r_1 + a_2 r_2 + a_3 r_3 = 2\Delta \qquad (\Delta = \text{ area of triangle})$$

with equality if and only if

$$x_i r_i^n = \lambda^n (a_i^n / x_i)^{1/(n-1)}$$
 for $i = 1, 2, 3$.

 λ^n is determined by solving for r_i and substituting in $\sum a_i r_i = 2\Delta$ or

$$\lambda = \frac{2\Delta}{\sum (a_i^n / x_i)^{1/(n-1)}}$$

and

$$r_i = \lambda (a_i/x_i)^{1/(n-1)}$$

Also,

$$S_{\min} = (2\Delta)^n \left\{ \sum (a_i^n / x_i)^{1/(n-1)} \right\}^{1-n}$$

The results for the proposed problem are gotten by setting $x_i = 1$, n = 2. The maximum of S is easily obtained in the following manner:

$$\frac{S}{(2\Delta)^n} = \sum \frac{x_i}{a_i^n} \left(\frac{a_i r_i}{2\Delta}\right)^n \le \max\left(\frac{x_1}{a_1^n}, \frac{x_2}{a_2^n}, \frac{x_3}{a_3^n}\right) \left(\frac{a_1 r_1 + a_2 r_2 + a_3 r_3}{2\Delta}\right)$$
since $1 \ge a_i r_i / 2\Delta \ge 0$. Thus,

$$S_{\max} = (2\Delta)^n \max_i x_i / a_i^n$$

and which is taken on for P coinciding with one of the vertices of the triangle. The above results can also be extended for simplexes in E^m in the same way. Two-Year Coll. Math. J., 13 No.1 (Jan., 1982) 65.

208. Proposed by M. S. Klamkin, University of Alberta, Canada

The following problem and solution appear in [1]: - "The sum of the roots of the equation

$$x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$$

is 4; explain why on attempting to solve the equation from the knowledge of this fact the method fails."

" $x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 5x + 5)(x^2 - 3x + 1)$; on putting x = 4 - y, the expressions $x^2 - 5x + 5$ and $x^2 - 3x + 1$ become $y^2 - 37 + 1$ and $y^2 - 5y + 5$, respectively, so that we merely reproduce the original equation."

Give a better explanation.

REFERENCE

1. H. S. Hall & S. R. Knight, Higher Algebra, Macmillan, London, 1932, pp.489, 551.

Two-Year Coll. Math. J., 14 No.3 (June, 1983) 261–262.

Composite of solutions by Floyd Barger, Youngstown State University, OH; C. Patrick Collier, University of Wisconsin–Oshkosh; and Clayton W. Dodge, University of Maine at Orono. If Hall & Knight has used more space, they mighthave made their point in the following way. Let $f(x) = x^4 - 8x^3 + 21x^2 - 20x + 5$. We can solve the equation f(x) = 0 without using the information that that the sum of two roots is 4 asfollows. Since the sum of the four roots is 8, to eliminate the third degree term, we compute $f(y+2) = y^4 - 3y^2 + 1$. Since the first degree term has also vanished, we continue the solution by completing the square:

$$(y^4 - 2y^2 + 1) - y^2 = (y^2 - 1 - y)(y^2 - 1 + y)$$

or

$$f(x) = (x^2 - 3x + 1)(x^2 - 5x + 5)$$
(1)

and the solution can be completed without difficulty.

This example has been given for the purpose of making the following comments. One sees easily from (1) that

$$gcd(f(x), f(3-x)) = x^2 - 3x + 1$$

$$gcd(f(x), f(5-x)) = x^2 - 5x + 5 \text{ and}$$

$$gcd(f(x), f(4-x)) = (x^2 - 3x + 1)(x^2 - 5x + 5) = f(x)$$

This shows that if the sum of two roots of f is a given rational number a, then the methodof obtaining a depressed equation with rational coefficients by using the Euclidean algorithm to compute the gcd of f(x) and f(a - x) would be successful in case a = 3 or a = 5, but it would fail in case a = 4. We note that the failure would not be due to the irreducibility of f(x) over the rational numbers, but to the fact that the substitution of 4 - x for x merely interchanges the factors $x^2 - 3x + 1$ and $x^2 - 5x + 5$.

A different method and one which will not fail to solve the equation f(x) = 0 when we are given that the sum of two roots is 4 begins by writing

$$f(x) = (x^2 - 4x + b)(x^2 - 4x + c)$$
(2)

and then b and c are easily determined by multiplying the factors on the right, and comparing the coefficients obtained with the coefficients given in the definition of f. In this way, we obtain the values $(-5 \pm \sqrt{5})/2$ for b and c, so that, in contrast to (1), the factorization (2) requires irrational numbers.

Two-Year Coll. Math. J., 13 No.1 (Jan., 1982) 66.

The Inequality
$$\sum (\sum a^r / \sum a^s) > \sum a^{r-s}$$

163. (Mar. 1981) (**Corrected**) Proposed by Wm. R. Klinger, Marion College, Marion, IN

Assume $a_i > 0$ (i = 1, 2, 3, 4), $a_5 = a_1$ and $a_6 = a_2$. Prove or disprove:

1.
$$\sum_{i=1}^{4} (a_i^3 + a_{i+1}^3 + a_{i+2}^3) / (a_i + a_{i+1} + a_{i+2}) \ge \sum_{i=1}^{4} a_i^2$$

and

2.
$$\sum_{i=1}^{4} (a_i^4 + a_{i+1}^4 + a_{i+2}^4) / (a_i + a_{i+1} + a_{i+2}) \ge \sum_{i=1}^{4} a_i^3$$

Solution by M. S. Klamkin, University of Alberta, Canada. Let n and k be integers that are > 1, let r > s > 0, let $a_i > 0$ (i = 1, 2, ...), let the a_i be not all equal to the same constant, and let $j \equiv m \pmod{n}$ imply $a_j = a_m$. We show more generally that

$$S \equiv \sum_{i=1}^{n-1} \frac{a_{i+1}^r + a_{i+2}^r + \dots + a_{i+k}^r}{a_{i+1}^s + a_{i+2}^s + \dots + a_{i+k}^s} > \sum_{m=1}^n a_m^{r-s}$$

By Chebyshev's inequality

$$\frac{1}{k} \sum_{j=1}^{k} a_{i+j}^{r} \ge \left(\frac{1}{k} \sum_{k=1}^{k} a_{i+j}^{s}\right) \left(\frac{1}{k} \sum_{k=1}^{k} a_{i+j}^{r-s}\right) \qquad i = 0, 1, 2, \dots$$
(*)

That is, the *i*th term in the sum which defines S is $\geq (1/k) \sum_{j=1}^{k} a_{i+j}^{r-s}$. Since the numbers $a_1, a_2 \dots$ are not all equal, at least one of the inequalities in (*) is strict. It follows that

$$S > \sum_{i=0}^{n-1} \frac{1}{k} \sum_{j=1}^{k} a_{i+j}^{r-s} = \frac{1}{k} \sum_{j=1}^{k} \sum_{i=0}^{n-1} a_{i+j}^{r-s}$$
$$= \frac{1}{k} \sum_{j=1}^{k} \sum_{m=1}^{n} a_m^{r-s} = \sum_{m=1}^{n} a_m^{r-s}$$

Two-Year Coll. Math. J., **13** No.2 (Mar., 1982) 151–152.

Probability of a Subradial Distance

178. (Nov. 1980) Proposed by Roger L. Creech, East Carolina University, Greenville, NC

If points A and B are selected at random in the interior of a circle, what is the probability that AB is less than the length of the radius of the circle?

Editor's Note: Klamkin noted that the problem appears in J. Edwards, *Treatise on Integral Calculus*, II. Chelsea, NY, 1954, p.852, and is credited to I. P. Ox, 1916.

Two-Year Coll. Math. J., 14 No.2 (Mar., 1983) 173.

248. Proposed by M. S. Klamkin, University of Alberta, Canada

(1) Given that a_1, a_2, \ldots and b_1, b_2, \ldots are non-constant, nonproportional arithmetic progressions. Determine the maximum number of consecutive terms in the sequence $(a_1/b_1), (a_2/b_2), \ldots$ which can be in (i) arithmetic progression, (ii) geometric progression.

(2) Given that a_1, a_2, \ldots and b_1, b_2, \ldots are non-constant, nonproportional geometric progressions. Determine the maximum number of consecutive terms in the sequence $(a_1 + b_1), (a_2 + b_2), \ldots$ which can be in (i) arithmetic progression, (ii) geometric progression.

Two-Year Coll. Math. J., 16 No.2 (Mar., 1985) 155–157.

Solution by Michael Vowe, Therwil, Switzerland. (1) Let $a_n = a + (n-1)d$, $b_n = b + (n-1)e$, n = 1, 2, ..., where $d \neq 0$, $e \neq 0$, [[not e = 0 as printed]] $ae - bd \neq 0$, since the progressions are nonconstant and nonproportional. Also $b_n \neq 0$, $n \geq 1$, for the ratios to be defined.

(i) Let f be the difference between successive terms in the sequence of ratios, so

$$f = \frac{a_2}{b_2} - \frac{a_1}{b_1} = \frac{bd - ae}{b(b+e)}$$

The difference between a third term and the first must be 2f, so

$$\frac{a_3}{b_3} - \frac{a_1}{b_1} = \frac{a+2d}{b+2e} - \frac{a}{b} = \frac{2(bd-ae)}{b(b+e)}$$

This yields e(ae - bd) = 0 which is impossible, so the maximum number of consecutive terms is two.

(ii) Clearly $a_n \neq 0$, $n \geq 1$, for the ratios to be in geometric progression. Let q be the ratio of successive terms in the sequence of ratios, so

$$q = \frac{a_2}{b_2} \bigg/ \frac{b(a+d)}{a(b+e)}$$

The ratio $a_3/b_3 = q^2(a_1/b_1)$ or $a(a+2d)(b+e)^2 = b(b+2e)(a+d)^2$. This may be written $((a+d)^2 - d^2)(b+e)^2 = ((b+e)^2 - e^2)(a+d)^2$, which yields $e^2(a+d)^2 = d^2(b+e)^2$ or $e(a+d) = \pm d(b+e)$. A three-term geometric progression which may be found by using this condition is a/b, k, $(b/a)k^2$, $k \neq 0$, formed of ratios of the arithmetic progressions a, $\frac{a+bk}{2}$, bk and b, $\frac{b+(a/k)}{2}$, a/k.

There cannot be fourth term in the sequence of ratios since this would give $a_4/b_4 = q^3(a_1/b_1)$, or $a^2(a+3d)(b+e)^3 = b^2(b+3e)(a+d)^3$. This may be written

$$((a+d)^3 - d^2(3a+d))(b+e)^3 = ((b+e)^3 - e^2(3b+e))(a+d)^3$$

which leads, after using $e^2(a+d)^2 = d^2(b+e)^2$, to ae = bd which is not allowed. Thus the maximum number of consecutive terms is three.

(2) Let $a_n = ap^{n-1}$, $b_n = bq^{n-1}$, n = 1, 2, ..., where $a \neq 0$, $b \neq 0$, $p \neq 1$, $q \neq 1$, $p \neq q$, since the progressions are nonconstant and nonproportional.

(i) Let d be the difference between successive terms in the sequence of sums, so $d = a_2+b_2-a_1-b_1 = a(p-1)+b(q-1)$. The difference between the third term and the first must be 2d. Therefore $a_3+b_3-a_1-b_1 = a(p^2-1)+b(q^2-1) = 2(a(p-1)+b(q-1))$, which leads to $a(p-1)^2+b(q-1)^2 = 0$. A three term arithmetic progression which may be found using this condition is 6, 10, 14, formed of sums of the geometric progressions 8, 16, 32 and -2, -6, -18.

There cannot be a fourth term in the sequence of sums since this would give $3d = a_4 + b_4 - a_1 - b_1$ or $3(a(p-1) + b(q-1)) = a(p^3 - 1) + b(q^3 - 1)$. This may be written $a((p-1)^3 + 3(p-1)^2) + b((q-1)^3 + 3(q-1)^2) = 0$, which gives $a(p-1)^2(p+2) + b(q-1)^2(q+2) = 0$. This leads, after using $a(p-1)^2 = -b(q-1)^2$, to p = q which is not allowed. Thus the maximum number of consecutive terms is three.

(ii) Let r be the ratio of successive terms in the sequence of sums, so $a_2+b_2 = r(a_1+b_1)$ or r(a+b) = ap+bq. A third term in the sequence of sums would give $a_3+b_3 = r^2(a_1+b_1)$ or $(a+b)(ap^2+bq^2) = (ap+bq)^2$. This implies $ab(p-q)^2 = 0$, which is impossible, so the maximum number of consecutive terms is two.

Two-Year Coll. Math. J., 14 No.3 (June, 1983) 261.

253. Proposed by M. S. Klamkin, University of Alberta, Canada

Assume that A, B and C are the angles of a triangle. Prove that

$$3(\cot A + \cot B + \cot C) \ge \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$$

with equality if and only if the triangle is equilateral.

Two-Year Coll. Math. J., 16 No.3 (June, 1985) 224–225.

Solution by Robert L. Young, Cape Cod Community College, West Barnstable, Mass. Using $\cot(A/2) = (1 + \cos A) / \sin A = \csc A + \cot A$ and the corresponding equations for B and C, the inequality can be reduced to

$$2(\cot A + \cot B + \cot C) \ge \csc A + \csc B + \csc C \tag{1}$$

The left-hand side of (1) can be written in the form $(\cot A + \cot B) + (\cot B + \cot C) + (\cot C + \cot A)$. If we then multiply each side of (1) by a factor equal to twice the area of triangle ABC, we obtain

$$ch_{c}(\cot A + \cot B) + ah_{a}(\cot B + \cot C) + bh_{b}(\cot C + \cot A)$$
$$\geq bc \sin A(\csc A) + ac \sin B(\csc B) + ab \sin C(\csc C)$$
(2)

where h_a , h_b and h_c represent the lengths of the altitudes of triangle ABC from the vertices A, B and C respectively. Then (2), and therefore also (1), is seen to be equivalent to

$$a^2 + b^2 + c^2 \ge bc + ca + ab \tag{3}$$

Doublung (3) gives us

$$(a^{2} + b^{2}) + (b^{2} + c^{2}) + (c^{2} + a^{2}) \ge 2ab + 2bc + 2ca$$

or, equivalently,

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$$

This last inequality is obviously correct, and it is obvious that equality holds if and only if a = b = c. This completes the solution.

Comment by Michael Vowe, Therwil, Switzerland. Let s be the semiperimeter of the triangle, R the circumradius, r the inradius, and r_a , r_b and r_c the radii of the excircles opposite A, B and C, respectively. Since $\cot A = (\cot(A/2) - \tan(A/2))/2$, the proposed inequality can be written in the form

$$\cot(A/2) + \cot(B/2) + \cot(C/2) \ge 3(\tan(A/2) + \tan(B/2) + \tan(C/2))$$

or

$$(s-a)/r + (s-b)/r + (s-c)/r \ge 3((r_a/s) + (r_b/s) + (r_c/s))$$

That is, $s^2 \ge 3r(r_a + r_b + r_c)$ or $s^2 \ge 3r(4R + r)$. This last inequality occurs in O. Bottema et al., *Geometric Inequalities*, Walters-Noordhoff, Groningen, 1968, p.49 (5.6). [[That shd be 'Wolters-Noordhoff' – R.]]

Two-Year Coll. Math. J., 14 No.5 (Nov., 1983) 439.

261. Proposed by M. S. Klamkin, University of Alberta, Canada

Explain the following numerical pattern which was obtained on a pocket calculator:

 $\begin{array}{rcl} \tan 89.999 &\approx & 57295.77951 \\ \tan 89.9999 &\approx & 572957.7951 \\ \tan 89.99999 &\approx & 5729577.951 \\ \tan 89.999999 &\approx & 57295779.51 \\ \tan 89.9999999 &\approx & 572957795.1 \\ \tan 89.99999999 &\approx & 572957795.1 \end{array}$

[[At this point the name of the journal changed to

College Math. Journal]]

Coll. Math. J., 16 No.4 (Sept., 1985) 306–307.

Composite of solutions by Dan Kalman, Augustana College, Sioux Falls, SD; and Jan Söderkvist (student), Royal Institute of Technology, Stockholm, Sweden. The calculator is operating correctly. Let $\theta_k = 10^{-k}\pi/180$, $k = 3, \ldots, 8$. The values given by the calulator for $\tan(\pi/2 - \theta_k)$ are, to 10 significant figures, the values of $1/\theta_k$. We will show that, to 10 significant figures, $\tan(\pi/2 - \theta_k) = 1/\theta_k$. Let $0 < \theta \le 10^{-3}\pi/180 = K$. By Taylor's formula with the Lagrange form of the remainder,

$$\tan(\pi/2 - \theta) = 1/\tan\theta = 1/(\theta + R)$$

where

$$R = (1/3)\theta^3 (1 + \tan^2 A)(1 + 3\tan^2 A)$$

for some A between 0 and θ . Also

$$1/\theta - R/\theta^2 < 1/(\theta + R) < 1/\theta$$

Since $(\tan x)/x$, considered for $0 < x < \pi/4$, is an increasing function of x, we have $\tan A < \tan K < 4K/\pi$ and

$$R/\theta^2 < (K/3)(1 + (4K/\pi)^2)(1 + 3(4K/\pi)^2)$$

Calculation shows that the right-hand side is less than $(1/2)10^{-5}$. Noting that the last figure given by the calculator for tan 89.999 is in the fifth decimal place completes the solution.

Two-Year Coll. Math. J., 15 No.1 (Jan., 1984) 68.

265. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine the extreme values of

$$S_i = \sin^i B + \sin^i C - \sin^i A \qquad (i = 1, 2)$$

where A, B and C are angles of a triangle.

Coll. Math. J., 16 No.4 (Sept., 1985) 310.

Composite of solutions by Richard Parris, Phillips Exeter Academy, Exeter, NH; and Doug Wiens, Dalhousie University, Halifax, Nova Scotia, Canada. If $B = C = \pi/2$ and A = 0, then $S_n = 2$, n = 1, 2. Since $\sin A \ge 0$, $S_1 > 2$ is impossible. It follows that $\sup S_n = 2$, n = 1, 2. Since the sines of the angles of a triangle are proportional to the lengths of the opposite sides, $S_1 > 0$ with equality if the triangle is degenerate. Therefore, $\inf S_1 = 0$. Finally,

$$S_2 = \sin^2 B + \sin^2 C - \sin^2 A = 2\sin B\cos C\cos A \tag{(*)}$$

which shows that negativevalues of S_2 are obtained if $\pi/2 < A < \pi$. With no loss in generality, let A satisfy these inequalities. By (*), using the arithmetic-geometric mean inequality and the concavity of the sine and cosine functions on the interval $(0, \pi/2)$, we have

$$S_{2} \geq 2((\sin B + \sin C)/2)^{2} \cos A \geq 2 \sin^{2}((B + C)/2) \cos A$$

= $2 \cos^{2}(A/2) \cos A = -2 \cos^{2}(A/2) \cos(\pi - A)$
 $\geq -2((2 \cos(A/2) + \cos(\pi - A))/3)^{3}$
 $\geq -2 \cos^{3}((2(A/2) + (\pi - A))/3) = -2 \cos^{3}(\pi/3) = -1/4$

with equality if and only if $A/2 = \pi - A$ and B = C. Then min $S_2 = -1/4$ and $S_2 = -1/4$ if and only if $A = (2/3)\pi$ and $B = C = \pi/6$. This completes the solution.

Coll. Math. J., 15 No.3 (June, 1984) 267.

278. Proposed by M. S. Klamkin, University of Alberta, Canada

If A is an $n \times n$ matrix such that $A^3 = pA^2 + gA + rI$, determine $(A - I)^{-1}$ assuming it exists.

Coll. Math. J., 17 No.2 (Mar., 1986) 185–186.

Solution by J. Suck, Essen, West Germany. More generally, let A satisfy the polynomial equation p(x) = 0, where $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ and the c_i are given elements of a field which also contains the components of $A, c_n \neq 0$. Let $P = (p_{ij})$ denote the $(n+1) \times (n+1)$ matrix such that

$$p_{ij} = \binom{j}{i}$$
 $(i = 0, 1, \dots, n; j = 0, 1, \dots, n)$

Let C be the $(n+1) \times 1$ column matrix defined as the transpose $(c_0, c_1, c_2, \ldots, c_n)'$ of the row $(c_0, c_1, c_2, \ldots, c_n)$. Let S = PC define $S = (s_0, s_1, s_2, \ldots, s_n)'$. Let $D = (d_{ij})$ be the $n \times n$ matrix such that

$$d_{ij} = c_{i+j-1}$$
 $(i = 0, 1, \dots, n; j = 0, 1, \dots, n)$

Let $X = (1, x, x^2, ..., x^{n-1})'$ and let P_i (i = 0, 1, ..., n-1) denote the $1 \times n$ row matrix whose j th component (j = 0, 1, ..., n-1) is the binomial coefficient $\binom{j}{i}$. Clearly, the result, indexed by i, of each one of the n successive divisions by x-1 occurring in the application of Horner's process to the problem of writing p(x) as a linear combination of powers of x-1 can be described by the equation

$$p(x) = P_i DX(x-1)^{i+1} + s_i(x-1)^i + \dots + s_2(x-1)^2 + s_1(x-1) + s_0$$
(1)

(i = 0, 1, 2, ..., n-1). The conclusion of the process, obtained for i = n-1, is, of course, the equation $p(x) = s_n(x-1)^n + \cdots + s_2(x-1)^2 + s_1(x-1) + s_0$. Substituting A for x, and A-I for x-1, in this equation shows that, in consideration of the existence of $(A-I)^{-1}$ and the fact that p(A) = 0,

$$u = \min\{i : s_i \neq 0\} < n$$

Substituting *u* for *i* in (1) gives us $p(x) = P_u DX(x-1)^{u+1} + s_u(x-1)^u$. From p(A) = 0and the existence of $(A - I)^{-1}$, we infer that *A* satisfies $P_u DX(x-1) + s_u = 0$ or, equivalently, $-s_u^{-1}P_u DV(x-1) = 1$. Therefore, the inverse of A - I is obtained by substituting *I*, *A*, A^2 , ..., A^{n-1} for the components of $X = (1, x, x^2, ..., x^{n-1})$ in the expression

$$-s_u^{-1}P_uDX$$

The solution of the given problem follows from the above discussion by setting n = 3and C = (r, q, p, -1). Then

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and S = P = (r+q+p-1, q+2p-3, p-3, -1). By (2), if $r+q+p \neq -1$, then

$$(A - I)^{-1} = -(r + q + p - 1)^{-1} \left((q + p - 1)I + (p - 1)A - A^2 \right)$$

If r + q + p = 1 and $q + 2p \neq 3$, then

$$(A - I)^{-1} = -(q + 2p - 3)^{-1}((p - 2)I - A)$$

If r + q + p = 1, q + 2p = 3 and $p \neq 3$, then

$$(A - I)^{-1} = -(p - 3)^{-1}(-I)$$

Our solution is completed by noting that r+q+p=1, q+2p=3 and p=3 contradict the existence of $(A-I)^{-1}$.

Coll. Math. J., 15 No.3 (June, 1984) 269–270.

Mean Values of Subsets of a Finite Set

230. (Sept. 1982) Proposed by Sydney Bulman-Fleming and Edward T. H. Wang, Wilfred Laurier University, Waterloo, Ontario, Canada

Define the average of a finite nonempty set T of numbers be the average of the elements of T. Is it true that the mean of the averages of all the nonempty subsets of a finite nonempty set W always equals the average of W?

Solution by M. S. Klamkin, University of Alberta, Canada. Let f be a continuous strictly monotonic function defined on an interval which contains all the elements of W. For all finite nonempty subsets T of this interval, we define the mean $M_j(T)$ by

$$M_j(T) = f^{-1} \left\{ \frac{1}{n(T)} \sum_{a \in T} f(a) \right\}$$

where n(T) denotes the number of elements in T.

Examples. Let T be a finite nonempty set of positive real numbers. If f(x) = x, then $M_f(T)$ is the arithmetic mean of the elements in T. If $f(x) = \ln x$, $M_f(T)$ is the geometric mean. If f(x) = 1/x, $M_f(T)$ is the harmonic mean.

We will show that if "average of T" is interpreted as $M_f(T)$, then the answer is in the affirmative. Let m = n(W). Let T and T_a denote, respectively, a nonempty subset of W, and a subset of W which contains the element a. We note that the T_a are either 1-element sets, 2-elementsets, ..., or m-element sets, and that the number of k-element sets of W which contain a given element a is $\binom{m-1}{k-1}$, k = 1, 2, ..., m. Then

$$\begin{aligned} f^{-1}\left\{\frac{1}{2^m - 1}\sum_T f(M_f(T))\right\} &= f^{-1}\left\{\frac{1}{2^m - 1}\sum_T \left(\frac{1}{n(T)}\sum_{a \in T} f(a)\right)\right\} \\ &= f^{-1}\left\{\frac{1}{2^m - 1}\sum_{a \in W} \left(f(a)\sum_{T_a}\frac{1}{n(T_a)}\right)\right\} \\ &= f^{-1}\left\{\frac{1}{2^m - 1}\sum_{a \in W} \left(f(a)\sum_{k=1}^m\frac{1}{k}\binom{m-1}{k-1}\right)\right)\right\} \\ &= f^{-1}\left\{\frac{1}{2^m - 1}\sum_{a \in W} \left(f(a)\sum_{k=1}^m\frac{1}{m}\binom{m}{k}\right)\right\} \\ &= f^{-1}\left\{\frac{1}{2^m - 1}\sum_{a \in W} f(a)\right\} \end{aligned}$$

as was to be shown.

Coll. Math. J., 15 No.4 (Sept., 1984) 346–347.

282. Proposed by M. S. Klamkin, University of Alberta, Edmonton, Canada and Gregg Patruno (student), Princeton University, New Jersey (jointly)

(a) It is immediate that

$$\frac{\cos^2 x \cdot \cos^2 y}{\cot^2 x \cdot \cot^2 y} = \sin^2 x \cdot \sin^2 y$$

Show that the identity remains valid even if the multiplication signs on the left-hand side are changed to subtraction signs.

- (b) Generalize the above result.
- Coll. Math. J., 17 No.3 (June, 1986) 250–251.

Solution by Harry Sedinger, St. Bonaventure University, St. Bonaventure, NY. The generalization will be shown first. The functional equation

$$\frac{f(x) \cdot f(y)}{g(x) \cdot g(y)} = \frac{f(x) - f(y)}{g(x) - g(y)}$$

may be written as

$$\frac{1}{f(x)} - \frac{1}{f(y)} = \frac{1}{g(x)} - \frac{1}{g(y)}$$

so [[that]] its general solution is given by any pair of functions satisfying (1/f(x)) - (1/g(x)) = k, where k is an arbitrary constant. Thus, if f(x) is given and g(x) = f(x)/(1-kf(x)), then the functional equation is identically satisfied. This identity is a generalization of the result in (a) as required by (b).

Specifically, to prove (a), choose k = 1 and $f(x) = \cos^2 x$, and let

$$g(x) = \frac{f(x)}{1 - kf(x)} = \frac{\cos^2 x}{1 - \cos^2 x} = \cot^2 x$$

The above argument then shows the result in (a).

Coll. Math. J., 15 No.4 (Sept., 1984) 349–351.

Generalization of $a/b + b/a \ge 2$

235. (Nov. 1982) Proposed by Norman Schaumberger, Bronx Community College, Bronx, NY

Let m_1, m_2, m_3, m_4 be positive real numbers with $S = \sum_{i=1}^4 m_i$. Prove that

$$\sum_{i=1}^{4} m_i / (S - m_i) \ge 4/3.$$

Comment by M. S. Klamkin, University of Alberta, Canada. Let m_1, m_2, \ldots, m_n be the lengths of the sides of an n-gon, $n \ge 3$. Then $S - m_i > m_i$, $i = 1, 2, \ldots, n$. In this case we have the companion inequality $\sum_{i=1}^{n} (m_i/(S - m_i)) < 2$ (Crux Mathematicorum, 7(1981) 28).

Proof:

$$S - m_i = (S - m_i)/2 + (S - m_i)/2 > (S - m_i)/2 + m_i/2 = S/2 \quad i = 1, 2, \dots, n$$

Then $2m_i/S > m_i/(S - m_i)$ and

$$2 = (2/S) \sum_{i=1}^{n} m_i > \sum_{i=1}^{n} m_i / (S - m_i).$$

Coll. Math. J., 15 No.5 (Nov., 1984) 446.

Thanks Again, Euler

239. (Jan. 1983) Proposed by Norman Schaumberger, Bronx Community College, Bronx, NY

Prove or disprove: The product of four distinct nonzero integers in arithmetic progression cannot be a fourth power.

[[in fact a (special case of a) classic problem. The result attributed to Euler was already given by Fermat. - R.]]

Solution by M. S. Klamkin, University of Alberta, Canada. It is well known that the product of four consecutive integers cannot be a square since otherwise the identity $(a-2)(a-1)a(a+1)+1 = (a^2-a-1)^2$ leads to the equation $x^2+1 = (a^2-a-1)^2$, $x \neq 0$, which has no solution. Similarly, the product of four distinct nonzero integers in arithmetic progression cannotbe a fourth power since otherwise the identity $(a - 2d)(a - d)a(a + d) + d^4 = (a^2 - ad - d^2)^2$ leads to the equation $x^4 + d^4 = y^2$, $dx \neq 0$, which was shown by Euler to have no solution. The result still holds if a, d are assumed rational: just multiply the identity by a common denominator. Note that the second identity follows from the first when a is replaced by a/d.

Editor's note: The solver noted that this problem appeared in *Crux Mathematicorum*, Problem 645(1981, p.177) where the above solution was initially sent. The solvers cited many different texts on Number Theory for the result by Euler.

Coll. Math. J., 16 No.2 (Mar., 1985) 153.

297. Proposed by M. S. Klamkin, University of Alberta, Canada

One is given a set of functions $H_i(x_1, x_2, ..., x_r)$, i = 0, 1, ..., s which are homogeneous functions of degrees n_i , respectively, and are functionally dependent, i.e.,

$$H_0 = F(H_1, H_2, \dots, H_s)$$

Show that H_0 is homogeneous in the functions J_1, J_2, \ldots, J_s , where

$$J_i = H_i^{1/n}$$

Coll. Math. J., 18 No.1 (Jan., 198) 72.

Solution by William P. Wardlaw, United States Naval Academy, Annapolis, MD. Let $X = (x_1, x_2, ..., x_r)$ and let $H_0 = F(H_1, H_2, ..., H_s) = G(J_1, J_2, ..., J_s)$ where $J_i = H_i^{1/n_i}$. The functions $J_i(X)$ are each homogeneous of degree 1 since $J_i(tX) = (H_i(tX))^{1/n_i} = tJ_i(X), i = 1, 2, ..., s$. The desired result follows since

$$G(tJ_1, tJ_2, \dots, tJ_s) = G(J_1(tX), J_2(tX), \dots, J_s(tX))$$

= $F(H_1(tX), H_2(tX), \dots, H_s(tX))$
= $H_0(tX) = t^{n_0}H_0(X) = t^{n_0}G(J_1, J_2, \dots, J_s)$

Thus $H_0 = G(J_1, J_2, \ldots, J_s)$ is homogeneous of degree n_0 in the functions J_1, J_2, \ldots, J_s as required.

Coll. Math. J., 16 No.3 (June, 1985) 230–231.

Estimation of a Product

259. (Sept. 1983) Proposed by Sanjukta Hota (student) and Kathy Williams (student), jointly, Southeast Missouri State University, Cape Giradeau

Let a, d > 0 and m be a positive integer. Prove:

$$\sqrt[m]{\frac{a}{a+mnd}} \le \prod_{k=1}^{n} \frac{a+(mk-1)d}{a+mkd} \le \sqrt[m]{\frac{a+(m-1)d}{a+(mn+m-1)d}}$$

Solution by M. S. Klamkin, University of Alberta, Canada. Let A = (a - d)/(md) and B = a/(md). The given inequalities are equivalent to

$$\left(\frac{B}{B+n}\right)^{B-A} \le \frac{\Gamma(A+n+1)\Gamma(B+1)}{\Gamma(A+1)\Gamma(B+n+1)} \le \left(\frac{A+1}{A+n+1}\right)^{B-A} \tag{1}$$

Equality occurs both when B = A + 1 and when B = A, corresponding to m = 1 and $m = \infty$, respectively. We will establish (1) for any number m, not necessarily an integer, which is ≥ 1 . Let

$$F(B) + (B - A) \ln \frac{A + 1}{A + n + 1} + \ln \frac{\Gamma(B + n + 1)}{\Gamma(B + 1)} - \ln \frac{\Gamma(A + n + 1)}{\Gamma(A + 1)}$$

Since F(A) = F(A+1) = 0, these cond inequality in (1), equivalent to $F(B) \ge 0$ for A < B < A+1, will be established if we show that F(B) is concave or that $F''(B) \le 0$. Since it is known that

$$\frac{d^2 \ln \Gamma(z)}{dz^2} = \sum_{k=0}^{\infty} (z+k)^{-2}$$

the demonstration is completed by observing that

$$F''(B) = \sum_{k=0}^{\infty} \left\{ \frac{1}{(B+n+1+k)^2} - \frac{1}{(B+1+k)^2} \right\} \le 0$$

Similarly, to prove the first inequality in (1), let

$$G(A) = \ln \frac{\Gamma(A+n+1)}{\Gamma(A+1)} - \ln \frac{\Gamma(B+n+1)}{\Gamma(B+1)} - (B-A)\ln \frac{B}{B+n}$$

We will prove the inequality $G(A) \ge 0$, equivalent to the desired inequality, for $B \ge A \ge B - 1$. Since G(B) = G(B - 1) = 0, it is sufficient to show $G''(A) \le 0$. The solution is completed by observing that

$$G''(A) = \sum_{k=0}^{\infty} \left\{ \frac{1}{(A+n+1+k)^2} - \frac{1}{(A+1+k)^2} \right\} \le 0$$

Editor's Note: Shafer proved that the lower bound for the product can be raised to

$$\left(\frac{2a+md-2d}{2a+2mnd+md-2d}\right)^{1/m}$$

for any real number m > 2.

Coll. Math. J., 16 No.4 (Sept., 1985) 304.

308. Proposed by M. S. Klamkin, University of Alberta, Canada Evaluate

$$\left\{\frac{d^{3n}}{dx^{3n}}(1-\sqrt[3]{2\sin x})^{3n}\right\}_{x=\pi/6}$$

Coll. Math. J., 18 No.3 (May, 1987) 250–251.

Solution by Michael Vowe, Therwil, Switzerland. The formula of Faà di Bruno, as given by Steven Roman, The Formula of Faà di Bruno, Amer. Math. Monthly, 87(1980) p.807, states: If f(t) and g(t) are functions for which all the necessary derivatives are defined, then

$$D^{n}f(g(t)) = \sum \frac{n!}{k_{1}!\cdots k_{n}!} (D^{k}f)(g(t)) \left(\frac{Dg(t)}{1!}\right)^{k_{1}} \cdots \left(\frac{D^{n}g(t)}{n!}\right)^{k_{n}}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all k_1, \ldots, k_n for which $k_1 + 2k_2 + \cdots + nk_n = n$. In this case replace n by 3n and let $f(t) = t^{3n}$ and $g(t) = 1 - (2 \sin t)^{1/3}$, then notice

$$\{(D^k f)(g(t))\}_{t=\pi/6} = 0 \text{ for } 1 \le k \le 3n-1$$

since $g(\pi/6) = 0$ and $D^k f(0) = 0$. Thus the only nonzero term in the formula has $k_1 = 3n$ and $k_2 = k_3 = \cdots = k_{3n} = 0$ (from

$$3n = k_1 + k_2 + \dots + k_{3n} = k_1 + 2k_2 + \dots + 3nk_{3n})$$

This gives the required evaluation as

$$\begin{cases} \frac{d^{3n}}{dx^{3n}} \left(1 - \sqrt[3]{2\sin x}\right)^{3n} \\ = \frac{(3n)!}{(3n)!} (D^{3n}f) \left(g\left(\frac{\pi}{6}\right)\right) \left(Dg\left(\frac{\pi}{6}\right)\right)^{3n} \\ = (3n)! (-1)^{3n} \left(\frac{2^{1/3}}{3} \left(\sin\frac{\pi}{6}\right)^{-2/3} \cos\frac{\pi}{6}\right)^{3n} \\ = \frac{(-1)^{3n}(3n)!}{3^{3n/2}} \end{cases}$$

Coll. Math. J., 16 No.5 (Nov., 1985) 416.

313. Proposed by M. S. Klamkin, University of Alberta, Canada

The surface z = F(x, y) is smooth and is tangent to the (x, y) plane at the origin. Also, all plane curves of the surface containing the z-axis have a minimum value at the origin. Must the origin be a minimum point of the surface?

Coll. Math. J., 18 No.5 (Nov., 1987) 426.

Solutions by Howard K. Hilton, Chicago, IL; Joseph D. E. Konhauser, MacalesterCollege, St. Paul, MN; and the proposer (independently). No. We give three examples.

Example by Konhauser: $z = f(x, y) = (y - x^2)(y - 2x^2)$. The value of z is negative for points (x, y) lying betteen the parabolas $y = x^2$ and $y = 2x^2$. The value is zero on the parabolas, and positive at all other points.

Example by Hilton: $z = f(x, y) = (x^2 + y^2 - 2y)(x^2 - 2y)$. The circle $x^2 + y^2 - 2y = 0$ is internally tangent to the parabola $x^2 - 2y$ at the origin. The factors of f will have the same sign at points which are either inside the circle or outside the parabola; the factors of f have opposite signs at each point that is simultaneously outside the circle and inside the parabola, and z is zero on the circle and on the parabola.

Example by the proposer: $z = f(x, y) = (x - y^2)^2 - y^6$. Here z is negative for points $\neq (0, 0)$ of the parabola $x = y^2$. For points (x, y) on the lines y = mx, m arbitrary, that are sufficiently close to the origin, z is a positive multiple of x^2 . For points of the (x, y)-plane on the line x = 0 that are sufficiently close to the origin, z is a positive multiple of y^4 .

Each of the three examples is an example of a surface that is smooth and tangent to the (x, y)-plane at the origin. In each example, the values of f on any line through the origin in the (x, y)-plane have a strict minimum equal to zero at the origin, and yet there are points (x, y) arbitrarily close to the origin at which the value of f is < 0.

Coll. Math. J., 17 No.1 (Jan., 1986) 92.

316. Proposed by M. S. Klamkin, University of Alberta, Canada

If $a_1, a_2, a_3, a_4, a_5 > 0$, prove that

$$\sum_{\text{cyclic}} \left(\frac{a_1 a_2 a_3}{a_4 a_5}\right)^4 \ge \sum_{\text{cyclic}} a_1 a_2^2 a_3$$

When is there equality?

Coll. Math. J., 18 No.5 (Nov., 1987) 428.

Solution by Beno Arbel, Tel Aviv University, Israel. The following inequality is obtained with the aid of the Arithmetic Mean-Geometric Mean Inequality: If $x_i > 0$, i = 1, ..., 5, then

$$\sum_{i=1}^{5} = \sum_{\text{cyclic}} \left(\frac{x_1 + x_2}{2} \right) \ge \sum_{\text{cyclic}} \sqrt{x_1 x_2}$$

with equality if and only if $x_1 = \cdots = x_5$. This inequality will be used three times to complete the following proof of the given inequality:

$$\sum_{\text{cyclic}} \left(\frac{a_1 a_2 a_3}{a_4 a_5}\right)^4 = \left(\frac{a_1 a_2 a_3}{a_4 a_5}\right)^4 + \left(\frac{a_3 a_4 a_5}{a_1 a_2}\right)^4 + \left(\frac{a_5 a_1 a_2}{a_3 a_4}\right)^4 + \left(\frac{a_2 a_3 a_4}{a_5 a_1}\right)^4 + \left(\frac{a_4 a_5 a_1}{a_2 a_3}\right)^4$$
$$\geq a_3^4 + a_5^4 + a_4^4 + a_1^4 = \sum_{i=1}^5 a_i^4 \geq \sum_{\text{cyclic}} a_1^2 a_2^2 \geq \sum_{\text{cyclic}} a_1 a_2^2 a_3$$

with equality if and only if $a_1 = \cdots = a_5$, which is the desired result.

Coll. Math. J., 17 No.2 (Mar., 1986) 184.

321. Proposed by M. S. Klamkin, University of Alberta, Canada

Prove that for x > 1,

$$s \ge \frac{(x-1)^{x-1}x^x}{(x-\frac{1}{2})^{2x-1}} > 1$$

Coll. Math. J., 19 No.1 (Jan., 1988) 84–85.

Composite of solutions by Chico Problem Group, California State University; Bjorn Poonen (student), Harvard College [[??]], Cambridge, MA; and Heinz-Jürgen Seiffert, Berlin, Germany.

Let $f(x) = x \ln x$ and note that $\lim_{x\to 0^+} f(x) = 0$ (use l'Hospital's rule), and that f(x) is convex for x > 0 since f''(x) = 1/x > 0. The given inequality is then obviously equivalent to $\ln 2 \ge f(x-1) + f(x) - 2f(x-1/2) > 0$ for x > 1. The right-hand inequality is then true by the convexity of f(x). To show the left-hand inequality, let g(x) = f(x-1) + f(x) - 2f(x-1/2): Then

$$\lim_{x \to 1^+} g(x) = 0 + 0 - 2(\frac{1}{2}\ln\frac{1}{2}) = \ln 2$$

It only remains to show that g(x) remains less than $\ln 2$ for x > 1. Differentiation yields

$$g'(x) = \ln(x-1) + \ln x - 2\ln(x-\frac{1}{2}) = \ln\left[1 - \frac{1/4}{(x-\frac{1}{2})^2}\right] < 0$$

for x > 1, so g(x) strictly decreases proving the left-hand side and hence the given inequality.

The result may be restated in various other forms: First, if u = x - 1/2, then it becomes

$$2 \ge (u - \frac{1}{2})^{u - 1/2} (u + \frac{1}{2})^{u + 1/2} u^{-2u} > 1$$
 for $u > \frac{1}{2}$

This may be written

$$2 \ge \left(1 - \frac{1}{4u^2}\right)^u \sqrt{\frac{2u+1}{2u-1}} > 1 \quad \text{for} \quad u > \frac{1}{2}$$

Next, a straightforward generalization gives the inequality

$$4c \ge \frac{(x-2c)^{x-2c}x^x}{(x-c)^{2x-2c}} > 1$$

for x > 2c > 0. Finally, replacing x by u + c gives

$$4c \ge (u-c)^{u-c}(u+c)^{u+c}u^{-2u} > 1$$
 for $u > c > 0$.

Coll. Math. J., 17 No.3 (June, 1986) 250.

330. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine the extreme values of

$$\frac{x^2}{x+yz} + \frac{y^2}{y+zx} + \frac{z^2}{z+xy}$$

given that x, y, z are positive numbers such that x + y + z = 1.

Coll. Math. J., 19 No.3 (May, 198) 291–292.

Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg, Capitol College, Middletown, PA. Let $P = \{(x, y, z) : x, y, z > 0, x + y + z = 1\}$ and

$$S = S(x, y, z) = \frac{x^2}{x + yz} + \frac{y^2}{y + zx} + \frac{z^2}{z + xy}$$

Clearly, for $(x, y, z) \in P$, $S < x^2/x + y^2/y + z^2/z = 1$. Since S = 1 on $\{(x, y, z) : x, y, z \ge 0, x + y + z = 1, \text{exactly one of } x, y \text{ or } z = 0\}$ we conclude, by continuity, that

l.u.b.
$$\{S(x, y, z) : (x, y, z) \in P\} = 1$$
 (*)

Define T = T(x, y, z) by S = 1 - T. By maximizing T, we will minimize S. For $(x, y, z) \in P$, we have

$$S = \frac{x(x+yz) - xyz}{x+yz} + \frac{y(y+zx) - xyz}{y+zx} + \frac{z(z+xy) - xyz}{z+xy}$$

= $1 - xyz \left(\frac{1}{1-y-z+yz} + \frac{1}{1-z-x+zx} + \frac{1}{1-x-y+xy} \right)$
= $1 - xyz \frac{(1-x) + (1-y) + (1-z)}{(1-x)(1-y)(1-z)}$
= $1 - \frac{2xyz}{(y+z)(z+x)(x+y)}$

so that

$$T = \frac{2xyz}{(y+z)(z+x)(x+y)} \le \frac{2xyz}{8\sqrt{yz}\sqrt{zx}\sqrt{xy}} = \frac{1}{4}$$

by the arithmetic-geometric mean inequality. It follows that $S \ge 3/4$ for $(x, y, z) \in P$. We conclude, since S(1/3, 1/3, 1/3) = 3/4, that

$$\min\{S(x, y, z) : (x, y, z) \in P\} = \frac{3}{4}$$

Together with (*), this completes the solution.

Coll. Math. J., 17 No.4 (Sept., 1986) 360.

332. Proposed by M. S. Klamkin, University of Alberta, Canada

Given that ABCD is an inscribed quadrilateral in a unit circle which is symmetric about \overline{AC} which is a diameter of the circle. Triangle ABD is rotated about \overline{BD} through an angle α . Determine the maximum value of the circumradius $R(\alpha)$ of the variable triangle $A(\alpha)CD$ for $0 \le \alpha \le \pi$.

Coll. Math. J., 19 No.3 (May, 1988) 294.

Solution by J. Foster, Weber State College. By the extended law of sines and the formula for the area of a triangle that is determined by two sides and the included angle, we obtain that the product of two sides of a triangle is equal to the altitude to the third side multiplied by twice the circumradius. Since the sides $\overline{A(\theta)D}$ and \overline{CD} of triangle $\underline{A(\theta)CD}$ have fixed length, we can maximize $R(\theta)$ by minimizing the altitude to side $\overline{A(\theta)C}$. This altitude goes from D to a point in the plane containing \overline{AC} and perpendicular to \overline{BD} . Hence, it is minimized when its length is equal to the distance from D to that plane. This occurs when $\theta = 0$ or π . Therefore, the maximum value of $R(\theta)$ is the radius of the unit circle, or 1.

Coll. Math. J., 17 No.5 (Nov., 1986) 442.

337. Proposed by M. S. Klamkin, University of Alberta, Canada

If a, b, c, d > 0, prove that

$$\frac{b^3c^3}{a^6} + \frac{c^3d^3}{b^6} + \frac{d^3a^3}{c^6} + \frac{a^3b^3}{d^6} \ge \max\left\{\frac{bc}{a^2} + \frac{cd}{b^2} + \frac{da}{c^2} + \frac{ab}{d^2}, \frac{a^2}{bc} + \frac{b^2}{cd} + \frac{c^2}{da} + \frac{d^2}{ab}\right\}.$$

Coll. Math. J., 19 No.5 (Nov., 1988) 450.

Solution by Benjamin G. Klein, Davidson College, NC, and V. N. Murty, Pennsylvania State University at Harrisburg, Capitol College, Middletown (independently).

Theorem. If $x_i > 0$, i = 1, 2, 3, 4 and $\prod_{i=1}^{4} x_i = 1$, then

$$\sum_{i=1}^{4} x_i^3 \ge \max\left(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} \frac{1}{x_i}\right)$$

Proof. It is known that

$$\frac{1}{4}\sum_{i=1}^{4}x_i^3 \ge \left(\frac{1}{4}\sum_{i=1}^{4}x_i\right)^3$$

Hence, to prove that $\sum_{i=1}^{4} x_i^3 \ge \sum_{i=1}^{4} x_i$, it is sufficient to show that

$$\frac{1}{16} \left(\sum_{i=1}^{4} x_i \right)^3 \ge \sum_{i=1}^{4} x_i$$

or, equivalently,

$$\left(\sum_{i=1}^{4} x_i\right)^2 \ge 1$$

The last inequality follows from the arithmetic-geometric mean inequality due to the fact that $\prod_{i=1}^{4} x_i = 1$. To prove that $\sum_{i=1}^{4} x_i^3 \ge \sum_{i=1}^{4} 1/x_i$ we note that, by the arithmetic-geometric mean inequality,

$$\sum_{i=1}^{4} x_i^3 = \sum_{i=1}^{4} \frac{1}{3} \sum_{j \neq i} x_j^3 \ge \sum_{i=1}^{4} \prod_{j \neq i} x_j = \prod_{j=1}^{4} x_j \sum_{i=1}^{4} \frac{1}{x_i} = \sum_{i=1}^{4} \frac{1}{x_i}$$

The problem is solved by substituting $x_1 = bc/a^2$, $x_2 = cd/b^2$, $x_3 = da/c^2$ and $x_4 = ab/d^2$.

Coll. Math. J., 17 No.5 (Nov., 1986) 448-449.

294. (Jan. 1985) Proposed by Russell Euler, Northwest Missouri State University, Maryville

Prove

$$\prod_{k=2}^{p} \left\{ r^2 - 2r \cos\left[\theta - \frac{2(k-1)\pi}{p}\right] + 1 \right\} = \frac{r^{2p} - 2r^p \cos p\theta + 1}{r^2 - 2r \cos \theta + 1}$$

Editor's Note: Klamkin stated that the result is a classical identity and can be found in a number of books on trigonometry; e.g., C. V. Durell & A. Robson, *Advanced Trigonometry*, G. Bell & Sons, London, 1953, p.226.

Coll. Math. J., 18 No.1 (Jan., 1987) 69.

343. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine the quotient when $2x^{n+2} - (n+2)x^2 + n$ is divided by $(x-1)^2$

Coll. Math. J., 20 No.2 (Mar., 1989) 166.

Solution by Tsz-Mei Ko (student), The Cooper Union, NewYork City. The quotient may be found directly by synthetic division. Alternatively, factor:

$$2x^{n+2} - (n+2)x^2 + n = 2(x^{n+2} - x^2) - n(x^2 - 1) = (x - 1)[2(x^{n+1} + x^n + \dots + x^2) - n(x + 1)]$$

This may in turn be factored:

$$(x-1)[2(x^{n+1}-x)+\dots+2(x^2-x)+n(x-1)] = (x-1)^2[2(x^n+\dots+x)+\dots+2(x^2+x)+2x+n]$$

Collecting like powers then gives the desired quotient as $2(x^n + 2x^{n-1} + \dots + nx) + n$.

Coll. Math. J., 19 No.1 (Jan., 1988) 81–82.

368. Proposed by M. S. Klamkin, University of Alberta, Canada

If w and z are complex numbers, it is a known identity that

$$|w| + |z| = \left|\frac{w+z}{2} - \sqrt{wz}\right| + \left|\frac{w+z}{2} + \sqrt{wz}\right|$$
(1)

Generalize to an identity involving three complex numbers.

Coll. Math. J., 20 No.5 (Nov., 1989) 443–444.

Composite of a joint solution by Kim McInturff and Peter Simon, Raytheon Corporation, Goleta, CA; and a solution by William P. Wardlaw, U.S. Naval Academy, Annapolis, MD. The identity (1) can be written in the form

$$2\left[|\sqrt{w}|^{2} + |\sqrt{z}|^{2}\right] = |\sqrt{w} - \sqrt{z}|^{2} + |\sqrt{w} + \sqrt{z}|^{2}$$
(2)

where \sqrt{w} is any one of the two square roots of w and \sqrt{z} is any one of the two square roots of z. Equation (2) is simply the parallelogram law applied to the parallelogram in the complex plane whose vertices are 0, \sqrt{w} , \sqrt{z} and $\sqrt{w} + \sqrt{z}$. The parallelogram law, ofcourse, states that the sum of the squares on the sides of any parallelogram is equal to the sum of the squares on its diagonals. For parallelepipeds, the corresponding law is that the sum of the squares on the twelve edges of any parallelepiped is equal to the sum of the squares on its four diagonals. Our generalization is, therefore, obtained as follows. Let s, w and z be any three complex numbers, and let \sqrt{s} , \sqrt{w} and \sqrt{z} represent a square root of s, a square root of w, and a square root of z, respectively. We think of the complex numbers \sqrt{s} , \sqrt{w} and \sqrt{z} as vectors from the origin that coincide with three concurrent edges of a flat, or flattened, parallelepiped. This parallelepiped has four edges parallel to \sqrt{s} , each of length $|\sqrt{s}|$; it has four edges parallel to \sqrt{w} . each of length $|\sqrt{w}|$; and it has four edges parallel to \sqrt{z} , each of length $|\sqrt{z}|$. Thus, in our generalization, the left side of (2) will be replaced by $4[|\sqrt{s}|^2 + |\sqrt{w}|^2 + |\sqrt{z}|^2]$. One diagonal of our flattened parallelepiped has an endpoint at the origin, and the same length and direction as the vector $\sqrt{s} + \sqrt{w} + \sqrt{z}$. The other diagonals have endpoints at the points \sqrt{s} , \sqrt{w} , \sqrt{z} , and their lengths and directions are those of the vectors $-\sqrt{s} + \sqrt{w} + \sqrt{z}$, $\sqrt{s} - \sqrt{w} + \sqrt{z}$ and $\sqrt{s} + \sqrt{w} - \sqrt{z}$, respectively. Therefore our generalization of (2) is

$$4[|\sqrt{s}|^{2} + |\sqrt{w}|^{2} + |\sqrt{z}|^{2}] = |\sqrt{s} + \sqrt{w} + \sqrt{z}|^{2} + |-\sqrt{s} + \sqrt{w} + \sqrt{z}|^{2} + |\sqrt{s} - \sqrt{w} + \sqrt{z}|^{2} + |\sqrt{s} + \sqrt{w} - \sqrt{z}|^{2}$$
(3)

Identity (3) can be proved by an obvious computation which begins with the expansion of the right sides of four equations (one equation for each of the four terms on the righthand side of (3)) similar in form to the equation

$$|\sqrt{s} + \sqrt{w} + \sqrt{z}|^2 = (\sqrt{s} + \sqrt{w} + \sqrt{z})(\overline{\sqrt{s}} + \overline{\sqrt{w}} + \overline{\sqrt{z}})$$

This proof of (3) is reminiscent of a familiar proof of the parallelogram law; it is in fact the same proof with complex numbers replacing vecors in 3-space, and the complex inner product $u\bar{v}$, defined for every two complex numbers u and v, replacing the inner product of vectors in real 3-dimensional space.

In order to obtain an identity whose relationship to (3) shall be like the relationship of (1) to (2), we rewrite (3) in the following way:

$$4(|s| + |w| + |z|) = |s + w + z + 2(\sqrt{s}\sqrt{w} + \sqrt{w}\sqrt{z} + \sqrt{z}\sqrt{s})| + |s + w + z + 2(\sqrt{s}\sqrt{w} - \sqrt{w}\sqrt{z} - \sqrt{z}\sqrt{s})| + |s + w + z + 2(-\sqrt{s}\sqrt{w} + \sqrt{w}\sqrt{z} - \sqrt{z}\sqrt{s})| + |s + w + z + 2(-\sqrt{s}\sqrt{w} - \sqrt{w}\sqrt{z} + \sqrt{z}\sqrt{s})|$$
(4)

or

$$4(|s| + |w| + |z|) = |s + w + z + 2(\sqrt{sw} + \sqrt{wz} + \sqrt{zs})| + |s + w + z + 2(\sqrt{sw} - \sqrt{wz} - \sqrt{zs})| + |s + w + z + 2(-\sqrt{sw} + \sqrt{wz} - \sqrt{zs})| + |s + w + z + 2(-\sqrt{sw} - \sqrt{wz} + \sqrt{zs})|$$
(5)

The square roots in (5) cannot be independently chosen. For example, if s = w = z = -1, and we choose $\sqrt{sw} = \sqrt{wz} = \sqrt{zs} = 1$, then, as is easily verified, (5) becomes false. When using (5) for arbitrary complex numbers s, w and z, the correct course is to impose the condition

$$\sqrt{sw}\sqrt{wz}\sqrt{zx} = swz \tag{6}$$

To verify that (5), when used in this way, is correct, we note, to begin with, that (5) reduces to (1) when s, w or z is zero. Let s, w and z not be zero. We note that we can then choose two of the square roots on the left of (6) arbitrarily, whereupon the value of the third square root is uniquely determined, and that, if \sqrt{sw} , \sqrt{wz} , \sqrt{zs} satisfy (6), then the only other systems of values of the three square roots that will satisfy (6) are \sqrt{sw} , $-\sqrt{wz}$, $-\sqrt{zs}$; and $-\sqrt{sw}$, \sqrt{wz} , $-\sqrt{zs}$; and $-\sqrt{sw}$, \sqrt{wz} , \sqrt{zs} . Direct examination of (5) shows that changing from one of these systems of values to another leaves (5) unaltered. Finally, we show that for any nonzero complex numbers s, w and z, if \sqrt{sw} , \sqrt{wz} , \sqrt{zs} are chosen so that (6) is satisfied, then the right of (5) will be equal to the right of (4) for some choice of the values of \sqrt{s} , \sqrt{w} and \sqrt{z} . In fact, if \sqrt{s} is any one of the two square roots of s and we define \sqrt{w} and \sqrt{z} by $\sqrt{w} = \sqrt{sw}/\sqrt{s}$ and $\sqrt{z} = \sqrt{sz}/\sqrt{s}$, then, by (6),

$$\sqrt{w}\sqrt{z} = \sqrt{sw}\sqrt{zs}/s = (swz/\sqrt{wz})/s = wz/\sqrt{wz} = \sqrt{wz}$$

from which it is apparent that the right sides of (5) and (4) are equal. Since we already know that (4) is correct, we conclude that (5) is correct when the square roots in (5)

are chosen in accordance with (6). Thus our generalization of (1) is the identity (5) with the values of its square roots subjected to (6), but otherwise arbitrary.

Editor's Notes: An extension of the above generalization of (1), a generalization in a different direction, and a combination of the two generalizations can be found in a classroom capsule, in this issue, by M. S. Klamkin & V. N. Murty, entitled "Generalization of a Complex Number Identity." The explicit statement of equation (6) is due to McInturff & Simon.

[[Here's the capsule just referred to. -R.]]

Coll. Math. J., 20 No.5 (Nov., 1989) 415–416.

Generalizations of a Complex Number Identity

M. S. Klamkin, University of Alberta, Edmonton, Alberta & V. N. Murty, Pennsylvania State University, Middletown, PA

A recurring exercise that appears in texts on complex variables is to show that if w and z are complex numbers, then

$$|w| + |z| = |(w+z)/2 - \sqrt{wz}| + |(w+z)/2 + \sqrt{wz}|$$

In problem 368, this journal, the first author asked for a generalization to any number of variables and to any dimensional Euclidean space by replacing the complex numbers by vectors.

First, we can simplify the identity by getting rid of the bothersome square roots. Letting $w = z_1^2$ and $z = z_2^2$ we get

$$2\{|z_1|^2 + |z_2|^2\} = |z_1 - z_2|^2 + |z_1 + z_2|^2$$
(1)

Geometrically, we now have that the sum of the squares of the edges of a parallelogram equals the sum of the squares of the diagonals. Consequently, by considering a parallelepiped, one generalization is that

$$4\{|z_1|^2 + |z_2|^2 + |z_3|^2\} = |z_1 + z_2 + z_3|^2 + |z_1 + z_2 - z_3|^2 + |z_1 - z_2 + z_3|^2 + |-z_1 + z_2 + z_3|^2$$
(2)

Here, z_1 , z_2 , z_3 can be complex numbers in the plane or vectors in space. For a proof, assuming the z_i are vectors, just note that

$$\begin{aligned} |z_1 + z_2 - z_3|^2 &= (z_1 + z_2 - z_3)^2 \\ &= z_1^2 + z_2^2 + z_3^2 + 2z_1 \cdot z_2 - 2z_1 \cdot z_3 - 2z_2 \cdot z_3 \text{ etc.} \end{aligned}$$

Geometrically we have that the sum of the squares of all the edges of a parallelepiped equals the sum of the squares of the four body diagonals. Also to be noted is that (1) is the special case of (2) when $z_3 = 0$. A generalization to *n*-dimensional space (for an *n*-dimensional parallelepiped) is immediate, i.e.,

$$2^{n} \sum z_{i}^{2} = \sum (\pm z_{1} \pm z_{2} \pm \dots \pm z_{n})^{2}$$
(3)

where the summation on the right is taken over all the 2^n combinations of the \pm signs.

For a generalization in another direction, note that (1) can be rewritten as

$$|z_1|^2 + |z_2|^2 = |(z_1 - z_2)/\sqrt{2}|^2 + |(z_1 + z_2)/\sqrt{2}|^2$$

In the real plane, the transformation $x' = (x - y)/\sqrt{2}$, $y' = (x + y)/\sqrt{2}$ represents a rotation of the coordinate axes by 45° and preserves all distances, i.e., $\sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}$. For the case here, the value of $|z_1|^2 + |z_2|^2$ is preserved under an orthogonal transformation. More generally (as is known), if z_1, z_2, \ldots, z_n are complex numbers (or vectors in space) and we make the transformation Z' = MZ where M is an arbitrary real orthogonal matrix and the transpose matrices of Z and Z' are

$$Z^T = (z_1, z_2, \dots, z_n)$$
 and $Z'^T = (z'_1, z'_2, \dots, z'_n)$

then

$$\sum |z_i'|^2 = \sum |z_i|^2$$

and its proof is quite direct:

$$\sum |z_i'|^2 = \sum z_i' \bar{z}_i' = Z'^T \bar{Z}' = (MZ)^T (\bar{M}\bar{Z}) = Z^T M^T M \bar{Z} = Z^T \bar{Z} = \sum |z_i|^2 \bar{Z}$$

(since M is orthogonal, $M^T M = I$). The proof for vectors is the same except that the multiplication of the two vector matrices Z^T and \overline{Z} is via the scalar dot product.

More generally, the matrix M can be replaced by a complex matrix U if it is unitary, i.e., $U^T \overline{U} = I$. Finally, the identity (3) can be generalized by replacing the z_i by z'_i and then letting Z' = UZ. For a simple example in (2), let

$$z'_1 = iz_1 \cos \theta + iz_2 \sin \theta, \qquad z'_2 = z_1 \sin \theta - z_2 \cos \theta, \qquad z'_3 = z_3$$

where θ is an arbitrary real angle.

Coll. Math. J., 19 No.2 (Mar., 1988) 186.

374. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine necessary and sufficient conditions on the consecutive sides a, b, c, d of a convex quadrilateral such that one of its configurations has an incircle, a circumcircle, and perpendicular diagonals.

Coll. Math. J., 20 No.5 (Nov., 1989) 448–449.

Solution by Brian Amrine, Goleta, CA; and Phil Clarke, Los Angeles Valley College, Van Nuys, CA (independently). Let a, b, c and d be the lengths of the consecutive sides of a convex quadrilateral. A necessary and sufficient condition for the quadrilateral to have an incircle is that

$$a + c = b + d \tag{1}$$

A necessary and sufficient condition for the diagonals of the quadrilateral to be perpendicular is that

$$a^2 + c^2 = b^2 + d^2 \tag{2}$$

Equations (1) and (2) imply

$$ac = bd$$
 (3)

Equations (1) and (3) imply

$$a = b$$
 and $c = d$ or $a = d$ and $c = b$ (4)

Conversely, (4) implies (1) and (2). Therefore (4) is a necessary and sufficient condition for a convex quadrilateral with consecutive sides a, b, c and d to have both an incircle and perpendicular diagonals. We claim that (4) is a solution to the proposer's problem. It is clear that (4) is necessary; conversely, if, say, a = b and c = d, then (1) and (2) assure the existence of an incircle and perpendicular diagonals, and we can certainly configure the quadrilateral so that side *a*meets side *d* at right angles, and side *b* meets side *c* at right angles, assuring the existence of a circumcircle. This completes the solution.

Also solved by the proposer, who referred to C. V. Durell & A. Robson, Advanced Trigonometry, G. Bell, London, 1953, pp.27–28.

Coll. Math. J., 19 No.3 (May, 1988) 290.

377. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine the maximum value of

$$P = (x - x^2)(1 - 2x)^2(1 - 8x + 8x^2)^2$$

over $0 \le x \le 1$.

Coll. Math. J., 20 No.5 (Nov., 1989) 450-451.

Solution I by C. Patrick Collier, University of Wisconsin, Oshkosh. Multiply and complete the square as follows:

$$P = \frac{1}{16} (1 - (8x^2 - 8x + 1)^2) (1 + (8x^2 - 8x + 1))(8x^2 - 8x + 1)^2$$

= $\frac{1}{16} (1 - (8x^2 - 8x + 1)^2) (8x^2 - 8x + 1)^2 = \frac{1}{16} \left(\frac{1}{4} - \left((8x^2 - 8x + 1)^2 - \frac{1}{2} \right)^2 \right)$

It is now clear that the maximum value of P is $\frac{1}{64}$, occurring when $(8x^2 - 8x + 1)^2 = \frac{1}{2}$ which determines $x = \frac{2\pm\sqrt{2\pm\sqrt{2}}}{4}$. These four values are between zero and one, which completes the solution.

Solution II by Philip D. Straffin, Beloit College, WI. The identity $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$ suggests the substitution $x = \cos^2 \theta$ which yields

$$P(\cos^2 \theta) = \cos^2 \theta \sin^2 \theta \cos^2 2\theta \cos^2 4\theta = \frac{1}{4} \sin^2 2\theta \cos^2 2\theta \cos^2 4\theta$$
$$= \frac{1}{16} \sin^2 4\theta \cos^2 4\theta = \frac{1}{64} \sin^2 8\theta$$

Thus for $0 \leq \theta \leq \frac{\pi}{2}$, the maximum of P is $\frac{1}{64}$, assumed for $\theta = \frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \frac{7\pi}{16}$. The corresponding values of $x = \cos^2 \theta$ are $x = \frac{2+\sqrt{2+\sqrt{2}}}{4}, \frac{2+\sqrt{2-\sqrt{2}}}{4}, \frac{2-\sqrt{2-\sqrt{2}}}{4}, \frac{2-\sqrt{2+\sqrt{2}}}{4}, \frac{2-\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4$

Solution III by Bill Mixon, Austin, TX; and Bert K. Waits, Ohio State University, Columbus (independently). A computer plot of P suggests that P maybe simply related to a Chebyshev polynomial. Following this hint, direct computation then verifies that $P = \frac{1}{128}(1 - T_8(2x - 1))$, where T_8 is the eighth Chebyshev polynomial: $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$. The domain $0 \le x \le 1$ corresponds to $-1 \le 2x - 1 \le 1$, so the well-known property $T_n(\cos \theta) = \cos(n\theta)$ gives $T_8(2x - 1) = \cos(8 \arccos 2x - 1)$. Thus P has a maximum of $\frac{1}{128}(2) = \frac{1}{64}$, assumed when $\cos(8 \arccos(2x - 1)) = -1$, thus giving the values $x = \frac{1}{2}(1 + \cos \frac{\pi}{8}), \frac{1}{2}(1 - \cos \frac{\pi}{8}), \frac{1}{2}(1 - \cos \frac{3\pi}{8}), \frac{1}{2}(1 - \cos \frac{3\pi}{8})$ corresponding to $x = \frac{2+\sqrt{2+\sqrt{2}}}{4}, \frac{2-\sqrt{2+\sqrt{2}}}{4}, \frac{2+\sqrt{2-\sqrt{2}}}{4}$ respectively. Solution IV by Mark Biegert, Honeywell Inc., Hopkins, MD. Complete squares to give

$$P = \left(\frac{1}{4} - \left(x - \frac{1}{2}\right)^2\right) \left(4\left(x - \frac{1}{2}\right)^2\right) \left(1 - 8\left(x - \frac{1}{2}\right)\right)^2$$
$$= \frac{1}{16} \left(2 - 8\left(x - \frac{1}{2}\right)^2\right) \left(8\left(x - \frac{1}{2}\right)^2\right) \left(1 - 8\left(x - \frac{1}{2}\right)\right)^2$$

Let $y = 8(x - \frac{1}{2})^2$, so $P(y) = \frac{1}{16}(2y - y^2)(1 - y)^2$, $0 \le y \le 2$. Then the AGM inequality gives

$$P(y) = \left(\sqrt{\left(\frac{2y-y^2}{4}\right)\left(\frac{(1-y)^2}{4}\right)}\right)^2 \le \left(\frac{1}{2}\left(\frac{1}{4}(2y-y^2) + \frac{1}{4}(1-2y+y^2)\right)\right)^2 = \frac{1}{64}$$

with equality when $2y - y^2 = 1 - 2y + y^2$, so $y = \frac{2\pm\sqrt{2}}{2}$, yielding the same values for x as above.

Coll. Math. J., 19 No.4 (Sept., 1988) 370.

383. Proposed by M. S. Klamkin, University of Alberta, Canada

In an article, "The Creation of Mathematical Olympiad Problems" in the Newsletter of the World Federation of National Mathematics Competitions, Feb. 1987, pp.18– 28, Arthur Engel, by using a sequence of transformations, "wipes out all traces of its origin" and ends up with the triangle inequality

$$a^{2} + b^{2} + c^{2} - 2bc - 2ca - 2ab + \frac{18abc}{(a+b+c)} \ge 4F\sqrt{3}$$

where a, b, c and F are the sides and area of the triangle, respectively. He notes that this inequality is now "a very difficult problem to prove". Prove or disprove the latter statement.

Coll. Math. J., **21** No.1 (Jan., 1990) 70.

Composite of solutions by Francisco Bellot, Valladolid, Spain and W. Weston Meyer, General Motors research Laboratories. Let s, r and R denote the semiperimeter, inradius and circumradius, resolution, of the triangle. With the well-known relations

$$a + b + c = 2s$$
 $bc + ca + ab = s^{2} + 4rR + r^{2}$ $abc = 4Rrs$

the given inequality becomes

$$2(s^2 - r^2 - 4Rr) - 2(s^2 + r^2 + 4Rr) + 36Rr \ge 4F\sqrt{3}$$

which in turn is equivalent to $5R - r \ge s\sqrt{3}$.

But an equilateral triangle is simultaneously the triangle of greatest primeter that can be inscribed in a circle of radius R, so that $R \ge 2s/(3\sqrt{3})$, and the triangle of least perimeter that can be circumscribed about a circle of radius r, so that $r \le s/(3\sqrt{3})$ [see Theorems 6.3a and 6.3c of Ivan Niven's Maxima and Minima Without Calculus, MAA, 1981]. The desired inequality follows. Coll. Math. J., 19 No.5 (Nov., 1988) 449.

389. Proposed by M. S. Klamkin, University of Alberta, Canada Factor $P^2 + PQ + Q^2$ into real polynomial factors where

$$P = x^2y + y^2z + z^2x$$
 and $Q = xy^2 + yz^2 + zx^2$

Coll. Math. J., 21 No.2 (Mar., 1990) ??.

Solution by Henry A. Williams, Carroll High School, Ozark, AL. Since

$$P^{3} - Q^{3} = x^{6}y^{3} - x^{3}y^{6} + y^{6}z^{3} - y^{3}z^{6} + z^{6}x^{3} - z^{3}x^{6}$$

= $(y^{3} - x^{3})(z^{3} - x^{3})(x^{3} - y^{3})$

and

$$P - Q = x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2$$

= $(y - x)(z - x)(x - y)$

we have

$$P^{2} + PQ + Q^{2} = \frac{P^{3} - Q^{3}}{P - Q} = \frac{(y^{3} - x^{3})(z^{3} - x^{3})(x^{3} - y^{3})}{(y - x)(z - x)(x - y)}$$
$$= (x^{2} + xy + y^{2})(y^{2} + yz + z^{2})(z^{2} + zx + x^{2})$$

Editor's Note: The factorization was also successfully accomplished by the symbolic algebra programs DeriveTM (Shippensburg U. Mathematical Problem Solving Group), MACSYMA[©] and *Mathematica*TM (Robert Weaver, Mount Holyoke C.) and MAPLETM (Robert Tardiff, Salisbury State U.).
Coll. Math. J., 20 No.1 (Jan., 1989) 68.

391. Proposed by M. S. Klamkin, University of Alberta, Canada

Determine all integral solutions of the Diophantine equation

$$2x/(1-x^2) + 2y/(1-y^2) + 2z/(1-z^2) = 8xyz/(1-x^2)(1-y^2)(1-z^2)$$

Coll. Math. J., 21 No.2 (Mar., 1990) 155–156.

Solution by Duane M. Broline, Eastern Illinois University, Charleston, IL. The only integral solutions are either of the form (0, a, -a), (a, 0, -a) or (a, -a, 0) where $a \neq \pm 1$, or of the form (r, s - r, t - r) where r, s and t are integers such that $r^2 + 1 = st$ and such that (r, s, t) is not in the following table.

r	-3	-3	-2	-2	-1	-1	-1	-1	0	0	1	1	1	1	2	2	3	3
s	-5	-2	-5	-1	-2	-1	1	2	-1	1	-2	-1	1	2	1	5	2	5
t	-2	-5	-1	-5	-1	-2	2	1	-1	1	-1	-2	2	1	5	1	5	2

[[This can surely be described more neatly ? How about: 'with |r| > 1 and, if |r| = 2 or 3, neither |s| nor |t| = 5"?? Let's see how these conditions emerge in the solution. - R.]]

Let (x, y, z) be an integral solution to the original equation. Clearly, none of |x|, |y| or |z| is equal to 1. [[Not clear to me why these should not be counted as solutions. - R.]] Multiplying both sides of the original equation by $(1 - x^2)(1 - y^2)(1 - z^2)$ abd simplifying gives

$$((x + y + z) - xyz)((yz + zx + xy) - 1) = 0$$

Thus, either x + y + z = xyz or yz + zx + xy = 1.

First consider the case that x + y + z = xyz. If any one of x, y or z is zero, then the other two add to zero. Hence (x, y, z) has the form (0, a, -a), (a, 0, -a) or (a, -a, 0), where $a \neq \pm 1$. If none of x, y or z is zero, then each of |x|, |y| and |z| is at least 2. Hence,

$$\frac{-3}{4} < \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \le \frac{3}{4}$$

[[I've added the possibility of equality to the second inequality. -R.]] which contradicts the assumption that

$$\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} = 1$$

If yz + zx + xy = 1, then $(x+y)(x+z) = 1+x^2$. Setting r = x, s = x+y and t = x+zwe see that (x.y.z) = (r, s - r, t - r), where $r^2 + 1 = st$. Conversely, if r, s and t are integers such that $r^2 + 1 = st$, then (x.y.z) = (r, s - r, t - r) provides a solution to the original equation except when one of r, s - r or t - r equals 1 or -1. If $r = \pm 1$ then it is easy to see that the possibilities for s and t are as given in the table. If $s - r = \pm 1$ then $r \pm 1$ divides $r^2 + 1$. Since $r^2 + 1 = (r + 1)^2 \mp 2r$, we see that $r \pm 1$ divides 2; hence $r \in \{-3, -2, -1, 0, 1, 2, 3\}$. The cases that $r = \pm 1$ have been treated previously, while the other cases give additional entries to the table. The case that $t - r = \pm 1$ is handled similarly.

[[This solution leaves much to be desired. The " \mp " should be "-". It seems to me that "possibilities" should be "impossibilities". But worst of all is that the symmetry has completely disappeared.]]

Coll. Math. J., 20 No.1 (Jan., 1989) 69.

395. Proposed by M. S. Klamkin, University of Alberta, Canada

If the altitudes of an acute triangle ABC are extended to intersect its circumcircle in points A', B', C' respectively, prove that

$$[A'B'C'] \le [ABC]$$

where [ABC] denotes the area of ABC.

Coll. Math. J., 21 No.3 (May, 1990) 249–250.

Composite of solutions by Walter Blumberg, Coral Springs, FL, and Dave Ohlsen, Santa Rosa Junior College, CA (independently). Let $\triangleleft C'CA = x$, $\triangleleft A'AB = y$ and $\triangleleft B'BC = z$. It follows that $x = \triangleleft C'A'A = \triangleleft ABB' = \triangleleft AA'B'$; $y = \triangleleft A'B'B = \triangleleft BCC' = \triangleleft BB'C'$; and $z = \triangleleft B'C'C = \triangleleft A'AC = \triangleleft AC'C$. Thus A = y + z, B = z + x, C = x + y, and A' = 2x, B' = 2y, C' = 2z. Since $2x + 2y + 2z = \pi$, $A' = \pi - 2A$, $B' = \pi - 2B$ and $C' = \pi - 2C$. Let R denote the radius of the circumcircle. Then $ABC = 2R^2 \sin A \sin B \sin C$ and $[A'B'C'] = 2R^2 \sin A' \sin B' \sin C' = 2R^2 \sin(2A) \sin(2A) \sin(2C)$. Hence the inequality $[A'B'C'] \leq [ABC]$ is equivalent to the well-known inequality $8 \cos A \cos B \cos C \leq 1$ (with equality if and only if $A = B = C = \pi/3$).

Editors' Note. Several solvers used properties of the *orthic triangle* of *ABC* (the triangle whose vertices are the feet of the altitudes). See David R. Davis, *Modern College Geometry*, Addison-Wesley, 1957.

Coll. Math. J., 20 No.2 (Mar., 1989) 164.

400. Proposed by M. S. Klamkin, University of Alberta, Canada

 A_i and B_i are two sets of points on an *n*-dimensional sphere with center O such that A_i and B_i are pairs of antipodal points for i = 1, 2, ..., n. It follows immediately that the volumes of the two simplexes

 $O, A_1, A_2, \ldots, A_r, B_{r+1}, \ldots, B_n$ and $O, B_1, B_2, \ldots, B_r, A_{r+1}, \ldots, A_n$

are equal since the simplexes are congruent. Show more generally that

 $vol[O, A_1, A_2, \dots, A_n] = vol[O, A_1, A_2, \dots, A_r, B_{r+1}, \dots, B_n]$

Coll. Math. J., 21 No.4 (Sept., 1990) 337–338.

Solution by the Siena Heights College Problem Solving Group, Adrian, MI. Fix $1 \leq r \leq n$ and let S be the simplex formed by the origin and the standard basis vectors e_1, e_2, \ldots, e_n in \mathbb{R}^n . Let T be the linear operator on \mathbb{R}^n that maps e_k to A_k when $1 \leq k \leq n$ and let \tilde{T} be the linear operator that maps e_k to A_k when $1 \leq k \leq r$ and maps e_k to B_k when $r + 1 \leq k \leq n$. Then det $T = \det(A_1, \ldots, A_n)$ and det $\tilde{T} = \det(A_1, \ldots, A_r, B_{r+1}, \ldots, B_n)$. Since $B_k = -A_k$ it follows that $|\det T| = |\det \tilde{T}|$.

The operator T maps S onto the simplex formed by $[O, A_1, \ldots, A_n]$ and the operator \tilde{T} maps S onto the simplex formed by $[O, A_1, \ldots, A_r, B_{r+1}, \ldots, B_n]$. By the change of variables theorem, we have

$$\operatorname{vol}[O, A_1, \dots, A_n] = \int_{T(S)} 1 = \int_S |\det T'| = \operatorname{vol}(S) \cdot |\det T'| \quad \text{and}$$
$$\operatorname{vol}[O, A_1, \dots, A_r, B_{r+1}, \dots, B_n] = \int_{\tilde{T}(S)} 1 = \int_S |\det \tilde{T'}| = \operatorname{vol}(S) \cdot |\det \tilde{T'}|$$

where T' and $\tilde{T'}$ denote matrices of mixed partials of T and \tilde{T} , respectively.

Since $|\det T'| = |\det T| = |\det \tilde{T}| = |\det \tilde{T}'|$, it follows that the volumes of the two simplexes are equal.

Coll. Math. J., 20 No.3 (May, 1989) 256.

404. Proposed by M. S. Klamkin, University of Alberta, Edmonton

If $a_n = (n+1)^{n+1}(n-1)^n/n^{2n+1}$, prove that the sequence (a_n) is increasing.

Coll. Math. J., 21 No.4 (Sept., 1990) 340-341.

Solution by Seung-Jin Bang, Seoul, Korea and Matthew Wyneken, University of Michigan-Flint (independently). Let $f(x) = \ln a_x$, x > 1. Then

$$f'(x) = \frac{1}{[x(x-1)] - \ln[1 + \frac{1}{x^2 - 1}]} > \frac{1}{[x(x-1)] - \frac{1}{x^2 - 1}]} = \frac{1}{[x(x^2 - 1)]} > 0$$

Hence the sequence (a_n) is increasing.

Coll. Math. J., 20 No.4 (Sept., 1989) 343.

406. Proposed by M. S. Klamkin, University of Alberta, Edmonton

Prove that

$$\frac{x^3}{y^3} + \frac{y^3}{x^3} + \frac{y^3}{z^3} + \frac{z^3}{y^3} + \frac{z^3}{x^3} + \frac{x^3}{z^3} \ge \frac{x^2}{yz} + \frac{yz}{x^2} + \frac{y^2}{zx} + \frac{zx}{y^2} + \frac{z^2}{xy} + \frac{xy}{z^2}$$

where x, y, z > 0.

Coll. Math. J., 21 No.5 (Nov., 1990) 424–425.

Solution I by Bijan Sadeghi, West Valley College, Saratoga, CA. The arithmetic meangeometric mean inequality yields

$$\frac{1}{3}\left(\frac{x^3}{y^3} + \frac{x^3}{y^3} + \frac{y^3}{z^3}\right) \ge \frac{x^2}{yz} \quad \text{and} \quad \frac{1}{3}\left(\frac{y^3}{x^3} + \frac{y^3}{x^3} + \frac{z^3}{y^3}\right) \ge \frac{yz}{x^2}$$

Adding the six inequalities obtained by cyclically permuting (x.y.z) in the above two inequalities produces the desired result.

Solution II by Michael Vowe, Therwil, Switzerland. The given inequality is equivalent to

$$(x^{3} + y^{3} + z^{3})\left(\frac{1}{x^{3}} + \frac{1}{y^{3}} + \frac{1}{z}^{3}\right) - 3 \ge (x^{3} + y^{3} + z^{3})\frac{1}{xyz} + xyz\left(\frac{1}{x^{3}} + \frac{1}{y^{3}} + \frac{1}{z^{3}}\right)$$

or

$$(x^{3} + y^{3} + z^{3} - xyz)\left(\frac{1}{x^{3}} + \frac{1}{y^{3}} + \frac{1}{z^{3}} - \frac{1}{xyz}\right) \ge 4$$

But

$$x^{3} + y^{3} + z^{3} \ge 3xyz$$
 and $\frac{1}{x^{3}} + \frac{1}{y^{3}} + \frac{1}{z^{3}} \ge \frac{3}{xyz}$

so that

$$x^{3} + y^{3} + z^{3} - xyz \ge 2xyz$$
 and $\frac{1}{x^{3}} + \frac{1}{y^{3}} + \frac{1}{z^{3}} - \frac{1}{xyz} \ge \frac{2}{xyz}$

from which the desired result follows (with equality if and only if x = y = z.)

Editor's Note. This inequality is a special case of Muirhead's Theorem: Let

$$[\alpha] = [\alpha_1, \alpha_2, \dots, \alpha_n] = \frac{1}{n!} \sum x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where the sum is taken over all n! permutations of the x_i . If

- i)
- ii)
- $\begin{aligned} \alpha_1' + \alpha_2' + \dots + \alpha_n' &= \alpha_1 + \alpha_2 + \dots + \alpha_n \\ \alpha_1' \geq \alpha_2' \geq \dots \geq \alpha_n' & \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \\ \alpha_1' + \alpha_2' + \dots + \alpha_\nu' \leq \alpha_1 + \alpha_2 + \dots + \alpha_\nu & (1 \leq \nu < n) \end{aligned}$ iii)

then $[\alpha'] \leq [\alpha]$ (see Inequalities, G. H. Hardy, J. E. Littlewood, G. Pólya, Cambridge, 1959, pp.44–48). The inequality in this problem is the case $[6, 3, 0] \ge \frac{1}{2}[5, 2, 2] + \frac{1}{2}[4, 4, 1]$ divided by $x^3y^3z^3$.

Coll. Math. J., 20 No.4 (Sept., 1989) 343.

407. Proposed by M. S. Klamkin, University of Alberta, Edmonton Evaluate

$$\int \left[(x^2 - 1)(x + 1) \right]^{-2/3} dx$$

Coll. Math. J., 21 No.5 (Nov., 1990) 425–426.

Solution by Benjamin M. Freed, Clarion University, Clarion, PA. Since

$$\int \left[(x^2 - 1)(x + 1) \right]^{-2/3} dx = \int \left(\frac{x - 1}{x + 1} \right)^{-2/3} (x + 1)^{-2} dx$$

we let $u = \frac{x-1}{x+1}$. Then $du = 2(x+1)^{-2} dx$ so that

$$\int \left[(x^2 - 1)(x + 1) \right]^{-2/3} dx = \frac{1}{2} \int u^{-2/3} du = \frac{3}{2} u^{1/3} + C = \frac{3}{2} \left(\frac{x - 1}{x + 1} \right)^{1/3} + C$$

Editors' Note. Many solvers used the same substitution. Other popular substitutions included 1/(x + 1); $((x - 1)/(x + 1))^k$ for $k = \frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}$ and -1; $\arctan x$; $\arctan \sqrt{x}$; $\arctan \frac{x-1}{2}$; $\arctan x$; $\arctan \frac{x-1}{2}$; $\operatorname{arccos} x$; and $\operatorname{arccosh} x$. Integration by parts and tables were also successfully employed. Several solvers and the proposer generalized the problem. Gonzáles-Torres remarks: "An integral of the form $\int [(x^2 - 1)(x + 1)]^p dx$, where p is a rational number, has an integrand which (by means of the substitution u = x + 1) becomes the differential binomial $u^2 p(u-2)^p du$. [[misprinted(?)as n-2.]] According to a theorem of P. Chebyshev, the integral can be expressed in terms of elementary functions if and only if at least one of the following three numbers is an integer: p, 2p or 3p. This shows that, among proper fractional exponents p, only $\pm \frac{1}{2}$, $\pm \frac{1}{3}$ and $\pm \frac{2}{3}$ will give nintegral of the above form that can be evaluated in finite terms."

Coll. Math. J., 20 No.5 (Nov., 1989) 442.

411. Proposed by M. S. Klamkin, University of Alberta, Edmonton, Canada

Let T_1, T_2, \ldots, T_n denote the elementary symmetric functions of $\{a_1, a_2, \ldots, a_n\}$, i.e., $T_1 = \sum_i a_i, T_2 = \sum_{i < j} a_i A_j, \ldots, T_n = a_1 a_2 \cdots a_n$ where the sums are symmetric over all the a_i . It is well known that the a_i are uniquely determined, aside from permutations, by the T_i . Let

$$S_r = a_1^r + a_2^r + \dots + a_n^r$$

Show that the a_i are determined uniquely, aside from permutations, if one is given $T_{k_1}, T_{k_2}, \ldots, T_{k_r}, S_{k_{r+1}}, S_{k_{r+2}}, \ldots, S_{k_n}$, where (k_1, k_2, \ldots, k_n) is any permutation of $(1, 2, \ldots, n)$.

Coll. Math. J., 21 No.5 (Nov., 1990) 429.

Solution by Daniel E. Otero, Xavier University, Cincinnati, OH. The result follows from Newton's identities [See L. E. Dickson, New First Course in the Theory of Equations, Wiley and Sons, 1939, p.147; or L. Weisner. Introduction to the Theory of Equations, Macmillan, 1938, p.115] which relate the elementary symmetric polynomials T_r and the power sums S_r for $1 \leq r \leq n$. Namely,

$$S_r - S_{r-1}T_1 + S_{r-2}T_2 - \dots + (-1)^{r-1}S_1T_{r-1} + (-1)^r rT_r = 0$$

For k = 1, 2, ..., r-1, let U_k represent either choice of S_k or T_k . Then it follows by induction from Newton's identities that S_r is a polynomial expression in $U_1, U_2, ..., U_{r-1}$ and S_r . In particular this implies that T_r is a polynomial expression in $U_1, U_2, ..., U_r$.

Since the a_i are uniquely determined, up to permutation, by the T_i , they are similarly determined by the U_i .

Coll. Math. J., 20 No.5 (Nov., 1989) 445–446.

Application of the Extended Chebyshev Inequality

390. (Jan. 1988) Proposed by Norman Schaumberger, Bronx Community College, NY Assume that a > 0, b > 0 and c > 0. Prove that, for n > 1,

$$\prod_{i=1}^{n} [a^{2i-1} + b^{2i-1} + c^{2i-1}] \le 3^{n-1} [a^{n^2} + b^{n^2} + c^{n^2}]$$

with equality if and only if a = b = c.

Coll. Math. J., 20 No.5 (Nov., 1989) 445–446.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Canada. The inequality is a special case of the following extended Chebyshev inequality. Let n be a positive integer, and let $q_1, q_2, q_3, \ldots, q_n$ be positive real numbers such that $q_1 + q_2 + q_3 + \cdots + q_n = 1$. If $0 \le a_1 \le a_2 \le a_3 \le \cdots \le a_n$, $0 \le b_1 \le b_2 \le b_3 \le \cdots \le b_n$, $\ldots, 0 \le k_1 \le k_2 \le k_3 \le \cdots \le k_n$, then

$$\left[\sum_{i=1}^{n} q_i a_i\right] \left[\sum_{i=1}^{n} q_i b_i\right] \cdots \left[\sum_{i=1}^{n} q_i k_i\right] < \sum_{i=1}^{n} q_i a_i b_i \cdots k_i$$

unless any one of the columns of the array

is a column of zeros, or each column, with at most one exception, is constant. This completes the statement of the extended Chebyshev inequality. To apply it in our case, with no loss of generality, let $a \leq b \leq c$. Set n = 3. Let $a_1 = a$, $a_2 = b$, $a_3 = c$; $b_1 = a^3$, $b_2 = b^3$, $b_3 = c^3$; $c_1 = a^5$, $c_2 = b^5$, $c_3 = c^5$; etc. The solution is completed by observing that $1 + 3 + 5 + \cdots + 2n - 1 = n^2$.

REFERENCE

D. S. Mitrinovic, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970, pp.36–37.

Comment by Eugene Levine, Adelphi University, Garden City, NY. We can just as easily, and in the same way, show that, for any positive real numbers r_1, r_2, \ldots, r_n and a_1, a_2, \ldots, a_k , where n > 1 and k > 1, the following inequality holds:

$$\prod_{i=1}^{n} \left[\sum_{j=1}^{k} a_j^{r_i} \right] \le k^{n-1} \sum_{j=1}^{k} a_j^N$$

in which $N = r_1 + r_2 + \cdots + r_n$. For example, using $\sum_{d|N} \phi(d) = N$, where ϕ is the Euler totient function, we obtain

$$\prod_{d|N} \left[\sum_{j=1}^k a_j^{\phi(d)} \right] \leq k^{\tau(N)-1} \sum_{j=1}^k a_j^N$$

where $\tau(N)$ is the number of distinct positive divisors of N.

Editor's Notes: Gebre-Egziabber, McInturff, Shan&Wang, and the proposer, using rational exponents and the power mean inequality, multiplied

$$\left[\frac{a^{n^2} + b^{n^2} + c^{n^2}}{3}\right]^{(2i-1)/n^2} \ge \frac{a^{2i-1} + b^{2i-1} + c^{2i-1}}{3}$$

for i = 1, 2, 3, ..., n. Murty referred to G. Chrystal, *Textbook of Algebra*, Vol.II, Dover, p.48.

Coll. Math. J., 21 No.1 (Jan., 1990) 68–69.

375. (March 1988) Proposed by Norman Schaumberger, Bronx Community College, NY

Assume $a_i > 0$, (i = 1, 2, 3, 4). Prove that

$$\sum_{i=1}^{4} a_i^{27} \left/ \prod_{i=1}^{4} a_i \right| \ge \sum_{i=1}^{4} \frac{1}{a_i}$$

Editor's Note: Klamkin, McInturff and Selby proved that the given inequality is valid for $a_i > 0$ if the exponent 27 is replaced by 3. Klamkin showed that, more generally, $a_1^m + a_2^m + \cdots + a_n^m \ge \sum a_1^{r_1} a_2^{r_2} \cdots a_s^{r_s}$ where the summation is cyclic over the subscripts and the r_i are arbitrary positive numbers whose sum is m. Klamkin also referred to a similar inequality, problem 6-3, Crux Mathematicorum, 5(1979) 198.

Coll. Math. J., 21 No.2 (Mar., 1990) 152.

387. (Nov. 1988) Proposed by Larry Hoehn, Austin Peay State University, Clarksville, TN

If a and b are distinct integers and n is a natural number, prove that

$$\frac{2^{2n-1}(a^{2n}+b^{2n})-(a+b)^{2n}}{(a-b)^2}$$

is an integer.

Editors' Note: Klamkin notes that a generalization is given in Problem 4, Crux Mathematicorum, 14(1988) pp.131, 139; viz., if P(x, y) is a symmetric polynomial in x and y and is divisible by $(x - y)^{2n-1}$, then it is also dividible by $(x - y)^{2n}$

[[I didn't notice any mention of Murray in 1991 – R.]]

Coll. Math. J., 23 No.1 (Jan., 1992) 69.

468. Proposed by Murray Klamkin and Andy Liu (jointly), University of Alberta, Edmonton, Canada

If

$$I_n = \int_1^\infty \frac{dx}{1+x^{n+1}} \qquad n > 0$$

show that

$$\frac{\log(2)}{n} + \frac{1}{4n^2} > I_n > \frac{\log(2)}{n}$$

where log(x) is the natural logarithm function.

Coll. Math. J., 24 No.1 (Jan., 1993) 97–98.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM. Let $u = x^{-1}$, so $du = -x^{-2} dx$. Then

$$I_n = \int_1^\infty \frac{dx}{1+x^{n+1}} = \int_0^1 \frac{u^{n-1} \, du}{1+u^{n+1}}$$

Now for $u \in (0, 1)$

$$\frac{u^{n-1}}{1+u^n} + u^{2n-1} - u^{2n} > \frac{u^{n-1}}{1+u^{n+1}} > \frac{u^{n-1}}{1+u^n}$$

which can be easily verified. Integrating from 0 to 1 gives

$$\frac{\log 2}{n} + \frac{1}{4n^2 + 2n} > I_n > \frac{\log 2}{n}$$

Editors' Note. Hongwei Chen and H.-J. Seiffert (independently) improved the lower bound to $\log(2)/n + (n+2)/6n(2n+1)(3n+2)$. Seiffert also gave the following generalization:

If

$$I^{m,n} = \int_{1}^{\infty} \frac{x^{m}}{1 + x^{n+1}} \, dx$$

with -1 < m < n, then

$$\frac{\log(2)}{n-m} + \frac{m+1}{2(n-m)(2n-m+1)} > I_{m,n} > \frac{m+1}{\log(2)n-m} + \frac{(m+1)(n+m+2)}{6(n-m)(2n-m-1)(3n-m+2)} + \frac{m+1}{2(n-m)(2n-m+1)} > I_{m,n} > \frac{1}{\log(2)} + \frac{1}{2(n-m)(2n-m+1)} + \frac{1}{2(n-m)(2n-m+1)(2n-m+1)} + \frac{1}{2(n-m)(2n-m+1)} + \frac{1}{2(n-m+1)} + \frac{1}{2(n-m)(2n-m+1)} + \frac{1}{2(n-m+1)} + \frac$$

Coll. Math. J., 23 No.4 (Sept., 1992) 340.

483. Proposed by Murray Klamkin, University of Alberta, Edmonton, Canada Let

$$S_r = a_1^r + a_2^r + \dots + a_n^r$$
 $r = 1, 2, \dots$

Determine S_{2n+2} given that $S_1 = 1$ and $S_2 = S_3 = \cdots = S_n = 0$.

Coll. Math. J., 24 No. (Sept., 1993) 380–382.

Solution by Prestonburg Community College Math Problem Solvers Group, Prestonburg, KY. Let $P(x) = \prod_{i=1}^{n} (x - a_i) = x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots + P_n$. Then we have Newton's formulas [1]

$$S_1 + P_1 = 0$$
$$S_2 + P_1 S_1 + 2P_2 = 0$$
$$\dots$$

$$S_n + P_1 S_{n-1} + P_2 S_{n-2} + \dots + P^{n-1} S_1 + n P_n = 0$$

and the recurrence relations [1]

$$S_{n+k} + P_1 S_{n+k-1} + \dots + P_n S_k = 0 \qquad k \ge 1$$

Since $S_1 = 1$ and $S_2 = \cdots = S_n = 0$, we get

$$P_k = -\frac{1}{k}P_{k-1} = \frac{(-1)^k}{k!} \qquad k \ge 1$$

Now we show by a recursive argument that

$$S_{n+k} = \frac{(-1)^{n+1}}{(k-1)!n!} \qquad 1 \le k \le n+1 \tag{(*)}$$

First, note that

$$S_{n+1} = -P_n = \frac{(-1)^{n+1}}{0!n!}$$

$$S_{n+2} = -P_1 S_{n+1} = \frac{(-1)^{n+1}}{1!n!}$$

$$S_{n+3} = -P_1 S_{n+2} - P_2 S_{n+1} = \frac{(-1)^{n+1}}{2!n!}$$

Now assume that (*) holds for S_{n+i} , $1 \le i \le k-1 < n+1$. Then

$$S_{n+k} = -\sum_{i=1}^{k-1} P_i S_{n+k-i}$$

= $-\sum_{i=1}^{k-1} \frac{(-1)^i}{i!} \frac{(-1)^{n+1}}{(k-i-1)!n!}$
= $\frac{(-1)^n}{(k-1)!n!} \sum_{i=1}^{k-1} (-1)^i \frac{(k-1)!}{i!(k-i-1)!}$
= $\frac{(-1)^n}{(k-1)!n!} \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i}$

It follows from $\sum_{j=0}^{m} (-1)^{j} {m \choose j} = 0$ that

$$S_{n+k} = \frac{(-1)^n}{(k-1)!n!} \left[-\binom{k-1}{0} \right] = \frac{(-1)^{n+1}}{(k-1)!n!}$$

Finally,

$$S_{2n+2} = -\sum_{i=1}^{n} P_i S_{2n-i+2}$$

= $-\sum_{i=1}^{n} \frac{(-1)^i}{i!} \frac{(-1)^{n+1}}{(n-i+1)!n!}$
= $\frac{(-1)^n}{n!(n+1)!} \sum_{i=1}^{n} (-1)^i \binom{n+1}{i}$
= $\frac{(-1)^n}{n!(n+1)!} \left[-\binom{n+1}{0} - (-1)^{n+1} \binom{n+1}{n+1} \right] = \frac{1-(-1)^n}{n!(n+1)!}$

1. J. V. Uspensky. *Theory of Equations*, McGraw-Hill, 1948.

Coll. Math. J., 23 No.4 (Sept., 1992) 347.

460. (Sept. 1991) Proposed by William V. Webb, Akron, OH

Solve the difference equation: $y_{n+1} = ay_n^x$ (n = 0, 1, ...), where a > 0; $x \neq 0$ and $y_0 > 0$.

Coll. Math. J., 20 No.5 (Nov., 1989) 445–446.

Solution II by Murray S. Klamkin, University of Alberta, Edmonton, Canada. By letting $y_n = a^{1/(1-x)}u_n^{x^n}$ we have $u_{n+1} = u_n$. Hence

$$y_n = a^{1/(1-x)} \left[a^{-1/(1-x)} y_0 \right]^{x^n}$$

[[Here one would need to treat x = 1 separately, in which case $y_n = a^n y_0 - R$.]]

Editors' Note: G. Ladas notes that this equation models the growth of a single-species population interacting with a maintained resource [see R. M.May, *et al.*, Time delays, density-dependence and single species oscillations, *J. Anim. Ecol.*, **43**(1974) 747–770].

Coll. Math. J., 23 No.5 (Nov., 1992) 435–436.

488. Proposed by Murray Klamkin and Andy Liu, University of Alberta, Edmonton

A is at the northeast corner and B is at the southwest corner of an n+1 by n+1 square lattice. In each move, A goes south or west to a neighboring lattice point while B simultaneously goes north or east to a neighboring lattice point. A and B stay within the lattice, and, when faced with two possible directions [[in which]] to move, each flips a fair coin to decide the direction. Determine the probability that A and B meet.

Coll. Math. J., 24 No.5 (Nov., 1993) 435–436.

Solution by John S. Summer and Kevin L. Dove (jointly), University of Tampa, Tampa, FL. Let B = (0,0) and A = (n,n). If A and B meet, then they must meet on the diagonal $\{(i, n - 1) : i = 0, 1, ..., n\}$. Suppose $0 \le i \le n$ and A and B meet at (i, n - 1). Since the number of ways for A and B to travel to this point is $\binom{n}{i}$ and $\binom{n}{i}$ respectively, with each path for A or B having probability 2^{-n} , it follows that the probability that A and B meet is

$$2^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2 = 2^{-2n} \binom{2n}{n}$$

Editor's note: Many solvers noted that the desired probability is asymptotic $1/\sqrt{\pi n}$. Several solvers generalized to n + 1 by m + 1 lattices, and to biased coins.

Coll. Math. J., 24 No.2 (Mar., 1993) 189–190.

475. (March 1992) Proposed by Seung-Jin Bang, Seoul, Korea

Prove that

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n+1} \right) \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$$

Editors' Note: M. S. Klamkin proved the following generalization: For r = 1, 2, ...,

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n+r} \right) \frac{1}{n(n+1)\cdots(n+r)} = \frac{1}{r^2 \cdot r!} \left(\frac{1}{r} + 1 + \frac{1}{2} + \dots + \frac{1}{r} \right)$$

Coll. Math. J., 24 No.4 (Sept., 1993) 378.

510. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada Which of the two integrals

$$S_n = \int_0^{\pi/2} \int_0^{\pi/2} \cdots \int_0^{\pi/2} \sin(x_1 + x_2 + \dots + x_n) \, dx_1 \, dx_2 \cdots dx_n$$
$$C_n = \int_0^{\pi/2} \int_0^{\pi/2} \cdots \int_0^{\pi/2} \cos(x_1 + x_2 + \dots + x_n) \, dx_1 \, dx_2 \cdots dx_n$$

is larger?

Coll. Math. J., 25 No.4 (Sept., 1994) 338–339.

Solution I by John S. Summer and Kevin L. Dove (jointly), University of Tampa, Tampa, FL. Using the identities for the sine and cosine of a sum, it is clear that $\{S_n\}$ and $\{C_n\}$ satisfy the system of difference equations

$$\begin{split} S_n &= C_{n-1} + S_{n-1} \quad C_n = C_{n-1} - S_{n-1} \quad n \geq 2\\ S_1 &= 1 \qquad \qquad C_1 = 1 \end{split}$$

The solution to this system is $S_n = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$ and $C_n = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$ for $n \ge 1$. Thus $S_n > C_n$ if $n \equiv 2, 3, 4 \pmod{8}$, $S_n < C_n$ if $n \equiv 0, 6, 7 \pmod{8}$ and $S_n = C_n$ if $n \equiv 1, 5 \pmod{8}$.

Solution II by Kevin Ford (student), University of Illinois at Urbana-Chanpaign, IL. We claim that $S_n = C_n$ when $n \equiv 1, 5 \pmod{8}$, $S_n < C_n$ when $n \equiv 2, 3, 4 \pmod{8}$ and $S_n < C_n$ when $n \equiv 0, 6, 7 \pmod{8}$. Since

$$C_n + iS_n = \int_0^{\pi/2} \int_0^{\pi/2} \cdots \int_0^{\pi/2} e^{i(x_1 + \dots + x_n)} dx_1 \cdots dx_n = \left(\int_0^{\pi/2} e^{ix} dx\right)^n$$

= $(1+i)^n$

we have $\arg(C_n + iS_n) = \pi n/4$ and the claim follows.

Coll. Math. J., 25 No.1 (Jan., 1994) 65.

520. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine all pairs (p,q) of positive integers (p > q) such that there exist geometric progressions in which

(i) the q th term is p, the p th term is q, and the (p+q)th term is an integer;

(ii) the q th term is q, the p th term is p, and the (p+q)th term is an integer.

Coll. Math. J., 26 No.1 (Jan., 1995) 71–72.

(i) Solution by John S. Summer and Kevin L. Dove (jointly), University of Tampa, Tampa, FL. Suppose (p,q) satisfies (i). Let $q = ar^{p-1}$ and $p = ar^{q-1}$. Then $k = ar^{p+q-1}$ is an integer. Note that

$$\left(\frac{k}{q}\right)^{1/q} = r = \left(\frac{k}{p}\right)^{1/p}$$
 and so $\left(\frac{q}{k}\right)^{p/k} = \left(\frac{p}{k}\right)^{q/k}$

It follows [Mathematics Magazine, (February 1990) 30] that there exists a positive integer u such that

$$\frac{p}{k} = \left(1 + \frac{1}{u}\right)^{u+1}$$
 and $\frac{q}{k} = \left(1 + \frac{1}{u}\right)^{u}$

Thus p/q = 1 + 1/u and p = q + q/u. Hence u divides q and we can put q = mu. We have now that p = m(1 + u) and

$$k = \frac{mu}{\left(1 + \frac{1}{u}\right)^u} = \frac{mu^{u+1}}{(1+u)^u}$$

Therefore $(1+u)^u$ divides m and again we put $m = M(1+u)^u$ for some positive integer M. Finally, we have $k = Mu^{u+1}$ so that

$$p = M(1+u)^{u+1}$$
 and $q = Mu(1+u)^u$

Conversely, it is easy to check that any pair (p,q) of such integers satisfies (i) with ratio $r = (k/p)^{1/p}$ and first term $a = q/r^{p-1}$.

(ii) Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark. We seek $p, q, n \in \mathcal{N}$ with $p > q \ge 1$ and such that there exist $a, r \in \mathcal{R}$ with

$$ar^{p} = p$$
 (1) $ar^{q} = q$ (2) and $n = ar^{p+q}$ (3)

Clearly (1)–(3) imply that

$$n = pr^q = qr^p$$
 (4) whereupon $r = \left(\frac{p}{q}\right)^{1/(p-q)}$ (5)

Now, from (4) and (5) we have

$$n = \frac{p^{p/(p-q)}}{q^{q/(p-q)}} \tag{6}$$

Letting k = gcd(p, q), setting p = Pk and q = Qk, so that gcd(P, Q) = 1, and putting d = P - Q, we get

$$Q^q \cdot n^d = P^P \cdot k^d \tag{7}$$

Fromgcd $(Q^Q, P^P) = 1$ it follows that $Q^Q \mid k^d$.

If the prime decomposition of Q is

$$Q = \prod_{i=1}^{t} Q_i^{\alpha_i} \tag{8}$$

then the prime factor Q_i (i = 1, ..., t) must occur in the prime decomposition of k, say with exponent β_i . Let $Q_i^{\gamma_i}$ be the maximum power of Q_i that divides n. Then by (7) we have

$$\alpha_i Q + \gamma_i d = \beta_i d \qquad (i = 1, \dots, t) \tag{9}$$

By (8), Q is then a d th power, say $Q = \hat{Q}^d$. Similarly, P is a d th power, say $P = \hat{P}^d$. We then have $d = P - Q = \hat{P}^d - \hat{Q}^d$ and, since \hat{P} and \hat{Q} are integers, this can happen only if d = 1 (since $\hat{P}^2 - \hat{Q}^2 > 2$, $\hat{P}^3 - \hat{Q}^3 > 3$, ...).

Consequently, P = Q + 1 and $Q^Q \mid k$. This implies that there is a positive integer m such that $k = Q^Q m$, from which it follows that

$$p = Pk = (Q+1)Q^Qm$$
 (10) and $q = Qk = QQ^Qm = Q^{Q+1}m$ (11)

As in (i), it is straightforward to check that if Q and m are chosen arbitrarily in \mathcal{N} , and p and q are defined by (10) and (11), then (p,q) is a solution our problem.

Coll. Math. J., 25 No.2 (Mar., 1994) 159–161.

497. (Mar. 1993) Proposed by Edward Aboufadel, Rutgers University, New Brunswick, NJ

In a book review in the September 1991 issue of *SIAM Review*, Nicholas Kazarinoff quotes from V. I. Arnol'd's *Huygens and Barrow*, Newton and Hooke: Pioneers in Mathematical Analysis and Catastrophe Theory from Evolvents to Quasicrystals:

Here is an example of a problem that people like Barrow, Newton and Huygens would have solved in a few minutes and which present-day mathematicians are not, in my opinion, capable of solving quickly: to calculate

$$\lim_{x \to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$$

Solve Arnol'd's problem. (it Editors' note: Although some computer algebra programs are able to evaluate this limit, we are not soliciting such solutions.)

 $Solution \ I \ based \ upon \ solutions \ submitted \ by \ most \ solvers$

Since

$$\sin(\tan x) = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{55}{1008}x^7 + O(x^9)$$

$$\tan(\sin x) = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{107}{5040}x^7 + O(x^9)$$

$$\arctan(\arctan x) = x - \frac{1}{6}x^3 + \frac{13}{120}x^5 - \frac{341}{5040}x^7 + O(x^9)$$

$$\arctan(\arcsin x) = x - \frac{1}{6}x^3 + \frac{13}{120}x^5 - \frac{173}{5040}x^7 + O(x^9)$$

it follows that

$$\lim_{x \to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)} = \lim_{x \to 0} \frac{-\frac{1}{30}x^7 + O(x^9)}{-\frac{1}{30}x^7 + O(x^9)} = 1$$

Solution II by M. S. Klamkin and L. Marcoux, University of Alberta, Edmonton, Canada. We provide the following generalization.

Let F(x) and G(x) be odd differentiable functions whose power series expansions are

$$F(x) = x + a_3 x^3 + a_5 x^5 + a_7 x^7 + \cdots$$

$$G(x) = x + b_3 x^3 + b_5 x^5 + b_7 x^7 + \cdots$$

Then by letting

$$F^{-1}(x) = x + c_3 x^3 + c_5 x^5 + c_7 x^7 + \cdots$$

$$G^{-1}(x) = x + d_3 x^3 + d_5 x^5 + d_7 x^7 + \cdots$$

it follows by substituting back in the series for F and G, expanding out, and equating like coefficients that

 $c_3 = -a_3$ $c_5 = 3a_3^2 - a_5$ $c_7 = 8a_3a_5 - 9a_3^3 - a_7$... $d_3 = -b_3$ $d_5 = 3b_3^2 - b_5$ $d_7 = 8b_3b_5 - 9b_3^3 - b_7$... and expansion that

$$F(G(x)) - G(F(x)) = \{2(a_5b_3 - a_3b_5) + 3a_3b_3(b_3 - a_3)\} x^7 + O(x^9)$$

$$F^{-1}(G^{-1}(x)) - G^{-1}(F^{-1}(x)) = \{2(c_5d_3 - c_3d_5) + 3c_3d_3(d_3 - c_3)\} x^7 + O(x^9)$$

$$= \{2(a_5b_3 - a_3b_5) + 3a_3b_3(b_3 - a_3)\} x^7 + O(x^9)$$

Hence if the coefficient of x^7 does not vanish, then

$$\lim_{x \to 0} \frac{F(G(x)) - G(F(x))}{F^{-1}(G^{-1}(x)) - G^{-1}(F^{-1}(x))} = 1$$

In particular, for $F(x) = \sin x$ and $G(x) = \tan x$, the coefficient of x^7 does not vanish, so the desired limit = 1.

The coefficient of x^7 can vanish if, say,

Case 1.
$$a_3 = a_5 = 0$$
 Case 2. $a_3 = b_3 = 0$

In both of these cases the limit is still 1 provided that, respectively,

$$a_7b_3 \neq 0$$
 $a_7b_5 - a_5b_7 \neq 0$

Here for case 1,

$$F(G(x)) - G(F(x)) = F^{-1}(G^{-1}(x)) - G^{-1}(F^{-1}(x)) = 4a_7b_3x^9 + O(x^{11})$$

and here for case 2,

$$F(G(x)) - G(F(x)) = F^{-1}(G^{-1}(x)) - G^{-1}(F^{-1}(x)) = 2(a_7b_5 - a_5b_7)x^{11} + O(x^{13})$$

Editors' note: ... this problem appeared as problem 7 in the *Missouri Journal of Mathematical Sciences*, Vol.1, No.1 (1989).

Coll. Math. J., 25 No.3 (May, 1994) 240.

527. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Let Q(x, y, z, w) be a polynomial that is symmetric with respect to each of its variables. Determine theminimum degree of Q if Q is divisible $(x^{m+1} - y^{m+1})^n$ where m and n are positive integers.

Coll. Math. J., 26 No.3 (May, 1995) 246.

Solution by the proposer. By symmetry, we must have a factor of the form $(x^{m+1} - y^{m+1})^n$ for each of the 6 pairs of variables. This would give the minimum degree for Q to be 6(m+1)n whenever n is even. Now suppose n is odd and observe that by hypothesis Q is divisible by $(x-y)^n$. We show that Q must be divisible by $(x-y)^{n+1}$. Note that

$$(x-y)^n R(x, y, z, w) = Q(x, y, z, w) = Q(y, x, z, w) = -(x-y)^n R(y, x, z, w)$$

where R is some polynomial. It then follows that

$$(x-y)^n \{ R(x, y, z, w) + R(y, x, z, w) \} = 0$$

for all x, y, z, w so that R(x, y, z, w) + R(y, x, z, w) is identically zero except possibly when x = y. But this sum is a polynomial, so a continuity argument implies that R(x, x, z, w) = 0 for all x, z, w. It now follows by the remainder theorem that R(x, y, z, w) must be divisible by x - y. Consequently, for n odd, the minimum degree for Q is 6(m + 1)n + 6.

Coll. Math. J., 25 No.4 (Sept., 1994) 334.

531. Proposed by Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Canada

If A, B, C and D are consecutive vertices of a quadrilateral such that $\angle DAC = 55^{\circ} = \angle CAB, \angle ACD = 15^{\circ}$ and $\angle BCA = 20^{\circ}$, determine $\angle ADB$.

Coll. Math. J., 26 No.4 (Sept., 1995) 330-331.

Solution I by Man-Keung Siu, University of Hong Kong, Hong Kong. We show that $\angle ADB = 40^{\circ}$.

Take a point E on AB such that $\angle ECB = 5^{\circ}$. Since the triangles ADC and AEC are congruent, AC and DE are perpendicular to each other. From this we see that $\angle ADE = 35^{\circ}$ and $\angle EDC = 75^{\circ}$. Since $\angle EBC = \angle ABC = 105^{\circ}$, EBCD is a cyclic quadrilateral. Hence $\angle EDB = \angle ECB = 5^{\circ}$. Finally, $\angle ADB = \angle ADE + \angle EDB = 35^{\circ} + 5^{\circ} = 40^{\circ}$.

Solution II by Michael H. Andreoli, Miami Dade Community College (North), Miami, FL. Since the sum of angles in $\triangle ACD$ and in $\triangle ABC$ is 180° , $\angle ADC = 110^{\circ}$ and $\angle ABC = 105^{\circ}$. Without loss of generality suppose AD = 1. Applying the law of sines to $\triangle ACD$ and $\triangle ABC$ and equating the resulting expressions for AC yields $AB = (\sin 20^{\circ} \sin 110^{\circ})/(\sin 15^{\circ} \sin 105^{\circ})$. Since $\sin 110^{\circ} = \cos 20^{\circ}$ and $\sin 105^{\circ} = \cos 15^{\circ}$, we can use the double angle formulas to write $AB = 2 \sin 40^{\circ}$.

Applying the law of cosines to $\triangle ABD$ yields $BD = [1 + (AB)^2 - 2(AB)\cos 110^\circ]^{1/2}$ where $AB = 2\sin 40^\circ$.

Applying the law of sines to $\triangle ABD$, using the above expressions for AB and BD gives

$$\sin(\angle ADB) = \sin 40^{\circ} \left[2\sin 110^{\circ} / (1 + 4\sin^2 40^{\circ} - 4\sin 40^{\circ} \cos 110^{\circ})^{1/2} \right]$$
(1)

This can be expressed in terms of sines and cosines of 40° using

$$\sin 110^{\circ} = \sin 70^{\circ} = \sin (40^{\circ} + 30^{\circ}) = (\sqrt{3}/2) \sin 40^{\circ} + (1/2) \cos 40^{\circ}$$
$$\cos 110^{\circ} = -\cos 70^{\circ} = -\cos(40^{\circ} + 30^{\circ}) = (1/2) \sin 40^{\circ} - (\sqrt{3}/2) \cos 40^{\circ} \text{ and}$$
$$1 = \sin^2 40^{\circ} + \cos^2 40^{\circ}$$

Making these substitutions in (1) and simplifying gives $\sin(\angle ADB) = \sin 40^{\circ}$. Since we know that $0 \leq \angle ABD \leq \angle ADC = 110^{\circ}$, it follows that $\angle ADB = 40^{\circ}$.

Coll. Math. J., 25 No.5 (Nov., 1994) 462.

536. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

Determine the minimum values of

- (i) $a^2 + T_2$
- (ii) $a^2 + b^2 + T_2$
- (iii) $a^2 + b^2 + c^2 + T_2$

where $T_2 = a(b + c + d) + b(c + d) + cd$, abcd = 1 and a, b, c, d > 0.

Coll. Math. J., 26 No.5 (Nov., 1995) 406-407.

Solution by David Zhu, Jet Propulsion Laboratory, Pasadena, CA.

(i) $a^2 + T_2 = a^2 + a(b + c + d) + bc + bd + cd$.

Note that $b + c + d \ge 2\sqrt[3]{bcd}$ and $bc + bd + cd \ge 3\sqrt[3]{b^2c^2d^2}$ where the inequalities hold if and only if b = c = d. Letting u = a and v = b = c = d it now suffices to minimize $u^2 + 3uv + 3v^2$ subject to $uv^3 = 1$. This is equivalent to minimizing $v^{-6} + 3v^{-2} + 3v^2$ which occurs when $v = [(\sqrt{5} + 1)/2]^{1/4}$. Thus, the minimum of $a^2 + T_2$ is

$$\frac{2\sqrt{2}(7+3\sqrt{5})}{(1+\sqrt{5})^{3/2}} = \sqrt{22+10\sqrt{5}}$$

(ii) $a^2 + b^2 + T_2 = (a^2 + b^2) + (a + b)(c + d) + ab + cd$

With the product ab fixed, $a^2 + b^2 + T_2$ is minimized if and only if a = b, and with the product cd fixed, $a^2 + b^2 + T_2$ is minimized if and only if c = d. Letting u = a = band v = c = d it now suffices to minimize $3u^2 + 4uv + v^2$ subject to uv = 1. But $3u^2 + v^2 \ge 2\sqrt{3}uv = 2\sqrt{3}$ with equality holding if and only if $\sqrt{3}u = v$ or $u = (1/3)^{1/4}$ so that the minimum value of $a^2 + b^2 + T_2$ is $2\sqrt{3} + 4$.

(iii)
$$a^2 + b^2 + c^2 + T_2 = (a^2 + b^2 + c^2) + (a + b + c)d + (bc + ca + ab).$$

[[the last paren was misprinted as (ab + ad + bc) - R.]]

With the product abc fixed, $a^2+b^2+c^2+T_2$ is minimized if and only if a = b = c. Letting u = a = b = c and v = d in this case, it now suffices to minimize $6u^2 + 3uv$ subject to $u^3v = 1$, which is equivalent to minimizing $6u^2 + 3u^{-2}$. But $6u^2 + 3u^{-2} \ge 6\sqrt{2}$ with equality holding if and only if $u = (1/2)^{1/4}$ so that the minimum value of $a^2+b^2+c^2+T_2$ is $6\sqrt{2}$.

Coll. Math. J., 25 No.5 (Nov., 1994) 462.

538. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

Determine the maximum area of the quadrilateral with consecutive vertices A, B, C and D id $\angle A = \alpha$, BC = b and CD = c are given.

Coll. Math. J., 26 No.5 (Nov., 1995) 407-409.

Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, OR. Notation. The area of a polygon is denoted by enclosing its name in brackets.

Case 1. $\alpha < \pi$. Construct two rays forming an angle α and originating at point A. Let points B and D move along these rays, with BC = b and CD = c and let $\angle BCD = \theta$. We seek to maximize [ABCD] by considering θ .

Clearly we may assume $\theta < \pi$ because $\theta > \pi$ would give a concave quadrilateral, and its area could be increased by reflecting $\triangle BCD$ in the line determined by B and D, producing a convex quadrilateral with a greater area.

Now [ABCD] = [BCD] + [ABD]. For a given θ , [BCD] is fixed, and so is the length x of BD. But [ABD] with a given angle α and a given side x opposite that angle is maximized when $\triangle ABD$ is isosceles with AB = AD. But $AB = AD \implies \angle ABD = \angle ADB = \pi/2 - \alpha/2$. By the law of sines applied to $\triangle ABD$ we have

$$AD = AB = \frac{x\sin(\pi/2 - \alpha/2)}{\sin\alpha} = \frac{x\cos(\alpha/2)}{\sin\alpha}$$

giving

$$[ABD] = \frac{1}{2}(AD)(AB)\sin\alpha = \frac{x^2\cos^2(\alpha/2)}{2\sin\alpha}$$

and

$$[ABCD] = \frac{1}{2}bc\sin\theta + \frac{(b^2 + c^2 - 2bc\cos\theta)\cos^2(\alpha/2)}{2\sin\alpha} \tag{1}$$

where we have used the law of cosines in $\triangle BCD$ to write $x^2 = b^2 + c^2 - 2bc \cos \theta$. We may now write

$$[ABCD] = \frac{(b^2 + c^2)\cos^2(\alpha/2)}{2\sin\alpha} + \frac{bc}{2}(\sin\theta - k\cos\theta)$$

where $k = [2\cos^2(\alpha/2)]/\sin \alpha$ is some positive constant. [ABCD] will be maximized when $(\sin \theta - k \cos \theta)$ is maximized, and it is easily determined that this occurs (for $0 < \theta < \pi$) when $\theta = \arctan(-1/k) + \pi$. Substituting back into (1) gives

$$[ABCD]_{\max} = \frac{(b^2 + c^2)\cos^2(\alpha/2) + 2bc\cos(\alpha/2)}{2\sin\alpha}$$

Case 2. $\alpha > \pi$. In this case there is no maximum value. Let rays \overrightarrow{AB} and \overrightarrow{AD} again form an angle α , with B and D sliding along these rays with fixed distances CB = b and CD = c, the angleat C being θ ($\theta < 2\pi - \alpha < \pi$) and BD having length x (which is a function of θ).

We have

$$[ABCD] = [CBD] - [ABD] = \frac{1}{2}bc\sin\theta - [ABD]$$

The first term [CBD] may have a maximum (if $\theta = \pi/2$ happens to be in the domain of θ), but the second term has no minimum ([ABD] has 0 as a greatest lower bound). Thus [ABCD] cannot be maximized.

Coll. Math. J., 25 No.5 (Nov., 1994) 463-464.

511. (Nov. 1993) Proposed by Zhang Zaiming, Yuxi Teachers' College, Yunnan, China

Let a_i, b_i, c_i and Δ_i be the lengths of the sides and the area, respectively, of triangle i, i = 1, 2. Prove that

$$a_1^2 a_2^2 + b_1^2 b_2^2 + c_1^2 c_2^2 \ge 16\Delta_1 \Delta_2$$

where equality holds if and only if the two triangles are equilateral.

Editors' note: Klamkin observes that the above result follows from the stronger inequality $a_1a_2 + b_1b_2 + c_1c_2 \ge 4\sqrt{3\Delta_1\Delta_2}$ in a problem of G. Tsintsifas [*Crux Mathematicorum*, **13**(1987) 185]. [[Gordon]] Bennet proved the stronger result

$$a_1^2 a_2^2 + b_1^2 b_2^2 + c_1^2 c_2^2 \ge 16\Delta_1 \Delta_2 + a_1^2 (b_2 - c_2)^2 + b_1^2 (c_2 - a_2)^2 + c_1^2 (a_2 - b_2)^2$$

[Univ. Beograd. Publ. Elektrotehn. Fak., (1977) 39–44]. Many solvers invoked Jensen's inequality, while others use[[d]] Pedoe's inequality [Amer. Math. Monthly, **70**(1963) 1012].

[[In the above I haven't been able to confirm the spelling of Bennet[t], who appears with both one tee and two. — R.]]

Coll. Math. J., 26 No.1 (Jan., 1995) 67.

544. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

Evaluate the sum

$$S = 1 - \frac{3n}{1!} + \frac{3n(3n-1)}{2!} - \frac{3n(3n-1)(3n-2)}{3!} + \dots + (-1)^N \frac{3n(3n-1)()\cdots(2n+1)}{n!}$$

Coll. Math. J., 27 No.1 (Jan., 1996) 77–78.

Composite of solutions submitted by most of the solvers.

$$S = \sum_{k=0}^{n} (-1)^{k} {3n \choose k} = 1 + \sum_{k=1}^{n} (-1)^{k} \left[{3n-1 \choose k} + {3n-1 \choose k-1} \right]$$
$$= \sum_{k=0}^{n} (-1)^{k} {3n-1 \choose k} + \sum_{k=0}^{n-1} (-1)^{k+1} {3n-1 \choose k} = (-1)^{n} {3n-1 \choose n}$$

Editors' note. Several solversused induction to establish the result. [[Eleven solvers]] provided the generalization $\sum_{0}^{n}(-1)^{k} {r \choose k} = (-1)^{n} {r-1 \choose n}$ [[Two solvers]] proved the generalization $\sum_{0}^{[n/3]} (-1)^{k} {r \choose k} = (-1)^{[n/3]} {n-1 \choose [n/3]}$. Several solvers noted that this problem (or one equivalent to it) appears as an exercise in several combinatorics texts. Callan noted the similarity to *Monthly* problem 6637 (1990,621; 1992,72).

[[This last was proposed by Herbert Wilf. - R.]]

Coll. Math. J., 26 No.5 (Nov., 1995) 405.

562. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada Evaluate the $(n + 2) \times (n + 2)$ symmetric determinant

0	1	1	1	 1
1	0	a_1	a_2	 a_n
1	a_1	0	$a_1 + a_2$	 $a_1 + a_n$
1	a_2	$a_2 + a_1$	0	 $a_2 + a_n$
:	÷	÷	:	:
1	a_n	$a_n + a_1$	$a_n + a_2$	 0

Coll. Math. J., 27 No.5 (Nov., 1996) 406.

Solution by Joe Howard, New Mexico Highlands University, Las Vegas, NM. Subtract row 2 from rows 3, 4, ..., (n + 2); expand along column 1; and factor out a_1, a_2, \ldots, a_n to obtain the $(n + 1) \times (n + 1)$ determinant

$$(-1)a_1a_2\cdots a_n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{vmatrix}$$

Subtract row 1 from rows 2, 3, ..., (n+1); and then expand along column 1 to obtain the $n \times n$ determinant

$$(-1)a_1a_2\cdots a_n \begin{vmatrix} -2 & 0 & 0 & \dots & 0\\ 0 & -2 & 0 & \dots & 0\\ 2 & 0 & -2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & -2 \end{vmatrix} = (-1)^{n+1}2^n(a_1a_2\cdots a_n)$$

[In connexion with the following problem, compare problem 629 below — R.]]

Coll. Math. J., 27 No.1 (Jan., 1996) 75.

570. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

In triangle ABC the angle bisectors of angles B and C meet the altitude AD at points E and F respectively. If $\overline{BE} = \overline{CF}$, prove that ABC is isosceles.

Coll. Math. J., 28 No.1 (Jan., 1997) 72.

Editors' note. As several solvers noted, if both B and C are acute, then this problem is identical to CMJ problem 546 (the solution to which appeard in the March 1996 issue). When B or C is obtuse, however, ABC need not be isosceles. For example, if the vertices of ABC are $A(0, \sqrt{3})$, B(0, 1) and $C(\sqrt[4]{12}, 0)$, then $E(0, -\sqrt{3})$ and $F(0, \sqrt{3}-1)$ so that $\overline{BE} = \overline{CF} = 2$, yet ABC is not isosceles. Coll. Math. J., 27 No.4 (Sept., 1996) 312.

583. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

A known property of a parabola is that if tangents are drawn at any two points P and Q of the curve, then the line from the point of intersection of the tangents and parallel to the axis of the parabola bisects the chord PQ. Does this property characterize parabolas? That is, if a curve has the above property where the line is drawn parallel to the y-axis, must the curve be a parabola whose axis is parallel to the y-axis?

Coll. Math. J., 28 No.4 (Sept., 1997) 319–320.

Solution by John D. Eggers, North Georgia College and State University, Dahlonega, Georgia. Yes, a curve with the above property must be a parabola with a vertical axis. Call the above property Property T, and let Γ be a curve satisfying Property T. First, observe that Γ is a differentiable curve, since by Property T it has a tangent at every point. Second, observe that Γ is the graph of a function; for, if it were not, there would exist a pair of points on Γ such that the segment joining them would be vertical, contradicting Property T. Thus, Γ has a parametrization of the form $t \mapsto (t, \gamma(t))$ with domain \mathcal{U} and open subset of \mathbb{R} . Let t_0 be a fixed element in \mathcal{U} and let t be in \mathcal{U} . The corresponding points of Γ are $P_0 = (t_0, \gamma(t_0))$ and $P = (t, \gamma(t))$. The equation of the line tangent to Γ at P_0 is

$$y = \gamma(t_0) + \gamma'(t_0)(x - t_0)$$

The equation of the line tangent to Γ at P is

$$y = \gamma(t) + \gamma'(t)(x-t)$$

The *x*-coordinate of the intersection of the tangent lines is

$$x = \frac{[\gamma'(t)t - \gamma'(t_0)t_0] - [\gamma(t) - \gamma(t_0)]}{\gamma'(t) - \gamma'(t_0)}$$

Since a vertical line through the point of intersection of the lines tangent to Γ at P_0 and at P must bisect the segment joining P_0 and P, it follows that $x = (t_0 + t)/2$. Thus

$$\frac{[\gamma'(t)t - \gamma'(t_0)t_0] - [\gamma(t) - \gamma(t_0)]}{\gamma'(t) - \gamma'(t_0)} = \frac{t_0 + t}{s}$$

After some calculation one finds that the above equation is equivalent to the linear first-order differential equation

$$\gamma'(t) - \frac{2}{t - t_0}\gamma(t) = -\frac{2\gamma(t_0)}{t - t_0} - \gamma'(t_0)$$

By standard techniques one finds the general solution to be

$$\gamma(t) = c(t - t_0)^2 + \gamma'(t_0)(t - t_0) + \gamma(t_0)$$

where c is a conctant of the integration. Observe that $c \neq 0$, for if c = 0, then Γ would be a line, contradicting Property T. Hence Γ is a parabola with a vertical axis. Coll. Math. J., 27 No.5 (Nov., 1996) 408–409.

New Perfect Boxes

565. (Nov. 1995) Proposed by K. R. S. Sastry, Dodballapur, India

Find the dimensions of all rectangular boxes with sides of integral [[integer]] length such that the volume is numerically equal to the sum of the lengths of the edges plus the surface area.

Solution by Murray S. Klamkin and Andy Liu (jointly), University of Alberta, Edmonton, Canada. The diophantine equation here is

$$abc = 4(a + b + c) + 2(bc + ca + ab)$$

where a, b, c are the lengths of the edges. Letting (a, b, c) = (x+2, y+2, z+2) we get the simpler equation xyz = 8(x + y + z) + 40 where x, y, z > 0. Assuming without loss of generality that $x \ge y \ge z$, we get the following cases.

(1) z = 1 and then $y = 8 + \frac{112}{(x - 8)}$ so that (x, y) = (22, 16), (24, 25), (36, 12), (64, 10) or (120, 9).

(2) z = 2 and then $y = 4 + \frac{44}{(x-4)}$ so that (x.y) = (15, 8), (26, 6) or (48, 5).

(3) z = 3 and then y = 2 + (2x + 80)/(3x - 8) so that (x, y) = (8, 8), (24, 4) or (88, 3).

(4) z = 4 and then $y = 2 + \frac{22}{(x-2)}$ so that (x, y) = (13, 4).

(5) z = 5 and then y = 1 + (3x + 88)/(5x - 8) so that there are no solutions.

(6) $z \ge 6$. Since $xyz \ge 36x$ and $8(x + y + z) + 40 \le 32x$, there are no more solutions. Finally all the solutions (a, b, c) are given by

Coll. Math. J., 28 No.2 (Mar., 1997) 145.

597. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada

Determine all pairs of integers (x, y) such that 19 | ax + by where (a, b) is any pair of integers such that 19 | 11a + 2b.

Coll. Math. J., 29 No.2 (Mar., 1998) 154.

Solution by Joseph Reznick (student), College of Charleston, Charleston, SC. Assume $11a + 2b \equiv 0 \pmod{19}$. Then $b \equiv 4a \pmod{19}$. Now, we want to find all pairs of integers (x, y) such that for every $a, 0 \le a \le 18$, we have $ax + by \equiv 0 \pmod{19}$. This is equivalent to

 $ax + 4ay \equiv 0 \pmod{19}$

For $a \not\equiv 0 \pmod{19}$ this implies

 $x + 4y \equiv 0 \pmod{19}$

Therefore the solution set is $\{(x, y) \mid x + 4y \equiv 0 \pmod{19}\}$.

In addition, it is interesting to note that if instead of 19 we take any other prime p, and instead of the coefficients 11 and 2 we take any other coefficients q and r, then the solution set is $\{(x, y) \mid x + qr^{-1}y \equiv 0 \pmod{p}\}$.

Coll. Math. J., 28 No.5 (Nov., 1997) 408-409.

Squared Cotangents

588. (Nov. 1996) Proposed by Can Anh Minh (student), University of Southern California, Los Angeles, CA

In $\triangle ABC$, denote the angles at A, B, C by 2α , 2β , 2γ , respectively. Prove that $\cot^2\beta\cot^2\gamma + \cot^2\gamma\cot^2\alpha + \cot^2\beta \ge 27$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. Let $S_n = \cot^n \beta \cot^n \gamma + \cot^n \gamma \cot^n \alpha + \cot^n \alpha \cot^n \beta$. We prove the more general result: $S_n \ge 3^{n+1}$ with equality if and only if $\alpha = \beta = \gamma$. From the Cauchy-Schwarz inequality,

 $(\cot\beta\cot\gamma + \cot\gamma\cot\alpha + \cot\alpha\cot\beta)(\tan\beta\tan\gamma + \tan\gamma\tan\alpha + \tan\alpha\tan\beta) \ge 9$

But since $\gamma = (\pi/2) - (\alpha + \beta)$, $\tan \gamma = \cot(\alpha + \beta) = (1 - \tan \alpha \tan \beta)/(\tan \alpha + \tan \beta)$ so that $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$. Hence $S_1 \ge 9$. By the power mean inequality, $(S_n/3)^{1/n} \ge S_1/3$, thus $S_n/3 \ge 3^n$ and the result follows. Coll. Math. J., **30** No.2 (Mar., 1999) 147–148.

Extending the Radius of Convergence

625. Proposed by Sining Zheng and Yuyue Song, Dalian University of Technology, Dalian, People's Republic of China

Let the power series $\sum a_n x^n$ have radius of convergence R > 0 and let $\lambda > R$. Does there exist a power series $\sum b_n$ such that

(1) $\sum b_n x^n$ has radius of convergence R, and

(2) $\sum (a_n + b_n) x^n$ has radius of convergence λ ?

Solution by James Duemmel, Western Washington University, Bellingham, WA and Murray S. Klamkin, University of Alberta, Canada (independently).

Consider any series $\sum c_n x^n$ with radius of convergence λ . Let $b_n = c_n - a_n$. Then $\sum b_n x_n$ has radius of convergence R and $\sum (a_n + b_n) x^n = \sum c_n x^n$ has radius of convergence λ .

Editors' Note: Most solvers made the unnecessary assumption that λ was finite.

[Compare the next item with problem 570 above. -R.]]

Coll. Math. J., **30** No.3 (May, 1999) 233.

An Isosceles Triangle

629. Proposed by David Beran, University of Wisconsin-Superior, Superior, WI

In triangle ABC the angle bisectors of angles B and C meed the median AD at points E and F respectively. If $\overline{BE} = \overline{CF}$, prove that $\triangle ABC$ is isosceles.

Editors' Note: Klamkin and Sastry point out that the identical problem with solution by Esther Szekeres appears in *Crux Mathematicorum*, **20**(1994) 264. [[John]] Graham and Sastry note that a special case of this problem appears with solution in *CMJ* problem **546** [1995,157; 1996, 150].

[[I made the following remark before I discovered that this wasn't Murray's last appearance in CMJ. Murray appears in the list of solvers of problem 677 on p.213 of Coll. Math. J., **32** No.3 (May, 2001). He also solved problems 695, 726, 742, the last mention being in the Jan., 2004 issue. — R.]]

Coll. Math. J., **32** No.5 (Nov., 2001) 381.

711. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

(a) Find all integers n such that there exists a polynomial $P_n(x)$, with integer coefficients which satisfies

$$P_n(n) = n^2$$
 $P_n(2n) = 2n^2$ and $P_n(3n) = n^2$

(b) Find all integers n such that there exists a polynomial $Q_n(x)$ with integer coefficients which satisfies

$$Q_n(n) = n^2$$
 $Q_n(2n) = 2n^2$ $Q_n(3n) = n^2$ and $Q_n(4n) = 2n^2$

Coll. Math. J., 33 No.5 (Nov., 2002) 412.

Solution by Li Zhou, Polk Community College, Winter Haven, FL.

(a) Let A denote this set of integers. Then $A = \mathbb{Z}$ since $P_n(x) = 2n^2 - (x - 2n)^2$ satisfies the conditions for any $n \in \mathbb{Z}$.

(b) Let B denote this set of integers. The $0 \in B$ since $Q_0(n) \equiv 0$ satisfies the conditions. Suppose that n is not zero and that $Q(x) \in \mathbb{Z}[x]$ satisfies $Q(n) = n^2$, $Q(2n) = 2n^2$ and $Q(4n) = 2n^2$. By the Remainder Theorem, there is a polynomial R(x) such that

$$Q(x) = (x - 2n)(x - 4n)R(x) + 2n^{2}$$

In fact, $R(x) \in F[x]$ by division and induction on the degree of Q(x). But then

$$n^2 = Q(n) = 3n^2 R(n) + 2n^2$$

which implies that $R(n) = -1/3 \notin \mathbb{Z}$. Thus, $B = \{0\}$, and note that we have only used three of the four conditions. (We note that the conditions at n, 3n and 4nforce $B = \{0\}$, the conditions at 2n, 3n and 4n give the same answer as (a) with $Q_n(x) = (x - 3n)^2 + n^2$) for each $n \in \mathbb{Z}$, and the conditions at n and 4n imply that $3 \mid n$ with $Q_{3k}(x) = kx + 6k^2$.

Excerpts from Murray Klamkin, Math. Mag.

Richard K. Guy

June 22, 2006

This file updated on 2006-04-28.

This is the second of a number of files listing problems, solutions and other writings of Murray Klamkin.

The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

Math. Mag., **24**(1951) 266

103. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N.Y.

A high-school student solved the linear differential equation dy/dx + Py = Q for y as if it were an ordinary algebraic equation. Under what conditions could this procedure have yielded a correct solution of the differential equation?

Math. Mag., 25(1951) 114.

II. Solution by R. E. Winger, Los Angeles City College. The high school student presumably cancelled the ds, getting y/x + Px = Q. If this is to be a correct solution of the differential equation, then y/x = dy/dx. That is, y = cx, an almost trivial result. When this is substituted into the original equation (in either form) we get Q = c(1 + Px), the necessary relation between P and Q.
Math. Mag., 25(1951) 47

103. Proposed by O. E. Stanaitis, St. Olaf College, Minnesota

Establish the convergence or divergence of

a)
$$1 - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{5\sqrt{5}} - \frac{1}{8} + \frac{1}{7\sqrt{7}} - \frac{1}{16} + \cdots$$

b) $1 - \frac{1}{\sqrt{2}} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{\sqrt{6}} + \frac{1}{7} - \frac{1}{2\sqrt{2}} + \cdots$

I. Solution by M. S. Klamkin's Sophomore Calculus Class, Polytechnic Institute of Brooklyn, N.Y. a) The given series may be written in the form

$$\left(1 + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)$$

thereby exhibiting it as the difference of two well-known convergent series. It follows that the given series is convergent.

b) Consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{\sqrt{2n}} \right) = \sum_{n=1}^{\infty} \frac{\sqrt{2n} - (2n-1)}{(2n-1)\sqrt{2n}}$$

Since the degree of the denominator exceeds that of the numerator by only $\frac{1}{2}$, the series diverges.

Math. Mag., 25(1952) 224

Q 57. Submitted by M. S. Klamkin

Prove that the derivative of an even function is odd and vice versa.

A 57. Since
$$E(x) = E(-x)$$
, $dE(x)/dx = dE(-x)/dx = [dE(-x)/d(-x)][d(-x)/dx] = -dE(-x)/d(-x) = \text{odd.}$ Since $O(x) = -O(-x)$, $dO(x)/dx = -dO(-x)/dx = [-dO(-x)/d(-x)][d(-x)/dx] = dO(-x)/d(-x) = \text{even.}$

Math. Mag., **25**(1952) 282–283

A Conic Unrolled

87. [Jan. 1951] Proposed by Leo Moser, Texas Technological College

A right circular cone is cut by a plane. The intersection, of course, is a conic. Find the equation of the curve thtthis conic goes into if the cone is unrolled on to a plane. In particular, if the cone is a cylinder and the plane cuts the axis of the cylinder at 45°, then the ellipse formed will unroll into a sine curve.

Solution by M. S. Klamkin's Sophomore Calculus Course, Polytechnic Institute of Brooklyn. Let the equation of the cone in cylindrical coordinates be $a^2r^2 = z^2$. Cut the cone along its intersection with the plane y = 0 and let that line become the x'-axis. Then the coordinates of the transform of a point (r, θ, z) on the cone are (r', θ') where

$$r' = \sqrt{r^2 + z^2} = r\sqrt{1 + a^2} \quad \text{and} \quad \theta' = r\theta/\sqrt{r^2 + z^2} = \theta/\sqrt{1 + a^2}$$

Now consider the transform of the intersection of the cone with a general surface, $F(r, \theta, z) = 0$. The equation of a cylinder with elements passing through the intersection curve and parallel to the z-axis is $F(r, \theta, ar) = 0$. Thus the equation of the transform curve will be

$$F(r'/\sqrt{1+a^2}, \theta'\sqrt{1+a^2}, ar'/\sqrt{1+a^2}) = 0$$

If $F(r, \theta, z) = 0$ is a plane, then

$$r(A\cos\theta + B\sin\theta) + Cz + D = 0$$

and the transform curve is

$$(r'/\sqrt{1+a^2})(A\cos\theta'\sqrt{1+a^2} + B\sin\theta'\sqrt{1+a^2} + Ca) + D = 0$$

If we use a cylinder, r = a, instead of the cone $a^2r^2 = z^2$ we find that the point (r, θ, z) transforms into (x', y') where x' = z and $y' = a\theta$. Thus if the curve of intersection is given by r = a and $F(r, \theta, z) = 0$, then upon development the intersection is transformed into F(a, y'/a, x') = 0. Now if the intersecting surface is the plane $r(A\cos\theta + B\sin\theta) + Cz + D = 0$, then

$$F(a, y'/a, x') = a(A\cos y'/a + B\sin y'/a) + Cx' + D = 0$$

which is a sine curve for all plane intersections except when A = B = 0 or when C = 0.

Math. Mag., **26**(1952) 54. **Q 70**. Submitted by M. S. Klamkin Find the maximum value of

$$\left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n - x_n)$$

A 70.

$$\left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n - x_n) = \left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n^2 - a_n x_n) \left/ \prod_{n=1}^{N} a_n x_n\right)$$

Clearly the sum of the factors of the numerator is a constant, $\sum_{n=1}^{N} a_n^2$. Now it follows from the theorem that the geometric mean \leq the arithmetic mean, that if the sum of k factors of a function is a constant, b, then the maximum value of the function is $(b/k)^k$. Therefore the maximum value sought is

$$\left(\sum_{n=1}^{N} a_n^2\right)^{N+1} / (N+1)^{n+1} \prod_{n=1}^{N} a$$

[[ARE THERE MISPRINTS IN THE PREVIOUS ITEM ??]]

Math. Mag., 26(1952) 115.

Q 15. [Sept. 1950] Find the sum of the squares of the coefficients in the expansion of $(a + b)^n$. *M. S. Klamkin* offers this alternative solution.

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \dots + \frac{n}{n}x^n$$
 and $(1+1/x)^n = \binom{n}{0} + \binom{n}{1}/x + \dots + \binom{n}{n}/x^n$.

Multiplying we have that the sum of the squares of the coefficients is the constant term (middle term) of $(1+x)^{2n}/x^n$. That is, $(2n)!/(n!)^2$ or $\binom{2n}{n}$.

[[This next item was among the first batch of Trickies, which Charles Trigg alternated with his Quickies.]]

Math. Mag., 26(1953) 168.

T 5. Submitted by M. S. Klamkin. A rich person who possessed a very expensive Swiss watch once bragged to a poor friend that not only was his watch an automatic winding one, but it lost only $1\frac{1}{4}$ sec. per day. The friend remarked that the watch would indicate the correct time only about once in a century. This annoyed the rich man who demanded to know of a better watch. The friend said that his four-year old daughter had just gotten a watch which though inexpensive at least did indicate the correct time twice a day. How accurate was the daughter's watch?

S 5. The cheap watch must have gained or lost 24 hours per day. It was a stationary toy watch.

Math. Mag., 26(1953) 221–222.

Euler's ϕ -function

145. [Sept. 1952] Proposed by Leo Moser, University of Alberta, Canada

It is well-known that n = 14 is the smallest even integer for which $\phi(n) = n$ is insolvable. Show for every positive integer, r, that $\phi(x) = 2(7)^r$ is insolvable.

II. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. In L. E. Dickson's History of the Theoryof Numbers, Volume I, p.135, there is a result due to Alois Pichler which states "When q is a prime > 3, $\phi(x) = 2q^n$ is impossible if $p = 2q^n + 1$ is not prime; while if P is prime it has the two solutions p and 2p." Since $2(7)^r + 1 = 2(6+1)^r + 1 = 6A + 3$ has the factor 3 iy follows that $\phi(x) = 2(7)^r$ has no solutions. The reference contains other results of a similar nature by Pichler.

Math. Mag., **27**(1953) 51.

177. Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn

If $w = z^n + a_1 z^{n-1} + \dots + a_n + b_1/z + b_2/z^2 + \dots + b_r/z^r$ maps into |w| = 1 for |z| = 1 show that $a_n = b_r = 0, n = 1, 2, 3, \dots$

Math. Mag., 27(1954) 222.

I. Solution by Alfredo Jones, University of Notre Dame. If

$$w = z^{n} + a_{1}z^{n-1} + \dots + a_{n} + b_{1}/z + b_{2}/z^{2} + \dots + b_{r}/z^{r}$$

maps |z| = 1 into |w| = 1 then

$$w' = w' \cdot z^{r+1} = z^{n+r+1} + a_1 z^{n+r} + \dots + b_r z$$

also satisfies that condition. The area of the image of the unit circle by w' will be $A = k\pi$ where k is the number of times w' traverses |w| = 1 while z traverses |z| = 1. Now $k \le n + r + 1$ as k is also the number of zeros of w' for |z| < 1, which is less than or equal to n + r + 1. Writing $z = Re^{i\theta}$ this area is also:

$$A = \pi \int_0^{2\pi} z f'(z) \overline{f(z)} \, d\theta = \pi [n+r+1+(n+r)|a_1|^2 + \dots + |b_r|^2]$$

But this implies that $a_1 = a_2 = \cdots = b_r = 0$.

[[I don't understand the effs in the last display. Are they a misprint? – R. There's a second solution by Walter B. Carver.]]

Math. Mag., 27(1953) 56–57.

Q 97. Submitted by Murray S. Klamkin

Solve $ax^2 + bx + c = 0$ without completing the square or using the quadratic formula.

A 97. Let x = y + h = y - b/2a. Then we have $ay^2 + y(2ah + b) + ah^2 + bh + c = 0$. Now since h = -b/2a, this equation becomes $ay^2 - b^2/4a + c = 0$, so $y = \pm \sqrt{(b^2 - 4ac)/4a^2}$ and $x = [-b \pm \sqrt{b^2 - 4ac}]/2a$. [This is the method of Vieta, for example, see D. E. Smith, *History of Mathematics*, Vol. II (Ginn) 1925, p.449.]

Math. Mag., 27(1953) 57.

Q 101. Submitted by Murray S. Klamkin

Prove that $2^{1/2} + 3^{1/3}$ is irrational.

A 101. Assume that $2^{1/2} + 3^{1/3} = R$, a rational number. Then $3 = (R - \sqrt{2})^3 = R^3 + 6R - \sqrt{2}(3R^2 + 2)$. It follows that $\sqrt{2}$ is rational. Since it is well-known that this is not true, the assumption is false.

[[There are misprints in each of the next two items. I hope that I've corrected more than I've made. - R.]]

Math. Mag., 27(1953) 106–107.

Q 100. Submitted by Murray S. Klamkin

Express $(a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2)$ as the sum of two squares.

A 100. $(a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3)$

$$= a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + a_2b_1) = i[b_3(a_1a_2 - b_1b_2) + a_3(a_1b_2 + a_2b_1)]$$

Equating the modules [[moduli ?]] of the two sides of the identity we have

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2)$$

= $[a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + a_2b_1)]^2 + [b_3(a_1a_2 - b_1b_2) + a_3(a_1b_2 + a_2b_1)]^2$

Math. Mag., 27(1953) 56-57.

Q 99. Proposed by Murray S. Klamkin Evaluate $\int_0^\infty \log x \, dx/(1+x^2)$. **A 99.** Let x = 1/y, then

$$I = \int_0^\infty \frac{\log x \, dx}{1 + x^2} = \int_0^\infty \frac{\log(1/y)(-1/y^2) \, dy}{1 + 1/y^2} = -\int_0^\infty \frac{\log y \, dy}{1 + y^2} = -I.$$

Thus I = 0.

[[There seem to be two 'Q101's, both submitted by Murray!!]]

Math. Mag., 27(1953) 106–107.

Q 101. Submitted by Murray S. Klamkin

Find integer solutions of $x^2 + y^2 = z^3$

A 101 In the result of A 100 let $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$, whence we have

$$(a^{2} + b^{2})^{3} = (a^{3} - 3ab^{2})^{2} + (3a^{2}b - b^{3})^{2}.$$

Hence a general solution is $x = a^3 - 3ab^2$, $y = 3a^2b - b^3$, $z = a^2 + b^2$, where the parameters a, b are integers.

[But is it THE general solution?? Are there solutions not of this form? - R.]]

Math. Mag., 27(1954) 148.

191. Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn Find the sum

$$\sum_{s=0}^{n} (-1)^{s} \left[\frac{1}{s+1} + \frac{1}{s+2} + \dots + \frac{1}{s+n} \right] \binom{n}{s}$$

Math. Mag., 28(1954) 35.

I. Solution by L. Carlitz, Duke University. Put

$$S_k = 1 + \frac{1}{2} + \dots + \frac{1}{k} \quad (k \ge 1) \qquad S_0 = 0$$

Then

$$S = \sum_{s=0}^{n} (-1)^{s} \left(\frac{1}{s+1} + \frac{1}{s+2} + \dots + \frac{1}{s+n} \right) \binom{n}{s} = \sum_{s=0}^{n} (-1)^{s} (S_{n+s} - S_{s}) \binom{n}{s}$$

Using the notation of finite differences

$$\triangle^n S_k = \sum_{s=0}^n (-1)^{n-s} \binom{n}{s} S_{k+s}$$

But

$$\Delta S_k = \frac{1}{k} \qquad \Delta^2 S_k = \frac{-1}{(k+1)(k+2)} \qquad \cdots \\ \Delta^n S_k = (-1)^{n-1} \frac{(n-1)!}{(k+1)(k+2)\cdots(k+n)}$$

so that

$$\sum_{s=0}^{n} (-1)^{n-s} (S_{k+s} - S_s) \binom{n}{s} = (-1)^{n-1} \frac{(n-1)!}{(k+1)\cdots(k+n)} - \frac{(n-1)!}{n!}$$

In particular for k = n we get

$$S = (-1)^n \sum_{s=0}^n (-1)^{n-s} (S_{n+s} - S_s) \binom{n}{s}$$
$$= (-1)^{-1} \left\{ \frac{(n-1)!}{(n+1)\cdots(2n)} - \frac{(n-1)!}{n!} \right\} = \frac{1}{n} \left\{ 1 - \binom{2n}{n}^{-1} \right\}$$

Math. Mag., 27(1954) 159–160.

T 10. Submitted by M. S. Klamkin.

Can a checker be moved from position (1,1) to position (8,8) moving one square at a time and never diagonally, in such a way that the checker enters each square once and once only?

S 10. Yes. The trick here is to move to (1,2) or (2,1) and then back to (1,1) which is then *entered* for the first time. One possible sequence of moves is (1,1), (1,2), (1,1), (2,1), (2,3), (1,3), (1,4), (2,4), (2,5), (1,5), (1,6), (2,6), (2,7), (1,7), (1,8), (3,8), (3,1), (4,1), (4,8), (5,8), (5,1), (6,1), (6,8), (7,8), (7,1), (8,1), (8,8).

Math. Mag., 27(1954) 218–219.

97. [March 1951] Proposed by Bruce Kellogg, Massachusetts Institute of Technology

Let $\{i_n\} = i_1, i_2, i_3, \ldots$ be a sequence of real numbers such that $\lim_{n\to\infty} i_n = 1$ and $i_n > 1$ for all n. Does the infinite series

$$\sum \frac{1}{n^{i_n}}$$

converge or diverge, or does the divergence or convergence depend upon the sequence $\{i_n\}$?

Solution by R. M. Foster and M. S. Klamkin, Polytechnic Institute of Brooklyn. Let

$$i_n = 1 + \frac{r \log \log n}{\log n}$$

where the logs are natural logarithms, then

$$\sum \frac{1}{n^{i_n}} = \sum \frac{1}{n^{(\log n)^2}}$$

Thus if r (constant) > 1 the series converges, and $r \leq 1$ the series diverges. See the solvers' note on the convergence of *p*-series in the American Mathematical Monthly, Nov. 1953, pp.625–626.

[[The last display is misprinted – the right side shd be a fn of r. Can someone correct it from the following article?

Amer. Math. Monthly, **60**(1953) 625–626.

ON THE CONVERGENCE OF THE p-SERIES

R. M. FOSTER and M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In quite a few textbooks, we find the statement that the *p*-series

$$\sum_{n=1}^{\infty} n^{-p} \tag{1}$$

converges for p > 1, and diverges for $p \le 1$. This is not quite correct. It should rather be stated, that the series converges for p (constant) > 1, and diverges for $p \le 1$. An example illustrating this point is given by the series (1) with p = 1 + (1/n). This series diverges since

$$\lim_{n \to \infty} \frac{n^{1+(1/n)}}{n} = 1$$

and $\sum_{n=1}^{\infty} n^{-1}$ diverges. This example was guiven as a problem in this MONTHLY (Nov. 1948, p.584). One of the solutions gave the following generalization of this problem:

If $\lim_{n\to\infty} \phi(n) = 0$, then $\sum_{n=1}^{\infty} n^{-[1+\phi(n)]}$ diverges. This latter statement, however, is not correct. For, let r be constant and let $n^{\phi(n)} = \psi(n) = (\ln n)^r$. Then

$$\lim_{n \to \infty} \phi(n) = \lim_{n \to \infty} \frac{\ln \psi(n)}{\ln n} = 0$$

but the series

$$\sum_{n=1}^\infty \frac{1}{n\psi(n)} = \sum_{n=1}^\infty \frac{1}{n(\ln n)^r}$$

converges for r (constant)> 1, and diverges for $r \leq 1$. similarly, other functions $\psi(n)$ can be chosen such that the series converges or diverges.]]

Math. Mag., 27(1954) 226–227.

Q 104. Submitted by M. S. Klamkin

Factor $\begin{vmatrix} 1+x & 1 & 1 & 1\\ 1+x & 1+x & 1 & 1+x\\ 1 & 1+x & 1+x & 1\\ 1 & 1 & 1+x & 1+x \end{vmatrix}$ [[This is evidently a misprint for $\begin{vmatrix} 1+x & 1 & 1 & 1+x\\ 1+x & 1+x & 1 & 1\\ 1 & 1+x & 1+x & 1\\ 1 & 1 & 1+x & 1+x \end{vmatrix}$]]

A 104. When x = 0 the four rows are identical, so x^3 is a factor. The sum of each of the rows is 4 + 2x which is therefore a factor. Thus $D = Ax^3(x+2)$, where A is a constant. Next x^4 appears in only two terms, each positive, so A = 1 + 1. Hence $D = 2x^3(x+2)$.

Math. Mag., 27(1954) 227.

Q 106. Submitted by M. S. Klamkin

Find the center of gravity of a semicircular area.

[[Sh'd've said it's of uniform surface density.]]

A 106. Clearly the C.G. falls on the radius which is perpendicular to the diameter at a distance \bar{y} from the diameter. When the area is rotated about the diameter a spherical volume is generated. Hence by Pappus's Theorem we have

$$2\pi\bar{y}(\pi r^2/2) = 4\pi r^3/3$$
 so $\bar{y} = 4r/3\pi$

Math. Mag., 27(1954) 226–228.

Q 108. Submitted by M. S. Klamkin

Find all the integral solutions of $x(x+1)(x+2)(x+3) + 1 = y^2$.

A 108. Since the left hand side of the equation is a perfect square, all integer values of x will satisfy the equation.

Math. Mag., 27(1954) 226–228.

Q 109. Submitted by M. S. Klamkin

If A, B and C are three vectors originating from a common point, prove that $(A \times B) + (B \times C) + (c \times A)$ is a vector perpendicular to the plane determined by the terminal points of A, B and C.

A 109. Since

$$(A \times B) + (B \times C) + (c \times A) = (B - A) \times (C - B)$$

the statement follows immediately.

[[The next item is an example of a "Falsie"]]

Math. Mag., 27(1954) 228.

F 15. Submitted by M. S. Klamkin

Let z be a complex number such that $\tan z = i$. Then

$$\tan(z+w) = \frac{1+\tan w}{i-\tan w} = i$$

Thus the tangent of every complex number is i.

E 15. The fallacy lies in the fact that no z exists such that $\tan z = i$.

Math. Mag., **27**(1954) 287.

Q 110. Submitted by M. S. Klamkin

If a, b and c are sides of a triangle such that $a^2 + b^2 + c^2 = bc + ca + ab$, prove that the triangle is equilateral.

[[From the answer, I guess that this sh'd've been 2(bc + ca + ab). - R.]]

A 110. The equation is equivalent to $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$. Thus a = b = c

Math. Mag., 27(1954) 287.

Q 112. Submitted by M. S. Klamkin

A billiard ball is hit (without any "English") so that it returns to its starting point after hitting four different cushions. Show that the distance travelled by the ball is the same regardless of the starting point.

A 112. Image that the four cushions are mirrors and we are using a light ray. By the principle of images it is easily shown that the distance equals twice the diagonal.

Math. Mag., 27(1954) 287.

Q 114. Submitted by M. S. Klamkin

Sum
$$\sum_{n=1}^{\infty} \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right)^{x^n}$$

A 114. If $F(x) = \sum_{n=1}^{\infty} a_n x^n$ then $F(x)/(1-x) = \sum_{n=1}^{\infty} (a_1 + a_2 + \dots + a_n) x^n$. Thus $F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$, so our sum is $S = \frac{e^x - 1}{1 - x}$.

[[Would someone check this ??]]

[[Next comes a wonderful Pólya original which has to go in!]]

Math. Mag., 28(1955) 27.

209. Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn

Show that
$$F(x,y) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{a^{n+1} + y}$$
 is symmetric in x and y.

Math. Mag., 28(1955) 235–236.

Solution by George Pólya, Stanford University.

(1) Heuristic consideration. Symmetry of F(x, y) means that x and y are interchangeable, yet the proposed expression does not render such interchangeability immediately clear. So, we desire another expression for F(x, y) which does render it immediately clear. What kind of expression? As the proposed expression is a power series in x, it is natural to think of a power series in x and y (the Maclaurin expansion in these two variables). How can one obtain it? Expand in powers of y.

(2) Proof. Expanding in geometric series we obtain:

$$F(x,y) = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{a^{m+1}} \left(1 + \frac{y}{a^{m+1}}\right)^{-1}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{a^{m+1}} \sum_{m=0}^{\infty} \left(-\frac{y}{a^{m+1}}\right)^n$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} x^m y^n}{a^{(m+1)(n+1)}}$$

in which expression x and y are obviously interchangeable.

(3) Critique. If $|a| \ge 1$, the foregoing transformations are easily justified provided that |x| < |a|, |y| < |a|. If, however, 0 < |a| < 1 the proposed series doesnot behave symmetrically in x and y. In fact, its sum is a regular (analytic) function in a certain neighborhood of the point x = -a, y = a, whereas it becomes infinite (has a pole) at the point x = a, y = -a. If a = 0, $F(x, y) = y^{-1}(1+x)^{-1}$ and the asymmetry is quite obvious.

Math. Mag., 28(1955) 37.

Q 118. Submitted by M. S. Klamkin

Find the three smallest integers such that the sum of the reciprocals of its divisors equals 2.

A 118. $\sum 1/d_r = 2$. But $\sum N/d_r = \sum d_r$. Thus $\sum d_r = 2N$ or N is a perfect number and the answer is 6, 28 and 496.

Math. Mag., 28(1955) 37-38.

Q 121. Submitted by M. S. Klamkin

If $|z^n + a_1 z^{n-1} + \dots + a_n + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_r}{z^r}| = 1$ for |z| = 1, what are the restrictions on the coefficients a_i and b_i ?

[[Compare 177 above.]]

A 121. The expression is equivalent to:

$$|b_r z^{r+n} + b_{r-1} z^{r+n-1} + \dots + a_1 z + 1| = 1$$

for |z| = 1. By the Maximum Modulus Theorem all the coefficients must be zero.

Math. Mag., 28(1955) 109–110.

A Fibonacci Expression

197. Proposed by A. S. Gregory, University of Illinois

Let the explicit expression for the *n*th term of a sequence K_n be known. Find an explicit expression for the *n*th term of a sequence $\{\phi_n\}$ which is defined as follows:

$$\phi_n = \phi_{n-1} + \phi_{n-2} + K_n \qquad n = 1, 2, 3, \dots$$

with ϕ_0 and ϕ_1 given.

Solution by Murray S. Klamkin, Polytechnic Institute of Brooklyn. From the definition we have:

$$\phi_2 = \phi_1 + \phi_0 + K_2
\phi_3 = 2\phi_1 + \phi_0 + K_2 + K_3
\phi_4 = 3\phi_1 + 2\phi_0 + 2K_2 + K_3 + K_4
\phi_5 = 4\phi_1 + 3\phi_0 + 3K_2 + 2K_3 + K_4 + K_5$$

Let A_r denote the r th term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$. Explicitly,

$$A_r = \frac{1}{2} \left[(1 + \sqrt{5})^{r-1} + (1 - \sqrt{5})^{r-1} \right]$$

[[This has several errors!! Instead read:

$$A_r = \frac{1}{2\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^r - \left(\frac{1-\sqrt{5}}{2} \right)^r \right]$$

]]

By induction it follows that:

$$\phi_n = A_n \phi_1 + A_{n-1} \phi_0 + A_{n-1} K_2 + A_{n-2} K_3 + \dots + A_1 K_n$$

Math. Mag., 28(1955) 113–114.

Q 123. Submitted by Murray S. Klamkin

If three generators of a right circular cone are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{x-3}{-3} \qquad \frac{x-1}{-3} = \frac{y-2}{3} = \frac{z+3}{3} \qquad \frac{x-1}{11} = \frac{y-2}{8} = \frac{z+3}{9}$$

[[That x - 3 should surely be z + 3?? And I think it should be $\frac{z+3}{1}$ in the second line. Make it:

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{-3} \qquad \frac{x-1}{-3} = \frac{y-2}{3} = \frac{z+3}{1} \qquad \frac{x-1}{11} = \frac{y-2}{8} = \frac{z+3}{9}$$

Then we have (1, 2, -3), (-3, 3, 1), (11, 8, 9) orthogonal?]]

show that the vertical semi-angle of the cone is $\arccos \sqrt{3}/3$.

A 123. Since the three generators are mutually orthogonal they may be replace by the x, y and z axes. By symmetry the axis of the cone can be written x = y = z. Thus $\cos \alpha = \sqrt{3}/3$.

Math. Mag., 28(1955) 113–114.

Q 127. Submitted by Murray S. Klamkin

Show that the volume of the solid $(x+y)^2 + (y+z)^2 + (z+x)^2 = 2$ equals $4\pi\sqrt{2}/3$.

A 127. By rotating the axes the equation can be transformed into $ax'^2 + by'^2 + cz'^2 = 1$ where a, b and c are the roots of the discriminating cubic $\begin{vmatrix} 1 - \lambda & 1/2 & 1/2 \\ 1/2 & 1 - \lambda & 1/2 \\ 1/2 & 1/2 & 1 - \lambda \end{vmatrix} = 0.$

1/2 is obviously a double root, and by adding the three rows the other root is 2. Thus the volume is $4\pi\sqrt{2}/3$.

Math. Mag., 28(1955) 114.

[[at this point Murray has a way of quickening \mathbf{Q} 105.]]

Math. Mag., 28(1955) 135–138.

Article by Murray:

On Barbier's Solution of the Buffon Needle Problem.

Math. Mag., 28(1955) 160.

223. Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn

Prove that there is no integral triangle such that $\cos A \cos B \cos C + \sin A \sin B \sin C = 1$.

Math. Mag., **29**(1955) 46.

Solution by Chi-yi Wang, University of Minnesota. The relation

 $(\cos A - \cos B)^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C)$

implies that $(\cos A - \cos B)^2 = -\cos^2 C \sin^2 A \sin^2 B$. This equation has the trivial solution $A = B = 0^\circ$, $C = 180^\circ$ and the only nontrivial solution $A = B = 45^\circ$, $C = 90^\circ$. Since the two legs and the hypotenuse of an isosceles right triangle are proportional to 1, 1, $\sqrt{2}$ the stated result follows.

Math. Mag., 28(1955) 163–164.

203. Proposed by Norman Anning, Alhambra, California

Prove that three of the intersections of $x^2 - y^2 + ax + by = 0$ and $x^2 + y^2 - a^2 - b^2 = 0$ trisect the circle through these three points.

I. is a solution by W. O. Moser, U of Toronto

II. is a solution by Husseyin Demir, Zonguldak, Turkey

III. Solution by Richard K. Guy, University of Malaya, Singapore. The curves are a rectangular hyperbola and a circle centre O. [[Yes!! It was spelt thus]] The circle through 3 of the points of intersection is therefore the circle $x^2 + y^2 = a^2 + b^2$. By inspection, (-a, b) is common to the two curves. Let P, Q, R be the other 3 points of intersection. Then, by well-known theorems, the orthocentre [[Yes!]] of PQR and the fourth point, (-a, b), of intersection of the rectangular hyperbola with the circle PQR lie at opposite ends of a diameter of the rectangular hyperbola. But the centre of the rectangular hyperbola is $(-\frac{1}{2}a, \frac{1}{2}b)$. Therefore the orthocentre of PQR is O. But this is also the circumcentre. Therefore PQR is equilateral.

IV. Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. Consider

$$z^{2} = (a^{2} + b^{2})^{3/2} e^{i\theta}$$
 where $\cos \theta = \frac{-a}{\sqrt{a^{2} + b^{2}}}$

and

$$\sin \theta = \frac{-b}{\sqrt{a^2 + b^2}}$$

Then

 $x^2 + y^2 = a^2 + b^2$

and

$$z^{2} = (a^{2} + b^{2})^{3/2} \frac{\cos \theta + i \sin \theta}{z}$$

Equating the real parts of this equation leads to $x^2 - y^2 = -ax - ay$. Thus the solution follows immediately.

Math. Mag., 28(1955) 170–172.

Q 128. Submitted by Murray S. Klamkin

Determine the probability that a random rational fraction a/b is irreducible.

A 128. Probability

$$P = \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \dots = \prod \left(1 - \frac{1}{p^2}\right)$$

where the infinite product is extended over all primes.

$$\frac{1}{P} = \frac{1}{\prod\left(1 - \frac{1}{p^2}\right)}$$
$$= \left(1 + \frac{1}{2^2} + \frac{1}{2^4} \cdots\right) \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\right) \cdots$$
$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Thus $P = 6/\pi^2$.

Math. Mag., **28**(1955) 170–171.

Q 132. Submitted by Murray S. Klamkin

If A, B, C are the angles of a triangle show that $\sin^2 A + \sin B \sin C \cos A$ is symmetric in A, B and C.

A 132. Since

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{bc \sin A}{abc} = \frac{2\Delta}{abc}$$

where Δ is the area of the triangle and as

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

the given sum equals

$$\frac{2\Delta^2(a^2 + b^2 + c^2)}{(abc)^2}$$

Math. Mag., **28**(1955) 241–242.

Q 136. Submitted by Murray S. Klamkin

Let [x] denote the greatest integer less than or equal to x, and let (x) denote the integer nearest to x. Express (X) as a function of [x].

A 136. By plotting y = [x] and y = (x) it is readily seen that $(x) = [x - \frac{1}{2}] + 1 = [x + \frac{1}{2}]$.

[[The next item is a chestnut, so, if we're pruning, it shd be one of the first to go. R.]]

Math. Mag., **28**(1955) 241–242.

Q 139. Submitted by Murray S. Klamkin

What is the smallest number of balance weights needed to weight every integral weight from 1 to 121 pounds?

A 139. If the weights can be placed only on one side of the balance, then the weights needed are 1, 2, 4, 8, 16, 32, 64. If they can be placed on both sides then we need only 1, 3, 9, 27, 81.

Math. Mag., 28(1955) 284.

240. Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn

Determine the value of

 $A \times (B \times C) + B \times (C \times A) + C \times (A \times B)$

without expanding any of the vector triple products.

Math. Mag., **29**(1956) 170.

Solution by Samuel Skolnik, Los Angeles City College. If A, B and C are coplanar vectors or if one of them is a null vector, the solution is trivial.

Assume that A, B and C are non-coplanar and let $P = A \times (B \times C) + B \times (C \times A) + C \times (A \times B)$. Then

$$\begin{aligned} A \cdot P &= A \cdot A \times (B \times C) + A \cdot B \times (C \times A) + A \cdot C \times (A \times B) \\ &= 0 + (A \times B) \cdot (C \times A) + (A \times C) \cdot (A \times B) \\ &= (A \times B) \cdot (C \times A) - (C \times A) \cdot (A \times B) = 0 \end{aligned}$$

Similarly $B \cdot P = 0$ and $C \cdot (= 0$. Since P could not be perperpendicular to three non-coplanar vectors A, B and C it follows that P = 0.

[[This next one only if you want an example of several unfamiliar sledgehammers cracking a fairly familiar nut.]]

Math. Mag., 28(1955) 285-287.

A FRACTIONAL SUM

216. [Nov. 1954] Proposed by Erich Michalup, Caracas, Venezuela

Prove that

$$\sum_{n=1}^{\infty} \frac{16n^2 + 12n - 1}{8(4n+3)(4n+1)(2n+1)(n+1)} = \frac{1}{24}$$

I. Solution by Dennis Russell

II. Solution by L. A. Ringenberg

III. Solution by Murray S. Klamkin, Polytechnic Institute of Brooklyn. Let S represent the sum ∞

$$\frac{1}{2}\sum_{1}^{\infty} \left[\frac{3}{4n+2} - \frac{1}{4n+1} - \frac{1}{4n+3} - \frac{1}{4n+4}\right]$$

Now

$$\sum_{1}^{N} \frac{1}{a+nb} = \frac{1}{b} \left[\chi \left(\frac{a}{b} + n + 1 \right) - \chi \left(\frac{a}{b} + 1 \right) \right]$$

where $\chi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Thus

$$S = \frac{1}{8} \left[-3\chi(\frac{3}{2}) + \chi(\frac{5}{4}) + \chi(\frac{7}{4}) + \chi(2) \right]$$

Since $\chi(x+1)-\chi(x) = 1/x$ and $\chi(1) = -\gamma$, $\chi(\frac{1}{2}) = -\gamma - 2\log 2$, $\chi(\frac{1}{4}) = \pi/2 - \gamma - 3\log 2$ it follows that $S = \frac{1}{24}$.

Math. Mag., 28(1955) 288–289.

A PRODUCT OF TWO BINOMIALS

218. [Nov. 1954] Proposed by Ben K. Gold, Los Angeles City, College Prove

$$\sum_{i=1}^{K} (-1)^{i} {\binom{K+i}{K}} {\binom{2K+1}{K-i}} = 1$$

II. Solution by Murray S. Klamkin, Polytechnic Institute of Brooklyn. We can establish the equality

$$\sum_{r=0}^{k} (-1)^r \binom{k+r}{k} \binom{2k+1}{k-r} = \frac{(2k+1)!}{(k!)^2} \sum_{r=0}^{k} \frac{\binom{k}{r}}{k+1+r}$$

A more general expression [[not 'expansion'?]] follows from the expansion of the Beta function (See author's note in *Scripta Math.*, Dec. 1953, p.275). If m and n are nonnegative integers

$$B(m+1, n+1) = \frac{m! \, n!}{(m+n+1)!} = \int_0^1 t^m (1-t)^n \, dt = \int_0^1 \sum_{r=0}^n (-1)^r \binom{n}{r} t^{m+r} \, dt$$

Thus

$$\sum_{r=0}^{n} \frac{(-1)^{r} \binom{n}{r}}{m+r+1} = \sum_{r=0}^{m} \frac{(-1)^{r} \binom{m}{r}}{n+r+1} = \frac{m! \, n!}{(m+n+1)!}$$

The proposed identity follows by setting m = n = k. (This identity was previously obtained by R. Gloden, *Scripta Math.*, 1952, p.178).

The identity can be extended to nonintegers m and n both ≥ 0 . In this case the limits are from 0 to ∞ , and

$$\frac{m!\,n!}{(m+n+1)!} \quad \text{becomes} \quad \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$$

Math. Mag., 28(1955) 292.

Q 142. Submitted by Murray S. Klamkin

Find the class of functions such that $\frac{1}{F(x)} = F(-x)$. One simple example is $F(x) = e^x$. A 142. Let F(x) = E(x) + O(x) where E(x) is even and O(x) is odd. Then

$$\frac{1}{E(x) + O(x)} = E(x) - O(x)$$

Thus $E(x) = \pm \sqrt{1 - O(x)^2}$ and $F(x) = O(x) \pm \sqrt{1 - O(x)^2}$. Let $O(x) = \tan x$, then $F(x) = \tan x \pm \sec x$.

[[but the following is better and more general - R.]]

Math. Mag., **29**(1955) 54.

[Alternate solution by Gaines Lang.] By taking absolute values and then logarithms of each side it is clear that $\ln F(x)$ is an odd function. Hence $F(x) = \pm e^{G(x)}$ where G(x) is an odd function.

Math. Mag., 28(1955) 292–293.

Q 145. Submitted by Murray S. Klamkin

Determine the equation of the cone through the origin passing through the intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

A 145. The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left[\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right]^2$$

is a cone through the origin and obviously passes through the given intersection.

[? Better also to give in the form

$$\frac{yz}{bc} + \frac{zx}{ca} + \frac{xy}{ab} = 0?]]$$

Math. Mag., **28**(1955) 293. **T 16.** Submitted by M. S. Klamkin.

Determine θ such that

$$\frac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta$$

S 16. As this is an identity it is satisfied for all θ for which the denominator is not zero.

Math. Mag., 28(1955) 293.

T 17. Submitted by M. S. Klamkin.

Find the relationship between A and B if

 $A = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots \text{ and } B = 2 - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \frac{10}{9!} - \cdots$

S 17. $A = 1 + \sin 2$ and $B = \sin 1 + \cos 1$. Therefore $A = B^2$

Math. Mag., 28(1955) 293.

T 18. Submitted by M. S. Klamkin.

Evaluate $\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ$.

S 18. The expression is the sum of the projections of a regular pentagon and therefore equals zero.

[[This next is a real chestnut.]]

Math. Mag., **29**(1955) 53.

Q 146. Submitted by Murray S. Klamkin

Sum the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$ A 146.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots = 1$$

Math. Mag., **29**(1955) 53.

Q 148. Submitted by Murray S. Klamkin Express $1/(1+x)(1+x^2)(1+x^4)(1+x^6)$ as a power series. [[That x^6 shd be x^8 — R.]] **A 148**. [[corrected]]

$$\frac{1}{(1+x)(1+x^2)(1+x^4)(1+x^8)} = \frac{(1-x)}{(1-x^{16})}$$
$$= \frac{(1-x)(1+x^{16}+x^{32}+x^{48}+\cdots)}{(1-x^{16}-x^{17}+x^{32}-x^{33}+\cdots)}$$

Math. Mag., **29**(1955) 53–54.

Q 150. Submitted by Murray S. Klamkin Show that $(a + b + c)^3 = 27abc$ if $a^{1/3} + b^{1/3} + c^{1/3} = 0$. **A 150**.

$$a+b+c-3(abc)^{1/3} = (a^{1/3}+b^{1/3}+c^{1/3})(a^{2/3}+b^{2/3}+c^{2/3}-a^{1/3}b^{1/3}-b^{1/3}c^{1/3}-c^{1/3}a^{1/3})$$

Now if $a^{1/3} + b^{1/3} + c^{1/3} = 0$ then we have $a + b + c = 3(abc)^{1/3}$ or $(a + b + c)^3 = 27abc$.

Math. Mag., **29**(1955) 107.

253. Proposed by Murray S. Klamkin and C. H. Pearsall Jr., Polytechnic Institute of Brooklyn

In Ripley's (New) "Believe It Or Not" the following statement appears (p.207). "The persistent number 526,315,789,473,684,210 may be mutiplied by any number. The original digits will always reappear in the result." Show that this statement is not correct.

Math. Mag., **29**(1956) 286.

Solution by M. A.Kirchberg, Milwaukee, Wiskonsin. Observing that twice this number equals one-tenth of it less 1 plus 10^{18} , we see that 19 times the number is $10^{19} - 10$ or 999999999999999990.

Also solved by R. K. Guy, University of Malaya, Singapore; [3 others] and the proposers.

Math. Mag., **29**(1955) 115.

Q 152. Submitted by Murray S. Klamkin

Prove that the average of the square of the velocity is greater than or equal to the square of the average velocity.

A 152. To prove $V^2 \ge (V)^2$ we have that

$$\frac{\int_a^b V^2 \, dp}{\int_a^b dp} \ge \left[\frac{\int_a^b V \, dp}{\int_a^b dp}\right]^2$$

follows from the Cauchy-Schwartz Inequality.

[[The notation needs improving, correcting and explaining — R.]]

Math. Mag., **29**(1955) 115–116.

Q 154. Submitted by Murray S. Klamkin

Show that $V_n/S_n = r/n$ where V_n and S_n are the volume and surface of an *n*-dimensional sphere.

A 154. From similar figures it follows that $V_n = K_1 r^n$ and $S_n = K_2 r^{n-1}$. But by dividing the spheres $dV_n = S_n dr$ so that $K_1 n = K_2$ and $\frac{V_n}{S_n} = \frac{K_1 r}{K_2} = \frac{r}{n}$.

[[Or, divide sphere into cones, whose vol. is (1/n)(base)(height)? — R.]]

Math. Mag., **29**(1955) 115–116.

Q 156. Submitted by Murray S. Klamkin

Solve $4x^3 - 6x^2 + 4x - 1 = 0$.

A 156. The given equation is equivalent to $(x-1)^4 = x^4$ or $(2x-1)[(x-1)^2 + x^2] = 0$ which has roots x = 1/2, $(1 \pm i)/2$.

Math. Mag., 29(1955) 115–116.

Q 158. Submitted by Murray S. Klamkin

Find the area of the ellipse $4x^2 + 2\sqrt{3}xy + 2y^2 = 5$.

A 158. Rotate the ellipse into the form $Ax^2 + Cy^2 = 5$. The A + C = 6 and $-4AC = (2\sqrt{3})^2 - 4(4)(2)$. That is A = 5 and C = 1 or A = 1 and C = 5. Thus the area is $\pi\sqrt{5}$.

[[¿¿ Better is to ask that

$$4x^2 + 2\sqrt{3}xy + 2y^2 = 5(x^2 + y^2)/r^2$$

represent a pair of coincident straight lines: $3 = (4 - 5/r^2)(2 - 5/r^2)$ and the product of the squares of the roots is 5 ?? — R.]]

Math. Mag., **29**(1956) 163.

261. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Determine the entire class of analytic functions F(x) so that Simpson's Quadrature Formula

$$\int_{-h}^{h} F(x) \, dx = \frac{h}{3} [F(-h) + 4F(0) + F(h)]$$

holds exactly.

Math. Mag., **30**(1956) 51.

Solution by Harry D. Ruderman, Bronx, New York. Let F(x) = E(x) + O(x), the sum of an even and odd function; that is

$$E(x) = \frac{F(x) + F(-x)}{2}$$
 and $O(x) = \frac{F(x) - F(-x)}{2}$

Assume that each is integrable. The Quadrature Formula is satisfied for any odd function that is integrable. The result is equal to 0. After replacing F(x) by E(x)+O(x) and using the property E(-x) = E(x), Simpson's formula becomes

$$\sum_{0}^{x} E(x) \, dx = \frac{x}{3} [E(x) + 2E(0)] \qquad 0 \le x \le h \tag{1}$$

This relation implies that E(X) has a derivative in this interval. Differentiate both members of (1).

$$E(x) = \frac{xE'(x)}{3} + \frac{E(x) + 2E(0)}{3}$$

This is a simple differential equation with the solution

$$E(x) = A + Bx^2$$
 with $A = E(0)$

If F(x) has two integrable components and satisfies Simpson's Formula in the interval $0 \le x \le h$, then $F(x) = A + Bx^2 + O(x)$. Thus F(x) is the sum of a quadratic and an odd function.

Math. Mag., **29**(1956) 173–174.

Q 160. Submitted by Murray S. Klamkin

Find the sum of 1+1+2+3+5+8+13+21+34+55+89+144.

A 160.

Math. Mag., 29(1956) 174.

Q 162. Submitted by M. S. Klamkin

(From the 1953 Putnam Competition) Six points are in general position in space, no three in line and no four in a plane. The fifteen line segments joining then in pairs are drawn and then painted, some segments red and some blue. Prove that some triangle has all its sides the same color.

A 162. There are five segments emanating from any point. Three of these must be of the same color, say red. No matter how we connect the ends of these three segments we get at least one triangle of the same color.

[[Fifty years on, some of these quickies seem awfully hackneyed, but perhaps they're new to a newer generation. — R.]]

Math. Mag., 29(1956) 174.

Q 163. Submitted by M. S. Klamkin

Prove that $1 + 1/2 + 1/3 + \cdots + 1/n$ is never an integer for n > 1.

A 163. Multiply the sum by one half the least common multiple. The there will be exactly one term equal to 1/2 and the remaining terms will be integers since there can be only one term which contains the highest power of 2 in the sequence $1, 2, 3, \ldots, n$.

Math. Mag., **29**(1956) 174.

Q 164. Submitted by M. S. Klamkin

If $\frac{F(m)+F(n)}{2} \ge F\left(\frac{m+n}{2}\right)$ is true for all real m and n, prove that $\frac{F^{-1}(m)+F^{-1}(n)}{2} \le F^{-1}\left(\frac{m+n}{2}\right)$ in a domain where the inverse function $F^{-1}(x)$ exists.

A 164. Geometrically this is equivalent to proving that if a function is convex its inverse is never convex. Plot the curves y = F(x) and $y = F^{-1}(x)$. These curves are mirror images in the line y = x. Thus the proof follows by symmetry.

Math. Mag., **29**(1956) 222.

269. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn Find the sum $\sum_{n=1}^{\infty} \left[\frac{n}{1!} + \frac{n-1}{2!} + \frac{n-2}{3!} + \dots + \frac{1}{n!} \right] x^n$ Math. Mag., **30**(1956) 108.

Solution by Maimouna Edy, Hull, P.Q., Canada.

$$\sum_{n=1}^{\infty} \left[\frac{n}{1!} = \frac{n-1}{2!} + \frac{n-2}{3!} + \dots + \frac{1}{n!} \right] x^n$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\right) \cdot \left(\sum_{n=0}^{\infty} nx^n\right) = \left(\frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}\right) \cdot \left(x\sum_{n=0}^{\infty} nx^{n-1}\right) \\ = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}\right) \left(\sum_{n=0}^{\infty} nx^{n-1}\right) = e^{x-1}(1-x)^{-2}$$

This result is valid for |x| < 1

Math. Mag., **29**(1956) 230.

Q 167. Submitted by M. S. Klamkin

Prove that $1 + 1/3 + 1/5 + \cdots + 1/(2n-1)$ for n > 1 can never be an integer.

A 167. Assume the sum S is integral and let p be the greatest prime less than or equal to 2n - 1. There will not be any other multiples of p less than 2n - 1 since between a and 2a there lies a prime. Then

$$1 + 1/3 + 1/5 + \dots + 1/(2n-1) - 1/p = S - 1/p = \frac{pS - 1}{p}$$

But the right side has a denominator p while the left side does not. This is a contradiction which proves the original statement. Math. Mag., **29**(1956) 231. **T 22**. Submitted by M. S. Klamkin Solve the simultaneous system

$$\cos A \cos B + \sin A \sin B \sin C = 1$$

$$\sin A + \sin B = 1$$

$$A + B + C = 180^{\circ}$$

S 22. From the first and third equations it follows that $cos(A - B) \ge 1$. Thus A = B and then $C = 90^{\circ}$. But this does not satisfy the second equation. Therefore the equations are inconsistent.

Math. Mag., **29**(1956) 290.

Q 176. Submitted by M. S. Klamkin

Sum
$$\sum_{n=0}^{\infty} \frac{n^4 - 6n^3 + 11n^2 - 6n + 1}{n!}$$

A 176.

$$\sum_{n=0}^{\infty} \frac{n^4 - 6n^3 + 11n^2 - 6n + 1}{n!} = \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3) + 1}{n!} = 2\sum_{n=0}^{\infty} \frac{1}{n!} = 2e$$

Math. Mag., **30**(1956) 47.

284. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Determine the envelope of convex polygons of n sides inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and having a maximum area.

[[This should be a quickie. Orthogonal projection changes area in a constant ratio; the max area of a cyclic *n*-gon is given by the regular *n*-gon, which envelops a circle which projects back to a homothetic ellipse of size $\cos(\pi/n)$ times the original.]]

Math. Mag., **30**(1957) 230.

[[Vindicated! This is Howard Eves's solution, but a second, longer, solution by Chih-yi Wang, was also published.]]

Math. Mag., **30**(1956) 53–54.

Q 177. Submitted by M. S. Klamkin

Prove

$$|\Sigma_1 + \sqrt{\Sigma_1^2 - \Sigma_2^2}| + |\Sigma_1 - \sqrt{\Sigma_1^2 - \Sigma_2^2}| = |\Sigma_1 + \Sigma_2| + |\Sigma_1 - \Sigma_2|$$

A 177. Let $\Sigma_1 + \Sigma_2 = w_1^2$ and $\Sigma_1 - \Sigma_2 = w_2^2$. Then $\Sigma_1^2 - \Sigma_2^2 = w_1^2 \cdot w_2^2$ or $|w_1 + w_2|^2 + |w_1 - w_2|^2 = 2|w_1|^2 + 2|w_2|^2$ This is equivalent to the theorem that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

Math. Mag., **30**(1956) 53–54.

Q 179. Submitted by M. S. Klamkin

Is i^i real or complex?

A 179. Here $i^i = e^{i \log i} = e^{i[i(\pi/2 + 2k\pi)]} = e^{-(\pi/2 + 2k\pi)}$, but this is real.

Math. Mag., **30**(1956) 53–54.

Q 180. Submitted by M. S. Klamkin

Solve $x^4 + 4x - 1 = 0$

A 180. $x^4 + 4x - 1 = 0$ is equivalent to $x^4 + 2x^2 + 1 - 2(x^2 - 2x + 1) = 0$. [[misprint corrected]] Thus $x^2 + 1 = \pm \sqrt{2}(x-1)$ and $x^2 \mp x\sqrt{2} + 1 \pm \sqrt{2} = 0$ so $x = \frac{\pm \sqrt{2} \pm \sqrt{2} - 4(1 \pm \sqrt{2})}{2}$

[[The following's a rather slow quickie!]]

Math. Mag., **30**(1956) 53–54.

Q 182. Submitted by M. S. Klamkin from Goursat-Hedrick, MATHEMATICAL ANALYSIS, Vol.1, p.32.

Determine a polynomial F(x) of seventh degree such that F(x) + 1 is divisible by $(x-1)^4$ and F(x) - 1 is divisible by $(x+1)^4$.

A 182. Let $F(x) + 1 = (x - 1)^4 P_1$ and $F(x) - 1 = (x + 1)^4 P_2$. F(x) not divisible by (x - 1)(x + 1). Multiplying we have $F(x)^2 - 1 = (x^2 - 1)^4 P_1 P_2$. Differentiating gives $2FF' = 8x(x^2 - 1)^3 P_1 P_2 + (x^2 - 1)^4 (P_1 P_2)'$. Thus F' is divisible by $(x^2 - 1)^3$ and since it is of sixth degree, $F' = (x^2 - 1)^3$. So $F(x) = k \left[\frac{x^7}{7} - \frac{3x^5}{5} + x^3 - x \right] + c$. The constants are determined from F(1) = -1, F(-1) = 1 so that $F(x) = \frac{1}{16} [5x^7 - 21x^5 + 35x^3 - 35x]$.

Math. Mag., **30**(1956) 103.

291. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Determine the minimum of

$$\frac{\sum_{r=1}^{s} x_r^s}{\prod_{r=1}^{s} x_r} \quad \text{where} \quad x_r > 0$$

Math. Mag., **30**(1957) 290.

Solution by Chih-yi-Yang, University of Minnesota. Since the arithmetic mean is never less than the geometric mean of any positive terms we have

$$s\left\{\sum_{r=1}^{s} \frac{x_r^s}{s}\right\} \ge s\prod_{r=1}^{s} x_r$$

whence the required minimum value is s, which is attained if all x_r are equal.

[[In connexion with the next item, it's interesting to compare SIAM Rev., 1(1959) 68–70, the issue where Murray initiated the famous Problems Section with:

Problem 59-1^{*}, *The Ballot Problem*, by Mary Johnson (American Institute of Physics) and M. S. Klamkin.

A society is preparing 1560 ballots for an election for three offices for which there are 3, 4 and 5 candidates, respectively. In order to eliminate the effect of the ordering of the candidates on the ballot, there is a rule that each candidate must occur an equal number of times in each position as any other candidate for the same office. what is the least number of different ballots necessary?]]

Math. Mag., **30**(1956) 110.

T 24. Submitted by M. S. Klamkin

A certain physical society is planning a ballot for the election of three officers. There being 3, 4 and 5 candidates for the three offices. There is a rule in effect (in order to eliminate the ordering of the candidates on the ballot as a possible influence on the election) that for each office, each candidate must appear in each position the same number of times as any other candidate. What is the smallest number of different ballots necessary?

S 24. [[misnumbered **S 25** in the original.]] Offhand one would say $3 \cdot 4 \cdot 5 = 60$. However, if one adds two fictitious names to the group of three and one fictitious name to the group of four, then only five different ballots are necessary. Not only will this method reduce the printing costs, but it will also give statistics on whether or not members vote by relative order [or not]. [[last two words redundant]] Math. Mag., **30**(1956) 111–112.

Q 184. Submitted by M. S. Klamkin

Simplify

$$I = \frac{\sin x + \sin 2x + \sin 3x + \dots + \sin nx}{\cos x + \cos 2x + \cos 3x + \dots + \cos nx}$$

A 184.

$$\sum_{1}^{n} \sin nx = \frac{\sin \frac{n-1}{2}x \sin \frac{n}{2}x}{\sin \frac{x}{2}}$$

and

$$\sum_{1}^{n} \cos nx = \frac{\cos \frac{n-1}{2}x \sin \frac{n}{2}x}{\sin \frac{x}{2}}$$

Therefore $I = \tan \frac{n-1}{2}x$

Math. Mag., **30**(1956) 111–112.

Q 185. Submitted by M. S. Klamkin

Prove that

$$\sum_{r=0}^{k} (-1)^r \binom{n+r}{s} \binom{k}{r} \quad \text{where} \quad s \le k-1$$

A 185.

$$x^{n}(1-x)^{k} = \sum_{r=0}^{k} (-1)^{r} \binom{k}{r} x^{n+r}$$

Since $D^{s}(x^{n})(1-x)^{k}|_{x=1} = 0$ the result follows.

Math. Mag., **30**(1956) 172.

Q 187. Submitted by M. S. Klamkin

Prove
$$\sqrt{\frac{ab+bc+ca}{3}} \ge \sqrt[3]{abc}$$
 where $a, b, c \ge 0$

A 187. Let ab + bc + ca = s then the maximum value of (ab)(bc)(ca) occurs when ab = bc = ca or a = b = c. Thus $\sqrt{\frac{ab+bc+ca}{3}} \ge \sqrt[3]{abc}$ for all $a, b, c \ge 0$.

Math. Mag., **30**(1956) 172–173. **Q 189**. Submitted by M. S. Klamkin

Evaluate
$$\sum_{r+1}^{s+1} \sum_{s+1}^{n} \sin[(2r-2s-1)\theta]$$

A 189. Since

$$(2r - 2s - 1)\theta + [2(n + 2 - r) - 2(n - 1 + s) - 1]\theta = 0$$

it follows that the given sum is zero.

Math. Mag., **30**(1956) 172–173.

Q 190. Submitted by M. S. Klamkin

Solve $y = x - \frac{x^2}{2}$ $x = \frac{1+y^2}{2}$ **A 190.** $y = \frac{1}{2} + \frac{y^2}{2} - \frac{x^2}{2}$ so $(y-1)^2 = x^2$ or $x = \pm(y-1)$. Thus $1 + y^2 = 2(1-y)$ or $1 + y^2 = 2(y-1)$ so we have $y = -1 \pm \sqrt{2}$ and $x = 2 \pm \sqrt{2}$ or $y = 1 \pm i\sqrt{2}$ and $x = i\sqrt{2}$

[[I've altered some ors and ands: the original didn't seem to have them right – R.]]

Math. Mag., **30**(1956) 173.

T 27. Submitted by M. S. Klamkin

A person travelling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north-east. What was the wind velocity?

S 27. The vector triangle is as follows, so $w = 3\sqrt{2}$ m.p.h. from the north-west.

$$3$$
 $_{45^\circ}$

w

3

Math. Mag., **30**(1957) 223.

305. Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn

Find
$$\lim_{z \to 1^+} (z-1) \sum_{n=0}^{\infty} \frac{2^n}{1+z^{2^n}}$$

Solution by L. Carlitz, Duke University, N.C. Logarithmic differentiation of the familiar identity

$$\prod_{j=1}^{\infty} (1+x^{2^n}) = \frac{1}{1-x} \qquad |x| < 1$$

gives

$$\sum_{0}^{\infty} \frac{2^n x^{2^n}}{1 - x^{2^n}} = \frac{x}{1 - x}$$

For $x = \frac{1}{z}$ this becomes

$$\sum_{0}^{\infty} \frac{2^n}{1 - z^{2^n}} = \frac{1}{z - 1} \qquad |z| > 1$$

and therefore

$$\lim_{z \to 1^+} (z-1) \sum_{0}^{\infty} \frac{2^n}{1-z^{2^n}} = 1.$$

Math. Mag., **30**(1956) 231–232.

Q 192. Submitted by M. S. Klamkin

Find the equation of the plane through the point (x_0, y_0, z_0) which cuts the least volume from the first octant.

S 192. Let the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. The *abc* is to be minimized subject to the restraint $\frac{x_0}{a} + \frac{y_0}{b} + \frac{z_0}{c} = 1$ or $\left(\frac{x_0}{a}\right) \left(\frac{y_0}{b}\right) \left(\frac{z_0}{c}\right)$ is to be maximized. Since the sum of the factors is 1, then $\left(\frac{x_0}{a}\right) = \left(\frac{y_0}{b}\right) = \left(\frac{z_0}{c}\right) = \frac{1}{3}$. Thus the plane is $\frac{x}{3x_0} + \frac{y}{3y_0} + \frac{z}{3z_0} = 1$.

Math. Mag., **30**(1956) 231–232.

Q 195. Submitted by M. S. Klamkin

If
$$\sum_{n=1}^{\infty} (a_n)^2$$
 converges, prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

S 195. $a_n^2 + \frac{1}{n^2} \ge \frac{2a_n}{n}$. But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Therefore $\sum_{n=1}^{\infty} \frac{a_n}{n}$ does also.

Math. Mag., **30**(1957) 290–291.

Q 197. Submitted by M. S. Klamkin

Find f(x, y) such that the family f(x, y) = c is orthogonal to the family f(x, y) = kA 197. Such a family can be obtained by solving any differential equation of the form

$$\frac{dy}{dx} = [G(x,y) + G(y,x)][H(x) - H(y)]$$

Math. Mag., **30**(1957) 290–291.

Q 201. Submitted by M. S. Klamkin

Determine a one-parameter solution of

$$\left(\frac{y''}{1+(y')^2}\right)^{\frac{1}{2}} = \sinh(y-x) - 1$$

A 201. By inspection a first integral is $y' = \sinh(y - x)$. Let z = y - x, then $z' + 1 = \sinh z$ which is integrable into a one-parameter solution.

Math. Mag., **30**(1957) 291.

Q 203. Submitted by M. S. Klamkin

Solve

$$\frac{dy}{dx} = x^2 + \sqrt{x^4 - 2xy}$$

A 203. To solve, let $y = x^3 z$, then $x^3 \frac{dz}{dx} + 3x^2 z = x^2(1 + \sqrt{1 - 2x})$. The variables are now separable.

Math. Mag., **31**(1957) 57.

Q 204. Submitted by M. S. Klamkin

If A, B, C and D are vectors and [ABC] is a scalar triple product, prove that

$$[BCD]A - [CDA]B + [DAB]C - [ABC]D = 0$$

A 204. Let $I = \begin{vmatrix} A & B & C & D \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$ Then $I \cdot i = I \cdot j = I \cdot k = 0$ so I = 0. Expanding by minors, we get the desired result.

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Math. Mag., **31**(1957) 57–58.

Q 206. Submitted by M. S. Klamkin

If x, y and z are positive and if x + y + z = 1 prove that $(1/x - 1)(1/y - 1)(1/z - 1) \ge 8$

A 206. The sum of the three factors on the left hand side of the inequality equals 1. Thus their minimum occurs when x = y = z = 1/3. Hence the inequality follows.

Math. Mag., **31**(1957) 57–58.

Q 208. Submitted by M. S. Klamkin

If $f(x) \equiv f(x+1) \equiv f(x+\sqrt{2})$ and $f(0) = \sqrt{2}$, find f(x).

A 208. Since f(x) has two independent periods, it must be the constant $f(x) = \sqrt{2}$.

Math. Mag., **31**(1957) 57.

Q 210. Submitted by M. S. Klamkin

If f(x) can be integrated in finite form, show that the inverse function $f^{-1}(x)$ can also be integrated in finite form.

A 210. $I = \int f^{-1}(x) dx$. Let $y = f^{-1}(x)$, then x = f(y) and dx = f'(y) dy. Then $I = \int y f'(y) dy = y f(y) - \int f(y) dy$

Math. Mag., **31**(1957) 119.

Q 211. Submitted by M. S. Klamkin

Show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

A 211. $3^{2n+1} + 2^{n+2} = 3(7+2)^n + 2^{n+2} = 7k + 3 \cdot 2^n + 4 \cdot 2^n = 7(k+2^n).$

Math. Mag., **31**(1957) 119.

Q 213. Submitted by M. S. Klamkin

Prove that every skew symmetric determinant of odd order has a value zero.

A 213. Interchanging rows and columns does not change the value, but is equivalent to changing the sign of every element, so

$$\triangle = (-1)^n \triangle = -\triangle$$
. Hence $\triangle = 0$
Math. Mag., **31**(1957) 119.

Q 215. Submitted by M. S. Klamkin

Determine the minimum value of the sum of the squares of the perpendiculars from a point in the plane of a triangle to its three sides.

A 215. Let x, y, z be the perpendiculars and a, b, c be the sides. Then $x^2 + y^2 + z^2$ is to be minimized subject to ax + by + cz = 2A where A is the area of the triangle. This is equivalent to finding the shortest distance from the origin to the plane ax + by + cz - 2A = 0. Thus $r = 2A/\sqrt{a^2 + b^2 + c^2}$ and the minimum $x^2 + y^2 + z^2$ is $4A^2/(a^2 + b^2 + c^2)$.

Math. Mag., **31**(1958) 177.

Q 216. Submitted by M. S. Klamkin

Find the sum of $1(1!) + 2(2!) + 3(3!) + \dots + n(n!)$

A 216. Since n(n!) = (n+1-1)(n!) = (n+1)! - n! we have

$$\sum_{1}^{n} n(n!) = (n+1)! - 1$$

Math. Mag., **31**(1958) 177.

Q 218. Submitted by M. S. Klamkin

Find an expression true for all n for the n th derivative of $\sin ax$.

A 218. $D^n \sin ax = a^n \sin(ax + \frac{n\pi}{2})$

Math. Mag., **31**(1958) 229.

338. Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts

The vector field $\frac{R}{r^3}$ satisfies the equations $\nabla \times \frac{R}{r^3} = 0$ and $\nabla \cdot \frac{R}{r^3} = 0$. Consequently, this field has a scalar potential and a vector potential. The scalar potential is well known to be $\frac{1}{r}$. Determine the vector potential.

Math. Mag., **32**(1958) 110.

Solution by D. A. Breault, Sylvania Electric Products Inc. Under the given conditions, we are asked to determine G = G(R), such that,

$$f(R) = R/r^3 = \operatorname{curl} G(R)$$

Since f(R) is solenoidal, G(R) is given by

$$G(R) = -R \times \int_0^1 f(tR)t \, dt$$

whenever the integral exists. In this case it does exist, and is in fact equal to $1/3 R \times R/r^3$, whence

$$G(R) = 1/3 R \times R/r^3 + \nabla \varnothing$$

where \emptyset is an arbitrary scalar function whose second partials exist (see Brand, AD-VANCED CALCULUS, pp.391ff).

Math. Mag., **31**(1958) 237.

Q 223. Submitted by M. S. Klamkin

Evaluate

$$I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx$$

A 223. Let $x = \pi - y$, then

$$I = -\int_0^\pi \frac{(\pi - y)\sin y}{1 + \cos^2 y} \, dy = -\frac{\pi}{2} \int_0^\pi \frac{d(\cos y)}{1 + \cos^2 y} = \frac{\pi^2}{4}$$

Math. Mag., **31**(1958) 294.

Q 224. Submitted by M. S. Klamkin

Find the sum to N terms of 1, 12, 123, 1234, ..., a_n , $a_n + 1111 \cdots 11$

A 224. The successive differences are

Thus

$$S = \sum_{r=1}^{N} \sum_{s=1}^{r} \frac{10^{s} - 1}{9} = \sum_{r=1}^{N} \frac{10(10^{r} - 1)}{9^{2}} - \frac{r}{9} = \frac{10^{2}(10^{N} - 1)}{9^{3}} - \frac{10N}{9^{2}} - \frac{N(N+1)}{18}$$

Math. Mag., **32**(1958) 33–34.

Note by Murray:

A note on an n th order linear differential equation.

Math. Mag., **32**(1958) 49.

A Composite of Conics

327 [January 1958]. Proposed by Chih-yi Wang, University of Minnesota Show that the curve

$$x^{6} + y^{6} - 18(x^{4} + y^{4}) + 81(x^{2} + y^{2}) - 108 = 0$$

consists of two ellipses and a circle.

I. Solution by M. S. Klamkin, AVCO, Lawrence, Massachusetts. The equation can be written in the form

$$(x^{2} + y^{2})^{3} - 18(x^{2} + y^{2})^{2} + 81(x^{2} + y^{2}) - 108 + 3x^{2}y^{2}(12 - x^{2} - y^{2}) = 0$$

[[misprint corrected]] or

$$(x^{2} + y^{2} - 12)\{(x^{2} + y^{2} - 3)^{2} - 3x^{2}y^{2}\} = 0$$

or

$$(x^{2} + y^{2} - 12)(x^{2} - xy\sqrt{3} + y^{2} - 3)(x^{2} + xy + y^{2} - 3) = 0$$

Whence, the curve consists of 2 ellipses and 1 circle.

Math. Mag., **32**(1958) 55–56.

Q 226. Submitted by M. S. Klamkin

Maximize $a \cos \alpha + b \cos \beta + c \cos \gamma$ where $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

A 226. If we let A = ai + bj + ck and $X = i \cos \alpha + j \cos \beta + k \cos \gamma$ we have to maximize $A \cdot X$ where |X|=1. Consequently we have $\max A \cdot X = |A| = \sqrt{a^2 + b^2 + c^2}$

Math. Mag., **32**(1958) 56.

Q 229. Submitted by M. S. Klamkin

Prove that $x^3 + y^3 + z^3 - 3xyz = a^3$ is a surface of revolution.

A 229. The general equation of a surface of revolution is $(x-a)^2 + (y-b)^2 + (z-c)^2 = F(rx + sy + tz)$ where the axis is $\frac{x-a}{r} = \frac{y-b}{s} = \frac{z-c}{t}$. Since the given equation may be written as

$$x^{2} + y^{2} + z^{2} = \frac{2a^{3}}{3(x+y+z)} + \frac{(x+y+z)^{2}}{3}$$

it follows that it is a surface of revolution about x = y = z.

Math. Mag., **32**(1958) 105.

356. Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts

Determine the shortest distance on the right circular cylinder r = R, z = 0, z = Hbetween the two points $P_1(r_1, \theta_1, 0)$; $P_2(r_1, \theta_2, H)$ and also between the two points $P_3(R, \theta_3, z_3)$ and $P_4(R, \theta_4, z_4)$.

[[Either no solution was published, or I missed it. - R.]]

Math. Mag., **32**(1958) 112–113.

Q 231. Submitted by M. S. Klamkin

Prove that N! cannot be a perfect square.

A 231. The proof follows from the fact that there is always a prime between r and 2r for all r > 1.

Math. Mag., **32**(1958) 171–172.

Q 236. Submitted by M. S. Klamkin

Find the sum of the series

$$S = 1 + x + 2x^{2} + 3x^{3} + 5x^{4} + \dots + ax^{n} + bx^{n+1} + (a+b)x^{n+2} + \dots$$

for |x| < 1.

A 236.

$$S = 1 + x + 2x^{2} + 3x^{3} + 5x^{4} + \dots + ax^{n} + bx^{n+1} + (a+b)x^{n+2} + \dots$$

$$xS = x + x^{2} + 2x^{3} + \dots$$

$$x^{2}S = x^{2} + x^{3} + \dots$$

so $S(1 - x - x^2) = 1$ or $S = 1/(1 - x - x^2)$.

Math. Mag., **32**(1958) 171. **Q 240**. Submitted by M. S. Klamkin

Show that

$$mz_1 + (1 - m)z_2 \le \max[|z_1|, |z_2|]$$

where $0 \le m \le 1$.

 $\mathbf{A}~\mathbf{240}.$ Proof:

A $m(z_1 - z_2)$ $z_2 \qquad B$ C $O \qquad z_1$

Geometrically it follows that $\overline{OB} \leq \max[\overline{OA}, \overline{OC}]$

Math. Mag., **32**(1959) 220.

369. Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts

An explorer travels on the surface of the earth, assume to be a perfect sphere, in the manner to be described. First he travels 100 miles due north. He then travels 100 miles due east. Next he travels 100 miles due south. Finally, he travels 100 miles due west, ending at the point from which he started. Determine all the possible points from which he could have started.

Math. Mag., **33**(1959) 110.

Solution by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts. The problem here is to choose the starting point so that the two East-West legs of the tour, even though they differ by 100 miles of latitude, span the same number of longitudinal units. This can easily be done if the starting point is anywhere on the circle of South latitude which is exactly 50 miles below the equator.

Math. Mag., **33**(1960) 226–228.

Editor's note: Since the statement of the problem does not exclude the possibility of the explorer retracing a portion of his path, a large family of solutions exists in addition to the one published in November. A number of such solutions have been received since that date.

Solution by Benjamin L. Schwartz, Technical Operations Inc., Honolulu, Hawaii. Let R denote the radius of the earth, t the length of each segment of the trip. Introduce a spherical coordinate system with origin at the earth's center, and θ and ϕ the longitude and colatitude, respectively. If the explorer starts at $P_0 = (R, \theta_0, \phi_0)$, then by elementary analytic geometry, the succeeding corners of his tour are:

$$P_{1} = (R, \theta_{0}, \phi_{0} - \frac{t}{R})$$

$$P_{2} = (R, \theta_{0} + \frac{t}{R}\sin(\phi_{0} - \frac{t}{R}), \phi_{0} - \frac{t}{R})$$

$$P_{3} = (R, \theta_{0} + \frac{t}{R}\sin(\phi_{0} - \frac{t}{R}), \phi_{0})$$

$$P_{4} = (R, \theta_{0} + \frac{t}{R}\sin(\phi_{0} - \frac{t}{R}) - \frac{t}{R}\sin\phi_{0}, \phi_{0})$$

and for P_4 to coincide with P_0 we require

$$\theta_4 = \theta_0 + \frac{t}{R}\sin(\phi_0 - \frac{t}{R}) - \frac{t}{R}\sin\phi_0$$

to be coterminal with θ_0 (not necessarily equal to θ_0 , as the other solvers have apparently assumed).

We have then

$$\frac{t}{R} \left[\frac{1}{\sin(\phi_0 - t/R)} - \frac{1}{\sin\phi_0} \right] = 2k\pi \tag{1}$$

for any integer k. For k - 0 we get the published solution.

Other solutions exist, however, for other values of k. Rewrite (1) in the form

$$\frac{\sin \phi_0 - \sin(\phi_0 - t/R)}{\sin \phi_0 \sin(\phi_0 - t/R)} = \frac{2k\pi R}{t}$$
(2)

In general, for any fixed integer value of k, this transcendental equation in ϕ_0 has a family of solutions, only a finite number satisfying $0 \le \phi_0 \le \pi$, which is implied since ϕ is the colatitude. These supplement the previously published partial solution to provide the general solution.

The solutions of (2) are not readily computed in closed form in general. To solve the equation numerically in any particular case, we can use some simple approximations. Since $R \gg t$, the right hand side is relatively large when $k \neq 0$, and the numerator of the left hand side is small. Hence, solutions exist only in the neighborhood of $\phi_0 = 0$ or $\phi_0 = \pi$, where the factors of the left denominator are small. Using first a neighborhood of $\phi_0 = 0$ (the North Pole) we can replace $\sin \phi$ approximately with ϕ , and the equation becomes

$$\frac{t/R}{\phi_0(\phi_0 - t/R)} = \frac{2k\pi R}{t}$$
(3)

which can easily be olved as a quadratic in ϕ_0 when numerical values are given to t, Rand k. For example, using R = 4000, t = 100, k = 2 we get

$$\phi_0^2 - 0.0250\phi_0 - 0.000049739 = 0 \tag{4}$$

which yields

$$\phi_0 = 0.026852$$

This is a circle of latitude 107.41 miles south of the North Pole. The explorer who starts here will proceed north to a point 7.41 miles from the pole; he will then turn eastward and encircle the pole twice, and go on an additional 6.883 miles. A southward journey will return him to his original latitude, exactly 100 miles east of his starting point, and his final westward leg will close the polygon. A similar analysis can be applied to determine the solutions with different numbers of windings around the pole, i.e., different k, as well as those in the neighborhood of the South Pole.

Math. Mag., **32**(1958) 171.

Q 246. Submitted by M. S. Klamkin

Determine the class of angles which can be trisected with a straight edge and compasses.

A 246. Since $\cos \theta = 4 \cos^3 \theta/3 - 3 \cos \theta/3$ it follows that angles of the form $\theta = \arccos(4x^3 - 3x)$ where x is constructible and $\cos \theta/3 = x$.

Math. Mag., **32**(1958) 285–286.

Q 247. Submitted by M. S. Klamkin

For what values of x is $m^2 + n^2 - a^2 - b^2 > (mn - ab)x$ where $0 \le a \le m$ and $0 \le b \le n$? A 247. Let $x = 2\cos\theta$. Then we have $m^2 + n^2 - 2mn\cos\theta > a^2 + b^2 - 2ab\cos\theta$.

$$egin{array}{c} A \ heta \end{array}$$

D

В

E

C

In the figure, AB = m, AC = n, AD = a and AE = b. So $\overline{BC}^2 = m^2 + n^2 - 2mn \cos \theta$ and $\overline{DE}^2 = a^2 + b^2 - 2ab\cos\theta$. In order that DE be less than BC we must have $\theta > 60^\circ$ or x < 1.

Math. Mag., **32**(1958) 286–287.

Q 252. Submitted by M. S. Klamkin

Find the sum of

$$= \binom{m}{r}\binom{n}{0} + \binom{m}{r-1} + \dots + \binom{m}{0}\binom{n}{r}$$

A 252. On equating coefficients of x^r on both sides of the identity

$$(1+x)^m (1+x)^n = (1+x)^{m+n}$$

we have the sum $S = \binom{m+n}{r}$.

Math. Mag., **32**(1958) 57–58.

Q 254. Submitted by M. S. Klamkin

Prove that $\int_{a}^{b} f(x) dx = \int_{a}^{b} f(a+b-x) dx$

A 254. Let a + b - x = y. Then we have $\int_{a}^{b} f(a + b - x) dx = -\int_{b}^{a} f(x) dx$.

Math. Mag., **33**(1959) 109.

393. Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts

Find a power series expansion of

$$P = \prod_{r=1}^{\infty} (1 + x^{2^r}) \text{ for } |x| < 1.$$

[[On *Math. Mag.*, 34(1960) 51 it was stated that this was originally submitted as a "Quickie".]]

Math. Mag., **33**(1960) 299.

Solution by Chih-yi Wang, University of Minnesota. Define

$$P_N = \prod_{r=1}^N (1 + x^{2^r})$$

Then

$$P_N = \frac{(1-x^2)P_N}{1-x^2} = \frac{1-x^{2^{N+1}}}{1-x^2}$$

And

$$P = \lim_{n \to \infty} P_N = (1 - x^2)^{-1} = \sum_{n=0}^{\infty} x^{2n}$$
 for $|x| < 1$.

[R. G.]Buschman gave as a reference Knopp, *Infinite Series*, 2nd English Edition, p.436, Example 3.

Math. Mag., **33**(1959) 118.

Q 262. Submitted by M. S. Klamkin

For what values of u_0 does the sequence $\{u_n\}$ diverge when $u_{n+1} = \frac{1}{u_n+2}$?

A 262. Consider the inverse sequence $a_n = \frac{1}{a_{n+1}+2}$ or $a_{n+1} = \frac{1}{a_n} - 2$. where $a_0 = -2$. Then $\{u_n\}$ diverges for $u_0 = a_r$, r arbitrary, since $u_r = -2$. $\{u_n\}$ can be shown to converge for all other real values of u_0 . Math. Mag., **33**(1959) 175–176. **Q 264**. Submitted by M. S. Klamkin Evaluate $\sum_{m=1}^{M} \sum_{n=1}^{N} \min(m, n)$ for $M \ge N$. **A 264**.

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \min(m, n) = \sum_{n=1}^{N} \left(\sum_{m=1}^{n} (m) + \sum_{m=n+1}^{M} (n) \right) = \sum_{n=1}^{N} \left[\frac{n}{2} (n+1) + n(m-n) \right]$$

Since $\sum_{n=1}^{N} n^2 = \frac{n(n+1)(2n+1)}{6}$, we have our sum equal to $\frac{N(N+1)(3M-N+1)}{6}$.

Math. Mag., **33**(1960) 226.

408. Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts

Three congruent ellipses are mutually tangent symmetrically. Determine the radius of the circumcircle.

Math. Mag., **34**(1960) 117.

Solution by Joseph D. E. Konhauser, HRB-Singer Inc., State College, Pennsylvania. Let C denote the intersection of the common tangents to the three ellipses. Extend the major axes of two of the ellipses and denote the point of intersection by P. Let T denote the point of tangency of these two ellipses. Denote the center of one of these ellipses by H, then angle $HPC = 30^{\circ}$. Let F and G be the foci of the ellipse with center H with points in the order F, H, G, P. Let I, J be the foci of the other ellipse with J between I and P.

If a, b, c have their usual meaning, then, by the law of cosines

$$(2a)^2 = \overline{GP}^2 + \overline{IP}^2 - \overline{GP} \cdot \overline{IP}$$

Using $\overline{IP} = \overline{GP} + 2c$, we obtain

$$\overline{GP} + c = \sqrt{4b^2 + c^2} = \sqrt{a^2 + 3b^2}$$

From triangle CHP

$$\overline{HC} = \frac{c + \overline{GP}}{\sqrt{3}} = \sqrt{\frac{a^2 + 3b^2}{3}}$$

Selecting coordinate axes with origin at H, with positive x-axis in direction HP and positive y-axis in direction HC, then the ellipse with center H has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and point C has coordinates

$$\left(0,\sqrt{\frac{a^2+3b^2}{3}}\right)$$

By means of the calculus, it can be shown that the points on the ellipse at maximum distance from C have coordinates

$$\left(\pm a\left\{1 - [b^2/c^4]\frac{a^2 + 3b^2}{3}\right\}, -\frac{b^2}{c^2}\sqrt{\frac{a^2 + 3b^2}{3}}\right)$$

The desired radius is then computed by the distance formula and is

$$\frac{2a^2}{\sqrt{3}c}$$

Math. Mag., **33**(1960) 236–237.

T 36. Submitted by M. S. Klamkin

Determine the greatest perimeter of all pentagons inscribed in a given circle.

S 36. The perimeter of the regular convex pentagon is $10r \sin \pi/5$, however the perimeter of the regular non-convex pentagon is $10r \sin 2\pi/5$ which is greater. For a (2n+1)-sided polygon the maximum perimeter wild be $(4n+2)r \sin 2\pi/(2n+1)$, whereas for the 2*n*-sided polygon the maximum perimeter would be 4nr. In this case the polygon has degenerated into the diameter.

Math. Mag., **33**(1960) 237–238.

F 17. Submitted by M. S. Klamkin

A student derived l'Hospital's rule in the following manner: Let

$$\frac{F(x)}{G(x)} = H(x) \quad \text{where} \quad F(a) = G(a) = 0 \quad \text{and} \quad G'(a) \neq 0.$$

Then F'(x) = G'(x)H(x) + G(x)H'(x) or

$$\frac{F'(x)}{G'(x)} = H(x) + \frac{G(x)}{G'(x)}H'(x).$$

Then

$$\lim_{x \to a} \frac{F'(x)}{G'(x)} = \lim_{x \to a} H(x)$$

since G(a) = 0 and $G'(a) \neq 0$.

E 17. Although G(a) = 0 and $G'(a) \neq 0$ it does not follow that

$$\lim_{x \to a} \frac{G(x)}{G'(x)} H'(x) = 0$$

Actually

$$\frac{G(x)}{G'(x)}H'(x) = \frac{F'(x)}{G'(x)} - \frac{F(x)}{G(x)}$$

so the student's assertion is equivalent to the trivial observation that if

$$\lim_{x \to a} \left[\frac{F'(x)}{G'(x)} - \frac{F(x)}{G(x)} = 0 \right] \quad \text{then} \quad \lim_{x \to a} \frac{F'}{G'} = \lim_{x \to a} \frac{F}{G}$$

Math. Mag., **33**(1960) 302.

Q 267. Submitted by M. S. Klamkin

Prove that the average velocity with respect to distance is greater than or equal to the average velocity with respect to time.

A 267. To prove $\bar{V}_s \geq \bar{V}_t$ we have

$$\frac{\int_{s_1}^{s_2} V \, ds}{\int_{s_1}^{s_2} ds} \ge \frac{\int_{t_1}^{t_2} V \, dt}{\int_{t_1}^{t_2} dt}$$

but this leads to

$$\frac{\int_{t_1}^{t_2} V^2 \, dt}{\int_{t_1}^{t_2} V \, dt} \ge \frac{\int_{t_1}^{t_2} V \, dt}{\int_{t_1}^{t_2} dt}$$

which follows from the Cauchy-Schwartz inequality.

Math. Mag., **34**(1960) 50.

422. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts Solve the differential equation

$$\{x(1-\lambda)D^2 + (x\phi'+1)D + x\phi'' + \phi'\}y = 0$$

where λ is a constant and ϕ is a given function of x.

Math. Mag., **34**(1960) 242.

Solution by P. D. Thomas, Coast and Geodetic Survey, Washington, D.C. Using primes to denote differentiation of y with respect to x, rearrange and collect the terms of the given differential equation to get

$$(xy\phi')' + (1-\lambda)(xy')' + \lambda y' = 0$$

a first integral being at once

$$xh\phi' + (1-\lambda)xy' + \lambda y = C$$
 (constant)

or

$$y' + \frac{y(x\phi' + \lambda)}{(1 - \lambda)x} = \frac{C}{(1 - \lambda)x}$$
(1)

Now (1) is linear and the known solution is

$$y = e^{-\int P \, dx \left(\int Q e^{\int P \, dx} \, dx + k\right)} \tag{2}$$

where from (1)

$$P = \frac{x\phi' + \lambda}{(1-\lambda)x}$$
 $Q = \frac{C}{(1-\lambda)x}$

and k is a constant.

$$\int P \, dx = \frac{1}{1 - \lambda} \int \left(\phi' + \frac{\lambda}{x}\right) \, dx = \frac{\phi + \lambda \ln x}{1 - \lambda} \tag{3}$$

$$\int Q e^{\int P \, dx} \, dx = \frac{C}{1-\lambda} \int e^{\phi/(1-\lambda)} \, x^{(2\lambda-1)/(1-\lambda)} \, dx \tag{4}$$

The solution may them be written from (2), (3) and (4) as

$$y = x^{\lambda/(\lambda-1)} e^{-\phi/(1-\lambda)} \left[\frac{CI}{1-\lambda} + k \right] \quad \text{where} \quad I = \int e^{\phi/(1-\lambda)} x^{(2\lambda-1)/(1-\lambda)} dx$$

Math. Mag., **34**(1960) 58–59.

Q 268. Submitted by M. S. Klamkin

Show that one cannot inscribe a regular polygon of more than four sides in an ellipse with unequal axes.

A 268. Assume that it can be done. Then there would exist a circle intersecting the ellipse in more than four points which is impossible.

Math. Mag., 34(1960) 58–59.

Q 269. Submitted by M. S. Klamkin

If a, b and c are positive numbers, give a geometrical interpretation for the inequality

$$2[a^2b^2 + b^2c^2 + c^2a^2] \ge a^4 + b^4 + c^4$$

A 269.

$$4a^2b^2 \ge (a^2 + b^2 + c^2)^2$$

or

$$(a+b)^2 - c^2][c^2 - (a-b)^2]$$

Now assume $a \ge b \ge c$, then $c \ge a - b$. Consequently, a, b and c form a triangle.

Math. Mag., **34**(1960) 109.

428. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts

The number N = 142857 has the property that 2N, 3N, 4N, 5N and 6N are all permutations of N. Does there exist a number M such that 2M, 3M, 4M, 5M, 6M and 7M are all permutations of M?

Math. Mag., **34**(1961) 3033–304.

I. Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey. Since we get all permutations of M by 1M, 2M, ..., 7M the number M, if it exists, is a seven-digit number.

Let M = abcdefg = Gg where G = abcdef and let $1 \le p \le 7$ such that $p \cdot Gg = gG$. Then

$$p(10G+g) = 10^6 g + G$$

or

$$G = \frac{(10^6 - p)g}{(10p - 1)} = N_p \cdot \frac{g}{Dp}$$

Now

p	N_p	D_p	N_p/D_p	$(n_p/3)/D_p$
1	999999	9	111111	• • •
2	999998	19	Irreducible	•••
3	999997	29	Irreducible	• • •
4	999996	$39 = 3 \cdot 13$	• • •	Irreducible
5	999995	49 = 7.7	Irreducible	
6	999994	59	Irreducible	• • •
7	999993	$69 = 3 \cdot 23$		Irreducible

Since the coefficient N_p/D_p is not an integer except when p = 1, there is no solution for G other than \overline{gggggg} . But $M = Gg = \overline{ggggggg}$ cannot be a solution.

Hence there is no solution to the problem.

II. Comment by Dermott A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts. The number M = 0588235294117647 has the property that kM is a permutation of M for k = 2, 3, ..., 16. The number

L = 0344827586206896551724137931

has the property that kL is a permutation of L for k = 2, 3, ..., 28. (*M* consists of the digits of one cycle of the decimal expansion of 1/17 and is 16 digits long, while L was similarly derived from 1/29. I believe that it it is correct when p is prime and 1/p = Q has cycle length p - 1, then kQ will be a permutation of Q for k = 2, 3, ..., p-1.)

[[The belief is correct, of course. In L & M above I've moved the zeros from the end to the beginning. In *The Book of Numbers*, p.160, Conway & Guy call these **long primes** and give the further examples:

 $23:\ 0434782608695652173913$

 $47:\ 0212765957446808510638297872340425531914893617$

 $59:\ 0169491525423728813559322033898305084745762711864406779661$

 $61:\ 016393442622950819672131147540983606557377049180327868852459$

97: 010309278350515463917525773195876288659793814432-989690721649484536082474226804123711340206185567

and indicate that 109, 131, 149, 167, 179, 181, 193, 223, 229, 223, 257, 269, \dots will also serve.]]

Math. Mag., **33**(1960) 120, 123.

T 39. Submitted by M. S. Klamkin

Find the sum of

$$S = \sum_{n=1}^{\infty} 1/p_n$$

where p_n is the *n* th prime in the sequence $n^5 + n + 1$.

S 39. Since

$$(n^{5} + n + 1) = (n^{2} + n + 1)(n^{3} - n^{2} + 1)$$

there is only one prime $p_1 = 3$. Whence S = 1/3.

Math. Mag., **34**(1961) 173.

435. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts

Determine the largest and the smallest equilateral triangles that can be inscribed in an ellipse.

Math. Mag., **34**(1961) 368–369.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey. Let $A_1A_2A_3$ be an equilateral triangle inscribed in the ellipse

$$(x^2/a^2) + (y^2/b^2) = 1$$
 (E) $a > b$ (1)

and let

$$(x-u)^{2} + (y-v)^{2} - r^{2} = 0 \qquad (\Omega)$$
(2)

be the circle circumscribed to $A_1A_2A_3$. It cuts (E) at the fourth point $A_4()x_4, y_4$.

Eliminating y between (1) and (2) we get an equation of fourth degree in x

$$c^4 \cdot y^4 - 4a^2c^2u \cdot x^3 + \dots = 0$$

of which the roots are x_1, x_2, x_3, x_4 .

If we elim[in] ate x between (1) and (2), the corresponding equation will be

$$c^4 \cdot y^4 + 4b^2c^2v \cdot y^3 + \dots = 0$$

and the roots are y_1, y_2, y_3, y_4 .

Since $A_1A_2A_3$ is an equilateral triangle, we have

$$\begin{array}{rcl} x_1 + x_2 + x_3 &=& 3u \\ y_1 + y_2 + y_3 &=& 3v \end{array}$$

and

$$x_{4} = \sum x_{i} - 3u = \frac{4a^{2}u}{c^{2}} - 3u = \frac{(a^{2} + 3b^{2})u}{c^{2}}$$
$$y_{4} = \sum y_{i} - 3v = -\frac{4b^{2}v}{c^{2}} - 3v = -\frac{(b^{2} + 3a^{2})v}{c^{2}}$$

These coordinates satisfy (1) so we obtain the relation

$$(u^2/\alpha^2) + (v^2/\beta^2) = 1$$
(3)

where

$$\alpha = \frac{ac^2}{a^2 + 3b^2} \qquad \beta = \frac{bc^2}{b^2 + 3a^2}$$

Hence the centers of the circles (Ω) lie on the ellipse (3) of which $\alpha > \beta$.

Now since the largest and the smallest triangles correspond to the greatest and the smallest values of the radius r of the circle (Ω), we write

$$r^{2} = (x_{4} - u)^{2} + (y_{4} - v)^{2}$$

= $\frac{(a - \alpha)^{2}u^{2}}{\alpha^{2}} + \frac{(b + \beta)^{2}v^{2}}{\beta^{2}}$
= $Au^{2} + (b + \beta)^{2} = Bv^{2} + (a - \alpha)^{2}$

dr/du = 0 gives

$$u = 0$$
 and $r_1 = b + \beta$

Similarly dr/dv = 0 gives

$$r_2 = a - \alpha$$

and one may readily verify that $r_1 > r_2$.

Hence, the largest (smallest) equilateral triangles inscribed in the ellipse are ones inscribed in the circles of center u = 0, $v = \pm \beta$ ($u = \pm \alpha$, v = 0) and radius $b + \beta(a - \alpha)$.

[[I think there's something wrong here — there should be at least two radii and the dimensions of the expression look wrong. — R.]]

There are four solutions, two for the largest and two for the smallest triangles.

Constructions: The largest (smallest) triangles inscribed in an ellipse have one of their vertices at the extremities of the minor (major) axis of the ellipse, the axis being an [[not 'the']] axis of symmetry of the triangle.

Math. Mag., **34**(1960) 182, 184.

Q 274. Submitted by M. S. Klamkin

Find the general solution of the Diophantine equation

$$(x^4 + y^4 + z^4)^2 = 2(x^8 + y^8 + z^8)$$

A 274. The equation can be factored into

$$(x^{2} + y^{2} + z^{2})(x^{2} + y^{2} - z^{2})(y^{2} + z^{2} - x^{2})(z^{2} + x^{2} - y^{2}) = 0$$

Consequently, the general solution is given by the complete solution to an integral right triangle. That is, x = 2mn, $y = m^2 - n^2$, $z = m^2 + n^2$ and permutations.

Math. Mag., **34**(1960) 182, 184.

T 43. Submitted by M. S. Klamkin

Determine integers a and b such that $x^{15} + ax + b$ and $5^{13} - 233x - 144$ have a common factor.

A 43. Assume the common factor has the form $x^2 - mx - n$. If m = n = 1, then $x^{13} - F_{13}x - F_{12}$ where F_n are the Fibonacci numbers 1, 1, 2, 3, 5, 8, ..., and $F_{13} = 233$ while $F_{12} = 144$. Consequently $-a = F_{15} = 610$ and $-b = F_{14} = 377$. Whether or not other solutions exist is a considerably more involved problem.

Math. Mag., **34**(1961) 244, 248.

Q 275. Submitted by M. S. Klamkin

Show that the moments of inertia about all centroidal axes of an area with *n*-fold $(n \ge 3)$ symmetry are the same

A 275. The ellipse of inertia must be circular since three diameters of a proper ellipse cannot all be equal.

Math. Mag., **34**(1961) 244, 248.

Q 276. Submitted by M. S. Klamkin

Show that $x^m D^{m+n} x^n \equiv D^n x^{m+n} D^m$ where D is the differential operator $\frac{d}{dx}$.

A 276. Since $D^n x = \sum a_r x^r D^r$, $x^n D^n = x D(xD-1) \cdots (xD_n+1)$. Consequently $x^m D^m$ and $D^n x^n$ commute.

Math. Mag., **34**(1961) 309–310.

T 45. Submitted by M. S. Klamkin

Determine the equation to the conic passing through the five points (-3, -2), (-2, 3), (1, 1), (-1, 1), (4, -1).

A 45. Since (-2,3), (1,1), and (4,-1) are collinear, the conic degenerates into the two straight lines

$$(2x + 35 - 5)(3x - 2y + 5) = .0$$

Math. Mag., **34**(1961) 363.

456. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts

Determine two-parameter solutions of the following "almost" Fermat Diophantine equations:

1.
$$x^{n-1} + y^{n-1} = z^n$$

2.
$$x^{n+1} + y^{n+1} = z^n$$

3.
$$x^{n+1} + y^{n-1} = z^n$$

Math. Mag., 35(1962) 124.

Solution by Leo Moser, University of Alberta. We will exhibit two-parameter solutions for the more general equation

$$x^{a} + y^{b} = z^{c}$$
 $(a, b, c) = 1$ (1)

Since (a, b, c) = 1 we can first find an m and n such that

$$abm + 1 = cn \tag{2}$$

Now let u and v be arbitrary integers and let

$$x = u^{bm} (u^{abm} + v^{abm})^{bm}$$

and

$$y = v^{am} (u^{abm} + v^{abm})^{am}$$

Then

$$x^a + y^b = (u^{abm} + v^{abm})^{abm+1}$$

By (2) we have

$$x^a + y^b = (u^{abm} + v^{abm})^{nc}$$

so that with $z = (u^{abm} + v^{abm})^n$ equation (1) is satisfied.

Math. Mag., **34**(1961) 372, 352.

Q 286. Submitted by M. S. Klamkin

Find the *n*th derivative of $\cos^3 x$.

A 286.

$$D^{n}\cos^{3}x = D^{n}\left[\frac{\cos 3x + 3\cos x}{4}\right] = \frac{3^{n}}{4}\cos(3x + n\pi/2) + \frac{3}{4}\cos(x + n\pi/2).$$

Math. Mag., **34**(1961) 435.

T 47. Submitted by M. S. Klamkin

Show that in any polygon there exist two sides whose ratio lies between 1/2 and 2.

A 47. Assume that it is not true. Then the largest side would be greater than the sum of all the other sides. That is

 $ar^n > a + ar + ar^2 + \dots + ar^{n-1}$ if $r \ge 2$.

Math. Mag., **34**(1961) 62–63.

Q 290. Submitted by M. S. Klamkin

A

Through a given point within a given angle, construct a line which will form a triangle of minimum area.

A 290. In order for the triangle ABC to be a minimum it follows that BP = PC. Consequently, draw PM parallel to AB, lay off MC = MA and draw CPB.

C



Math. Mag., **35**(1962) 117.

474. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts

The three polynomials x - x, $x_2 + y^2 - 2xy$ and $x^3 + y^3 + z^3 - 3xyz$ can each be factored into real polynomials. Which if any of the higher prder analogous polynomials

$$\sum_{r=1}^{n} x_r^n - nx_1 x_2 \cdots x_n$$

are reducible?

Math. Mag., **35**(1962) 310–311.

Solution by J. A. Tyrell, King's College, London. None of the higher order polynomials are reducible (into either real or complex factors). To see this, observe that a factorization of

$$x_1^n + y_1^n + z_1^n \tag{1}$$

could be obtained from any factorization of the given polynomial (for $n \ge 4$) merely by setting x_4, \ldots, x_n equal to zero. As (1) is well-known to be irreducible (for all positive integers n) our assertion follows. The following proof that (1) is irreducible may be of interest. (The impossibility of *linear* factors is trivial to demonstrate.) To prove the more general assertion, interpret x_1, x_2, x_3 as the homogeneous coordinates of a point in a projective plane; the vanishing of (1) then represents a plane curve of order n and, since the partial derivatives of (1) with respect to the x_i [[not x_1]] do not vanish simultaneously at any point of the plane, the curve is non-singular (i.e. it has no multiple points). If the expression (1) were factorable, the curve would be reducible and then would necessarily possess multiple points (at the points of intersection of any two components). It follows that the expression (1) is irreducible. (*Note*: the geometrical facts used here may be looked up in any elementary treatise on Higher Plane Curves.)

Math. Mag., **35**(1962) 126, 128.

Q 294. Submitted by M. S. Klamkin

Show that the vector expression

 $A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A$

would be the same in an English or an Israeli (reading from right to left) article.

A 294. Both expressions are just expressions of $\nabla(A \cdot B)$

Math. Mag., **35**(1962) 126, 128.

Q 296. Submitted by M. S. Klamkin

If two ellipsoids have an ellipse in common, all their other points of intersection, if real, lie on another ellipse.

A 296. Let the equations of the common ellipse be

$$C \equiv ax^2 + bxy + cy^2 + dx + ey + f = 0$$

The most general equations of two ellipsoids which pass through this ellipse are

$$C + z(a_1x + b_1y + c_1z + d_1) = 0$$

$$C + z(a_2x + b_2y + c_2z + d_2) = 0$$

Al points on both ellipsoids which are not on z = 0 must satisfy

$$a_1x + b_1y + c_1z + d_1 = a_2x + b_2y + c_2z + d_2$$

which is another plane intersecting in another ellipse.

Math. Mag., 35(1962) 252–253.

A Random Probability

469. [January 1962]. Proposed by J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela.

A random straight line is drawn across a regular hexagon. What is the probability that is intersects two opposite sides? [[the first published solution, by W. W. Funkenbusch, Michigan College of Miningand Technology, gave the answer $\frac{1}{3} + (\sqrt{3}/\pi) \ln \frac{3}{4} \approx 0.174725894.$]]

II. Solution by Murray S. Klamkin, AVCO, Wilmington, Massachusetts. The problem is not uniquely soluble for no definition of the random straight line distribution was given. We will obtain two answers by assuming two different distributions both of whaich are invariant under the group of motions in the plane (will give the same answer to all observers). **1**. We assume that two points (which define the random line) are taken at random with a uniform distribution on the sides of the hexagon with no two points on the same side. There is no loss of generality in assuming that one of the points is on a fixed side and the other is on one of the five other sides. Whence, probabibility of intersecting two opposite sides is 1/5. **2**. If we assume that the two points as before can also be on the same side, then probility is 1/6. Math. Mag., **35**(1962) 257–258. **Q 300**. Submitted by M. S. Klamkin Integrate

$$I = \int \frac{d\theta}{a + b\cos\theta}$$

without recourse to the usual substitution $z = \tan \theta/2$. [[In fact Murray's solution does exactly that!]] A 300.

$$I = \int \frac{d\theta}{(a-b)\sin^2\frac{\theta}{2} + (a+b)\cos^2\frac{\theta}{2}}$$

= $2\int \frac{d\tan\theta/2}{a+b+(a-b)\tan^2\theta/2}$
= $\frac{2}{\sqrt{a^2+b^2}}\arctan\sqrt{\frac{a-b}{a+b}}\tan\theta/2$ for $a > b$ or
= $\frac{2}{\sqrt{a^2+b^2}}\operatorname{arctanh}\sqrt{\frac{b-a}{b+a}}\tan\theta/2$ for $a < b$.

Math. Mag., **35**(1962) 309.

497. Proposed by M. S. Klamkin, AVCO, Wilmington, Massachusetts Show that

$$4\sum_{r=0}^{n} r^{3} \binom{n}{r}^{p} = 6n\sum_{r=0}^{n} r^{2} \binom{n}{r}^{p} - n^{3}\sum_{r=0}^{n} \binom{n}{r}^{p}$$

Math. Mag., **36**(1963) 201.

Solution by Francis D. Parker, University of Alaska. This problem is equivalent to showing that

$$\sum_{r=0}^{n} [4r^3 - 6nr^2 + n^3] \binom{n}{r}^p = 0$$

Let $f(r) = 4r^3 - 6nr^2 + n^3$ and $g(r) = {n \choose r}^p$.

It follows easily that f(r) = -f(n-r) and that g(r) = g(n-r). From these results the conclusion is immediate.

Math. Mag., **35**(1962) 317, 272.

Q 303. Submitted by M. S. Klamkin

If every r th term is removed from the series $1 - 1/2 + 1/3 - 1/4 + \cdots$, find the resulting sum.

A 303. If r is even the resulting series obviously diverges. If r is odd, then

$$S_{rn-r} = \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{rn}\right) - \frac{1}{r}\left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{n}\right)$$
$$S = \lim_{n \to \infty} S_{rn-r} = \left(1 - \frac{1}{r}\right) \ln 2$$

Math. Mag., **35**(1962) 317, 272.

T 55. Submitted by M. S. Klamkin

Find all the solutions of the Diophantine equation

$$y = \frac{x^{11}}{11} + \frac{x^{10}}{2} + \frac{5x^9}{2} - x^7 + x^5 - \frac{x^3}{2} + \frac{5x}{66}$$

S 55. Since $y = \sum_{r=1}^{x} r^{10}$, y is integral for all integers x.

Math. Mag., **35**(1962) 317, 272.

T 56. Submitted by M. S. Klamkin

Describe the family of curves whose equation is $(x+y+1)^2 = 3(x+y-xy-a^2)$ where x and y are real and as a real parameter.

S 56. The equation can be rewritten as $()^2 + ()^2 + ()^2 = -6a^2$. Whence the family consists of the single point (1,1).

Math. Mag., **35**(1962) 77–78.

Q 306. Submitted by M. S. Klamkin

Show that

$$\frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \dots + \frac{1}{10} = \frac{1}{1100} + \frac{1}{111000} + \dots$$

[[This was originally misprinted with one too few zeroes in the last fraction. There is a correction at *Math. Mag.*, **37**(1964) 203.]]

A 306. Since

$$\frac{1}{11}\left(1-\frac{1}{100}\right) + \frac{1}{111}\left(1-\frac{1}{1000}\right) + \dots = \frac{1}{9}\left[\frac{1}{100} + \frac{1}{1000} + \dots\right] = \frac{1}{10}$$

the result follows immediately.

Math. Mag., **35**(1962) 77–78.

Q 309. Submitted by M. S. Klamkin

Determine

$$\prod_{n=2}^{\infty} \left[1 - \frac{2}{1+n^3} \right]$$

A 309.

$$\prod \left[1 - \frac{2}{1+n^3} \right] = \prod \left(\frac{n-1}{n+1} \right) \cdot \prod \left(\frac{n^2+n+1}{n^2-n+1} \right) = (2) \left(\frac{1}{3} \right) = \frac{2}{3}.$$

[[something wrong here, though answer is right. The *n* th partial product of the third product is $\frac{n^2-n+11}{3}$, which tends to infinity. If the original product is written as $\prod \frac{n^3-1}{n^3+1}$ then it is seen to converge, since the *n* th partial product is $\frac{2(n^2+n+1)}{3n(n+1)}$ which tends to 2/3 (from above!) In fact see the following:]]

Math. Mag., **37**(1964) 282–283.

Comment by Alan Sutcliffe, Knottingley, Yorkshire, England. There appear to be two compensating errors in this rather abbreviated solution. The first is the assumption that

$$\prod_{n=2}^{\infty} f(n)g(n) = \prod_{n=2}^{\infty} f(n) \prod_{n=2}^{\infty} g(n)$$

which is not true.

The second is in the evaluation of the two products, where cancellation is used to show that

$$\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \dots = 2$$
$$\frac{7}{3} \cdot \frac{13}{7} \cdot \frac{21}{13} \cdot \frac{31}{21} \cdot \frac{43}{31} \cdot \dots = \frac{1}{3}$$
ies by

and

$$\frac{3}{1} \cdot \frac{6}{3} \cdot \frac{10}{6} \cdot \frac{15}{10} \cdot \frac{21}{15} \cdots$$

(= 1, by the solver's method of cancellation) we can prove that 1 = 2 as follows:

$$1 = \prod_{n=2}^{\infty} 1 = \prod_{n=2}^{\infty} \left(\frac{n-1}{n+1}\right) \cdot \prod_{n=2}^{\infty} \left(\frac{\frac{1}{2}n(n+1)}{\frac{1}{2}n(n-1)}\right) = (2)(1) = 2.$$

A valid proof of the original proposition, suggested by the editor, may be given in the following way:

$$\begin{split} \prod_{n=2}^{N} \left[1 - \frac{2}{1+n^3} \right] &= \prod_{n=2}^{N} \left(\frac{n-1}{n+1} \right) \cdot \prod_{n=2}^{N} \left(\frac{n^2+n+1}{n^2-n+1} \right) \\ &= \frac{2}{N(N+1)} \cdot \frac{N^2+N+1}{3} \\ &= \frac{2}{3} \left(1 + \frac{1}{N^2+N} \right) \\ \prod_{n=2}^{N} \left[1 - \frac{2}{1+n^3} \right] &= \lim_{n \to \infty} \frac{2}{3} \left(1 + \frac{1}{N^2+N} \right) = \frac{2}{3} \end{split}$$

Math. Mag., 36(1963) 142, 108.

Q 310. Submitted by M. S. Klamkin

Show that $\sin \theta > \tan^2 \theta/2$ for $0 < \theta < \pi/2$.

A 310. Let $\cos \theta = x$. Then

$$\sqrt{1-x^2} > \frac{1-x}{1+x}$$
 or $(1+x)^{3/2} > (1-x)^{1/2}$

which is obviously true.

Math. Mag., 36(1963) 142, 108.

Q 312. Submitted by M. S. Klamkin

Show that no equilateral triangle which is either inscribed in or circumscribed about an ellipse (excluding the circular case) can have its centroid coinciding with the center of the ellipse.

A 312. Orthogonally project the ellipse into a circle. The equilateral inscribed or circumscribed triangles will become inscribedor circumscribed non-equilateral triangles whose centroids cannot coincide with the center of the circle. Since centroids transform into centroids, the proof is completed.

[[See also Math. Mag., 42(1969) 287. 816 which is identical.]]

Math. Mag., 36(1963) 197.

518. Proposed by M. S. Klamkin, State University of New York at Buffalo

Show that an integer is determined uniquely from a knowledge of the product of all its divisors.

Math. Mag., **37**(1964) 57–59.

I. Solution and comments by Leo Moser, University of Alberta. By pairing a divisor d of n with its complementary divisor n/d (and leaving \sqrt{n} , if it is a divisor, unpaired) we see that the geometric mean of the divisors of n is \sqrt{n} and hence, if $\tau(n)$ denotes the number of divisors of n,

$$\prod_{d|n} d = n^{\tau(n)/2}$$

We therefore need to show that

 $n^{\tau(n)} = m^{\tau(m)}$ implies that n = m

We will show more generally that if f(n) is an arithmetic function for which

 $m \mid n \text{ implies } f(m) \le f(n) \text{ then}$ (1)

$$n^{f(n)} = m^{f(m)} \quad \text{implies} \quad n = m \tag{2}$$

Proof of (2): Clearly n and m have the same prime factors. Suppose that

$$n = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$$
 and $m = P_1^{\beta_1} \cdots P_k^{\beta_k}$

are the prime power representations of n and m. Comparing the exponents of P_1 in n and m we find

$$\alpha_1 f(n) = \beta_1 f(m) \tag{3}$$

Similarly

$$\alpha_2 f(n) = \beta_2 f(m) \tag{4}$$

From (3) and (4) we find

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$$

and similarly we find that

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \dots = \frac{\alpha_k}{\beta_k}$$

If this common ratio is 1 we are done. If not, assume without loss of generality that it exceeds 1. Then $m \mid n$ and by (1) $f(m) \leq f(n)$. Also m < n so that $n^{f(n)} > m^{f(m)}$ and the result is established.

We note that special cases of suitable f(n) include

$$f(n) = \phi(n) = \sum_{(a,n)=1} 1$$
 $f(n) = \sigma(n) = \sum_{d|n} d$

and $f(n) = \omega(n)$, where $\omega(n)$ is the number of distinct prime divisors of n.

Somewhat related to the fact that $n^{\phi(n)} = m^{\phi(m)}$ implies n = m is the fact that $n\phi(n) = m\phi(m)$ implies n = m. This appears as a problem in An Introduction to the Theory of Numbers by Niven & Zuckerman. On the other hand we note that the corresponding result is not true for ϕ replaced by σ . In fact $12\sigma(12) = 14\sigma(14)$ and more generally, if (a, 42) = 1 then $12a\sigma(12a) = 14a\sigma(14a)$.

[[This last formula is garbled in the original. -R.]]

Let us call a solution of $n\sigma(n) = m\sigma(m)$ primitive if it cannot be obtained from a smaller solution by mutiplying through by some factor. We have not been able to decide whether $n\sigma(n) = m\sigma(m)$, $n \neq m$ has infinitely many solutions.

[[Forty years later Moser's problem is still unsolved at **B11** in Unsolved Problems in Number Theory. It is almost certain that the 'We' in his last sentence is Erdős & Moser. — R.]]

Math. Mag., **36**(1963) 206, 156.

Q 314. Submitted by M. S. Klamkin

It follows by symmetry that the line joining the centers of two congruent, parallel, intersecting ellipse bisects the common chord. Show that the result holds if the ellipses are no longer congruent but similar.

A 314. The result is obvious true for two intersecting circles. Consequently it is true for two similar paralle ellipses by orthogonal projection.

Math. Mag., 36(1963) 206, 156.

Q 317. Submitted by M. S. Klamkin

A determinant whose elements are either 0 or 1 has a value ± 1 . What is the maximum and minimum number of ones?

A 317. Obviously the minimum number is n. The maximum number is $n^2 - n + 1$ which occurs in the determinant $|A_{rs}|$ where $A_{rs} = 1 - \delta^{1,r-s}$ and $\delta_{m,n} = 0$ for $m \neq n$, and $\delta_{m,m} = 1$.

Math. Mag., 36(1963) 266–267.

An Eccentric Orbit

504. [January 1963] Proposed by M. S. Demos, Drexel Institute of Technology

The orbit of the earth about the sun is an ellipse with the sun at the focus. Astronomy textbooks say that the mean distance of the sun from the earth is the major semi-axis a.

Show that the correct mean distance with respect to time is $(1 + e^2/2)a$, where e is the eccentricity.

Solution by M. S. Klamkin, SUNY at Buffalo, New York. For an elliptic orbit where

$$r = a(1 - e\cos E)$$
$$dE = \frac{k}{r}\frac{dt}{\sqrt{a}}$$

("Theoretical Mechanics", Vol.1, Macmillan, p.279.) Whence,

$$\bar{r} = \frac{\int r \, dt}{\int dt} = \frac{\int_0^{2\pi} r^2 \, dE}{\int_0^{2\pi} r \, dE} = a(1 + e^2/2)$$

by a simple integration. Also it is easily shown that the average r with respect to polar angle is $a\sqrt{1-e^2}$. Both of these results are well known and in particular they are both posed as a problem in the aforementioned reference p.304, problem 25. Also

the problem as posed is given in "Theoretical Mechanics", C. J. Coe, p.149. i.e., "In elliptic orbits the major semi-axis a of the ellipse is known in astronomy as the mean distance of the planet from the sun. Show that the actual average distance relative to the time is not a but $a(1 + e^2/2)$."

Note: The arithmetic average of the perihelion distance and aphelion distance is a.

Math. Mag., 36(1963) 270, 239.

Q 318. Submitted by M. S. Klamkin

N perfectly elastic balls of equal mass are moving on the same straight line. What arrangement of velocities will produce the maximum number of collisions?

A 318. When two balls collide they will just exchange velocities. A simpler way of looking at this is to imag[in]e the balls passing through each other. If we arrange the velocities in monotonic order, we will obtain $\binom{N}{2}$ collisions. That this is maximum follows by considering the world lines of the balls (*s* vs. *t*). The maximum number of points of intersection of *N* straight lines is $\binom{N}{2}$. If we have an elastic wall at one point of the line, the maximum number of collisions will be doubled.

[[The original read 'worldliness' which would be nice to preserve! - R.]]

Math. Mag., **36**(1963) 320.

497. Proposed by M. S. Klamkin, State University of New York, Buffalo

It is known that if a family of integral curves of the linear differential equation y' + P(x)y = Q(x) is cut by the line x = a, then the tangents at the points of intersection are concurrent. Prove, conversely, that if for a first order equation y' = P(x, y) the tangents (as above) are concurrent, then F(x, y) is linear in y.

Math. Mag., **37**(1964) 203.

Solution by Roop N. Kesarwani, Wayne State University, Michigan. Let the point of intersection with the line x = a of a typical member of the family of integral curves of y' = F(x, y) be (a, y_0) . If a is fixed, y_0 clearly depends on the parameter of the family.

The tangent at the point of intersection to the integral curve is then $y-y_0 = F(a, y_0)(x-a)$. All such tangents pass through the same point, say (A, B). Therefore $B - y_0 = F(a, y_0)(A - a)$, or

$$F(a, y_0) = \frac{B - y_0}{A - a}$$

proving that F(x, y) is linear in y.

Klamkin pointed out that this result provides a basis for a graphical solution of the given differential equation. See H. Betz, P. B. Burcham & G. M. Ewing, *Differential Equations with Applications*, 1954; and M. S. Klamkin, On a graphical solution of a first order linear differential equation, *Amer. Math. Monthly*, **61**(1954) 565–567.

Math. Mag., 36(1963) 328, 280.

Q 323. Submitted by M. S. Klamkin

Determine the maximum number of consecutive terms of the coefficients of a binomial expansion which are in arithmetic progression.

A 323. Three. For three terms to be in A.P., we must have

$$2\binom{m}{n} = \binom{m}{n-1} + \binom{m}{n+1} \quad \text{or} \quad (m-2n)^2 = m+2$$

whence

$$m = a^2 - 2$$
 $2n = a^2 \pm a - 2$

In order to have four terms in A.P., $(a^2 - a)/2 = (a^2 + a - 2)/2$ or a = 1 and impossible. (See Note of Th. Motzkin, *Scripta Math.*, March, 1946, p.14.)

Math. Mag., **37**(1964) 55.

543. Proposed by M. S. Klamkin, State University of New York, Buffalo

If $\int_a^b [F(x) - x^r]^2 dx = \lambda^2$, find upper and lower bounds for $\int_a^b [F(x)]^2 dx$.

(Note: For a class of similar problems, see J. L. Synge, *The Hypercircle in Mathematical Physics*, p.82.)

Math. Mag., 37(1964) 281–282.

Solution by Martin J. Cohen, Beverly Hills, California. I will prove a more general statement: Let F, G, H be functions such that F(x) = G(x) + H(x). Let

$$A = \left[\int_{a}^{b} F^{2}(x) dx\right]^{1/2}$$
$$B = \left[\int_{a}^{b} G^{2}(x) dx\right]^{1/2}$$
$$C = \left[\int_{a}^{b} H^{2}(x) dx\right]^{1/2}$$

 $A \ge 0, B \ge 0, C \ge 0$. Then $(B - C)^2 \le A^2 \le (B + C)^2$.

All we need is the form of the Minkowski integral inequality which states that

$$\left[\int_{a}^{b} f^{2}(x) \, dx\right]^{1/2} + \left[\int_{a}^{b} g^{2}(x) \, dx\right]^{1/2} \ge \left[\int_{a}^{b} (f(x) \pm g(x))^{2} \, dx\right]^{1/2}$$

$$B + C = \left[\int_{a}^{b} G^{2}(x) dx\right]^{1/2} + \left[\int_{a}^{b} H^{2}(x) dx\right]^{1/2} \ge \left[\int_{a}^{b} (G(x) + H(x))^{2} dx\right]^{1/2}$$
$$= \left[\int_{a}^{b} F^{2}(x) dx\right]^{1/2} = A$$

so that $A^2 \leq (B+C)^2$.

$$A + B = \left[\int_{a}^{b} F^{2}(x) dx\right]^{1/2} + \left[\int_{a}^{b} G^{2}(x) dx\right]^{1/2} \ge \left[\int_{a}^{b} (F(x) - G(x))^{2} dx\right]^{1/2}$$
$$= \left[\int_{a}^{b} H^{2}(x) dx\right]^{1/2} = C$$

and similarly $A + C \ge B$ so that $A \ge |B - C|$ and $A^2 \ge (B - C)^2$ Letting $G(x) = x^r$ we see that

$$B = \left[\frac{b^{2r+1} - a^{2r+1}}{2r+1}\right]^{1/2}$$

so that

$$\int_{a}^{b} F^{2}(x) \, dx \leq \left[\left[\int_{a}^{b} (F(x) - x^{r})^{2} \right]^{1/2} + \left[\frac{b^{2r+1} - a^{2r+1}}{2r+1} \right]^{1/2} \right]^{2}$$

and

$$\int_{a}^{b} F^{2}(x) \, dx \ge \left[\left[\int_{a}^{b} (F(x) - x^{r})^{2} \right]^{1/2} - \left[\frac{b^{2r+1} - a^{2r+1}}{2r+1} \right]^{1/2} \right]^{2}$$

Math. Mag., **37**(1964) 62, 53.

Q 327. Submitted by M. S. Klamkin

Determine all the triangles such that

$$a^{2} + b^{2} - 2ab\lambda \cos C = b^{2} + c^{2} - 2bc\lambda \cos A$$
$$= c^{2} + a^{2} - 2ca\lambda \cos B$$

A 327. $a^2 + b^2 - 2ab\lambda \cos C = (1 - \lambda)a^2 + (1 - \lambda)b^2 + \lambda c^2$. Consequently, a = b = c unless $\lambda = 1/2$ for which case the equations are identically satisfied.

Math. Mag., **37**(1964) 62, 53.

Q 328. Submitted by M. S. Klamkin

Determine

$$\int_0^\infty \frac{1 - e^{-t}}{t^m} dt \quad \text{where} \quad (2 > m > 1).$$

A 328. Let

$$\phi(a) = \int_0^\infty \frac{1 - e^{-at}}{t^m} dt$$

then

$$\phi'(a) = \int_0^\infty \frac{e^{-at}}{t^{m-1}} dt = \frac{1}{a^{2-m}} \Gamma(2-m)$$

Hence

$$\phi(1) = \frac{1}{m-1}\Gamma(2-m) = -\Gamma(1-m).$$

This procedure can be extended to integrals of the form

$$\int_0^\infty \left(1 - t + \frac{t^2}{2!} - \dots - e^{-t}\right) \frac{dt}{t^r}$$

Math. Mag., **37**(1964) 119.

549. Proposed by M. S. Klamkin, SUNY at Buffalo, New York

The solution of the Clairaut equation y = xy' + F(y') is obtained by setting y' = c which gives y = cx + F(c). Determine the most general first order differential equation in which the solution can be obtained in this manner.

[[Compare Math. Mag., 45(1972) 102, 112. Q 537.]]

Math. Mag., **37**(1964) 358.

Solution by Josef Andersson, Vaxholm, Sweden. (Translated by the editor.) If the equation is written $y = \Phi(x, y')$ it follows that

$$y' = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y'} \cdot y''$$

The solution y' = C, y'' = 0 gives

$$\frac{\partial \Phi}{\partial x} = y'$$

and $\Phi = xy' + F(y')$.

The Clairaut equation is therefore unique.

Math. Mag., **37**(1964) 126, 83. **Q 332**. Submitted by M. S. Klamkin Factor $x^5 - 5x^2 + 2$. **A 332**.

so $x^5 - 5x^2 + 2 = (x^2 - x - 1)(x^3 + x^2 + 2x - 2).$ [[Is this really a quickie? — R.]]

Math. Mag., **37**(1964) 126, 83.

Q 334. Submitted by M. S. Klamkin

Determine the ratio

$$\sum_{r=0}^{n} r \binom{n}{r}^{p} \div \sum_{r=0}^{n} \binom{n}{r}^{p}$$

A 334. The ratio n/2 follows immediately from

$$\sum_{r=0}^{n} r \binom{n}{r}^{p} = \sum_{r=0}^{n} (n-r) \binom{n}{r}^{p} \quad \text{or from}$$
$$\sum_{r=0}^{n} r^{2} \binom{n}{r}^{p} = \sum_{r=0}^{n} (n-r)^{2} \binom{n}{r}^{p}$$

Math. Mag., **37**(1964) 126, 82.

T 59. Submitted by M. S. Klamkin

Determine a function $\phi(x, y)$ such that the set of points (x, y) satisfying $\phi(x, y) = 0$ has area 1.

S 59. $\phi(x,y) = |x-1| + |x+1| + |y-1| + |y+1| - 4 = 0$. This set consists of all points in and on the square with vertices $(\pm 1, \pm 1)$.

[[This seems to give area 4, rather than area 1. - R.]]

Math. Mag., **37**(1964) 126, 82.

T 60. Submitted by M. S. Klamkin

A person was directed to the downtown side of an unfamiliar subway station. He desired to get on the first car. Which end of the platform should he walk to, assuming that there are no signs, signal lights or trains in the station to cue him?

S 60. In the United Stes, he should walk in a direction such that the uptown tracks are kept on his left. Presumably, in London, it would be in the opposite direction. That is, if the trains run the same was as the automobile.

Math. Mag., **37**(1964) 206, 176.

Q 336. Submitted by M. S. Klamkin

Show that the only factorization of homogeneous polynomials into polynomials is into homogeneous ones.

A 336. Proof for three variables.

Assume

$$H(x, y, z) = F(x, y, z)G(x, y, z)$$

But H can be expressed in the form

$$x^n P\left(\frac{y}{x}, \frac{z}{x}\right)$$

Let r = y/x and s = z/x, then

$$x^{n}P(r,s) = F(x,rx,sx)G(x,rx,sx)$$

Now it follows that

$$F(x, rx, sx) = x_{n_1} P_1(r, s)$$
$$G(x, rx, sx) = x_{n_2} P_2(r, s)$$

Since the only factorizations of x^n are of the form $x^{n_1} \cdot x^{n_2}$ where $n_1 + n_2 = n$. Whence F and G are homogeneous.

[[Not a very quickie? Does the proof automatically extend to any number of variables? - R.]]
Q 263. [January 1960]. Submitted by Melvin Hochster & Jeff Cheeger. Solve $\arctan \frac{p}{x} + \arctan \frac{q}{x} + \arctan \frac{r}{x} = \pi$ for x where p, q and r are fixed.



 $\arctan \frac{p}{x} + \arctan \frac{q}{x} + \arctan \frac{r}{x} = \alpha + \beta + \gamma = (\pi/2) - (A/2) + (\pi/2) - (B/2) + (\pi/2) - (C/2) = (3\pi/2) - \pi/2 = \pi.$ Thus x is the radius of the incircle of a triangle of sides p + q, q + r and r + p and has the value $\sqrt{pqr/(p+q+r)}.$

Math. Mag., **37**(1964) 282.

Comment by M. S. Klamkin, State University of New York at Buffalo. The proof submitted by the proposers, although elegant, is only valid if p + q, q + r and r + p form a triangle. The solution $x = \sqrt{pqr/(p+q+r)}$ is still correct even if a triangle is not formed.

This follows from

$$\arctan \frac{p}{x} + \arctan \frac{q}{x} + \arctan \frac{r}{x} = \arctan \frac{x^2(p+q+r) - pqr}{x(x^2 - pq - qr - rp)}$$

Math. Mag., **37**(1964) 283.

Comment on Q319

Q 319. [September 1963]. Submitted by C. W. Trigg. Factor $a^3 + b^3 + c^3 - 3abc$.

A 319. By symmetry, one factor must be (a + b + c) and another facot must contain squared terms and terms of the form -ab so that in the product, terms of the form a^2b will vanish, so $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

[[There's an obvious misprint here, as well as the relevant following comment by Murray. — R.]]

Comment by M. S. Klamkin, SUNY at Buffalo, New York. The argument used in obtaining the factorization of Q 319, i.e.,

$$a^3 + b^3 + c^3 - 3abc$$

is invalid in general. It works here since the given polynomial, coincidentally, has a pair of symmetric factors. While it is easy to establish that the only factorizations of homogeneous polynomials are into homogeneous polynomials, it is not true that symmetric polynomials factor into symmetric polynomials. Two obvious counter-examples are

$$x^{2}y^{2} + x^{3} + y^{3} + xy = (x^{2} + y)(y^{2} + x) \text{ and}$$
$$xy^{2} + x^{2}y + yz^{2} + y^{2}z + zx^{2} + z^{2}x + 2xyz = (x + y)(y + z)(z + x)$$

Another method which is often useful for finding symmetric but not necessarily homogeneous factors is the following:

Let a, b, c be the roots of

$$x^3 - px^2 + qx - r = 0$$

Then

$$\sum a^{2} = p^{2} - 2q$$
$$\sum a^{3} = p \sum a^{2} - q \sum a + 3r = p^{3} - 3pq + 3r$$

Whence

$$a^{3} + b^{3} + c^{3} - 3abc = p^{3} - 3pq = p(p^{2} - 3q) = \left(\sum a\right) \left(\sum (a^{2} - bc)\right)$$

Math. Mag., **37**(1964) 286, 252.

Q 340. Submitted by M. S. Klamkin

Determine $\{a_r\}$ such that

$$[a_0 + a_1 + a_2 + \cdots]^x = a_0 + a_1 x + a_2 x^2 + \cdots$$

A 336.

$$[a_0 + a_1 + a_2 + \cdots]^x = e \log S = 1 + \frac{x \log S}{1!} + \frac{x^2 (\log x)^2}{2!} + \cdots$$

where $S = \sum a_r$. Consequently we have

$$a_0 = 1$$
 $a_2 = \frac{a_1^2}{2!}$... $a_r = \frac{a_1^r}{r!}$

and $S = e^{a_1}$

Math. Mag., 37(1964) 360, 321.

Q 344. Submitted by M. S. Klamkin and D. J. Newman [[Not 'J. D.']]

If \bar{r} denotes the mean distance between two random points in a sphere of radius r (with uniform distribution with respect to volume), show that $3r/2 > \bar{r} > 3r/4$.

A 348. Let A' denote the image of A with respect to O, the center of the sphere. Then OA + OB > AB (in general). Whence

$$\overline{OA} = \overline{OB} = \int_0^r r \cdot 4\pi r^2 \, dr \div \int_0^r 4\pi r^2 \, dr = 3r/4$$

Also $\overline{AB} + |overlineAB' > 2\overline{OA}$. Since $\overline{AB} = \overline{AB'}$ the inequalities follow. [[not 'follows']]

Math. Mag., 37(1964) 360, 321.

Q 348. Submitted by M. S. Klamkin

If three forces are in equilibrium they must be coplanar and concurrent.

A 348. If two of the forces are skew, it would be possible to get a nonzero moment about an axis intersecting these two axes. Consequently, these two forces must lie in a plane and intersect (possibly at infinity). Then the third force (by moments) must lie in this plane and be concurrent to the other two.

[[It may not be immediately apparent that you can get your nonzero moment — e.g., if the three forces were in three members of one family of generators of a hyperboloid of one sheet, then you might keep trying generators from the other family and get a zero moment. Also the case of three parallel, coplanar forces doesn't seem to be dismissed. — R.]] Math. Mag., 38(1965) 53.

577. Proposed by M. S. Klamkin, SUNY at Buffalo and L. A. Shepp, Bell Telephone Laboratories

[[Shepp's name was added in an erratum at *Math. Mag.*, **39**(1966) 127.]]

Show that if $x_n \ge x_{n-1} \ge \cdots \ge x_2 \ge x_1 \ge 0$ then $x_1^{x_2} x_2^{x_3} \cdots x_n^{x_1} \ge x_2^{x_1} x_3^{x_2} \cdots x_1^{x_n}$ for $n \ge 3$, with equality holding only if n-1 of the numbers are equal.

Math. Mag., 38(1965) 249–250.

Solution by L. Carlitz, Duke University. We may assume that $x_1 > 0$. Then the stated inequality is equivalent to

$$\left(\frac{x_2}{x_1}\right)^{x_2} \cdots \left(\frac{x_n}{x_1}\right)^{x_1} \ge \left(\frac{x_2}{x_1}\right)^{x_1} \left(\frac{x_n}{x_1}\right)^{x_{n-1}}$$

[[I'm a bit suspicious of this. — R.]]

We may therefore assume that $x_n \ge \cdots \ge x_2 \ge x_1 = 1$.

For n = 3 put $x_2 = 1 + a$, $x_3 = 1 + b$, where $b \ge a$. The stated inequality becomes $(1+a)^{1+b}(1+b) \ge (1+a)(1+b)^{1+a}$, that is,

$$(1+a)^b \ge (1+b)^a$$
 (1)

This is an immediate consequence of Bernoulli's inequality. Moreover, we have equality if and only if a = b or a = 0.

In the general case, we wish to show that

$$\prod_{s=2}^{n-1} x_s^{x_{s+1}} \cdot x_n \ge x_2 \prod_{s=3}^n x_s^{x_{s-1}}$$

If we put $x_s = 1 + a_s$, $\frac{1}{2} \le s \le n$, this inequality becomes

$$\prod_{s=2}^{n-1} (1+a_s)^{1+a_{s+1}} (1+a_n) \ge (1+a_2) \prod_{s=2}^n (1+a_s)^{1+a_{s-1}}$$
(2)

where $a_n \ge a_{n-1} \ge \cdots \ge a_2 \ge 0$. Then by (1), the left member of (2) is greater than or equal to

$$\prod_{s=2}^{n-1} (1+a_{s+1})^{a_s} \cdot \prod_{s=2}^{N} (1+a_s) = \prod_{s=2}^{n} (1+a_s)^{a_{s-1}} \cdot \prod_{s=2}^{n} (1+a_s)$$
$$= (1+a_2) \prod_{s=2}^{n} (1+a_s)^{1+a_{s-1}}$$

This proves (2).

The condition for equality in (1) is either a = b or a = 0. Thus the condition for equality in (2) is either $a_s = a_{s+1}$ or $a_s = 0$, (s = 2, ..., n-1). Assume that

$$a_2 = \dots = a_k = 0 < a_{k+1} = \dots = a_n \tag{3}$$

then (2) becomes

$$(1+a_n)^{(n-k-1)(1+a_n)+1} = (1+a_2)(1+a_n)^{(n-k-1)(1+a_n)+1}$$

Provided $2 \le k < n$. This gives $a_n = a_2$ which contradicts (3). Hence either $a_2 = \cdots = a_{n-1} = 0$ or $a_2 = \cdots = a_{n-1} = a_n$.

Also solved by the proposer.

[Compare the following item from the MONTHLY:

Amer. Math. Monthly, **76**(1969) 1138.

E 2203^{*}. Proposed by M. S. Klamkin, Ford Scientific Laboratory

It is known that if $0 \le x_1 \le x_2 \le \cdots \le x_n$, $(n \ge 3)$, then

$$x_1^{x_2}x_2^{x_3}\cdots x_n^{x_1} \ge x_2^{x_1}x_3^{x_2}\cdots x_1^{x_n}$$

Are there any other nontrivial permutations $\{a_1\}$ and $\{b_i\}$ of the $\{x_i\}$ such that

$$a_1^{a_2}a_2^{a_3}\cdots a_n^{a_1} \ge b_2^{b_1}b_3^{b_2}\cdots b_1^{b_n}$$
?

Amer. Math. Monthly, 77(1970) 1008–1009.

Solution (adapted) by G. L. Watson, University College, London, England. For n = 3 there is no other nontrivial permutation of the x_i of the form required. For n = 4 there are other solutions. For one such solution, note that $x_3/x_1 \ge 1$, $x_4/x_3 \ge 1$, $x_3 - x_2 \ge 0$, $x_3 - x_1 \ge 0$ imply

$$(x_3/x_1)^{x_3-x_2}(x_4/x_3)^{x_3-x_1} \ge 1$$

whence (upon multiplying both sides by $x_2^{x_4}/x_3^{x_2}$)

$$x_1^{x_2} x_2^{x_4} x_4^{x_3} x_3^{x_1} \ge x_1^{x_3} x_3^{x_2} x_2^{x_4} x_4^{x_1}$$

For n > 4, the possibilities increase rapidly. For example, with n = 5,

$$(x_5/x_2)^{x_4-x_3}(x_2/x_1)^{x_5-x_2} \ge 1$$

implies

$$x_1^{x_2} x_2^{x_3} x_3^{x_5} x_5^{x_4} x_4^{x_1} \ge x_1^{x_5} x_5^{x_3} x_3^{x_2} x_2^{x_4} x_4^{x_1} \quad]]$$

Math. Mag., **38**(1965) 60, 47.

Q 349. Submitted by M. S. Klamkin

Four spheres whose centres are at (x_n, y_n, z_n) . n = 1, 2, 3, 4 are mutually tangent externally. Find their radii.

A 349. It follows that

$$r_i + r_j = \sqrt{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)}$$

[[the last two minuses were printed as pluses. There seem to be other misprints as well. Perhaps the next 'sub i' shd be 'sub 1' ? Can someone check ? Thanks. — R.]]

Whence $2r_i + \sum_{1}^{4} r_i = a_{12} + a_{13} + a_{14}, \sum_{1}^{4} r = \frac{1}{3} \sum_{r,s} a_{rs}, \ 6r_1 = 2(a_{12} + a_{13} + a_{14}) - (a_{23} + a_{34} + a_{42})$

Math. Mag., 38(1965) 60, 47.

Q 353. Submitted by M. S. Klamkin

Solve the functional equation

$$f(x+y)f(x-y) = \{f(x) + f(y)\}\{f(x) - f(y)\}$$

given f has a second derivative.

A 353. Differentiating with respect to x and then with respect to y yields

$$\frac{f''(x+y)}{f(x+y)} = \frac{f''(x-y)}{f(x-y)} =$$
constant.

Whence $f(x) = a \sin mx$, ax, or $a \sinh mx$. This is a sort of a converse to Trickie T 52 by C. F. Pinzka (Vol.35, No.2, March, 1962).

Math. Mag., 44(1971) 113.

Comment by Sid Spital, California State College, Hayward. Answer A 353 is correct but fails to point out that all solutions must be odd. This results from first setting x = 0 in the functional equation: $f(y)f(-y) = f(0)^2 - f(y)^2$ and then setting y = 0: f(0) = 0. Hence f(-y) = -f(y). Math. Mag., **38**(1965) 188, 159.

Q 359. Submitted by M. S. Klamkin

Minimize $\int_0^1 F'(x)^2 dx$ where F(0) = 0 and F(1) = 1.

A 359. By the Schwartz inequality

$$\int_0^1 F'(x)^2 \, dx \int_0^1 \, dx \ge \left\{ \int_0^1 F'(x) \, dx \right\}^2 = 1$$

Math. Mag., 44(1971) 53.

Comment by Sidney Spital, California State College at Hayward. An alternative solution is obtained by letting G(x) = F(x) - x. Then clearly since G(0) = G(1) = 0, we have

$$\int_0^1 (G'(x) + 1)^2 \, dx = \int_0^1 (G'(x))^2 \, dx + 1 \ge 1$$

Math. Mag., 38(1965) 188, 159.

Q 361. Submitted by M. S. Klamkin

Find a geometrical solution for the functional equation $F(2\theta) = F(\theta) \cos \theta/2$.

A 361. $F(\theta)$ denotes the distance the C.G. of a sector of angle 2θ is from the center. Consequently $F(\theta) = \sin \theta / 2\theta$.

Math. Mag., 38(1965) 188, 159.

Q 363. Submitted by M. S. Klamkin

Factor $x^{1}1 + x^{4} + 1$.

A 363. If ω is a primitive cube root of unity, it follows immediately that $\omega^{3m+2} + \omega^{3n+1} = 0$. [[Is it really??]] Consequently $x^2 + x + 1$ is a factor of $x^{3m+2} + x^{3n+1} + 1$. To find other factors, just divide.

[[this needs cleaning up a bit. -R.]]

Math. Mag., **38**(1965) 252, 211. **Q 366**. Submitted by M. S. Klamkin Solve

$$\begin{array}{rcrcrc} x+y+z &=& 3\\ x^2+y^2+z^2 &=& 7/2\\ x^3+y^3+z^3 &=& 9/2 \end{array}$$

A 366. Let x, y, z be the roots of $s^3 + a_1s^2 + a_2s + a_3 = 0$. Then

$$a_1 = 3$$
 $\sum x^2 = \left(\sum x\right)^2 - 2\sum xy$ and $a_2 = 11/4$

Next

$$\sum x^3 + a_1 \sum x^2 + a_2 \sum x + 3a_3 \quad \text{and} \quad a_3 = -3/4$$

The roots of the cubic are 1/2, 2/2 and 3/2.

Math. Mag., **38**(1965) 326, 302.

 \mathbf{Q} 369. Submitted by M. S. Klamkin

Find

$$I_n = D^n \left\{ \arctan \frac{2x^3}{1+3x^2} \right\}_{x=0}$$

A 369. Since

$$\arctan\frac{2x^3}{1+3x^2} = 2\arctan x - \arctan 2x$$

we have $I_{2n} = 0$ and

$$I_{2n+1} = \frac{(-1)^n}{2n-1} (2^{2n-1} - 2)$$

Math. Mag., **38**(1965) 326, 302.

Q 373. Submitted by M. S. Klamkin

Show that e^x is a transendental function.

A 373. Assume that e^x is algebraic, then

$$a_0(x)e^{nx} + a_1(x)e^{(n-1)x} + \dots + a_n(x) = 0$$

where $a_r(x)$ are polynomials. Consequently

$$-a_0(x)e^{x/2} = \frac{a_1(x)e^{(n-1)x} + \dots + a_n(x)}{e^{(n-1/2)x}}$$

Letting $x \to 0$ we obtain a contradiction, whence e^x is transcendental.

 $[[{\rm No}\ {\rm doubt}\ {\rm my}\ {\rm stupidity},\ {\rm but}\ {\rm I}\ {\rm don't}\ {\rm get}\ {\rm this.}\ -{\rm R.}]]$

Math. Mag., **39**(1966) 71–73.

588. [May, 1965]. Proposed by Joseph D. E. Konhauser, HRB-Singer Inc., State College Pennsylvania

Show that the operators $(D-1)^n \times (D-1)$ and $x(D-1)^{n+1} + n(D-1)^n$ are equivalent for $n = 1, 2, 3, \ldots$, where $D \equiv d/dx$.

II. Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. Let L(D) designate any linear differential operator with variable coefficients (it could even be just a function of x). Then by Leibniz's rule for the n th derivative,

$$D^n\{xL(D)\} \equiv xD^nL(D) + nD^{n-1}L(D)$$

Now multiply by $\exp \int p \, dx$ (p is an arbitrary function of x) and use the exponential shift theorem, i.e.,

$$\exp \int p \, dx \, L(D) \equiv L(D-p) \exp \int p \, dx$$

This yields:

$$\{(D-p)^n x L(D-p) - x(D-p)^n L(D-p) - n(D-p)^{n-1} L(D-p)\} \exp \int p \, dx \equiv 0$$

or equivalently

$$(D-p)^n x L(D-p) \equiv x(D-p)^n L(D-p) + n(D-p)^{n-1} L(D-p)$$

The proposed problem corresponds to the special case L(D) + D, p = 1.

Math. Mag., **39**(1966) 76, 42–43. **Q 377**. Submitted by M. S. Klamkin Solve the difference equation

$$P_{n+1} - 2P_n + (1+x^2)P_{n-1} = 0$$

where $P_0 = a(x)$ and $P_1 = b(x)$. A 377. Let $xQ_n = P_{n+1} - P_n$, then

$$Q_{n+1} = Q_n - xP_n$$

Now let $F_n = P_n + iQ_n$, which gives us

$$F_{n+1} = (1 - ix)F_n$$

 $F_0 = b(x) + i[b(x) - a(x)]/x$

and

$$F_n = (1 - ix)^n F_0$$

where P equals the real part of F_n .

Math. Mag., **39**(1966) 134, 111.

Q 378. Submitted by M. S. Klamkin and W. J. Miller

Find the average area of all triangles which can be inscribed in a given triangle. (It is assumed that the vertices are uniformly distributed over the sides of the given triangle.)

A 378. (1) Analytic solution.

$$\bar{A} = \frac{1}{abc} \int_0^a \int_0^b \int_0^c \{A - \frac{1}{2} [z(b-y)\sin A + x(c-z)\sin B + y(a-x)\sin C]\} \, dx \, dt \, dz$$

 $\overline{A} = A/4$ where A is the area of the given triangle.

(2) Geometric solution. If a series of triangles have a common base and their vertices be in a given finite straight line which wholly on the same side of the base, the average [[area]] of all triangles thus formed is that [[of]] whose vertex is at the middle of the line segment; since for every triangle which exceeds this, there is obviously another just as much less than it. Consequently the mean-inscribed triangle is one joining the midpoints of the sides, and $\bar{A} = A/4$.

Math. Mag., **39**(1966) 187.

624. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan

Show that a sufficient condition for a sphere to exist which intersects each four given spheres in a great circle is that the centers of the four given spheres be noncoplanar.

Math. Mag., 40(1967) 47–48.

Solution by P. N. Bajaj, Western Reserve University. Let the given spheres have equations

$$x^{2} + y^{2} + z^{2} + 2u_{i}x + 2v_{i}y + 2w_{i}z + d_{i} = 0 \qquad i = 1, 2, 3, 4$$

referred to rectangular coordinates. Sphere $x^2 + y^2 + z^2 + 2Ux + 2Vy + 2Wz + P = 0$ cuts these in the circles lying in the planes

$$2(U - u_i)x + 2(V - v_i)y + 2(W - w_i)z + () = 0 \qquad i = 1, 2, 3, 4$$

If the circles of intersection are great circles, then

$$-2(U - u_i)u_i - 2(V - v_i)v_i - 2(W - w_i)w_i + (D - d_i) = 0 \qquad i = 1, 2, 3, 4$$

or
$$2Uu_i + 2Vv_i + 2Ww_i - D = 2u_i^2 + 2v_i^2 + 2w_i^2 - d_i \qquad i = 1, 2, 3, 4$$

A sufficient condition for these equations to determine U, V, W, D is

$$\det \begin{vmatrix} u_1 & v_1 & w_1 & 1 \\ u_2 & v_2 & w_2 & 1 \\ u_3 & v_3 & w_3 & 1 \\ u_4 & v_4 & w_4 & 1 \end{vmatrix} \neq 0$$

i.e., centers of the given spheres are nonplanar. Hence the result.

Math. Mag., 39(1966) 257, 226.
Q 388. Submitted by M. S. Klamkin
Evaluate in closed form the integral

$$\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}}$$

A 388. Consider

$$I = \int \frac{dx}{x + x^m} = \int frac x^{-m} dx x^{1-m} + 1$$

= $\frac{1}{1 - m} \log(x^{1-m} + 1)$

Thus

$$\int_{\sqrt{2}}^{\infty} \frac{dx}{x + x^{\sqrt{2}}} = (\sqrt{2} - 1) \log[1 + 2^{(1 - \sqrt{2})/2}]$$

Math. Mag., **39**(1966) 315, 281.

Q 393. Submitted by M. S. Klamkin Sum

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{2^n}+1}$$

[[the exponent 2^n was misprinted as 2n — R.]]

A 393. Here

$$\frac{2^n}{3^{2^n}+1} = \frac{2^n}{3^{2^n}-1} - \frac{2^{n+1}}{3^{2^{n+1}}-1}$$

So the sum S = 1/2

Math. Mag., **39**(1966) 315, 281.

Q 397. Submitted by M. S. Klamkin

Determine

$$\lim_{n \to \infty} \frac{1}{n} \sqrt[n]{(n+1)(n+2)\cdots(n+n)}$$

A 397.

$$\log L = \lim_{n \to \infty} \frac{1}{n} \left[\log \left(1 + \frac{1}{n} \right) + \log \left(1 + \frac{2}{n} \right) + \dots + \log \left(1 + \frac{n}{n} \right) \right]$$

$$\log L = \int_0^1 \log(1+x) \, dx = 2 \log 2 - 1$$

from the definition of a definite integral. Therefore L = 4/e.

Math. Mag., **41**(1968) 49.

Comment by S. Spital, California State Polytechnic College, Pomona. An alternative solution is provided by a generating power series. Let $a_n = (n+1)(n+2)\cdots(n+n)/n^n$ and consider

$$\sum_{n=0}^{\infty} a_n x_n$$

From the ratio and root tests,

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} a_n + 1/a_n$$

=
$$\lim_{n \to \infty} \frac{(2n+1)(2n+2)}{(n+1)^2(1+1/n)^n}$$

= $4/e$

Math. Mag., 41(1968) 294–295.

Comment by Eckford Cohen, Manhattan, Kansas. This limit can also be evaluated by applying a weak form of Stirling's formula. We may write

$$c_n \stackrel{def}{=} \frac{1}{n} ((n+1)(n+2)\cdots(n+n))^{1/n} = 4\left(\frac{a_{2n}^2}{a_n}\right)$$

where $a_n = \sqrt[n]{n!}/n$. It follows that

$$\lim_{n \to \infty} c_n = \frac{4}{e}$$

from the well-known result, $\lim_{n\to\infty} a_n = 1/e$. The latter result can be proved in a number of ways. For a simple proof based on the exponential function, we refer to S. Saks & A. Zygmund, *Analytic Functions*, Chapter 7, Section 5.

Math. Mag., 40(1967) 52–53.

Comment on Problem 612

612. [January & September, 1966]. Proposed by M. B. McNeil, University of Missouri at Rolla

The integral

$$I_{1} = \frac{1}{\pi^{3}} \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} \frac{du \, dv \, dw}{1 - \cos u \cos v \cos u}$$

occurs in the study of ferromagnetism and in the study of lattice vibrations. Prove that

$$I_1 = (4\pi^3)^{-1} [\Gamma(1/4)]^4$$

Comment by William D. Fryer, Cornell Aeronautical Laboratory, Buffalo, N.Y., and Murray S. Klamkin, Scientific Laboratory, Ford Motor Company, Dearborn, Michigan. The sum

$$S = \sum_{n=0}^{\infty} \left\{ \frac{1}{2^{2n}} \binom{2n}{n} \right\}^3$$

occurs in a combinatorial probability problem [1]. We evaluate the sum by two methods and obtain as a by-product some interesting equivalent expressions.

Since

$$\binom{2n}{n} = \frac{2}{\pi} \int_0^{\pi/2} (2\cos\theta)^{2n} d\theta$$
(1)
$$S = \frac{4}{\pi^2} \sum_{n=0}^\infty \frac{1}{2^{6n} \binom{2n}{n}} \int_0^{\pi/2} \int_0^{\pi/2} (4\cos\theta\cos\psi)^{2n} d\theta d\psi$$
(2)

By using

$$\sum_{0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

(2) becomes

$$S = \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{d\theta \, d\phi}{\sqrt{1 - \cos^2 \theta \cos^2 \phi}}$$
$$= \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{d\theta \, d\phi}{\sqrt{1 - \sin^2 \theta \sin^2 \phi}}$$

or, in terms of the complete elliptic function of the first kind,

$$S = \frac{4}{\pi^2} \int_0^{\pi/2} K(\sin\theta) \, d\theta$$
$$= \frac{4}{\pi^2} \int_0^1 \frac{K(k) \, dk}{\sqrt{1-k^2}}$$

The last integral is given in [2, p.637] as

$$S = \frac{4}{\pi^2} K \left(\frac{1}{\sqrt{2}}\right)^2$$

Identities leading to equivalent hypogeometric or gamma functions may be found in the same reference (pp.905, 909). Whence, also,

$$S = \frac{1}{4\pi^2} \Gamma(\frac{1}{4})^4 = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2})^2 = 1.393203929685^+$$

The sum S was also obtained as a by-product in establishing

$$I = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{du \, dv \, dw}{1 - \cos u \cos v \cos w} = \frac{1}{4\pi^2} \Gamma(\frac{1}{4})^4$$

which is problem 612 in the *Mathematics Magazine* (January-February, 1966) due to M. B. McNeil.

1. W. Feller, An Introduction to Probability Theory and its Applications, Wiley, New York, 1950.

2. I. M. Ryshik & I. S. Gradstein, *Tables of Series, Products and Integrals*, Academic Press, New York, 1965.

Math. Mag., 40(1967) 54, 30.

Q 400. Submitted by M. S. Klamkin

Find the general solution to the differential equation

$$\{D^n x^{2n} D^n - x^n D^{2n} x^n + \lambda^{2n-1}\}y = 0$$

A 400. The only solution is y = 0 since $D^n x^{2n} D^n \equiv x^n D^{2n} x^n$. This follows from $D^m x^m = x^m D^m + a_1 x_{m-1} + \dots + a_m$ by Leibniz's theorem, $x^r D^r = x D(xD-1\cdots(xD-r+1))$. Since $xD - k_1$ commutes with $xD - k_2$, $D^m x^m$ commutes with $x^n D^n$ or $D^m x^{m+n} D^n \equiv x^n D^{m+n} x^m$.

Math. Mag., 40(1967) 101.

655. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan

It is easy to show that any two spheres are homothetic, regardless of their orientation. Show that this property characterizes spheres; that is, if two bounded figures are homothetic, regardless of their orientation, then they both must be spheres.

Math. Mag., 40(1967) 287–288.

Solution by Pierre Bouchard, Université de Montréal, Canada. It is easy to show that $\{x : |x| \in (1,2), x \in \mathbb{R}^3\}$ and $\{x : |x| \in (3,6), x \in \mathbb{R}^3\}$ are not spheres and are homothetc, regardless of their orientation. This negates the proposal as stated. However, we can prove that the given figures must have a frontier which is the union of a set S of concentric spheres, the cardinality of S being the same in each figure. But "seen" from "outside the bounds" they look like spheres. We proceed to prove this last fact.

Let F_1 and F_2 be the "exterior frontiers" of the given figures in a given position:more precisely $F_i = \{x : \exists y \text{ such that } |y| = 1 \text{ and } x = \sup z_{z=cxy \in \mathcal{F}_i}\}$ where i = 1, 2 and \mathcal{F}_i is the *i*th figure $c \in R$. First remark that for every *y* on the unit sphere there is a corresponding *x* (because of the "regardless of orientation"; otherwise the figures would be unbounded or void). Since \mathcal{F}_1 and \mathcal{F}_2 are homothetic regardless of orientation, so are F_1 and F_2 (since affine homothety is translation or central homothety we may restrict ourselves to central homothety).

Let P_1 , P_2 be in F_1 . Then there is an α in R such that αP_1 , αP_2 are in F_2 . Let r be a rotation such that $r(\alpha P_1)$ is on the line OP_2 and $r(\alpha P_2)$ is on the line OP_1 (i.e., r is the rotation of π with respect to the axis passing through O and $\frac{1}{2}(P_1/|P_1| + P_2/|P_2|)$). Then F_1 and $r(F_2)$ are homothetic and since $\sup z_{z\in\mathcal{F}} z = cy$ is unique we must have $r(\alpha P_1) = \beta P_2$, $r(\alpha P_2) = \beta P_1$. Since r preserves distances,

[[I hope that I haven't made more typos than I've corrected — especially since I'm not sure that I understand the notation or the argument. Would someone check? Thanks! — R.]]

$$\beta = |r(\alpha P_1)|/|P_2| \\ = |\alpha(P_1)|/|P_2| \\ = \alpha|P_1|/|P_2|$$

and

$$\beta = |r(\alpha P_2)|/|P_1| \\ = |\alpha(P_2)|/|P_1| \\ = \alpha|P_2|/|P_1|$$

Whence $|P_1|/|P_2| = |P_2|/|P_1|$, or $|P_2| = |P_1|$. That is F_1 is a sphere so is F_2 (homothetic image of a sphere).

Klamkin suggested that the counterexample exhibited by Bouchard could be eliminated by adding to the statement of the problem the qualifying statement, "bounded closed convex figures".

Math. Mag., 40(1967) 110, 85.

Q 405. Submitted by M. S. Klamkin

It is apparent that a bounded figure need not have a unique chord of maximum length. Show, however, that two such maximum chords cannot be parallel.

A 405. The proof is indirect. Assume two congruent and parallel chords of maximum length. The endpoints of these chords are the vertices of a parallelogram, one of whose diagonals, at least, is larger than all the sides. This contradicts our initial assumption and, consequently, we obtain our stated result.

Math. Mag., 40(1967) 164.

Greatest Divisors of Even Integers

636. [November, 1966]. Proposed by Vassili Daiev, Sea Cliff, New York

The greatest divisors of the form 2^k of the numbers of the sequence 2, 4, 6, 8, 10, 12, 14, ..., are 2, 2^2 , 2, 2^3 , 2, 2^2 , 2, ... Find the *n* th term of this sequence.

II. Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. If a_n denotes the nth term, then it follows immediately that

$$\begin{array}{rcrcrcr} a_{2n+1} & = & 2^1 \\ a_{4n+2} & = & 2^2 \\ a_{8n+4} & = & 2^3 \\ \vdots & \vdots \end{array}$$

In general

$$a_{2^rm+2^{r-1}} = 2^r$$

Note that every number n can be expressed uniquely in the form $2^r m + 2^{r-1}$.

Math. Mag., 40(1967) 167.

A Convex Curve Property

641. [November, 1966]. Proposed by Yasser Dakkah, S.S. Boys' School, Qalqilya, Jordan

Prove that if

$$\sum_{i=1}^{n} x_i = S$$

and $0 < x_i$ (i = 1, 2, ..., n), then

$$\sum_{i=1}^{n} \cosh x_i \ge \cosh(S/n)$$

I. Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. The result follows immediately from the well known inequality for convex functions, i.e.,

If $\phi(x)$ is convex then

$$\frac{\Phi(x_1) + \phi(x_2) + \dots + \phi(x_n)}{n} \ge \phi\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$$

Since $\cosh x$ is everywhere convex $(D^2 \cosh x > 0)$, just replace $\phi(x)$ by $\cosh x$ to give the desired result. Note that there is noncessity for the restriction $x_i > 0$.

Math. Mag., 40(1967) 232, 199.

Q 414. Submitted by M. S. Klamkin

How many primes p exist such that p, p + 2d and p + 4d are all primes where d is not divisible by 3?

A 414. Now p must be of the form 3, 3m + 1 or 3m + 2, while d must be of the form 3n + 1 or 3n + 2. Going through the six possibilities, we find there is only one prime, p = 3.

Math. Mag., 40(1967) 289, 254.

Q 416. Submitted by M. S. Klamkin

Determine the range of the function I(t) where

$$I(t) = \int_0^\infty \frac{dx}{(x^2 + 1)(x^t + 1)}$$

A 416.

$$I(t) = \int_0^1 \frac{dx}{(x^2+1)(x^t+1)} + \int_1^\infty \frac{dy}{(y^2+1)(y^t+1)}$$

In the second integral let y = 1/x. We then obtain

$$I(t) = \int_0^1 \frac{dx}{x^2 + 1} = \pi/4$$

Thus the range of I(t) = 0. This integral appears in Induction and Analogy in Mathematics, by G. Pólya.

Math. Mag., **41**(1968) 43.

683. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan

Two triangles have sides $\sqrt{a^2 + b^2}$, $\sqrt{b^2 + c^2}$, $\sqrt{c^2 + a^2}$ and $\sqrt{p^2 + q^2}$, $\sqrt{q^2 + r^2}$, $\sqrt{r^2 + p^2}$. Which triangle has the greater area if in addition we have $a^2b^2 + b^2c^2 + c^2a^2 = p^2q^2 + q^2r^2 + r^2p^2$ and a > p, b > q?

Math. Mag., **41**(1968) 221.

I. Solution by Michael Goldberg, Washington, D.C. The tri-rectangular tetrhedron whose right-angled edges have lengths a, b, c has the lengths $\sqrt{a^2 + b^2}$, $\sqrt{b^2 + c^2}$ and $\sqrt{c^2 + a^2}$ for its other edges. Since the squares of the areas of the right-triangle faces add to the square of the area of the fourth face, the square of the area of the fourth face is $(a^2b^2 + b^2c^2 + c^2a^2)/4$. Similarly, for a tri-rectangula tetrahedron whise right-angled edges have lengths p, q, r, the square of the area of the fourth face is $(p^2q^2 + q^2r^2 + r^2p^2)/4$. But since we are told that $a^2b^2 + b^2c^2 + c^2a^2 = p^2q^2 + q^2r^2 + r^2p^2$, the two triangles have the same area, regardless of the relations between a, p, b and q.

Math. Mag., 41(1968) 50, 42.

Q 423. Submitted by M. S. Klamkin

Can one find a number (to base 10) which doubles itself on reversing its digits?

A 423. No. Let the number be of the form $a \cdots b$ then $b \cdots a = 2(a \cdots b)$. Now a can be 0, 1, 2, 3 or 4 and corresponding to these values b can be (0), (2,3), (4,5), (6,7) or (8,9) respectively. By comparing the last digits, none of these are possible.

Math. Mag., 41(1968) 166, 133.

Q 430. Submitted by M. S. Klamkin

Find a "simple" n th term formula for the sequence 0, 1, -1, 0, 0, -1, 1, o, 0, 1, -1, 0, 0, -1, 1, 0, 0, 1, -1,

A 430. One possible answer is; $\sin \pi (n^2 - n)/4$.

Math. Mag., 41(1968) 223, 191.

Q 435. Proposed by Irving Gerst and M. S. Klamkin

Evaluate the ratio

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n^2}}{(1-x)(1-x^3)\cdots(1-x^{2n+1})}$$

divided by

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n^2+n}}{(1-x^2)(1-x^4)\cdots(1-x^{2n+2})}$$

A 435. The ratio is one, since each sum is one. This follows from the known simple summation

$$1 = \frac{1}{1 - a_1} - \frac{a_1}{(1 - a_1)(1 - a_2)} + \frac{a_1 a_2}{(1 - a_1)(1 - a_2)(1 - a_3)} - \cdots$$

[[I'm not sure about the signs here — would someone check? Thanks! — R.]]

Math. Mag., 41(1968) 175–181.

[[There's an article by Murray:

On the volume of a class of truncated prisms and some related centroid problems.]]

Math. Mag., 41(1968) 285-286.

Fermat's Principle

685. [March, 1968]. Proposed by Jack M. Elkin, Polytechnic Institute of Brooklyn

Prove Fermat's Principle for a circular mirror. That is, given two points A and B inside a circle, locate P such that AP + PB is an extremum.

I. Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. The problem is not formulated correctly. For the case of a constant refractive medium, Fermat's principle states that the path of light is such that AP + PB is an extremum. Thus we will not be proving Fermat's principle by locating P such that AP + PB is an extremum. Perhaps the proposer wishes to establish the law of reflection from Fermat's principle or conversely. In either case, this is a well-known problem and a simple solution follows by the use of level lines. Also, for greater generality, we can just as easily use an arbitrary convex closed curve with continuous curvature.

Consider the family of curves AP + PB = k (constant). These are ellipses having A and B as foci. Clearly, the minimum occurs (possibly at more than one point) for the smallest ellipse of the family which is tangent to the given curve. The point (or points) of tangency will correspond to the minimizing point P. Then since the focal radii make equal angles with any tangent line, we obtain the law of reflection. Similarly, the maximum occurs for the largest ellipse of the family which is tangent to the given curve.

The converse theorem follows just as easily. IF \overline{AP} and \overline{PB} make equal angles with the given curve, then the ellipse with foci at A and B and which passes through Pmust be tangent to the given curve at P. Either the ellipse will be locally inside the given curve at P or outside of it. In the former case AP + PB will a local minimum and in the latter case a local maximum.

II. Comments by Leon Bankoff, Los Angeles, California. This is essentially the Billiard Problem of Alhazan (965–1039 A.D.), which appears as Problem 41 in Dörrie's "100 Great Problems of Elementary Mathematics" (Dover Reprint, N.Y., 1965). In its optical application, the problem is associated with Fermat's principle that "nature always acts by the shortest path". A solution to an analogous problem is given on p.73 of the 1869 issue of the "Lady's and Gentleman's Diary", a source of reference somewhat less accessible than the work of Dörrie. The European reader may prefer to consult the French counterpart of Dörrie's book, "Célèbres Problèmes Mathématiques", by Édouard Callandreau, Éditions Albin Michel, Paris, 1949, p.305, Problem 71, or Dörrie's German text, "Triumph der Matematik", Physica-Verlag, Würzburg, 1958.

Scholarly enthusiasts who are not allergic to the dust of obscure library shelves may enjoy delving into Volume I of Leybourn's "Diary Questions", pp.167–169, which gives three solutions originally published in the "Ladies' Diary" for 1727–1728.

Most of the published solutions involve one of the four intersections of the given circle with the equilateral hyperbola whose diameter is AB and whose ordinate axis is parallel to the line connecting the inverses of A and B with respect to the given circle.

One of the solutions in the 1869 "Diary" locates the point P as the point of tangency of the given circle with one of the family of confocal ellipses whose foci are A and B. In the proposed problem, both A and B lie within the circle. Hence the required ellipse lies entirely within the circle and touches the circle at the point P on the circumference for which AP + PB is a minimum.

The location of P by means of conic sections precludes the possibility of a construction with Euclidean tools, except in the trivial case where A and B lie on a circle concentric with the given circle. In that case, P lies on the perpendicular bisector of the line joining A and B, and is easily found by ruler and compass or by a Mascheroni construction with compass alone.

An interesting sidelight mentioned in the 1869 'Diary" is that "this question occurs in the construction of steam boilers. The brace in the form of A'P, B'P, OP (where A" and B' are the inverses of A and B with respect to the given circle whose center is O) is stronger when the angle A'PB' is bisected by OP." Math. Mag., 41(1968) 287–288.

Central Symmetry

687. [March, 1968]. Proposed by Sidney H. L. Kung, Jacksonville University, Florida

Prove that if the perimeter of a quadrilateral ABCD is cut into two portions of equal length by all straight lines oassing through a fixed point O in it, the quadrilateral is a parallelogram.

II. Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. We consider a more general problem wherewe have a closed curve which is starlike with respect to the fixed point O and has the same perimeter property as the quadrilateral.

By the perimeter property,

$$R \, d\theta = S \, d\theta$$

or the curve must be centro-symmetric with respect to O. If C is a quadrilateral, it follows that it must then be a parallelogram.

$R d\theta$
R
0
S

Math. Mag., 42(1969) 223. $S d\theta$

Comment by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. 1. In the statement of the problem, "quadrilateral" should be replaced by "simple quadrilateral"; otherwise we could have Figure 1 as a solution. O C R O S

2. The second solution given by myself is erroneous. The sophomoric error is in the equation $R d\theta = S d\theta$. This equation should have been

$$R^{2} + \left(\frac{dR}{d\theta}\right)^{2} = S^{2} + \left(\frac{dS}{d\theta}\right)^{2}$$

It now does not necessarily follow that R = S to give a centrosymmetric figure. As a nice counterexample, consider Figure 2 made up from three semicircles. Every line through O bisects the perimeter. It would be of interest to find a noncentrosymmetric convex counterexample. However, if we restrict the figure to be a simple polygon, then Goldberg's solution implies that the polygon is centrosymmetric. Math. Mag., **41**(1968) 295.

Comment on Q 426

Q 426. Without using calculus, determine the least value of the function f(x) = (x+a+b)(x+a-b)(x-a+b)(x-a-b), where a and b are real constants. [Submitted by Roger B. Eggleton]

Comment by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. A more direct solution can be obtained by noting that

$$f(x) = (a+b+x)(a+b-x)(a-b+x)(a-b-x)$$

= $((a+b)^2 - x^2)((a-b)^2 - x^2)$
= $(a^2 + b^2 - x^2)^2 - 4a^2b^2$

Thus the minimum is $-4a^2b^2$ for $x^2 = a^2 + b^2$.

[[On p.224 there's an almost identical comment by S. Spital, California State College at Hayward.]]

Math. Mag., 42(1969) 221–222.

Integral Distances

718. [January, 1969]. Proposed by A. H. Lumpkin, East Texas State University

In $R \times R$ with the usual metric, if G is an infinite subset of $R \times R$ such that for all x, y in G, d(x, y) is an integer, then $G \subseteq l$ for some line l.

II. Comment by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. This is a known result due to Paul Erdős, Integral distances, Bull. Amer. Math. Soc., 51(1945) 996. See also H. Hadwiger, H. Debrunner & V. Klee, Combinatorial Geometry in the Plane, Holt Reinhart & Winston, New York, 1964, p.5.

As shown in the latter reference, the result does not imply the existence of a number k_0 such that the conclusion also holds when the number k of points with exclusively integral distances is greater than k_0 .

Additionally, there's no need for the "symbolic" language of the proposal. It could have been stated simply as: "If an infinite set of points in the plane is such that all of its points are at integral distances from each other, then all the points lie on a single line." Math. Mag., **42**(1969) 268.

746. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, and Morris Morduchow, Polytechnic Institute of Brooklyn

Determine the extreme values of $S_1/r + S_2/(n-r)$ where n is a fixed integer, $S_1 = p_1 + p_2 + \cdots + p_r$

$$S_1 + S_2 = \sum_{i=0}^{n-1} i$$

and the ps are distinct integers in the interval [0, n-1].

Math. Mag., 43(1970) 171–172.

Solution by L. Carlitz, Duke University. We may assume without loss of generality that $r \leq n/2$. We will show that

$$\frac{1}{2}(n-2r-2) \le A \le \frac{1}{2}(3n-2r-2) \qquad \left(1 \le r \le \frac{n}{2}\right) \tag{(*)}$$

where $A = S_1/r + S_2/(n-r)$.

Proof. If

$$S_1 = 0 + 1 + 2 + \dots + (r - 1) = \frac{1}{2}r(r - 1)$$

$$S_2 = r + (r + 1) + \dots + (n - 1) = \frac{1}{2}n(n - 1) - \frac{1}{2}r(r - 1)$$

$$= \frac{1}{2}(n - r)(n + r - 1)$$

then

$$A = \frac{1}{2}(n+2r-2)$$
 (1)

If

$$S_{1} = (n-r) + (n-r+1) + \dots + (n-1)$$

= $\frac{1}{2}n(n-1) - \frac{1}{2}(n-r)(n-r-1)$
= $\frac{1}{2}r(n-2r-1)$
$$S_{2} = 0 + 1 + \dots + (n-r-1) = \frac{1}{2}(n-r)(n-r-1)$$

then

$$A = \frac{1}{2}(3n - 2r - 2) \tag{2}$$

Now let

$$S_1 = p_1 + \dots + p_r$$
 $(p_1 < p_2 < \dots < p_r)$ $S_1 + S_2 = \sum_{k=0}^{n-1} k$ (3)

where p_1, p_2, \ldots, p_r are any r distinct numbers in $[0, 1, \ldots, n-1]$. Assume that the corresponding values of A satisfies (*). Let $a \in S_1, b \in S_2$ and put

$$S'_1 = S_1 - a + b$$
 $S'_2 = S_2 + a - b$

Then

$$A' = \frac{1}{r}(S_1 - a + b) + \frac{1}{n - r}(S_2 + a - b) = A + \frac{n - 2r}{r(n - r)}(b - a)$$

Hence if b > a it follows that $A' \ge A$ (with strict inequality except when n = 2r). Thus setting out with

$$S_1 = 0 + 1 + \dots + (r - 1)$$
 $S_2 = r + (r + 1) + \dots + (n - 1)$

it is clear that after a number of interchanges $a \leftrightarrow b$ we get S_1 , S_2 as in (3) and that the corresponding A satisfies

$$A \ge \frac{1}{2}(n+2r-2)$$

Similarly, starting with

$$S_1 = (n-r) + (n-r+1) + \dots + (n-1)$$

$$S_2 = 0 + 1 + \dots + (n-r-1)$$

then again after a number of interchanges $a \leftrightarrow b$ we get S_1, S_2 as in (3) and that the corresponding A satisfies

$$A \le \frac{1}{2}(3n - 2r - 2)$$

Math. Mag., 42(1969) 277, 244.

Q 466. Submitted by M. S. Klamkin

If AB and BA are both identity matrices, then A and B are both square matrices.

A 466. Let A be $m \times n$ and B be $n \times m$ with $m \ge n$. The rank of AB is m. But since the rank of a product of two matrices cannot exceed the rank of either of the two matrices which is at most n, we must have $m \le n$. Thus m = n. The result is also valid if AB and BA are both nonsingular scalar matrices, e.g., if ABC, CAB and BCA are all nonsingular scalar matrices, then they are all square matrices.

Math. Mag., 43(1970) 290.

Comment by J. L. Brenner, University of Arizona. An alternative proof is the following. We first note the equations

$\begin{bmatrix} \lambda I \\ 0 \end{bmatrix}$	$\begin{bmatrix} A \\ \lambda I \end{bmatrix} \begin{bmatrix} \lambda I \\ -B \end{bmatrix}$	$\begin{bmatrix} -A\\ \lambda I \end{bmatrix} \begin{bmatrix} \lambda I\\ B \end{bmatrix}$	$\begin{bmatrix} 0\\ \lambda I \end{bmatrix}$	=	$\begin{bmatrix} \lambda^3 I - AB \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0\\ \lambda^3 I \end{bmatrix}$
$\begin{bmatrix} \lambda I \\ B \end{bmatrix}$	$\begin{array}{c} 0\\ \lambda I \end{array} \begin{bmatrix} \lambda I\\ -B \end{array}$	$\begin{bmatrix} -A\\\lambda I \end{bmatrix} \begin{bmatrix} \lambda I\\0 \end{bmatrix}$	$\begin{bmatrix} A \\ \lambda I \end{bmatrix}$	=	$\begin{bmatrix} \lambda^3 I & 0 \\ 0 & \lambda^3 I - \end{bmatrix}$	$\cdot BA$

[[One of the above λ^3 s was misprinted as λ^2]]

If $AB = I_r$, $BA = I_s$ then using $\det(XYZ) = \det X \det Y \det Z = \det(ZYX)$, $(\lambda^2 - 1)^r (\lambda^3) s = (\lambda^3)^r (\lambda^2 - 1) s$.

This cannot be valid for all λ except if r = s, which is what was to be shown. The only thing this proof uses is $\det(XY) = \det X \det Y = \det Y \det X$. The proof is therefore slightly more powerful than the one given on page 244.

[[Math. Mag., 43(1970) 165. Murray is listed as Associate Editor of the Problems Section.]]

Math. Mag., 43(1970) 236, 185.

Q 483. Submitted by M. S. Klamkin

If A, B, C are the angles of a triangle such that

$$\tan(A-B) + \tan(B-C) + \tan(C-A) = 0$$

then the triangle is isosceles.

A 483. Expanding out and replacing $\tan A$, $\tan B$ and $\tan C$ by a, b and c respectively we get

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0$$

On combining fractions and factoring we obtain

$$(a-b)(b-c)(c-a) = 0$$

and thus the triangle is isosceles. Note [that] the condition that $A + B + C = \pi$ is not necessary.

Math. Mag., 43(1970) 272–275.

[[There's an article by Murray:

On some soluble N th order differential equations.]]

Math. Mag., 44(1971) 55, 16.

Q 494. Submitted by M. S. Klamkin

In a given sphere APB, CPD and EPF are three mutually perpendicular and concurrent chords. If AP = 2a, BP = 2b, CP = 2c, DP = 2d, EP = 2e and FP = 2f, determine the radius of the sphere.

A 494. If we choose a rectangular coordinate system whose axes are along the three given chords, then the center is is at the point (b-a, d-c, f-e) where we are assuming without loss of generality that $b \ge a$, $d \ge c$ and $f \ge e$. Then [[The eff in the next equation was misprinted as bee]]

$$R^{2} = (b - a - 2b)^{2} + (d - c - 0)^{2} + (f - e - 0)^{2}$$

= $a^{2} + b^{2} + c^{2} + d^{2} + e^{2} + f^{2} - 2ef$

(since ab = cd = ef).

It is to be noted that the result is easily extended for the case of an *n*-dimensional sphere. For the special case of the circle (n = 2), the cross terms disappear.

Math. Mag., 44(1971) 55, 17.

Q 498. Submitted by M. S. Klamkin

Given a cake in the form of a triangular layer (prism) which is covered with a thin layer of icing on its top and sides. Show how to divide the cake into eleven portions so that each portion contains the same amount of cake and icing.

A 498. Divide the triangle perimeter into eleven equal parts and make vertical cuts emanating from the center of the inscribed circle to these points of division. This problem for a square appears in H. S. M. Coxeter, *Introduction to Geometry* and the method is valid for any polygonal layer cake having an incircle.

Math. Mag., 44(1971) 106.

792. Proposed by Murray S. Klamkin, Ford Scientific Laboratory

It is a known result that a necessary and sufficient condition for a triangle inscribed in an ellipse to have a maximum area is that its centroid coincide with the center of the ellipse. Show that the analogous result for a tetrahedron inscribed in an ellipsoid is not valid.

Math. Mag., 45(1972) 53.

Solution by the proposer. By means of an affine transformation, it suffices to consider a sphere instead of an ellipsoid.

For a sphere, it is a known result that the inscribed regular tetrahedron has the maximum volume and for this case its centroid coincides with the center of the sphere. However, the converse is not valid, i.e., if the centroid of an inscribed tetrahedron in a sphere coicides with the center of the sphere, the tetrahedron need not be regular but it must be isosceles (one whose pairs of opposite edges are congruent). This latterresult can be obtained vectorially as follows:

Let **A**, **B**, **C**, **D** denote the four vertices on a unit sphere with center at *O*. Then if the two centroids coincide, we have $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$ in addition to $\mathbf{A}^2 = \mathbf{B}^2 =$ $\mathbf{C}^2 = \mathbf{D}^2 = 1$. Whence $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{C} + \mathbf{D})^2$ or $\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{D} = 0$. Thus

$$(\mathbf{A} - \mathbf{B})^2 = (\mathbf{C} - \mathbf{D})^2$$

and similarly

$$(\mathbf{A} - \mathbf{C})^2 = (\mathbf{D} - \mathbf{B})^2$$
$$(\mathbf{A} - \mathbf{D})^2 = (\mathbf{B} - \mathbf{C})^2$$

and the tetrahedron is isosceles.

Conversely, if the tetrahedron is isosceles, then $(\mathbf{A} - \mathbf{B})^2 = (\mathbf{C} - \mathbf{D})^2$, $(\mathbf{A} - \mathbf{C})^2 = (\mathbf{D} - \mathbf{B})^2$, $(\mathbf{A} - \mathbf{D})^2 = (\mathbf{B} - \mathbf{C})^2$ in addition to $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = \mathbf{D}^2 = 1$. Whence,

 $\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{D}, \ \mathbf{A} \cdot \mathbf{C} = \mathbf{D} \cdot \mathbf{B}, \ \mathbf{A} \cdot \mathbf{D} = \mathbf{B} \cdot \mathbf{C} \ \text{and} \ (\mathbf{A} + \mathbf{B})^2 = (\mathbf{C} + \mathbf{D})^2.$ Then

$$(\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D}) \cdot (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 0$$

and similarly

$$(\mathbf{A} + \mathbf{D} - \mathbf{B} - \mathbf{C}) \cdot (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 0$$

Thus

$$(\mathbf{A} - \mathbf{C}) \cdot (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 0$$

and similarly

$$(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 0$$
$$(\mathbf{A} - \mathbf{D}) \cdot (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}) = 0$$

Finally the last three equations imply $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$ or that the two centroids coincide. (A geometric proof appears in N. Altshiller-Court, *Modern Solid Geometry*, MacMillan, New York, 1935, p.95).

Math. Mag., 44(1971) 170–171.

Another Triangle Property

773. [September, 1970]. Proposed by Norman Schaumberger, Bronx Community College

Let M be an arbitrary point not necessarily in the plane of triangle $A_1A_2A_3$. If B_i is the midpoint of the side opposite A_i prove

$$\sum_{i=1}^{3} MA_i^2 - \sum_{i=1}^{3} MB_i^2 = \frac{1}{3} \sum_{i=1}^{3} A_i B_i^2$$

II. Solution by M. S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan. We Prove a more general result, i.e., if $A_1, A_2, \ldots, A_{n+1}$ benote any n+1 points in any E_r and if B_j $(j = 1, 2, \ldots, n+1)$ denotes the centroid of all the A_i with the exception of A_j then for any arbitrary point M

$$\sum_{i=1}^{n+1} MA_i^2 - \sum_{i=1}^{n+1} MB_i^2 = \frac{n-1}{n+1} \sum_{i=1}^{n+1} A_i B_i^2 \tag{1}$$

Let \mathbf{A}_i , \mathbf{B}_i , \mathbf{M} denote vectors from the centroid of all the A_i to A_i , B_i , M respectively. Then

$$\sum_{i=1}^{n+1} \mathbf{A}_i = 0 \quad \text{and} \quad -\mathbf{B}_i = \mathbf{A}_i / n$$

The l.h.s. of (1) is now

$$\sum_{i=1}^{n+1} (\mathbf{M} - \mathbf{A}_i)^2 - \sum_{i=1}^{n+1} (\mathbf{M} + \mathbf{A}_i/n)^2$$

or

$$\frac{n^2 - 1}{n^2} \sum_{i=1}^{n+1} \mathbf{A}_i^2$$

Since the r.h.s. of (1) is now

$$\frac{n-1}{n+1}\sum_{i=1}^{n+1}\mathbf{A}_i^2(1+1/n)^2$$

identity (1) follows. The proposed problem corresponds to the special case n = 2.

Math. Mag., 44(1971) 178, 140.

Q 522. Submitted by M. S. Klamkin

Determine all triangles XYZ satisfying

$$\frac{\sin 2X}{\sin A} = \frac{\sin 2Y}{\sin B} = \frac{\sin 2Z}{\sin C}$$

where ABC is a given triangle.

A 522. First note that XYZ must be acute. Now let $2X = \pi - R$, $2y = \pi - S$ and $2Z = \pi - T$. Thus RST is a triangle and

$$\frac{\sin R}{\sin A} = \frac{\sin S}{\sin B} = \frac{\sin T}{\sin C}$$

Whence $RST \sim ABC$ and

$$2X = \pi - A$$
 $2Y = \pi - B$ $2Z = \pi - C.$

Math. Mag., 44(1971) 229, 240.

Q 527. Submitted by M. S. Klamkin

Evaluate the determinant

$$D_n = |a_r - b_s|$$
 $r, s = 1, 2, \dots, n$

A 527. Since *D* vanishes for $a_p = a_q$, $p \neq q$, and is linear in a_r , it must identically vanish for n > 2. Also $D_1 = a_1 - b_1$ and $D_2 = (a_1 - a_2)(b_1 - b_2)$.

Math. Mag., 44(1971) 238–239.

Comment on Q 503

Q 503. [January, 1971]. Submitted by A. K. Austin, University of Sheffield. A boy walks 4 mph, a girl walks 3 mph and a dog walks 10 mph. They all start together at a certain place on a straight road and the boy and girl walk steadily in the same direction. The dog walks back and forth between the two of them, going repeatedly from one to the other and back again. After one hour where is the dog and which direction is he facing?

I. Comment by M. S. Klamkin, Ford Motor Company. I disagree with the proposer's solution. While I agree that the motion is reversible from any initial starting position in which the participants are not at the same location, it is not possible to start the motion when all three start fro the same location. The dog would have a nervous breakdown attempting to carry out his program. If one id not convinced, let the initial

starting distance between the boy and the girl be ϵ (arbitrarily small), then one can show that the number of times the dog reverses becomes arbitrarily large in a finite time.

An analogous situation occurs in the well known problem of the four bugs pursuing each other cyclically with the same constant speed and starting initially at the vertices of a square. At any point of their motion (except when together), the motion is reversible by reversing the velocities. However, when together, the directions of the velocities are indeterminate and thus they cannot reverse without further instructions.

[Parallel comments were also made by Leon Bankoff, Charles Trigg, Lyle E. Pursell]]

Math. Mag., 42(1969) 287.

816. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan

Show that no equilateral triangle which is either inscribed in or circumscribed about a noncircular ellipse can have its centroid coincide with the center of the ellipse.

[[This is identical with *Math. Mag.*, **36**(1963) 142, 108.

 \mathbf{Q} 312. Submitted by M. S. Klamkin

Show that no equilateral triangle which is either inscribed in or circumscribed about an ellipse (excluding the circular case) can have its centroid coinciding with the center of the ellipse.

A 312. Orthogonally project the ellipse into a circle. The equilateral inscribed or circumscribed triangles will become inscribedor circumscribed non-equilateral triangles whose centroids cannot coincide with the center of the circle. Since centroids transform into centroids, the proof is completed.]]

Math. Mag., 45(1972) 236.

Solution by Leon Bankoff, Los Angeles, California. If an equilateral triangle and a *circumscribed* ellipse were to share the same centroid, the ellipse and the circumcircle of the triangle would be concentric. Consequently the four intersections of the circle and the ellipse would be vertices of a rectangle. Since the vertices of an equilateral triangle cannot lie on three vertices of a rectange, the initial assumption regarding a common centroid is untanable.

The same assumption for the *inscribed* ellipse would mean that each chord of contact of the ellipse would be bisected by a corresponding internal angle bisector of the tangential equilateral triangle. This could only occur if each vertex of the triangle lay on an extended principle axis of the ellipse. This, in turn, would necessitate two vertices on one axis—an obvious impossibility for a circumscribed triangle.

Math. Mag., 44(1971) 297.

Comment on Q 505

 ${\bf Q}$ 505. [January, 1971]. Submitted by Gregory Wulczyn. Solve the differential equation

$$(x-a)(x-b)y'' + 2(2x-a-b)y' + 2y = 0$$

Comment by M. S. Klamkin, Ford Motor Company. The problem can be easily extended and solved to the differential equation

$$(x-a)(x-b)y'' + n(2x-a-b)y' + n(n-1)y = 0 \qquad (n=2,3,4,\ldots)$$

Letting $y = D^{n-2}z$, the differential equation can be rewritten as

$$D^n\{z(x-a)(x-b)\} = 0$$

Whence

$$z = \frac{1}{(x-a)(x-b)} \{A_0 + A_1 x + \dots + A_{n-1} x^{n-1}\}$$

= $B_0 + B_1 x + \dots + B_{n-3} x^{n-3} + \frac{A}{x-a} + \frac{B}{x-b}$

Finally

$$y = D^{n-2} \left\{ \frac{A}{x-a} + \frac{B}{x-b} \right\} = \frac{A'}{(x-a)^{n-1}} + \frac{B'}{(x-b)^{n-1}}$$

[[By now it's rather a slow quickie. — R.]]

Math. Mag., 45(1972) 238–239.

[[Further !]] Comment by Murray S. Klamkin, Ford Motor Company. The still more general equation

$$(x-a)(x-b)y'' + n(2x-a-b)y' = F(x) \qquad (n \text{ arbitrary})$$

can also be solved easily by first noting the factorization

$$\{(x-a)D+n\}\{(x-b)D+n-1\}y = F(x)$$

Then by the exponential shift theorem,

$$y = (x-b)^{1-n} \int \frac{(x-b)^{n-2} dx}{(x-a)^n} \int F(x)(x-a)^{n-1} dx$$

On letting F(x) = 0, we find on comparison with my previous comment (Nov.-Dec., 1971) that

$$\int \frac{(x-b)^{n-2} dx}{(x-a)^n} = A' \left\{ \frac{x-b}{x-a} \right\}^{n-1} + B'$$
(1)

which at first glance is somewhat surprising since one would expect a series by expanding out $\{(x-a) + (a-b)\}^{n-2}$ (here A' = 1/(n-1)(b-a)). This leads to the summation

$$\sum_{r=0}^{n} \frac{\binom{n}{r}}{r+1} \left\{ \frac{a-b}{x-a} \right\}^{r+1} = \frac{1}{n+1} \left\{ \left(\frac{x-b}{x-a} \right)^{n+1} - 1 \right\}$$

A further extension to an r th order equation is given by the following: If S_i (i = 0, 1, ..., r) denote the elementary symmetric functions of $x - a_i$ (i = 1, 2, ..., r), i.e.,

$$\prod_{i=1}^{r} (\lambda + x - a_i) \equiv \sum_{i=0}^{r} S_i \lambda^{r-1}$$

then the solution of the differential equation

$$\sum_{j=0}^{r} j! \binom{n}{j} S_{r-j} D^{r-j} y = 0$$
 (2)

is given by

$$y = \sum_{i=1}^{r} A_i (x - a_i)^{r-n-1} \qquad (A_i \text{ arbitrary constants})$$

The latter follows since it can be shown by induction that (2) factorizes into

$$\{(x-a_1)D+n\}\{(x-a_2)D+n-1\}\cdots\{(x-a_r)D+n-r+1\}y=0$$

for any ordering of the a_i . The nonhomogeneous equation corresponding to (2) can also be solved by quadratures by means of the exponential shift theorem. Also, corresponding to (1) for r = 3, we have

$$\int \frac{(x-c)^{n-3} dx}{(x-b)^{n-1}} \int \frac{(x-b)^{n-2} dx}{(x-a)^n} = A \left\{ \frac{x-c}{x-a} \right\}^{n-2} + B \left\{ \frac{x-c}{x-b} \right\}^{n-2} + C$$

There are analogous results for r > 3.

[[This must be the world's slowest quickie ! — R.]]
Math. Mag., 45(1972) 47, 56.

Q 536. Submitted by M. S. Klamkin

Show that the square roots of three distinct prime numbers cannot be terms of a common geometric progression.

A 536. If they were then

 $ar^{n_1} = \sqrt{p_1}, \quad ar^{n_2} = \sqrt{p_2}, \quad ar^{n_3} = \sqrt{p_3}, \quad (n_1, n_2, n_3 \text{ distinct integers})$

Eliminating a and r yields

$$(p_1/p_2)^{n_2-n_3} = (p_2/p_3)^{n_1-n_2}$$

[[exponents have been corrected(?) please check! - R.]]

which is clearly impossible (by the unique factorization theorem). The result holds for any integral roots.

Math. Mag., 45(1972) 239.

Comment by William Wernick, City College of New York. If three terms are in geometric progression then the product of the first and last must equal the square of the second, thus in this case $\sqrt{ac} = b$ or $b^2 = ac$ which is clearly impossible with distinct primes.

[[This assumes that the terms were consecutive — this was not the intention — Later, on *Math. Mag.*, 46(1973) 174–175 one reads]]

Comment by the proposer. In his comment on Q 536 (September 1972) Wernick does not solve the given problem since he assumes that the three terms are consecutive terms of a geometric progression.

Math. Mag., 45(1972) 102, 112.

Q 537. Submitted by M. S. Klamkin

Determine solutions to

$$xF'(x) - F(x) = F'(F'(x))$$

other than F(x) = a(x-1).

A 537. This is a Clairaut equation. Consequently we differentiate obtaining xF''(x) = F''(x)F''(F'(x)). One solution is F''(x) = 0 or F(x) = a(x-1). The other solutions are derivable from x = F''(F'(x)). The general solution of this latter equation seems difficult to derive. However, it does have the power solution $F(x) = ax^{n+1}$ where $n = (1 \pm \sqrt{5})/2$ and $a = n^{-1/n}/(n+1)$.

[[sqrt sign missing from penult equation. And compare *Math. Mag.*, **37**(1964) 119. **549**.]]

Math. Mag., 45(1972) 167, 176.

Q 542. Submitted by Murray S. Klamkin

If a, b, c, d and x, y denote respective lengths of four consecutive sides and both diagonals of a quadrilateral having both an incircle and a circumcircle, show that $(a + b + c + d)^2 \ge 8xy$, with equality if and only if the quadrilateral is a square.

A 542. Since the quadrilateral has an incircle, a + c = b + d. Since the quadrilateral is inscribable, xy = ac + bd. Thus we must show equivalently that

$$a^2 + c^2 \ge 2b(a + c - b)$$

For a given a + c the r.h.s. has a maximum value of $(a + c)^2/2$ when b = (a + c)/2. Since

$$2(a^{2} + c^{2}) - (a + c)^{2} = (a - c)^{2} \ge 0$$

our inequality is established. The stated inequality is also equivalent to

$$(a+b+c+d)^2 \ge 8(ac+bd)$$

for circumscribable quadrilaterals (a, b, c, d are lengths of consecutive sides).

Math. Maq., 45(1972) 168–169.

Minimum of an Exponential Function

803. [September, 1971]. Proposed by Kenneth Rosen, University of Michigan

Let x and y be positive real numbers with x + y = 1. Prove hat $x^x + y^y \ge \sqrt{2}$ and discuss conditions for equality.

III. Solution by Murray S. Klamkin, Ford Motor Company. It is well known that if F(x) is strictly convex for $0 \le x \le a$, then

$$F(x_1) + F(x_2) + \dots + F(x_n) \ge nF\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$$

with equality if and only if $x_1 = x_2 = \cdots + x_n$. Since $D^2 x^x = x^x (1 + \log x)^2 + x^{x-1}$, x^x is strictly convex for $x \ge 0$. Thus for $x_1 + x_2 + \cdots + x_n = nb$,

$$\sum_{i=1}^{n} x_i^{x_i} \ge nb^b.$$

The given problem corresponds to the special case $n = 2, b = \frac{1}{2}$.

Math. Mag., 45(1972) 172–173.

Arithmetic-Geometric Mean Inequality

807. [September, 1971]. Proposed by Norman Schaumberger, Bronx Community College

Let $(x_i), i = 1, 2, 3, ...$ be an arbitrary sequence of positive real numbers and set

$$\Delta_{k} = (1/k) \sum_{i=1}^{k} x_{i} - \left(\prod_{i=1}^{k} x_{i}\right)^{1/k}$$

If $n \ge m$, prove that $n\Delta_n \ge m\Delta_m$.

II. Solution by Murray S. Klamkin, Ford Motor Company. It suffices just to prove the case m = n - 1 $(n \ge 2)$, i.e.,

$$(n-1)(x_1x_2\cdots x_{n-1})^{1/(n-1)} \ge n(x_1x_2\cdots x_n)^{1/n} - x_n \tag{1}$$

Let

$$x_n = \lambda^n (x_1 x_2 \cdots x_{n-1})^{1/(n-1)}$$

so that (1) becomes

$$\lambda^n - 1 \ge n(\lambda - 1)$$

which is a known elementary result (just factor $\lambda^n - 1$) with equality if and only if $\lambda = 1$.

REMARK. Since $\Delta_1 = 0$, the above solution provides an apparently new elementary inductive proof of the Arithmetic-Geometric mean inequality ($\Delta_n \ge 0$).

Math. Mag., 46(1973) 52–53.

[[Further !]] Comment by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan. The result here is known and is contained in a class of inequalities which are sometimes called Rado type inequalities (see D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Heidelberg, 1970, pp.94, 98–102). Related to these inequalities are the ones analogous to

$$\left\{\frac{G_n(x)}{A_n(x)}\right\} \le \left\{\frac{G_{n-1}(x)}{A_{n-1}(x)}\right\}^{n-1}$$

which are sometimes called Popoviciu type inequalities. Here, G_n and A_n denote the gepmetric and arithmetic means of x_1, x_2, \ldots, x_n , respectively. A similar proof can also be given for the latter inequality. Just let

$$x_n = \lambda(x_1 + x_2 + \dots + x_{n-1})$$

giving

$$\frac{\lambda}{(1+\lambda)^n} \le \frac{(n-1)^{n-1}}{n^n}$$

It follows easily that the r.h.s. is the maximum value of the l.h.s. which is taken on for $\lambda = 1/(n-1)$.

Math. Mag., 45(1972) 228–229.

841. Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan Solve the following generalization of Clairaut's equation:

$$y = xp + F(p)\{1 + \sqrt{1 + xG(p)}\}$$

where p = dy/dx.

Math. Mag., 46(1973) 170-171.

Solution by the proposer. Let $r = 1 + \sqrt{1 + xG(p)}$ and differentiate with respect to x, giving

$$xp' + rp'F' + r'F = 0$$

Now replace r' by $p'\frac{dr}{dp}$ and x by $(r^2 - 2r)/G$ to give

$$p'\left\{(r^2 - 2r)/G + rF' + F\frac{dr}{dp}\right\} = 0$$

If the first factor is zero, i.e., p' = 0, we get $y = cx + F(c)\{1 + \sqrt{1 + xG(c)}\}$ provided that F(c)G(c) = 0.

The other factor can be rewritten as

$$\left\{D_p + \frac{2}{FG} - \frac{F'}{F}\right\}\frac{1}{r} = \frac{1}{FG}$$

Whence

$$\frac{1}{r} = F \exp\left\{-2\int \frac{dp}{FG}\right\} \int \frac{\exp\left\{2\int \frac{dp}{FG}\right\} dp}{F^2G}$$

which gives x as a function of P. The original equation gives y also as a function of p. These two latter equations give the solution in parametric form.

Math. Mag., 46(1973) 42.

853. Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan

It is a well-known theorem that all quadric surfaces which pass through seven given points will also pass through an eighth fixed point. (a) If the seven given points are (0,0,0), (0,0,1), (0,1,0), (2,0,0), (1,1,0), (1,0,1) and (1,1,1), determine the eighth fixed point. (b) Determine the eighth fixed point explicitly as a function of the seven general given points (x_i, y_i, z_i) , i = 1, 2, 3, ..., 7.

Math. Mag., 46(1973) 288–289.

Solution by the proposer. If the equation of a quadric surface be

$$ax^{2} + by^{2} + cz^{2} + dxy + eyz + fzx + gx + hy + iz + j = 0$$

then the equations must satisfy the 7 equations

j = 0	a+b+d+g+h=0
c+i=0	a + c + f + g + i = 0
b+h=0	a + b + c + d + e + f + g + h + i = 0
4a + 2g = 0	

Thus, the equation reduces to the form

$$a(x^{2} + xy - zy + zx - 2x) + by(y - 1) + cz(z - 1) = 0$$

and the eighth fixed point is (-1, 1, 1).

[[Part (b) not solved??]]

Math. Mag., 46(1973) 43, 54.

Q 558. Submitted by Murray S. Klamkin

It is known that if a ray of light is reflected off three successive faces of a triangular corner mirror, the final direction of the ray is parallel but opposite to that of the incoming ray. Show that the same property holds more generally for n successive reflections off the n faces of an n-rectangular corner mirror in E^n .

A 558. Let $R = (\cos a_1, \cos a_2, \dots, \cos a_n)$ denote the unit vector corresponding to the incoming ray. Here a_j denotes the angle R makes with the axis x_j . After reflection off any face (say the one normal to x_1), the new ray is given by

$$R = (-\cos a_1, \cos a_2, \dots, \cos a_n), \quad \text{etc.}$$

Consequently, after n reflections, the final ray is given by

$$R_n = (-\cos a_1, -\cos a_2, \dots, -\cos a_n) = -R.$$

Math. Mag., 46(1973) 43, 54.

Q 559. Submitted by Murray S. Klamkin

If $a_{n+1} = 5a_n + \sqrt{24a_n^2 - 1}$, n = 0, 1, 2, ... and $a_o = 0$, show that the sequence $\{a_n\}$ is always integral.

A 559. Squaring $a_{n+1}^2 - 10a_n + a_n^2 = 1$. Solving for a_n : $a - n = 5a_{n+1} - \sqrt{24a_{n+1}^2 + 1}$. Reducing *n* by one in the latter equation and adding it to the given equation we get $a_{n+1} = 10a_n - a_{n-1}$. Since $a_0 = 0$ and $a_1 = 1$ all the a_i are integers.

Math. Mag., 46(1973) 53–54.

Comment on Q 546

Q 546. [May, 1972]. Submitted by Erwin Just. If n is an integer greater than 2, prove that n is thesum of the n th powers of the roots of $x^n - kx - 1 = 0$.

Comment by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan. One can obtain further results in a similar fashion. If T_1, T_2, \ldots, T_n denote the elementary symmetric functions of x_1, x_2, \ldots, x_n , i.e.,

$$P(x) = \prod (x - x_i) = x^n - T_1 x^{n-1} + T_2 x^{n-2} - \dots + (-1)^n T_n$$

and if

$$S_k = \sum_{i=1}^n x_i^k$$

then the Newton formulae are given by

$$S_{k} - T_{1}S_{k-1} + T_{2}S_{k-2} - \dots + (-1)^{k-1}T_{k-1}S_{1} + (-1)^{k}T_{k} = 0 \qquad (k \le n)$$
(A)
$$S_{k} - T_{1}S_{k-1} + T_{2}S_{k-2} - \dots + (-1)^{n}T_{n}S_{k-n} = 0 \qquad (k > n)$$
(B)

If $P(x) \equiv x^n - ax - 1$, then $T_{n-1} = (-1)^n a$, $T_n = (-1)^{n-1}$ and $T_1 = T_2 = \cdots = T_{n-2} = 0$. It then follows that $S_m = 0$ for m = rn + 1, rn + 2, ..., (r+1)n - r - 2 $(1 \leq r \leq n-3)$. The nonvanishing power sums are given by

$$S_{n-1} = (n-1)a \qquad S_n = n \qquad S_{2n-2} = (n-1)a^2$$
$$S_{2n-1} = (2n-1)a \qquad S_{2n} = n \qquad S_{3n-3} = (n-1)a^3$$
$$S_{3n-2} = (3n-2)a^2 \qquad S_{3n-1} = (3n-1)a \qquad S_{3n} = n \quad \text{etc.}$$

Math. Mag., 46(1973) 104, 112.

Q 564. Submitted by Murray S. Klamkin

If $A_i B_i C_i D_i$ (i = 1, 2, 3, 4) denote four given quadrilaterals in space such that the four vector sums

$$A_{i}B_{i+1} + C_{i}B_{i-1} + A_{i+1}B_{i+2} + C_{i+1}D_{i+2} + A_{i+2}B_{i} + C_{i+2}D_{i}$$

(i = 1, 2, 3, 4) and $A_i = A_{i+4}$, etc. are zero, show that the sums remain zero for any changes of the lorientations of the quadrilaterals.

A 564. Let $A_i B_i = R_{\sim i}$, $B_i C_i = S_{\sim i}$, $C_i D_i = T_{\sim i}$, $OA_i = U_{\sim i}$ (i = 1, 2, 3, 4) then the given vectors can be shown to reduce to

[[these rather curious subscripts are all printed as ~ 1 , as is that on T in the next display, the I'm sure they shd be eyes]]

$$R_{\sim i} + T_{\sim i} + R_{\sim i+1} + T_{\sim i+1} + R_{\sim i+2} + T_{\sim i+2} \qquad (i = 1, 2, 3, 4)$$

Consequently $R_{\sim i} + T_{\sim i} = 0$ which is invariant under rigid body motions. Also the quadrilaterals must be parallelograms.

Math. Mag., 46(1973) 104–105, 112.

Q 565. Submitted by Murray S. Klamkin

Determine the trihedral angles OA'B'C' such that if one picks an arbitrary point A, B, C, respectively on the open rays OA', OB', OC', then ABC is always an acute triangle.

A 565. It follows by continuity that none of the face angles can be acute or obtuse. Thus the only possibility is a trirectangular angle. If OA = a, OB = b, OC = c, then

$$AB^2 = a^2 + b^2$$
 $BC^2 = b^2 + c^2$ $AC^2 = c^2 + a^2$

Since the sum of any two is greater than the third, ABC is acute.

Math. Maq., 46(1973) 111–112.

Comment on Q 543

Q 543. [May, 1972]. Submitted by Alexander Zujus. Sow that for all natural numbers $n \ge 4$, $(n-1)^n > n^{n-1}$

II. Comment by Murray S. Klamkin, Ford Motor Company. More generally, $x^{1/x}$ is a monotonic decreasing function for $x \ge e$. This follows since

$$Dx^{1/x} = x^{1/x}(1 - \log x)/x^2$$

Math. Mag., 46(1973) 167, 112.

Q 568. Submitted by Murray S. Klamkin

Solve the equation $x^{2n} + x^{2n-2} + \dots + x^{2r} + \dots + x^2 + 1 = x^n$

A 568. Summing the left hand side, we get $\frac{x^{2n+2}-1}{x^2-1} = x^n$ or equivalently $(x^n-1)(x^{n+2}+1) = 0$. Thus $x = e^{i\theta}$ where

$$\theta = 2\pi m/n$$
 $m = 1, 2, ..., n-1$ (excluding $\theta = \pi$ if m is even)

 $\theta = \pi (2m+1)/n$ $m = 1, 2, \dots, n+1$ (excluding $\theta = \pi$ if n is odd)

More generally one can treat

$$\sum_{i=0}^{n} x^{ri} = x^{rn/2}$$

in a similar way.

Math. Mag., 46(1973) 230.

875. Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan If $\{a_i\}, \{b_i\}$ denote two sequences of positive numbers and n is a positive integer, show that:

$$\sum_{i} a_i^{2n} \cdot \sum_{j} b_j^{2n} \ge \sum_{i} a_i^{2n-1} b_i \cdot \sum_{j} a_j b_j^{2n-1} \ge \dots \ge \sum_{i} a_i^n b_i^n \cdot \sum_{j} a_j^n b_j^n$$

Math. Mag., 47(1974) 170–171.

Solution by Robert M. Hashway, West Warwick, Rhode Island. Since the inequalities are trivially true when either the a_i or the b - i are all zero, the a_i and the b_i may be either positive or zero. Hence $\{a_i\}$ and $\{b_i\}$ can be made of the same dimension by inserting zeroes. What must be shown is that: if $\{a_i\}$, $\{b_i\}$ are sequences consisting of positive real numbers or zero, and n is a positive number, then

$$\sum_{i} a_{i}^{2n-k} b_{i}^{k} \sum_{j} a_{j}^{k} b_{j}^{2n-k} \ge \sum_{i} a_{i}^{2n-k-1} b_{i}^{k-1} \sum_{j} a_{j}^{k-1} b_{j}^{2n-k-1}$$

where k is an integer such that $n-1 \ge k \ge 0$

Proof: We need to determine if the expression below is positive or zero:

$$\sum_{i} a_{i}^{2n-k} b_{i}^{k} \sum_{j} a_{j}^{k} b_{j}^{2n-k} - \sum_{i} a_{i}^{2n-k-1} b_{i}^{k-1} \sum_{j} a_{j}^{k-1} b_{j}^{2n-k-1}$$

[[in the original, the last two exponents k-1 were printed as n+1 and k+1 but I don't think that can be right. — R.]]

We can easily see that:

$$\sum_{i} a_{i}^{2n-k} b_{i}^{k} \sum_{j} a_{j}^{k} b_{j}^{2n-k} = \sum_{i} a_{i}^{2n} b_{i}^{2n} + \sum_{i} \sum_{j \neq i} a_{i}^{2n-k} a_{j}^{k} b_{i}^{k} b_{j}^{2n-k}$$

and

$$\sum_{i} a_{i}^{2n-k-1} b_{i}^{k-1} \sum_{j} a_{j}^{k-1} b_{j}^{2n-k-1} = \sum_{i} a_{i}^{2n} b_{i}^{2n} + \sum_{i} \sum_{j \neq i} a_{i}^{2n-k-1} a_{j}^{k+1} b_{i}^{k+1} b_{j}^{2n-k-1}$$

Hence what remains to prove is that

$$s_k = \sum_i \sum_{j \neq i} (a_i^{2n-k} a_j^k b_i^k b_j^{2n-k}) - a_i^{2n-k-1} b_i^{k+1} a_j^{k+1} b_j^{2n-k-1} \ge 0$$

By interchanging the indices i and j and summing over all j, i we have

$$s_{k} = \sum_{i} \sum_{j < i} \left(a_{i}^{2n-k} a_{j}^{k} b_{i}^{k} b_{j}^{2n-k} + a_{j}^{2n-k} a_{i}^{k} b_{j}^{k} b_{i}^{2n-k} - a_{i}^{2n-k-1} b_{i}^{k+1} a_{j}^{k+1} b_{j}^{2n-k-1} - a_{j}^{2n-k-1} b_{j}^{k+1} a_{i}^{k+1} b_{i}^{2n-k-1} \right)$$

By factoring out similar terms we have:

$$s_k = \sum_{i} \sum_{j < i} a_j^k b_i^k a_i^k b_j^k (b_i a_j - b_j a_i) ((a_j b_i)^m - (a_i b_j)^m)$$

where m = 2(n - k) - 1.

Since each term of the series positive, the result is clear.

Math. Mag., 46(1973) 232, 240.

Q 576. Submitted by Murray S. Klamkin

Given an *n*-dimensional simplex $OA_1A_2 \cdots A_n$ whose edges emating from O are mutually orthogonal. Show that the square of the content of the (n-1)-dimensional face opposite O is equal to the sum of the squares of the contents of the remaining faces.

A 576. Let the tetrahedron [[sic]] be given by the *n* coordinate planes of an *n*-dimensional rectangular coordinate system x_1, x_2, \ldots, x_n and the hyperplane P: $\sum x_i/a_i = 1$. If *V* and *B* denote the contents of the simplex and the face opposite *O* (the origin), respectively, then V = dB/n where *d* denotes the distance from *O* to *P*. Since $V = \frac{1}{n!}\pi a_2$ and $P = \{\sum a_i^{-2}\}^{-1/2}$

[[something weird about that formula for V but I can't figure out what it is. And ditto near the end of the next display. - R.]]

$$B^{2} = \{(n-1)!\}^{-2} \{\sum a_{i}^{-2}\} \pi a_{i}^{2}$$

Now each term in the summation for B^2 corresponds to the square of the content of the remaining faces (the (n-1)-dimensional version of the expression for V). The case n = 2 corresponds to the Pythagorean theorem but we have not proved it since it was used implicitly in the solution. However, the result does generalize the known result for the tetrahedron (n = 3).

Math. Mag., 46(1973) 286–287, 296.

Q 580. Submitted by Murray S. Klamkin

Determine the extreme values of

$$\frac{2\sin A\cos B\cos C + 2\sin B\cos C\cos A + 2\sin C\cos A\cos B}{\sin 2A + \sin 2B + \sin 2C}$$

where A, B, C denote the angles of a triangle.

A 580. The expression is equivalent to

$$\frac{\sum (a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{\sum a^2(b^2 + c^2 - a^2)} = 1$$

dron (n = 3).

Math. Mag., 46(1973) 48, 60.

Q 584. Submitted by Murray S. Klamkin

If A, B, C are nonnegative angles satisfying the triangle inequality and with a sum $\leq \pi$ show that

$$2\sum \sin^2 B \sin^2 C \ge 4\prod \sin^2 A + \sum \sin^4 A$$

with equality if and only if $A + B + C = \pi$.

A 584. The inequality can be rewritten as $4\Delta \ge 8 \sin A \sin B \sin C$ or $1 \ge R$ where Δ and R are area and circumradius, respectively, of a triangle of sides $2 \sin A$, $2 \sin B$ and $2 \sin C$. If we inscribe the triangle in a unit sphere, we obtain a corresponding spherical triangle of sides 2A, 2B, 2C whose circumradius is then ≤ 1 . The equality occurs when the spherical triangle corresponds to a great circle.

Math. Mag., 47(1974) 107.

900. Proposed by Murray S. Klamkin, Ford Motor Company, and Seymour Papert, Massachusetts Institute of Technology

A long sheet of rectangular paper ABCD is folded such that D falls on AB producing a smooth crease EF with E on AD and F on CD (when unfolded). Determine the minimal area of triangle EFD by elementary methods.



Math. Mag., 48(1975) 121–122.

Solution by Michael Goldberg, Washington, D.C. If AD = 1 and K denotes the area of the triangle EFD, then

$$K = (DD')(EF)/4 = (1/\cos\theta)(1/2\cos^2\theta\sin\theta)/4 = \{1/(\sin 2\theta + \frac{1}{2}\sin 4\theta)\}/2$$
$$= 1/2M \text{ where } M = \sin 2\theta + \frac{1}{2}\sin 4\theta$$

Then $dM/d\theta = 2\cos 2\theta + 2\cos 4\theta = 0$. Hence $-\cos 4\theta = \cos 2\theta$

$$\pi - 4\theta = 2\theta, \qquad \theta = \pi/6 = 30^{\circ}, \qquad K = \{1/(\sqrt{3}/2 + \sqrt{3}/4)\}/2 = 2\sqrt{3}/9 \approx 0.385$$

The following demonstration can serve as an elementary kinematic solution or verification of the foregoing result. The triangle D'EF attains its extremal area when the line EF intersects its neighboring position at its midpoint G; then the area added by moving the triangle is equal to the area subtracted. As the point D' moves along the straight line AB, the instantaneous center of rotation of the triangle D'EF is on a line through D' perpendicular to AB. Hence, the perpendicular must pass through G. Hence $AD' = \frac{1}{2}(DF)$ and this occurs only when $\theta = 30^{\circ}$.

Math. Mag., 46(1973) 108, 114.

Q 590. Submitted by Murray S. Klamkin

O - ABCDE is a regular pentagonal pyramid such that $\angle AOB = 60^{\circ}$. Find $\angle AOC$.

A 590. By symmetry $\angle AOC = \angle ABC = 108^{\circ}$.

Math. Mag., 47(1974) 109–110.

Property of an Interior Point

867. [May, 1973]. Proposed by L. Carlitz, Duke University

[[Compare **910** below. — R.]]

Let P be a point in the interior of the triangle ABC. Let R_1 , R_2 , R_3 denote the distances of P from the vertices of ABC and let r_1 , r_2 , r_3 denote the distances from P to the sides of ABC. Show that

$$\sum r_1 R_2 R_3 \geq 12r_1 r_2 r_3$$
$$\sum r_1 R_1^2 \geq 12r_1 r_2 r_3$$
$$\sum r_2^2 r_3^2 R_2 R_3 \geq 12r_1^2 r_2^2 r_3^2$$

In each case there is equality if and only if ABC is equilateral and P is the center of ABC.

II. Solution by Murray S. Klamkin, Ford Motor Company. The three inequalities are special cases of

$$\sum_{\text{cyclic}} \frac{R_1^i R_2^j R_3^k}{r_i^u r_2^v r_3^w} \ge 3 \left\{ \frac{R_1 R_2 R_3}{r_1 r_2 r_3} \right\}^{m/3} \ge 3 \cdot 2^m$$

where $i + j + k = u + v + w = m \ge 0$. The left hand inequality follows immediately from the A.M.-G.M. inequality while the right hand inequality follows from the known inequality $R_1R_2R_3 \ge 8r_1r_2r_3$ with equality if and only if *ABC* is equilateral and *P* is the center [see O. Bottema et al., *Geometric Inequalities*, Walters-Noordhoff, Groningen, 1969, p.111].

REMARK: We can obtain a stronger identity by using (loc. cit.)

$$R_1 R_2 R_3 \ge r_1 r_2 r_3 / \prod \sin A/2$$

Also, by using $(x+y+z)/3 \geq \{\sum yx/3\}^{1/2}$ we can augment the proposed inequalities to

$$\left\{ \sum \frac{R_1}{r_1} \right\}^2 \geq 3 \sum \frac{R_2 R_3}{r_2 r_3} \geq 36$$
$$\left\{ \sum \frac{1}{r_1 R_1^2} \right\}^2 \geq 3 \sum \frac{1}{r_2 r_3 R_2^2 R_3^2} \geq 36$$
$$\left\{ \sum r_1^2 R_1 \right\}^2 \geq 3 \sum r_2^2 r_3^2 R_2 R_3 \geq 36 r_1^2 r_2^2 r_3^2$$

Math. Mag., 46(1973) 167, 178.

Q 597. Submitted by Murray S. Klamkin

Prove that

$$\frac{(n+1)^{n+1}}{n^n} > \frac{n^n}{(n-1)^{n-1}}$$

for n = 1, 2, 3, ... (here $n^n = 1$ for n = 0).

A 597. The inequality can be rewritten as

$$\frac{n+1}{n-1}\left\{1-\frac{1}{n^2}\right\}^n > 1$$

By Bernoulli's inequality

$$\left\{1 - \frac{1}{n^2}\right\}^n \ge 1 - \frac{1}{n}$$

whence

$$\frac{n+1}{n-1}\left\{1-\frac{1}{n^2}\right\}^n \ge \frac{n+1}{n} > 1$$

Math. Mag., 47(1974) 177–178.

Comment on Q 572

Q 572. [May, 1973]. Submitted by Norman Schaumberger

Show that if n and k are positive integers then $x^n + y^n = z^{n+1/k}$ always has solutions in integers x, y, z.

Comment by Murray S. Klamkin, Ford Motor Company. Since z must be a k th power, we can replace the equation by $x^n + y^n = z^{nk+1}$. One can show more generally that $x^a + y^b = z^c$ always has solutions in integers x, y, z if a, b, c are positive integers with ab, c relatively prime. Just let $x = 2^{bt} \cdot u^{bc}, y = 2^{at} \cdot u^{ac}, z = 2^s \cdot u^{ab}$. Then $2^{abt+1} = 2^{cs}$.

Since (ab, c) = 1, there are infinitely many positive integers s, t satisfying abt + 1 = cs.

Math. Mag., 47(1974) 240.

914. Proposed by Murray S. Klamkin, Ford Motor Company

If for any n of a given n+1 integral weights, there exists a balance of them on a two pan balance where a fixed number of weights are placed on one pan and the remainder on the other pan, prove that the weights are all equal.

Math. Mag., 48(1975) 247.

Solution by Thomas E. Elsner, General Motors Institute. Let $w_1, w_2, \ldots, w_{n+1}$ be the n+1 integral weights. Since any n of the weights balance, the sum of any n weights must be even. This implies further that all the weights have the same parity (congruent (mod 2)). Now the balancing properties of the initial weights must be shared by the integers $w_i/2$ or $(w_i - 1)/2$ (depending on whether the w_i are all even or odd).

Hence the w_i must be congruent (mod 4). Continuing in the same way, the w_i are congruent (mod 2^k) for every k and this implies that the weights are equal and further, that n is even.

Editor's comment. Several solvers noted that this problem is a generalization of problem B-1 on the 1973 William Lowell Putnam Exam. The proposer referenced the USSR Olympiad Problem Book, W. H. Freeman and Co., 1962, p.8.

Math. Mag., 49(1976) 254.

Editors' Comment. James A. Davis and Richard A. Gibbs point out that the published solution (September 1975) is incomplete. The argument that the balancing property of the initial weights w_i must be shared by $w_i/2$ or $(w_i - 1)/2$ fails just when all w_i are odd and the the two pans contain unequally many weights. For example, 3+3+3=9, but $1+1+1 \neq 4$. It should also be noted that the fixed number in the problem must be the same for every choice of n of the n+1 weights. The necessity of this is seen for the set $\{1, 1, 1, 1, 3\}$ of weights.

II. Solution by James A. Davis, Sandia Laboratories: We assume that the result holds if equally many weights are placed in the two pans, as proved in Solution I.

Let $S = \{w_1, w_2, \ldots, w_{n+1}\}$ be the given set of n+1 weights such that any n of them can be balanced with a suitable choice of k weights on one pan and n-k weights on the other. Now consider a set $T = \{w_1, w_2, \ldots, w_{n+1}, w_1, w_2, \cdots, w_n\}$; that is, Tconsists of 2n+1 weights with two copies of the weights w_1, w_2, \cdots, w_n . Now if weight w_{n+1} is removed from T, the remaining weights balance with n weights on each pan. If weight w_i $(1 \le i \le n)$ is removed from T, then T can be viewed as the union of $T_1 = \{w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \cdots, w_{n+1}\}$ and $T_2 = \{w_1, w_2, \ldots, w_n\}$. Since both T_1 and T_2 can be divided into k and n-k weights which balance, there is a balance of the weights of T with n weights on each pan. Thus, from Solution I, we conclude that all the weights of T, and hence of S, are equal and that n must be even. *Editors' Note.* The problem generalizes to n+1 weights with real, positive values. A very nicesolution (using linear algebra) to the problem with equally many weights on the two pans has been given by C. C. Clever and K. L. Yocom, this MAGAZINE, **49** 135–136.

[[I've copied out the article here. — was Murray on the Putnam Committee at that time? — R.]]

A Generalization of a Putnam Problem

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The following problem appeared on the 1973 Putnam Examination: Let $a_1, a_2, \ldots, a_{2n+1}$ be integers such that, if any one of them is removed, those remaining can be divided into two sets of n having equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$. A proof may be based on special properties of integers. (Show that the given integers are either all even or all odd. The if they are all even they may be divided by 2 while if they are allodd they may may be increased by 1 without destroying the property of the problem.) In generalizing the problem, we developed a different proof which is an interesting application of linear algebra. We begin with two generalizations of the problem, which we prove by means of a lemma concerning matrices. Then we state and prove a further generalization as our main theorem.

GENERALIZATION 1. Let $x_1, x_2, \ldots, x_{2n+1}$ be complex numbers such that, if any one of them is removed, those remaining can be divided into two sets of n having equal sums; then $x_1 = x_2 = \cdots = x_{2n+1}$.

GENERALIZATION 2. Let x_1, x_2, \ldots, x_{2n} be complex numbers such that, if any one of them is removed, those remaining can be divided into two sets having equal sums; then $x_1 = x_2 = \cdots = x_{2n} = 0$.

LEMMA. If A is an n by nmatrix having zeros on the main diagonal and all ± 1 off the diagonal, then A is nonsingular if n is even and the rank of A is at least n-1 if n is odd.

Proof. In the expansion of det A, each term -s 0, 1 or -1 and the number, d_n , of nonzero terms in the expansion is the number of permutations of order n which leave no element fixed. Such permutations are commonly called derangements and it is well known [1, p.31] that $d_1 = 0$, $d_2 = 1$ and $d_{n+2} = (n+1)(d_n + d_{n+1})$ for $n \ge 1$. It follows inductively that d_n is even for n odd and d_n is odd for n even. Thus if n is even, det $A \ne 0$ while if n is odd, each principal submatrix of A of order n-1 has a nonzero determinant. This completes the proof of the lemma.

Proof of 1. Let $x = col(x_1, x_2, ..., x_{2n+1})$ and let A be a 2n+1 by 2n+1 matrix having zeros on the main diagonal and exactly n entries equal to 1 and n equal to -1 in each row. Then the components of a solution vector x of Ax = 0 satisfy the hypotheses of Generalization 1. Since $x_0 = col(1, 1, ..., 1)$ is one such solution vector, A is singular and by the lemma, A has rank 2n. Thus all solutions are of the form $x = cx_0 = col(c, c, ..., c)$.

Proof of 2. Let $x = \operatorname{col}(x_1, x_2, \ldots, x_{2n})$ and let A be a 2n by 2n matrix with zeros on the main diagonal and ± 1 off the diagonal. Then A is nonsingular by the lemma and hence Ax = 0 has only the trivial solution x = 0.

THEOREM. Let k and nbe positive integers satisfying n > 2 and $1 \le k \le n - 2$.

(a) If n-k = 2m, an even integer, and x_1, x_2, \ldots, x_n is a sequence of complex numbers such that, if any k of them are removed, those remaining can be divided into two sets of m having equal sums, then $x_1 = x_2 = \cdots = x_n$.

(b) If n - k = 2m + 1, an odd integer, and x_1, x_2, \ldots, x_n is a sequence of complex numbers such that, if any k of them are removed, those remaining can be divided into two sets having equal sums, then $x_1 = x_2 = \cdots = x_n = 0$.

Proof. The theorem is true for k = 1 by 1 and 2 above. Now proceed inductively on k, assuming the theorem true for k = 1, 2, ..., K-1 < n-2. First suppose n - K = 2m in which case we are to establish (a) for k = K. Let x_i and x_j be any two designated elements of thesequence with $i \neq j$. Remove any K - 1 elements of the sequence, but leave x_i and x_j (this is possible since $n - K + 1 \geq 3$). Then we are left with a sequence of length 2m + 1 satisfying the hypotheses of 1 and hence $x_i = x_j$. Thus $x_1 = x_2 = \cdots = x_n$. Similarly if n - K = 2m + 1 we must establish (b) for k = K. Again, remove K - 1 elements of the sequence of length $n - K + 1 \geq 2m$ staisfies the hypotheses of 2 and hence $x_i = 0$. Thus $x_1 = x_2 = \cdots = x_n = 0$.

1. H. J. Ryser, Combinatorial Mathematics, MAA Carus Monograph, No.14, 1963.

Math. Mag., 47(1974) 241, 224.

Q 599. Submitted by Murray S. Klamkin

If a(n) denotes the exponent of the prime p in the factorization of n, determine the sum

$$S(m) = a(1) + a(2) + \dots + a(p^m)$$

A 599. $S(m+1) - S(m) = a(p^m+1) + a(p^m+2) + \dots + a(p^{m+1})$. Since a(pq) = 1 + a(q) and a(r) = 0 if $p \nmid r$,

$$S(m+1) - S(m) = (p-1)p^{m-1} + a(p^{m-1}+1) + a(p^{m-1}+2) + \dots + a(p^{m-1}+(p-1)p^{m-1})$$

or

$$S(m+1) - 2S(m) + S(m-1) = (p-1)p^{m-1}$$

It now follows easily that

$$S(m) = \frac{p^m - 1}{p - 1}$$

Remark: The special case p = 2 was given as a problem on a recent Dutch Mathematical Olympiad.

Math. Mag., 47(1974) 241, 224.

Q 601. Submitted by Murray S. Klamkin

If $\{z_i\}$ and $\{w_i\}$ $(w_1 \neq z_1 \neq z_2)$ denote complex numbers such that

$$\begin{vmatrix} z_1 & z_4 & 1 \\ z_2 & z_3 & 1 \\ w_1 & w_2 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & z_3 & 1 \\ w_1 & w_3 & 1 \end{vmatrix} \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & w_2 & 1 \\ w_1 & w_3 & 1 \end{vmatrix} = 0$$

prove that $z_2 - z_1 = z_3 - z_4$.

[[The entries in the top row of the second determinant look fishy. Also, after looking at the answer, I believe that the inequalities should have been given in the order $(z_1 \neq w_1 \neq z_2)$ — or should it be $(z_1 \neq w_1 \neq z_2 \neq z_1)$? — R.]]

A 601. Solving for w_2 and w_3 from the first two determinants being zero and substituting into the third one, we obtain

$$(z_2 + z_4 - z_1 - z_3)(w_1 - z_1)(w_1 - z_2) = 0$$

Thus, it is necessary and sufficient that

$$z_2 - z_1 = z_3 - z_4$$

The sufficiency condition is equivalent to the following geometric theorem: If ABCD is a parallelogram and ABX, DCY and ACZ are directly similar triangles, then also $XYZ \sim ABX$ (this is given as an exercise in T. M. MacRobert, Functions of a Complex Variable, Macmillan, London, 1950, p.277).

Math. Mag., 47(1974) 286.

919. Proposed by Murray S. Klamkin, Ford Motor Company

An (n+1)-dimensional simplex with vertices $O, A_1, A_2, \ldots, A_{n+1}$ is such that the (n+1) concurrent edges OA_i are mutually orthogonal. Show that the orthogonal projection of O onto the *n*-dimensional face opposite to it coincides with the orthocenter of that face (this generalizes the known result for n = 2).

Math. Mag., 48(1975) 299.

Solution by Leon Gerber, St. John's University. Let O be the origin of an (n+1)dimensional coordinate system and let A_i lie on the *i* th coordinate axis at a distance a_i from O. The equation of the face F opposite O is $\sum_{i=1}^{n+1} x_i/a_i = 1$. Let $a^{-1} = \sum_{i=1}^{n+1} a_i^{-2}$ and let $H = (a/a_i)$. Clearly H lies in F and OH is perpendicular to F. Also,

$$A_1H \cdot A_2A_3 = [a/a_1 - a_1, a/a_2, a/a_3, \dots, a/a_{n+1}] \cdot [0, -a_2, a_3, 0, \dots, 0]$$

= $-a + a = 0$

so A_1H is perpendicular to A_2A_3 and similarly for any three distinct subscripts.

REMARK. This result and its converse, namely that if an *n*-simplex is orthocentric there exist numbers a_i , i = 1, ..., n+1 such that $A_iA - j^2 = a_i^2 + a_j^2$, $i \neq j$, is in the literature:

W. J. C. Sharp, On the properties of simplicissima, *Proc. London Math. Soc.*, (1886-7) 325–359 (footnote on p.358).

E. Egervary, On orthocentric simplexes, Acta Litt. Sci. Szeged, 9(1940) 218–226.

Math. Mag., 47(1974) 287, 297.Q 606. Submitted by Murray S. KlamkinShow elementarily that

$$(x+y+z)^{x+y+z} \ge x^x y^y z^z$$

for positive x, y, z.

A 606. More generally, it will follow by induction that

$$\{\sum x_i\}^{\sum x_i} \ge \prod x_i^{x_i} \qquad (x_i > 0)$$

if we first show that

$$(x+y)^{x+y} \ge x^x y^y$$

Letting y = kx we get the obvious inequality

$$(1+k)(1+k)^k \ge k^k$$

Another solution, but not as elementary, follows from the concavity of $\log x$:

$$\frac{\sum x_i \log x_i}{\sum x_i} \le \log \frac{\sum x_i^2}{\sum x_i} \le \log \sum x_i$$

There is equality if and only if all the x_i but one are zero.

ALGEBRA

Inequalities: fractions

Math. Mag., 47(1974) 52, 58.

Q 608. Submitted by Murray S. Klamkin

If x, y, z are nonnegative and are not sides of a triangle, show that

$$1 + \frac{x}{y + z - x} + \frac{y}{z + x - y} + \frac{z}{x + y - z} \le 0$$

A 608. We will show more generally that if x_i (i = 1, 2, ..., n) are nonnegative and are not sides of an *n*-gon, then

$$1 + \sum_{i=1}^{n} \frac{x_i}{S - 2x_i} \le 0$$

where $S = \sum x_i$. Assume that $2x_n > S$ and let $x_n = T + 2a$ where $T = x_1 + x_2 + \cdots + x_{n-1}$ and a > 0. Then we have to show equivalently that

$$\frac{T+2a}{2a} \ge 1 + \sum_{i=1}^{n-1} \frac{x_i}{2(a+T-x_i)}$$

which follows from

$$\frac{x_i}{2a} \ge \frac{x_i}{2(a+T-x_i)}$$

ALGEBRA

Systems of equations: n variables

Math. Mag., 48(1975) 115.

930. Proposed by Murray S. Klamkin, University of Waterloo

Solve the system of equations $(x_i - a_{i+1})(x_{i+1} - a_{i+3}) = a_{i+2}^2$ (i = 1, 2, ..., n) for the x_i , where $a_{n+i} = a_i$, $x_{n+i} = x_i$ and $a_1 a_2 \cdots a_n \neq 0$.

Math. Mag., **49**(1976) 97–98.

Solution by Paul Y. H. Yiu, University of Hong Kong. Let $y_i = x_i - a_{i+1} \ (\neq 0)$, $i = 1, 2, \ldots, n$. The given equations reduce to

$$y_{i+1} - a_{i+3} = -\frac{a_{i+2}}{y_i}(y_i - a_{i+2}) \qquad y_i = y_{n+i} \quad i = 1, 2, \dots, n \tag{(*)}$$

Suppose $y_i = a_{i+2}$ for some i = 1, 2, ..., n; then $y_i = a_{i+2}$ for all i = 1, 2, ..., n. A solution is therefore furnished by $y_i = a_{i+2}$, i.e.,

$$x_i = a_{i+1} + a_{i+2}$$
 $i = 1, 2, \dots, n$

If $y_i \neq a_{i+2}$ for all $i = 1, 2, \ldots, n$, then we have

$$\prod_i y_i = (-1)^n \prod_i a_i$$

This indicates that we may have solutions of the form

$$y_i = a_{i+2} + k_i$$

where $k_i \neq 0, k_i = k_{n+i} \ i = 1, 2, ..., n$. (*) gives

$$\frac{1}{k_i} + \frac{1}{k_{i+1}} = -\frac{1}{a_{i+2}} \tag{**}$$

We distinguish bewteen the following two cases:

(1) n is odd. From (**)

$$\frac{2}{k_{i}} = \left(\frac{1}{k_{i}} + \frac{1}{k_{i+1}}\right) - \left(\frac{1}{k_{i+1}} + \frac{1}{k_{i+2}}\right) + \dots - \left(\frac{1}{k_{i-2}} + \frac{1}{k_{i-1}}\right) + \left(\frac{1}{k_{i-1}} + \frac{1}{k_{i}}\right) \\
= -\frac{1}{a_{i+2}} + \frac{1}{a_{i+3}} - \dots + \frac{1}{a_{i}} - \frac{1}{a_{i+1}} \\
k_{i} = 2\left\{\sum_{p=2}^{n+1} \frac{(-1)^{p+1}}{a_{i+p}}\right\}^{-1}$$

(2) *n* is even. (**) is consistent if and only if $\sum_{i=1}^{n} ((-1)^{i}/a_{i}) = 0$. Under this condition we have

$$\frac{1}{k_i} = (-1)^{i+1}\lambda + \frac{1}{n}\sum_{p=1}^{n-1}\frac{(-1)^p(n-p)}{a_{i+p-1}}$$

where λ is a parameter. (A standard algorithm leads to a solution in which $x_n = \lambda_n$ is a parameter; the above expression is obtained by symmetrizing this parametric solution.) Thus, if n is even and if

$$\sum_{i=1}^{n} \frac{(-1)^i}{a_i} = 0$$

the given system admits besides the obvious solution $x_i = a_{i+1} + a_{i+2}$ an infinite number of solutions given by

$$x_{i} = a_{i+1} + a_{i+2} + \left\{ (-1)^{i+1}\lambda + \frac{1}{n} \sum_{p=1}^{n-1} \frac{(-1)^{p}(n-p)}{a_{i+p+1}} \right\}^{-1}$$

where λ is an arbitrary parameter except that it cannot be chosen to make any $1/k_i$ vanish.

ALGEBRA

Inequalities: fractions

Math. Mag., 47(1974) 117, 122.

Q 618. Submitted by M. S. Klamkin

If $1 \ge x, y, z \ge -1$, show that

$$\frac{1}{(1-x)(1-y)(1-z)} + \frac{1}{(1+x)(1+y)(1+z)} \ge 2$$

with equality if and only if x = y = z = 0.

A 618. More generally, we have

$$S = \prod_{i=1}^{m} (1 - x_i)^{n_i} + \prod_{i=1}^{m} (1 + x_i)^{n_i} \ge 2$$

where $-1 \le x_i \le 1$, $n_i < 0$ for i = 1, 2, ..., m. Since $a + b \ge 2\sqrt{ab}$ for $a, b \ge 0$ we have $S \ge 2 \prod_{i=1}^{m} (1 - x_i^2)^{n_i/2} \ge 2$ with equality if and only if $x_i = 0$.

ANALYSIS

Maxima and minima: constraints

Math. Mag., 48(1975) 181.

942. Proposed by M. S. Klamkin, University of Waterloo

Determine the maximum value of

$$S = \sum_{1 \le i < j \le n} \left(\frac{x_i x_j}{1 - x_i} + \frac{x_i x_j}{1 - x_j} \right)$$

. where $x_i \ge 0$ and $x_1 + x_2 + \dots + x_n = 1$.

Math. Mag., **49**(1976) 153–154.

Solution by Joseph Silverman, Student, Brown University. We have

$$2S = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{x_i x_j}{1 - x_i} + \frac{x_i x_j}{1 - x_j} \right) - sum_{i=1}^{n} \frac{2x_i^2}{1 - x_i} = 2\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{x_i x_j}{1 - x_i} - \sum_{i=1}^{n} \frac{2x_i^2}{1 - x_i}$$
$$= 2\left(\sum_{j=1}^{n} x_j\right) \left(\sum_{i=1}^{n} \frac{x_i}{1 - x_i}\right) - 2\sum_{i=1}^{n} \frac{x_i^2}{1 - x_i}$$
$$= 2\sum_{i=1}^{n} \frac{x_i - x_i^2}{1 - x_i} = 2\sum_{i=1}^{n} x_i = 2$$

Thus S = 1.

ANALYSIS

Integral inequalities

Math. Mag., 48(1975) 182, 186.

Q 622. Submitted by M. S. Klamkin

If G, F are integrable, a > 0, $G(x) \ge F(x) \ge 0$ and $\int_0^1 xF(x) dx = \int_0^a xG(x) dx$, show that $\int_0^1 F(x) dx \le \int_0^a G(x) dx$.

A 622. Since

$$\int_{a}^{1} xF(x) \, dx = r \int_{a}^{1} F(x) \, dx = \int_{0}^{a} x \{G(x) - F(x)\} \, dx = s \int_{0}^{a} \{G(x) - F(x)\} \, dx$$

where $1 \ge r \ge a$ and $a \ge s \ge 0$, we have that $\int_a^1 F(x) dx \le \int_0^a \{G(x) - F(x)\} dx$ which is equivalent to the desired result.

Remarks. The problem arose in showing that the time of vertical ascent of a particle subject to gravity and air resistance is less than the time of descent. One can give another proof by showing that the speed of ascent is greater than the speed of descent at corresponding heights. h equality if and only if $x_i = 0$.

GEOMETRY

Butterfly problem

Math. Mag., 48(1975) 238–239.

949. Proposed by P. Erdős, Hungarian Academy of Science, and M. S. Klamkin, University of Waterloo

In a circle with center O, OXY is perpendicular to chord AB (as shown).



Prove $DX \leq CY$.

[This problem also appeared as

OSSMB 75-5 Ontario Secondary School Math. Bull., 11(1975/1) 15 by Paul Erdős and as CRUX 75 Crux 1(1975) 71 by R. Duff Butterill]

Math. Mag., 49(1976) 217–218.

I. Solution by Mark Kleiman, Student, Stuyvesant High School, New York, N.Y. Draw DY and choose H on DY so that XH is perpendicular to DY. We have that $\angle XDH = \angle ZDY = \pi/2 - \angle XYZ$ since the intercepted arcs form a semicircle. Thus, right triangle XDH is similar to right triangle XCY and so XH : DX = XY : CY. Since $XH \leq XY, DX \leq CY$.

II. Solution by Donald Batman, Socotto, New Mexico. Let the end of the diameter be P and let PD intersect AB at E. Then, EX = XC by the "butterfly problem". (See, for example, Steven R. Conrad, Another simple solution to the butterfly problem, this MAGAZINE, **46**(1973) 278–280.) Applying the law of sines to triangles DEX and

YXC to obtain

$$\frac{DX}{\sin DEX} = \frac{EX}{\sin D} = \frac{XC}{\sin Y} = \frac{CY}{\sin YXC}$$

we find $DX = CY \sin DEX \le CY$.

Math. Mag., 48(1975) 242–243.

Inequalities for a Triangle

910. [September, 1974]. Proposed by L. Carlitz, Duke University

[[Compare 867 above. — R.]].

Let P be a point in the interior pf the triangle ABC and let r_1 , r_2 , r_3 denote the distances from P to the sides of ABC. Let a, b, c denote the sides and r the radius of the incircle of ABC. Show that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \ge \frac{2s}{r}$$
(1)
$$ar_1^2 + br_2^2 + cr_3^2 \ge 2r^2s$$
(2)
$$(s-a)r_2r_3 + (s-b)r_3r_1 + (s-c)r_1r_2 \le r^2s$$
(3)
$$ar_1^2 + br_2^2 + cr_3^2 + (s-a)r_2r_3 + (s-b)r_3r_1 + (s-c)r_1r_2 \ge 3r^2s$$
(4)

where 2s = a + b + c. In each case there is equality if and only if P is the incenter of ABC.

Solution by M. S. Klamkin, University of Waterloo. Since $ar_1 + br_2 + cr_3 = 2rs = 2\Delta$ (Δ = area of ABC), it follows from Cauchy's inequality that

$$\left(\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3}\right)(ar_1 + br_2 + cr_3) \ge (\sqrt{ax} + \sqrt{by} + \sqrt{cz})^2 \tag{5}$$

$$\left(xr_1^2 + yr_2^2 + zr_3^2\right)\left(\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}\right) \ge (ar_1 + br_2 + cr_3)^2 \tag{6}$$

for all $x, y, z \ge 0$. Thus,

$$\frac{x}{r_1} + \frac{y}{r_2} + \frac{z}{r_3} \ge (\sqrt{ax} + \sqrt{by} + \sqrt{cz})^2 / 2\Delta$$
(7)

$$xr_1^2 + yr_2^2 + zr_3^2 \ge 4\Delta^2 \left/ \left\{ \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \right\}$$
(8)

with equality in (7) and (8) respectively, if and only if

$$\frac{ar_1^2}{x} = \frac{br_2^2}{y} = \frac{cr_3^2}{z}$$

$$\frac{xr_1}{a} = \frac{yr_2}{b} = \frac{zr_3}{c}$$

It is to be noted that (6) is valid for all real r_1 , r_2 , r_3 . For the special case (x, y, z) = (a, b, c), (7) and (8) rduce to (1) and (2). Incidentally, (2) will also follow immediately from (3) and (4).

We now show that (3) and (4) are special cases corresponding to n = 1 of the known master triangle inequality

$$u^{2} + v^{2} + w^{2} \ge (-1)^{n+1} \{2vw \cos nA + 2wu \cos nB + 2uv \cos nC\}$$
(9)

[[misprint of $2uc \cos nC$ in last term has been corrected. — R.]]

where u, v, w are arbitrary real numbers; A, B, C are angles of an arbitrary triangle. There is equality if and only if $u/\sin nA = v/\sin nB = w/\sin nC$ (M. S. Klamkin, Asymmetric triangle inequalities, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, No.357–No.380(1971) 33–44). Letting u = ax, v = by, w = cz, (9) for n = 1 is also equivalent to

$$a^{2}x^{2} + b^{2}y^{2} + c^{2}z^{2} \ge (b^{2} + c^{2} - a^{2})yz + (c^{2} + a^{2} - b^{2})zx + (a^{2} + b^{2} - c^{2})xy$$
(10)

The latter inequality can be traced back to Wolstenholme (ibid.).

By multiplying (3) and (4) by 4s and using $2rs = \sum ar_1$ they can be rewritten respectively as

$$\sum_{\text{cyclic}} \{a^2 r_1^2 - (b^2 + c^2 - a^2) r_2 r_3\} \ge 0 \tag{11}$$

$$\sum_{\text{cyclic}} \{a(2b + 2c - a) r_1^2 + [(b + c)^2 - a^2 - 6bc] r_2 r_3\} \ge 0 \tag{12}$$

Now noting that if $a_1^2 = a(2b + 2c - a)$, $b_1^2 = b(2c + 2a - b)$ and $c_1^2 = c(2a + 2b - c)$, then a_1 , b_1 , c_1 are sides of a triangle, it follows that (11) and (12) are valid for all real r_1 , r_2 , r_3 with equality if and only if $r_1 = r_2 = r_3$.

One can also obtain (8) as a special case of (10).

A further generalization of (2) is given by

$$\left(xr_1^m + yr_2^m + zr_3^m\right)^{1/m} \left\{ \left(\frac{a^m}{x}\right)^{1/(m-1)} + \left(\frac{b^m}{y}\right)^{1/(m-1)} + \left(\frac{c^m}{z}\right)^{1/(m-1)} \right\}^{(m-1)/m} \ge 2\Delta$$

where x, y, z, m-1 > 0; this follows from Hölder's inequality.

ANALYSIS

Limits: sequences

Math. Mag., 48(1975) 293–294.

958. Proposed by Murray S. Klamkin, University of Waterloo

Give direct proofs of the following two results:

a. If $\operatorname{Re}(z_0) > 0$ and the sequence $\{z_n\}$ is defined for $n \ge 1$ by

$$z_n = \frac{1}{2} \left(z_{n-1} + \frac{A}{z_{n-1}} \right)$$

where A is real and positive, then $\lim_{n\to\infty} = \sqrt{A}$.

b. Suppose $\{x_n\}$ is a real sequence defined for $n \ge 1$ by

$$x_n = \frac{1}{2} \left(x_{n-1} - \frac{A}{x_{n-1}} \right)$$

where A is positive. Show that if p is a given integer greater than 1, then the initial term x_0 can be chosen so that $\{x_n\}$ is periodic with period p. (These results are contained implicitly in K. E. Hirst, A square root algorithm giving periodic sequences, J. London Math. Soc., (2) 6(1972) 56–60.)

Math. Mag., **50**(1977) 48–49.

Solution by L. van Hamme, Vrije Universiteit Brussel, Brussels, Belgium. a. If w is a complex number defined by

$$z_0 = \sqrt{A} \, \frac{1+w}{1-w}$$

then

$$\operatorname{Re}(z_0) = \frac{z_0 + \bar{z_0}}{2} = \frac{\sqrt{A}}{2} \left(\frac{1+w}{1-w} + \frac{1+\bar{w}}{1-\bar{w}} \right) = \sqrt{A} \frac{1-|w|^2}{|1-w|^2}$$

The condition $\operatorname{Re}(z_0) > 0$ is therefore equivalent to |w| < 1. Now

$$z_1 = \frac{1}{2} \left(\sqrt{A} \, \frac{1+w}{1-w} + \sqrt{A} \, \frac{1-w}{1+w} \right) = \sqrt{A} \, \frac{1+w^2}{1-w^2}$$

Using repeatedly the relation

$$z_n = \frac{1}{2} \left(z_{n-1} + \frac{A}{z_{n-1}} \right)$$

we find in the same way

$$z_n = \sqrt{A} \, \frac{1 + w^{2^n}}{1 - w^{2^n}}$$

Since |w| < 1 it follows that $\lim_{n \to \infty} z_n = \sqrt{A}$.

Remark. If $\operatorname{Re}(z_0) < 0$ one finds $\lim_{n \to \infty} z_n = -\sqrt{A}$.

b. Take $x_0 = \sqrt{A} \cot(\pi/(2^p - 1))$. We will show that $x_n = \sqrt{A} \cot(\pi \cdot 2^n/(2^p - 1))$ for $n \ge 0$. This follows, by induction, from

$$x_{n+1} = \frac{\sqrt{A}}{2} \left[\cot\left(\frac{\pi \cdot 2^n}{2^p - 1}\right) - \tan\left(\frac{\pi \cdot 2^n}{2^p - 1}\right) \right] = \sqrt{A} \cot\left(\frac{\pi \cdot 2^{n+1}}{2^p - 1}\right)$$

Hence

$$x_{n+p} = \sqrt{A}\cot\left(\frac{\pi \cdot 2^{n+p}}{2^p - 1}\right) = \sqrt{A}\cot\left(\pi \cdot 2^n + \frac{\pi \cdot 2^n}{2^p - 1}\right) = \sqrt{A}\cot\left(\frac{\pi \cdot 2^n}{2^p - 1}\right) = x_n$$

and $\{x_n\}$ has period p.

SOLID GEOMETRY

Skew quadrilaterals

Math. Mag., 48(1975) 295, 303.

Q 630. Submitted by M. S. Klamkin and M. Sayrafiezadeh

Suppose a skew quadrilateral ABCD with diagonal AC perpendicular to diagonal BD is transformed into the quadrilateral A'B'C'D' so that the corresponding lengths of the sides are preserved. Prove that A'C' is perpendicular to B'D'.

A 630. The result is a consequence of the following, which is elementary but apparently not widely known:

THEOREM. Given vectors a, b, c and d such that a + b + c + d = 0, then (a+b) is perpendicular to (a+d) if and only if $|a|^2 + |c|^2 = |b|^2 + |d|^2$.

Proof. Using dot products,

$$c \cdot c = (-1)^2 (d + a + b) \cdot (d + a + b) = |a|^2 + |b|^2 + |d|^2 + 2(a \cdot b + a \cdot d + d \cdot b)$$
$$= |b|^2 + |d|^2 - |a|^2 + 2(a \cdot a + a \cdot b + a \cdot d + d \cdot b)$$

Hence

$$|a|^{2} + |c|^{2} = |b|^{2} + |d|^{2} + 2(a+b) \cdot (a+d)$$

and the theorem follows.

Note. A more geometric proof can be given by considering spheres centered at some of the vertices of the figures and the powers of certain points with respect to them.

Math. Mag., 48(1975) 296–297.

Trilinear Coordinates

916. [November, 1974]. Proposed by H. Demir, M.E.T.U., Ankara, Turkey

Let XYZ be the pedal triangle of a point P with respect to the triangle ABC. The find the trilinear coordinates x, y, z of P such that

$$YA + AZ = ZB + BX = XC + CY.$$

Solution by M. S. Klamkin, University of Waterloo. By drawing segments from P parallel to AB and AC respectively and terminating on BC it follows that

$$BX = x \cot B + z \csc B$$
 $CX = x \cot C + y \csc C$

The other distances CY, AY, AZ, BZ follow by cyclic interchange. From the hypothesis,

$$(y+z)(\cot A + \csc A) = (z+x)(\cot B + \csc B) = (x+y)(\cot C + \csc C) = \frac{2s}{3}$$

where s = semiperimeter. Solving:

$$x = \frac{s}{3} \left(\tan \frac{B}{2} + \tan \frac{C}{2} - \tan \frac{A}{2} \right)$$
$$y = \frac{s}{3} \left(\tan \frac{A}{2} + \tan \frac{C}{2} - \tan \frac{B}{2} \right)$$

and

$$z = \frac{s}{3} \left(\tan \frac{A}{2} + \tan \frac{B}{2} - \tan \frac{C}{2} \right)$$

ANALYSIS

Differential equations: order 4

Math. Mag., 49(1976) 44, 48.

Q 631. Submitted by M. S. Klamkin, University of Waterloo

Solve the differential equation $(xD^4 - axD + 3a)y = 0.$

A 631. We solve the more general problem $(xD^{n+1} - k^n xD + k^n)y = 0$. The equation of factored into $(D^n - k^n)(xD - n)y = 0$. Thus,

$$(xD-n)y = \sum_{i=0}^{n-1} A_i e^{k\omega^i x}$$

where ω is a primitive *n* th root of unity and the A_i are arbitrary constants. Integrating again, we get

$$y = A_n x^n + x^n \sum_{i=0}^{n-1} A_i \int frac e^{k \omega^i x} dx x^{n+1}$$

ANALYSIS

Rate problems

Math. Mag., **49**(1976) 95–101.

The Longest Swim

January 1975

926. Proposed by Melvin F. Gardner, University of Toronto

A swimmer can swim with speed v in still water. He is required to swim for a given length of time T in a stream whose speed is r < v. If he is also required to start and finish at the same point, what is the longest path (total arc length) that he can complete? Assume the path is continuous with piecewise continuous first derivatives.

Solution by M. S. Klamkin, University of Waterloo. If $\theta(t)$ denotes the angle heading of the swimmer with respect to the stream velocity, then

$$\dot{x} = \frac{dx}{dt} = v\cos\theta + r$$
 $\dot{y} = \frac{dy}{dt} = v\sin\theta$

The length L of a closed path swum in time T is then given by

$$L = \int_0^T \{\dot{x}^2 + \dot{y}^2\}^{1/2} dt = \int_0^T \{v^2 + 2vr\cos\theta + r^2\}^{1/2} dt$$
$$= \int_0^T \{v^2 - r^2 + 2r\dot{x}\}^{1/2} dt$$

Applying the Schwarz-Buniakowski inequality and noting that $\int_0^T \dot{x} dt = 0$

$$L^{2} \leq \int_{0}^{T} \{v^{2} - r^{2} + 2r\dot{x}\} dt \cdot \int_{0}^{T} dt = T^{2}(v^{2} - r^{2})$$

with equality if and only if $\dot{x} = \text{constant}$. Thus $L_{\text{max}} = T(v^2 - r^2)^{1/2}$ for a back and forth segment path perpendicular to the stream velocity.

We can also find the closed path of minimum length for a given time T. Since

$$v^{2} + 2vr + r^{2} \ge (v + r\cos\theta)^{2}$$
$$L \ge \int_{0}^{T} (v + r\cos\theta) dt = \int_{0}^{T} \{v + r(\dot{x} - r)/v\} dt = (v^{2} - r^{2})T/v$$

with equality if and only if $\cos^2 \theta = 1$. Thus $L_{\min} = (v^2 - r^2)T/v$ for a back ad forth segment path parallel to the stream velocity.

The above reults are generalized for the flight of an aeroplane in a three-dimensional irrotational wind field in a paper *On extreme length flight paths* submitted for publication.

NUMBER THEORY

Forms of numbers: sums of squares

Math. Mag., 49(1976) 96, 101.

Q 634. Submitted by M. S. Klamkin, University of Waterloo

If a, b, c, d are positive integers where ab = cd, show that $a^2 + b^2 + c^2 + d^2$ is always composite.

A 634. Since d = ab/c, a = mn, b = rs, c = mr, d = ns. The,

$$a^{2} + b^{2} + c^{2} + d^{2} = (m^{2} + s^{2})(n^{2} + r^{2})$$

This problem appeared on a West German Olympiad.

GEOMETRY

Equilateral triangles: orthogonal projection

Math. Mag., 49(1976) 211.

988. Proposed by Murray S. Klamkin, University of Waterloo

A given equilateral triangle ABC is projected orthogonally from a given plane P to another plane P'. Show that the sum of the squares of the sides of triangle A'B'C' is independent of the orientation of the triangle ABC in plane P.

Math. Mag., **51**(1978) 71–72.

Solution by W. Weston Meyer, General Motors Research Laboratories. We associate complex variables z = x + iy and z' = x' + iy' with P and P' respectively. Let Π be a closed n-sided polygom in P with vertices $z_1, z_2, \ldots, z_n, z_{n+1}$ ($z_{n+1} = z_1$), and let Π' be the image of Π in P' under an affine transformation x' = ax + by + c, y' = dx + ey + f. In terms of z' and z the transformation can be written $z' = \alpha z + \beta \overline{z} + \gamma$ where α, β , γ are complex constants and \overline{z} is the conjugate of z. A side $\Delta_j = z_{j+1} - z_j$ of Π will transform: $\Delta'_j = \alpha \Delta_j + \beta \overline{\Delta_j}$. Applying the cosine law identity

$$|u+v|^2 = |u|^2 + |v|^2 + 2\operatorname{Re}(u\bar{v})$$

one obtains

$$|\Delta'_j|^2 = (|\alpha|^2 + |\beta|^2)|\Delta_j|^2 + 2\operatorname{Re}(\alpha\bar{\beta}\Delta_j^2)$$

for the squared length pf Δ'_i . Let

$$S = \sum_{j=1}^{n} |\Delta_j|^2 \qquad S' = \sum_{j=1}^{n} |Delta'_j|^2 \quad \text{and} \quad \sigma_k = \sum_{j=1}^{n} \Delta_j^k \quad (k = 1, 2)$$
Then $S' = (|\alpha|^2 + |\beta|^2)S + 2\operatorname{Re}(\alpha\bar{\beta}\sigma_2)$. Rotation of Π through an angle θ in plane P will leave S unchanged while causing each Δ_j^2 , and hence σ_2 , to rotate through an angle 2θ . The sum S' will be fixed for all θ if and only if the same is true of $\operatorname{Re}(\alpha\bar{\beta}\sigma_2)$, in other words if and only if $\alpha\bar{\beta}\sigma_2 = 0$. We have the following theorem:

Under any affine transformation, other than a similarity transformation, for S' to be independent of the orientation of Π in P it is necessary and sufficient that $\Delta_1, \Delta_2, \ldots, \Delta_n$ be roots of $\Delta^n - p_{n-3}(\Delta) = 0$, where p_{n-3} is a polynomial of degree n-3 at most.

It is generally true that

 $(\Delta - \Delta_1)(\Delta - \Delta_2) \cdots (\Delta - \Delta_n) = \Delta^n - \sigma_1 \Delta^{n-1} + \frac{1}{2}(\sigma_1^2 - \sigma_2)\Delta^{n-2} - p_{n-3}(\Delta)$. In the present case, $\sigma_1 = 0$ because Π is closed; by excluding similarity (i.e., conformal) transformations we deny the value 0 to $\alpha \bar{\beta}$ so that $\alpha \bar{\beta} \sigma_2 = 0$ only if $\sigma_2 = 0$. Thus the theorem.

When n = 3, p_{n-3} must be a constant. This establishes the equilateral triangle as the only triangle with the O-I property (meaning that S' is orientation-independent). By reason of the cyclotomic equation, $\Delta^n = 1$, all regular polygons have the O-I property; but a polygon of more than three sides need not be regular to have it. Indeed, by the addition of a single vertex, any *n*-gon lacking the property can be expanded into an (n+1)-gon possessing it. We should mention, finally, that orthogonal projection from P to P' is a special case of affine transformation, non-conformal if the planes are not parallel.

GEOMETRY

Triangle inequalities: medians and sides

Math. Mag., 49(1976) 212, 218.

Q 638. Submitted by M. S. Klamkin, University of Waterloo

Let a, b and c denote the sides of an arbitrary triangle with respective medians m_a , m_b and m_c . Determine all integral p and q so that

$$\left(\frac{\sqrt{3}}{2}\right)^{p} \left(a^{p} m_{a}^{q} + b^{p} m_{b}^{q} + c^{p} m_{c}^{q}\right) \ge \left(\frac{\sqrt{3}}{2}\right)^{q} \left(a^{q} m_{a}^{p} + b^{q} m_{b}^{p} + c^{q} m_{c}^{p}\right)$$

A 639. If *a* is any integer, $(-1)^{|a|} = (-1)^{a}$. Hence,

$$(-1)^{\sum |k_i - m_i|} = \prod (-1)^{|k_i - m_i|} = \prod (-1)^{(k_i - m_i)} = (-1)^{\sum (k_i - m_i)} = (-1)^0 = 1$$

Thus $\sum |k_i - m_i|$ must be even.

ALGEBRA

Means

Math. Mag., 49(1976) 253.

1000. Proposed by Murray S. Klamkin, University of Waterloo

T denotes a cyclic permutation operator actingon the indices of a sequence $\{a_i\}$, that is, $T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_2x_1 + a_3x_2 + \cdots + a_1x_n$. If, for all $i, a_i \ge 0$ and $x_i > 0$, show that

$$\left\{\sum_{i=1}^{n} \frac{a_i}{n}\right\}^n \ge \prod_{i=1}^{n} T^i \left\{\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{x_1 + x_2 + \dots + x_n}\right\} \ge \prod_{i=1}^{n} a_i$$

Math. Mag., **51**(1978) 201.

Solution by Jerry Metzger, University of North Dakota. Let

$$w_i = T^i \left\{ \frac{a_1 x_1 + \dots + a_n x_n}{x_1 + \dots + x_n} \right\}$$

Since the geometric mean of the w_i never exceeds their arithmetic mean, we have

$$\sum_{i=1}^{n} \frac{a_1}{n} = \sum_{i=1}^{n} \frac{w_i}{n} \ge \left(\prod_{i=1}^{n} w_i\right)^{1/n}$$

which yiels the left-hand inequality. Also, from the arithmetic mean-geometric mean inequality,

$$T^{i}\left\{\frac{\sum a_{i}x_{i}}{\sum x_{i}}\right\} \geq \left\{a_{i}^{x_{1}}\cdots a_{n}^{x_{n-i+1}}a_{1}^{n-i+2}\cdots a_{i-1}^{x_{n}}\right\}^{1/\sum x_{i}}$$

so that

$$\prod_{i=1}^{n} T^{i} \left\{ \frac{\sum a_{i} x_{i}}{\sum x_{i}} \right\} \ge \prod_{i=1}^{n} a_{i}$$

ALGEBRA

Rate problems: rivers

Math. Mag., 50(1977) 46.

1004. Proposed by Murray S. Klamkin, University of Alberta

A river flows with constant speed w. A motorboat cruises with a constant speed v with respect to the river, where v < w. If the path travelled by the boat is a square of side L with respect to the ground, the time of traverse will vary with the orientation of the square. Determine the maximum and minimum time for the traverse.

Math. Mag., **51**(1978) 248.

Solution by Paul Y. H. Yiu, University of Hong Kong.

To travel along the side of a square making an angle $\theta \leq \pi/2$ with the direction of the current, the motorboat must be set in an appropriate direction, as shown in the diagram below, and the resultant speed is

$$u_{1} = \sqrt{v^{2} - w^{2} \sin^{2} \theta} + w \cos \theta \qquad v \qquad v$$
$$\theta$$
$$u_{2} \qquad u_{1}$$

The same diagram shows that the resultant speed along the opposite side is

$$u_2 = \sqrt{v^2 - w^2 \sin^2 \theta} - w \cos \theta$$

Replacing θ by $(\pi/2) - \theta$ we obtain the resultant speeds u_3 and u_4 along the other two sides. The time T of the traverse is $L(u_1^{-1} + u_2^{-1} + u_3^{-1} + u_4^{-1})$. Thus we find that

$$T = \frac{2L}{v^2 - w^2} \left[2v^2 - w^2 + \sqrt{(2v^2 - w^2)^2 - (w^2 \cos 2\theta)^2} \right]^{1/2}$$

We see that the minimum occurs when $\theta = 0$ and $T_{\min} = 2L(v + \sqrt{v^2 - w^2})/(v^2 - w^2)$. The maximum occurs when $\theta = \pi/4$ and $T_{\max} = 2L(4v^2 - 2w^2)^{1/2}/(v^2 - w^2)$.

[[On Math. Mag., 50(1977) 47, in connexion with Problem 955 there is quoted:

M. S. Klamkin, An identity for simplexes and related inequalities, *Simon Stevin*, **48**(1974-1975) 57–64.]]

SOLID GEOMETRY

Space curves

Math. Mag., 50(1977) 165–166.

Coplanar Points

November 1975

962. Proposed by Curt Monash, The Ohio State University

Consider the space curve C(t) defined by $C(t) = (t^k, t^m, t^n)$ for $t \ge 0$ and k, m, n integers.

a. Show that if (k, m, n) equals (1,2,3) or (-2, -1, 1), then C(t) does not contain four coplanar points.

b. Show that for (k, m, n) = (1, 3, 4), C(t) does contain four coplanar points.

c^{*}. Find a characterization of (k, m, n) so that C(t) does not contain four coplanar points.

I. Solution by Vaclav Konecny, Jarvis Christian College. Consider the equation of a plane in usual notation $A \equiv ax + by + cz + d = 0$. C(t) has common points with A if $at^k + bt^m + ct^n + d = 0$ ($t \ge 0$). As k, m, n are integers this equation can be always rewritten as polynomial equation in t. The number of changes of coefficients can be made maximum 3. Thus the number of positive roots is not greater than 3. Therefore C(t) does not contain four coplanar points except in some special cases (e.g., k = m; k = n; m = n; k = m = n) when the curve is in a plane.

[[the above solution, where the English could be improved, has been included, since it may be that Murray's solution (where '1/6' should ? be replaced by '6 times') refers to conditions therein. I'm not clear about the 'characterization' — why does 'b.' have four coplanar points ?]]

II. Solution by M. S. Klamkin, University of Alberta. We shall show that C(t) never contains four coplanar points under the given constraints.

[[what are these ?]]

Let the four points correspond to a, b, c, d. Then there will be no four coplanar points if the alternant determinant

$$\begin{vmatrix} 1 & a_1^k & a_2^m & a_3^n \\ 1 & b_1^k & b_2^m & b_3^n \\ 1 & c_1^k & c_2^m & c_3^n \\ 1 & d_1^k & d_2^m & d_3^n \end{vmatrix} \equiv |a^0 b^k c^m d^n| \neq 0$$

(here the determinant is 1/6 of the volume of the tetrahedron spanned by the four points).

It is a known result [1,2] that the generalized Vandermonde determinant $|a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n}|$, where a_1, a_2, \ldots, a_n are positive and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers, is equal to zero if and only if either amongst the numbers a_i or amongst the numbers α_i some are equal. Note that, if say a = 0, the determinant reduces to the lower order one $|b^k c^m d^n|$. For the case where any of the (k, m, n) are negative, none of a, b, c, d can be zero. Then by clearing of fractions, we are back to the previous cases.

In particular, it is known that

$$\begin{split} |a^{0}b^{1}c^{3}d^{4}| &= D\sum ab \\ |a^{0}b^{1}c^{3}d^{5}| &= D\{\sum a^{2}b+2\sum abc\} \\ |a^{0}b^{2}c^{3}d^{5}| &= D\{\sum a^{2}bc+3abcd\} \\ |a^{0}b^{1}c^{4}d^{5}| &= D\{\sum a^{2}b^{2}+\sum a^{2}bc+2abcd\} \\ |a^{0}b^{1}c^{3}d^{6}| &= D\{\sum a^{3}b+\sum a^{2}b^{2}+2\sum a^{2}bc+3abcd\} \end{split}$$

where the summations are symmetric sums over a, b, c, d and

$$D = |a^{0}b^{1}c^{2}d^{3}| = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

1. A. P. Mishina, I. V. Proscuryakov, Higher Algebra, Pergamon, Oxford, 1965, pp.14–15.

2. F. R. Gantmacher, M. G. Krein, Oscillating Matrices and Kernels and Small Vibrations of Mechanical Systems, Gostekhizat, Moscow, 1950, p.88. [[Gostekhizdat ?]]

GEOMETRY

Triangle inequalities: interior point

Math. Mag., 50(1977) 212–213.

A Geometric Inequality: Completed

November 1975

959. Proposed by L. Carlitz, Duke University

Let P be a point in the interior of the triangle ABC and let r_1 , r_2 , r_3 denote the distances from P to the sides of the triangle.

Let R denote the circumradius of ABC. Show that

$$\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \le 3\sqrt{R/2}$$

with equality if and only if ABC is equilateral and P is the center of ABC.

Comment by Murray S. Klamkin, University of Alberta. The n-dimensional extension of this problem (Jan. 1977) is not entirely complete. The verification of the extreme point was said to be easy and consequently was not done. However, since this is a maximum problem subject to the constraints $\sum e_i/h_i = 1$, $e_i \ge 0$, one has to check for extrema on all the boundaries of the constraint domain, which consists of very many faces of dimensions 0 to n - 1. Here we give a still further extension with a simple (non-calculus) proof using Hölder's inequality.

We will show that

$$\left\{\sum x_i^{2p/(2p-3)}\right\}^{(2p-3)/2p} \cdot \left\{\frac{R^2(n+1)^3}{n^2}\right\}^{1/2p} \ge \sum x_i r_i^{1/p} \tag{1}$$

where x_i (i = 1, 2, ..., n + 1) are arbitrary non-negative numbers, r_i are the distances from an interior point P of an n-simplex to the (n-1)-dimensional faces, R is the circumradius of the simplex, and p is any number greater than 3/2. Letting $x_i = 1$ and $p \to 3/2$, we recapture the extension given previously by Gerber. For n = 2 and p = 2 we obtain

$$\frac{27R^2}{4}(x_1^4 + x_2^4 + x_3^4) \ge \{x_1\sqrt{r_1} + x_2\sqrt{r_2} + \sqrt{r_3}\}^4$$

There is equality in (1) if and only if the simplex is regular, P is the centroid and the x_i are equal.

Proof. By Hölder's inequality,

$$\left\{\sum r_i/h_i\right\}^{1/p} \left\{\sum x_i^q h_i^{q/p}\right\}^{1/q} \ge \sum x_i r_i^{1/p}$$
(2)

and

$$\left\{\sum x_i^{2p/(2p-3)}\right\}^{(2p-3)/2p} \cdot \left\{\sum h_i^2\right\}^{q/2p} \ge \sum x_i^q h_i^{q/p} \tag{3}$$

where 1/p + 1/q = 1 and p > 3/2. Combining (2) and (3), using $\sum r_i/h_i = 1$ ($r_i = e_i$ in Gerber's notation and h_i = altitude of simplex from vertex *i*), we get

$$\left\{\sum x_i^{2p/(2p-3)}\right\}^{(2p-3)/2p} \cdot \left\{\sum h_i^2\right\}^{1/2p} \ge \sum x_i h_i^{1/p}$$

Finally, using

$$\sum h_i^2 \leq \sum m_i^2$$

where m_i is the median of the simplex from vertex i and

$$n^2 \sum m_i^2 \le R^2 (n+1)^3$$

we obtain (1).

Although Gerber notes that (4) is an immediate consequence of Lagrange's identity (which may have been known to Leibniz), we include a proof for completeness.

Let \mathbf{V}_i and \mathbf{G} denote vectors from the circumcenter O to the vertices V_i and to the centroid, respectively, of the simplex. Then

$$\sum \mathbf{V}_i^2 = \sum \{(\mathbf{V}_i - \mathbf{G}) + \mathbf{G}\}^2$$
$$= \sum |V_i G|^2 + (n+1)|OG|^2$$

Since $|\mathbf{V}_i|^2 = R^2$ and $|V_iG| = nm_i/(n+1)$ we obtain (4). Further applications of this polar moment of inertia identity are given in this MAGAZINE, **48**(1975) 44–46.

[[the above comment includes references to the following article:

Geometric Inequalities via the Polar Moment of Inertia

M. S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

and to the solution by Gerber, which is given below:

Math. Mag., **50**(1977) 50–51.

II. Solution (generalization) by Leon Gerber,. Let P be a point in an n-simplex \mathcal{A} with inradius r and circumradius R. Let the distances of P from the vertices and faces of \mathcal{A} be respectively d_i and e_i for $i \in \mathcal{I} = \{0, 1, \ldots, n\}$. Berkes [1] proved that

$$\left(\frac{1}{n+1}\sum d_i^p\right)^{1/p} \ge nr \tag{1}$$

for p = 1. Since the left side is a power mean, which increases with p. the result follows for all $p \ge 1$. In [2, Theorem 4.4] we proved (1) for $p \ge 2/[1 + \log(n+1)]$ and also

$$\left(\frac{1}{n+1}\sum e_i^p\right)^{1/p} \le R/n \tag{2}$$

for $p \leq 0$. The present problem is that of proving (2) for n = 2 and p = 1/2. We shall prove (2) for p = 2/3 and hence obtain:

THEOREM. Inequality (2) isvalid for all $p \leq 2/3$. Equality holds if and only if P is the center of a regular simplex. (We conjecture that the best possible exponent exceeds 2/3 and approaches 1 as n increases.)

Proof. Let h_i be the altitude to face i, V_i the *n*-dimensional volume of the *n*-simplex with vertex P and opposite face i, V the volume of the given simplex, and K_i the (n-1)-dimensionalarea of face i. Then

$$\sum \frac{e_i}{h_i} = \sum \frac{e_i K_i / (n+1)}{h_i K_i / (n+1)} = \sum \frac{V_i}{V} = 1$$

Hence the problem becomes

maximize
$$\sum e_i^{2/3}$$
 subject to $\sum \frac{e_i}{h_i} = 1$

The method of Lagrange multipliers yields

$$\frac{2}{3}e_i^{-1/3} - \lambda h_i^{-1} = 0 \qquad i \in \mathcal{I}$$
(3)

for the extreme point; it is easy to verify that this yields a maximum. Then

$$e_i = \left(\frac{2h_i}{3\lambda}\right)^3$$
 $1 = \sum \frac{e_i}{h_i} = \left(\frac{2}{3\lambda}\right)^3 \sum h_i^2$ and $\lambda = \frac{2}{3} \left(\sum h_i^2\right)^{1/3}$

Multiplying (3) by $3e_i/(2(n+1))$ and summing, we get

$$\left[\frac{1}{n+1}\sum e_i^{2/3}\right]^{3/2} = \left[\frac{3\lambda}{2(n+1)}\right]^{3/2} = (n+1)^{-3/2} \left(\sum h_i^2\right)^{1/2}$$

for the extreme point. Since this last expression is independent of λ and e_i we have

$$\left[(n+1)^{-1} \sum e_i^{2/3} \right]^{3/2} \le (n+1)^{-3/2} \left(\sum h_i^2 \right)^{1/2} \tag{4}$$

for all points of \mathcal{A} .

Let G be the centroid and C the circumcenter of \mathcal{A} , and let A_iG_i $(i \in \mathcal{I})$ be the medians. Clearly $h_i \leq A_iG_i$ $(i \in \mathcal{I})$, with equality if and only if \mathcal{A} is regular, in which

case equality holds in (4) if and only if P is the center. Further, it is an immediate consequence of Lagrange's identity that

$$0 \le CG^2 = R^2 - (n+1)^{-1} \sum A_i G^2 = R^2 - n^2 (n+1)^{-3} \sum A_i D_i^2$$

with equality if and only if C = G, which holds if and only if \mathcal{A} is regular. Thus

$$\left[(n+1)^{-1} \sum e_i^{2/3} \right]^{3/2} \leq (n+1)^{-3/2} \left(\sum h_i^2 \right)^{1/2}$$
$$\leq (n+1)^{-3/2} \left(\sum A_i G_i^2 \right)^{1/2} \leq R/n$$

 J. Berkes, Einfacher Beweis und Verallgemeinerung einer Dreiecksungleichung, *Elem. Math.*, 22(1967) 135–136; *MR* 20 #1254.

2. L. Gerber, The orthocentric simplex as an extreme simplex, *Pacific J. Math.*, **56**(1975) 97–111; *MR* **51** #12717.

[[The 50-year index contains the following Murray items:]]

A note on an *n*-th order differential equation, 32(1958) 33–34.

A probability of more heads, 44(1971) 146–149.

(with R. G. McLenaghan) An ellipse inequality, 50(1977) 261–263.

An extension of the butterfly problem, 38(1965) 206–208; Sequels, 42(1969) 17–21, 21–23; 46(1973) 278, 280; 49(1976) 86–87.

Circle through three given points, 44(1971) 279–282.

(with Ernest C. Schlesinger) Diamond inequalities, $\mathbf{50}(1977)$ 96–98; Comment, $\mathbf{50}(1977)$ 276.

Duality in spherical triangles, 46(1973) 208–211.

Extensions of some geometric inequalities, 49(1976) 28–30.

(with D. J. Newman) Extensions of the Weierstrass product inequalities, 43(1970) 137–141.

Geometric Inequalities via the polar moment of inertia, 48(1975) 44–46.

On Barbier's solution of the Buffon needle problem, 28(1955) 135–138.

On ruled and developable surfaces of revolution, 27(1954) 207–209.

On some problems in gravitational attraction, 41(1968) 130–132.

On some soluble N th order differential equations, 43(1970) 272–275.

On the volume of a class of truncated prisms and some related centroid problems, 41(1968) 175–181; Correction, 42(1969) 23.

Perfect squares of the form $(m^2 - 1)a_n^2 + t$, **42**(1969) 111–113.

(with D. J. Newman) Some combinatorial problems of arithmetic, 42(1969) 53-56.

The equation of a sphere, 42(1969) 241–242.

(with Ralph P. Boas) Extrema of polynomials, 50(1977) 75–78.

(with G. D. Chakerian) Minimal covers for closed curves, 46(1973) 55–61.

(with G. D. Chakerian and G. T. Sallee) On the butterfly problem, 42(1969) 21–23.

(with Leon Bankoff and Paul Erdős) The asymmetric propeller, 46(1973) 270–272.

(with Robert W. Gaskell and P. Watson) Triangulations and Pick's theorem, **49**(1976) 35–37; Comment, **49**(1976) 105, 158.

 $[[{\rm I~just~tried~'Klamkin'~on~MathSciNet-115~hits,~all~of~them~Murray.}$ Would be easy to pick them all up, if wanted. - R.]]

NUMBER THEORY

Digit problems: primes

Math. Mag., **51**(1978) 69.

1029^{*}. Proposed by Murray S. Klamkin, University of Alberta

Does there exist any prime number such that if any digit (in base 10) is changed to any other digit, the resulting number is always composite?

Math. Mag., **52**(1979) 180–182.

Solution by Paul Erdős, Hungarian Academy of Science. We prove a slightly stronger result and in the end make some comments and state a few more problems.

For every $k > k_0$ there are primes

$$p = \sum_{i=0}^{k} a_i 10^i \qquad a_0 > 0, \ a_k > 0, \ 0 \le a_i \le 9$$

so that all the integers

$$p + t \cdot 10^i \qquad |t| \le 10 \qquad 0 \le i \le k \tag{1}$$

are composite. In fact there are 21(k + 1) integers of the form (1). Put $x = 10^{k+1}$. We will determine $p \mod q_i$ where the q_i will be suitably chosen primes whose product is less than x^{ϵ} where ϵ is small but fixed. Then by Linnik's theorem (the smallest $p \equiv a \pmod{b}$ is less than b^c for an absolute constant c if (a, b) = 1) there is a p < xwhich satisfies all these congruences. The congruences will be chosen so that all of the numbers will be multiples of one of the q_i ; thus they are all composite.

By a well-known theorem of Bang-Birkhoff-Vandiver, there is always a q_j so that $10^j \equiv 1 \mod q_j$ and $10^i \not\equiv 1 \pmod{q_j}$ $1 \leq i < j$. (The theorem states: for every a and j [except $2^6 - 1$] there is a q_j so that $a^j - 1 \equiv 0 \pmod{q_j}$ and $a_i - 1 \not\equiv 0 \pmod{q_j}$ for every $1 \leq i < j$.) Consider these primes so that

$$\prod_{j=1}^{r} q_j \le x^{\epsilon/2} < \prod_{j=1}^{r+1} q_j$$

since $\prod_{j=1}^{r} q_j < 10^{r^2}$, r can be chosen as $[\epsilon \sqrt{\log x}]$. Now we determine the congruences. Suppose that the congruences

$$p \equiv u_m \pmod{q}_m \qquad 1 \le m \le j-1$$

have already been determined. Let $b_1, \ldots, b_{s_{j-1}}$ be the integers of the form $t \cdot 10^i$, $|t| \leq 10, 0 \leq i \leq k$ which do not satisfy any of the congruences

$$t \cdot 10^i \equiv -u_m \pmod{q}_m \qquad 1 \le m \le j-1$$

The numbers $t \cdot 10^i$ determine at most 21j residues mod q_j (since 10^i takes exactly j distinct values by $10^j \equiv 1 \pmod{q_j}$). Therefore there is a u_j for which $b_L \equiv -u_j \pmod{q_j}$, $1 \leq L \leq j$ is satisfied by at least $\{b_{s_{j-1}}\}$ values of L where $\{N\}$ denotes the least integer $\geq N$. Put $p \equiv u_j \pmod{q_j}$. This determines the congruences

$$p \equiv u_j \pmod{q}_j \qquad 1 \le j \le r = [\epsilon \sqrt{\log x}]$$

$$\tag{2}$$

The number of integers $t \cdot 10^{j}$, $|t| \leq 10$, $1 \leq j \leq k$ for which $p + t \cdot 10^{j}$ (p satisfying the congruences (1)) is not a multiple of one of the q_{j} $1 \leq j \leq r$ is at most

$$21(k+1)\prod_{j=2}^{r} \left(1 - \frac{1}{j}\right) < \frac{21\log x}{r} < \frac{21\sqrt{\log x}}{\epsilon}$$

Let $v_1, v_2, \ldots, v_x, x < \frac{21 \log x}{\epsilon}$ be those integers of the form $t \cdot 10^j$. Let Q_1, Q_2, \ldots, Q_x be the consecutive primes which are not q_i . Put

$$p \equiv -v_i \pmod{Q}_i \qquad i = 1, \dots, x$$
 (3)

There are r + x congruences (2) and (3). The product of the moduli equals

$$\prod_{j=1}^{r} q_j \prod_{i=1}^{x} Q_i < x^{\epsilon/2} x^{\epsilon/2} = x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} = x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} = x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} = x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} x^{\epsilon/2} = x^{\epsilon/2} x^$$

 $(\prod Q_i < x^{\epsilon/2} \text{ is trivial from the prime number theorem or a much more elementary result.) This completes our proof since the primes <math>p$ satisfying (2) and (3) satisfy(1) and as stated by Linnik there are primes p < x.

Denote by $l_a(p)$ the exponent of $a \mod p$.

I can prove

$$\sum_{p < x} \frac{1}{l_a(p)} > x^c \tag{4}$$

and in (4) probably c can be taken to be $1 - \epsilon$, but this seems very difficult.

From (4) we can deduce by the methods used here that there are infinitely many primes $p, 10^k , so that if we simultaneously alter <math>(\log k)^{\epsilon}$ digits we always get a composite number.

Is it true that if $a_m > c^m/m$, c > 1, m = 1, 2, ... then there are always primes, in fact infinitely manyof them, so that all the numbers $p + a_m$, $a_m < p$ are composite? I do not know.

Editor's Comment. Individual computer searches by Allan Wm. Johnson, Harry L. Nelson and Stanley Rabinowitz found six-digit primes which provide a solution. The complete list of such six-digit primes supplied by Nelson is: 294001, 505447,584141, 604171 and 971767. Rabinowitz supplied the table below which shows a divisor for each number that can be formed from 294001 by changing one digit.

digit	replacement digit									
changed	0	1	2	3	4	5	6	7	8	9
100000	23	3	1	47	3	73	7	3	587	239
10000	7	173	3	29	17	3	227	7	3	1
1000	3	397	29	3	1	7	3	43	11	3
100	1	19	3	7	83	3	151	11	3	29
10	1	41	3	29	11	3	157	409	3	7
1	2	1	2	3	2	5	2	7	2	3

TRIGONOMETRY

Inequalities: tan and sec

Math. Mag., 51(1978) 128, 132.

Q 652. Submitted by M. S. Klamkin, University of Alberta

Show that $\sum_{i=1}^{n} (1 + \tan \alpha_i) \leq \sqrt{2} \sum_{i=1}^{n} \sec \alpha_i$ when $\sec \alpha_i > 0$. When does equality hold? (This is a generalization of Q472, March 1970.)

A 652. We see that

$$\sum_{i=1}^{n} (1 + \tan \alpha_i) = \sum_{i=1}^{n} \frac{\sin \alpha_i + \cos \alpha_i}{\cos \alpha_i} = \sqrt{2} \sum_{i=1}^{n} \frac{\sin(\alpha_1 + \pi/4)}{\cos \alpha_i} \le \sqrt{2} \sum_{i=1}^{n} \sec \alpha_i$$

Equality holds if and only if $\sin(\alpha_i + \pi/4) = 1$ for all *i*. In particular, if α_1 , α_2 and α_3 are the angles of a triangle, then $3 + \sum \tan 3\alpha_i/4 \le \sqrt{2} \sum \sec 3\alpha_i/4$ with equality if and only if the triangle is equilateral.

In the original answer to Q472 it was also shown that $\sec \alpha + \sec \beta \ge 2$. Again, this isn't sharp. Since $\sec x$ is convex in $(-\pi/2, \pi/2)$,

$$\frac{1}{n}\sum_{i=1}^{n}\sec\alpha_i \ge \sec\left(\sum_{i=1}^{n}\alpha_i/n\right)$$

with equality if and only if $\alpha_i = \text{constant}$.

GEOMETRY

Triangle inequalities: radii

Math. Mag., **51**(1978) 193.

1043. Proposed by M. S. Klamkin, University of Alberta

If (a_i, b_i, c_1) are the sides, R_i the circumradii, r_i the inradii. and s_i the semi-perimeters of two triangles (i = 1, 2), show that

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\right\}^{1/2} \ge 3\left\{\frac{1}{\sqrt{a_1a_2}} + \frac{1}{\sqrt{b_1b_2}} + \frac{1}{\sqrt{c_1c_2}}\right\}$$
(1)

with equality if and only if the two triangles are equilateral.

Also show that the analogous three triangle inequality

$$\left\{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\frac{s_3}{r_3R_3}\right\}^{1/2} \ge 9\left\{\frac{1}{\sqrt{a_1a_2a_3}} + \frac{1}{\sqrt{b_1b_2b_3}} + \frac{1}{\sqrt{c_1c_2c_3}}\right\}$$
(2)

is invalid.

Math. Mag., **52**(1979) 320–321.

Solution by Paul Bracken, undergraduate, University of Toronto. We prove first that

$$(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) \ge 3\left(\sqrt{a_1 a_2 b_1 b_2} + \sqrt{b_1 b_2 c_1 c_2} + \sqrt{a_1 a_2 c_1 c_2}\right)$$
(3)

Apply the Schwarz inequality $x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$ to $L \equiv \sqrt{a_1 + b_1 + c_1}\sqrt{a_2 + b_2 + c_2}$ to obtain

$$L \ge \sqrt{a_1 a_2} + \sqrt{b_1 b_2} + \sqrt{c_1 c_2} \quad \text{and}$$
$$L^2 \ge a_1 a_2 + b_1 b_2 + c_1 c_2 + 2\left(\sqrt{a_1 a_2 b_1 b_2} + \sqrt{b_1 b_2 c_1 c_2} + \sqrt{a_1 a_2 c_1 c_2}\right)$$

But certainly for positive x, y, z we know that $x^2 + y^2 + z^2 \ge xy + yz + zx$. Set $x = \sqrt{a_1 a_2}$, $y = \sqrt{b_1 b_2}$ and $z = \sqrt{c_1 c_2}$ in this inequality and apply it in the previous expression for L^2 . This gives (3) immediately.

Now L^2 is $4s_1s_2$ and we have the well-known relation $4r_iR_i = (a_ib_ic_i)/s_i$ (i = 1, 2). Hence

$$4s_1s_2 = \sqrt{s_1s_2/(r_1R_1r_2R_2)} \cdot \sqrt{a_1a_2b_1b_2c_1c_2}$$

By using this in (3) and dividing both sides by $\sqrt{a_1a_2b_1b_2c_1c_2}$, (1) is proved. Equality will hold in (3) if and only if we have $a_1 = a_2 = b_1 = b_2 = c_1 = c_2$; that is if both triangles are equilateral, sice only then will equality hold in both of the inequalities that were used above.

By using $4r_iR_i = a_ib_ic_i/s_i$ equation (2) becomes analogous to (3):

$$p_1 p_2 p_3 \ge 9 \left(\sqrt{a_1 a_2 a_3 b_1 b_2 b_3} + \sqrt{b_1 b_2 b_3 c_1 c_2 c_3} = \sqrt{c_1 c_2 c_3 a_1 a_2 a_3} \right)$$

This is not always true for triangles. Consider sides a_1 , $a_1/100$, $99a_1/100$ as a_1 , b_1 , c_1 ; a_2 , $a_2/100$, $99a_2/100$ as a_2 , b_2 , c_2 ; a_3 , $a_3/100$, $99a_3/100$ as a_3 , b_3 , c_3 . The left hand side is $8a_1a_2a_3$. The right side is bigger than $(8.8)a_1a_2a_3$.

[[This needs checking. The triangles are degenerate and I'm sure that non-degenerate ones can be found. — R.]]

Comment by M. S. Klamkin, University of Alberta. As a companion inequality we also have

$$4\left\{\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{b_1}} + \frac{1}{\sqrt{c_1}}\right\}\left\{\frac{1}{\sqrt{a_2}} + \frac{1}{\sqrt{b_2}} + \frac{1}{\sqrt{c_2}}\right\} \ge \left\{\frac{s_1}{r_1R_1} \cdot \frac{s_2}{r_2R_2}\right\}^{1/2}$$

or equivalently

$$\{2\sum \sqrt{a_1b_1}\}\{2\sum \sqrt{a_2b_2}\} \ge \{\sum a_1\}\{\sum a_2\}$$
(4)

The latter follows from the area of a triangle of sides $a^{1/4}$, $b^{1/4}$, $c^{1/4}$ being non-negative, i.e.,

$$2\sum \sqrt{ab} - \sum a \ge 0$$

There's equality in (4) if both triangles are degenerate (each having a vanishing side).

Math. Mag., **51**(1978) 198–199.

Lott's Problem

November 1976

997. Proposed by John Lott, Student, Southwest High School, Kansas City, Missouri

Let P be a polynomial of degree n, $n \ge 2$ with simple zeros z_1, z_2, \ldots, z_n . Let $\{g_k\}$ be the sequence of functions defined by $g_1 = 1/P'$ and $g_{k+1} = g'_k/P'$. Prove for all k that $\sum_{j=1}^n g_k(z_j) = 0$.

Editors' Comment. M. S. Klamkin gives an extension by starting with the partial fraction expansion

$$\frac{1}{(P(z))^m} = \sum_{i=1}^m \left\{ \frac{A_{i1}}{z - z_i} + \frac{A_{i2}}{(z - z_i)^2} + \dots + \frac{A_{im}}{(z - z_i)^m} \right\}$$

where the A_{ij} are functions of the roots. By expanding both sides of this equation in powers of 1/z and equating the coedificients of like powers of z, he obtains the following identities:

$$\sum_{i} A_{i1} = 0 \qquad \sum_{i} (A_{i1}z_{i}) = 0 \qquad \sum_{i} (A_{i1}z_{i}^{2} + 2A_{i2}z_{i} + A_{i3}) = 0 \quad \text{etc.}$$

He then shows $(m-1)!A_{i1} = g_m(z_i)$ and derives other recurrence relations for the remaining A_{ij} .

ANALYSIS

Curves: normals

Math. Mag., **52**(1979) 113.

1067. Proposed by M. S. Klamkin, University of Alberta

Problem P.M.11 on the first William Lowell Putnam Competition, April 16, 1938, was to find the length of the shortest chord that is normal to the parabola $y^2 = 2ax$, a > 0, at one end. A calculus solution is quite straightforward. Give a completely "non-calculus" solution.

Math. Mag., **53**(1980) 185.

Solution by J. M. Stark, Lamar University. Let t be a real parameter, t > 0. Then $P[t^2/(2a), t]$ is a point on that part of the parabola $y^2 = 2ax$ which lies above the x-axis.

A line with slope m passing through P has equation

$$y - t = m[x - (t^2/2a)]$$
(1)

Without use of the calculus, the tangent line to the parabola at point P is found by requiring that line (1) have a double point of intersection with the parabola at P. Solving (1) for y and substituting into $y^2 - 2ax$ gives a quadratic in x whose discriminant can be written in the form $(2a - 2mt)^2$. Requiring that this discriminant be zero gives m = (a/t) as the slope of the tangent line to the parabola $y^2 = 2ax$ at the point $P[t^2/(2a), t], t > 0$. By trigonometry the slope of the normal line to this parabola at P is -t/a, and so the normal line at P has equation

$$y - t = (-t/a)[x - (t^2/2a)]$$
(2)

Solving simultaneously (2) and $y^2 = 2ax$ gives that the normal line to the parabola $y^2 = 2ax$ at the point $P[t^2/(2a), t]$, t > 0, also intersects the parabola at the point $Q[(t^2 + 2a^2)^2/(2at^2), -t - (2a^2)/t]$.

Denote by L the length of the chord \overline{PQ} , normal to $y^2 = 2ax$ at P. Using the formula for the distance between two points and factoring, it is easily obtained that

$$L^{2} = (a^{6}/t^{4})[4(t/a)^{2} + 1][(t/a)^{2} - 2]^{2} + 27a^{2}$$
(3)

Now (3) shows that $L^2 \ge 27a^2$ and that for t > 0 the shortest chord normal to the parabola at one end is obtained when $t = \sqrt{2}a$, and the shortest chord is $3\sqrt{3}a$.

ALGEBRA

Inequalities: exponentials

Math. Mag., **52**(1979) 114, 118.

Q 658. Submitted by M. S. Klamkin, University of Alberta

If a, b > 0, prove that $a^b + b^a > 1$.

A 658. Since the inequality is obviously valid if either a or $b \ge 1$, it suffices to consider a = 1 - x and b = 1 - y, where 0 < x, y < 1. Our inequality now becomes

$$\frac{1-x}{(1-x)^y} + \frac{1-y}{(1-y)^x} > 1$$

By the mean value theorem

$$(1-x)^y = 1 - \frac{xy}{(1-\theta x)^{1-y}} \le 1 - xy$$

Hence

$$\frac{1-x}{(1-x)^y} + \frac{1-y}{(1-y)^x} \ge \frac{1-x+1-y}{1-xy} = \frac{2-x-y}{1-xy} - 1 + 1 = \frac{(1-x)(1-y)}{1-xy} + 1 > 1$$

The stated inequality appears in D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p.281, where a more involved proof is given.

[This problem also appeared as CMB P261 by R. Schramm

Canad. Math. Bull., 20(1977) 148]

GEOMETRY

Regular polygons: inscribed polygons

Math. Mag., **52**(1979) 258.

1076. Proposed by M. S. Klamkin, University of Alberta

Let \mathcal{B} be an *n*-gon inscribed in a regular *n*-gon \mathcal{A} . Show that the vertices of \mathcal{B} divide each side of \mathcal{A} in the same ratio and sense if and only if \mathcal{B} is regular.

Math. Mag., **53**(1980) 249–250.

Solution by Imelda Yeung, student, Oberlin College. Let A_i , B_i (i = 1, 2, ..., n) denote the vertices of a regular *n*-gon \mathcal{A} and the vertices of an inscribed *n*-gon \mathcal{B} , respectively. Let *a* be the length of each side of \dashv . If B_i divides $\overline{A_i A_{i+1}}$ into segments of length *c* and *d*, then by the law of cosines, we deduce that

$$\overline{B_{i-1}B_i}^2 = c^2 + d^2 - 2cd\cos\theta = \text{ constant, where } \theta = \frac{(k-2)\pi}{k}$$

Thus $\triangle B_{i-1}A_iB_i \simeq \triangle B_iA_{i+1}$ and therefore angle $(B_{i-1}B_iB_{i+1}) = 180^\circ - (180^\circ - \theta) = \theta$. Hence the vertices B_i form a regular *n*-gon.

Conversely, if \mathcal{B} is a regular *n*-gon, then angle $(B_{i-1}B_iB_{i+1}) = \text{angle } (A_{i-1}A_iA_{i+1}) = \theta$. It follows that

angle $(A_i b_i B_{i-1}) +$ angle $(A_i B_{i-1} B - i) =$ angle $(A_i b_i B_{i-1}) +$ angle $(A_{i+1} B_i B_{i+1})$

and so angle $(A_i B_{i-1} B_i) = \text{angle}(A_{i+1} B_i B_{i+1})$. Therefore $\triangle B_{i-1} A_i B_i \simeq \triangle B_i A_{i+1} B_{i+1}$. Thus $\overline{B_{i-1} A_i} = \overline{B_i A_{i+1}}$ and $\overline{A_i B_i} = \overline{A_{i+1} B_{i+1}}$. Therefore the vertices of \mathcal{B} divide each side of \mathcal{A} in the same ratio and sense.

ANALYSIS

Maxima and minima: unit circle

Math. Mag., 52(1979) 259, 265.

Q 662. Submitted by M. S. Klamkin, University of Alberta

Determine the maximum of

$$R = \frac{|z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_1|^3}{|z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_5 + z_4 z_5 z_1 + z_5 z_1 z_2|^2}$$

where z_1 , z_2 , z_3 , z_4 and z_5 are complex numbers of unit length.

A 662. Divide the denominator by $|z_1z_2z_3z_4z_5|$ and use $1/z = \overline{z}$ to obtain (with $z_6 = z_1$) $R = |\sum z_i z_{i+1}| \leq \sum |z_i z_{i+1}| = 5$. The idea of this problem is based on Problem E3528 of A. A. Bennett in the *Amer. Math. Monthly*, **39**(1932) p.115.

LINEAR ALGEBRA

Matrices: orthogonal matrices

Math. Mag., **52**(1979) 259.

Really Orthogonal

March 1978

1035. Proposed by H. Kestelman, University College, London

A is a real $n \times n$ matrix. Do there exist orthogonal matrices B such that A + B is real orthogonal?

Solution by M. S. Klamkin, University of Alberta. Since there are $n + \binom{n}{2} = (n^2 + n)/2$ conditions for an $n \times n$ matrix to be orthogonal, we have $n^2 + n$ conditions on B. Consequently, we do not expect to find solutions. In particular, let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Then it would be necessary that $(2 + \cos \theta, 1 + \sin \theta)$ and $(1 - \sin \theta, 2 + \cos \theta)$ are orthogonal vectors or that $\cos \theta = -2$.

GEOMETRY

Maxima and minima: shortest paths

Math. Mag., **52**(1979) 316.

1083. Proposed by M. S. Klamkin and A. Liu, University of Alberta

Given an equilateral point lattice with n points on a side, it is easy to draw a polygonal path of n segments passing through all the n(n+1)/2 lattice points. Show that it cannot be done with less than n segments.

Math. Mag., 53(1980) 303.

n = 5

Solution by Richard Beigel, Stanford University. Proof by induction. This obvious for n = 2. Assume the proposition for n-1. If any segment is contained completely in an edge, then at least n-1 segments are required in order to cover the remaining points by the inductive hypothesis. Otherwise, each segment intersects an edge in at most one point. Since each edge has n points, the proposition holds. Clearly, the proof shows that the segments need not be connected.

ALGEBRA

Inequalities: finite sums

Math. Mag., 52(1979) 317, 323.

Q 664. Submitted by M. S. Klamkin, University of Alberta

Prove that

$$\sum_{k=1}^{n} (x_k + 1/x_k)^a \ge \frac{(n^2 + 1)^a}{n^{a-1}}$$

where $x_k > 0$ (k = 1, 2, ..., n), a > 0 and $x_1 + x_2 + \cdots + x_n = 1$.

[[I've stuck in a label '(1)'as it's referred to several times – is it in the right place? Later: I've moved the label down below. Is that better? This is another of Murray's not-so-quickies. — R.]]

A 664. The inequality is an immediate consequence of Jensen's inequality for convex functions F, i.e.,

$$\sum_{k=1}^{n} F(x_k)/n \ge F\left\{\sum_{k=1}^{n} x_k/n\right\}$$

With equality if and only if $x_k = \text{constant}$ and followed by Cauchy's inequality:

$$\sum_{k=1}^{n} (x_k + 1/x_k)^a \ge n \left\{ \sum_{k=1}^{n} (x_k + 1/x_k)/n \right\}^a \ge \frac{(n^2 + 1)^n}{n^{a-1}}$$
(1)

It now remains to show that the function $y = (x+1/x)^a$ is convex in the given domain 0 < x < 1 or equivalently that $y'' \ge 0$. Here

$$y'' = a(x+1/x)^{a-2} \left\{ a(1-1/x^2)^2 + 1/x^4 + 4/x^2 - 1 \right\}$$

Since 1/x > 1, $y'' \ge 0$ for $a \ge 0$. It also follows that y'' < 0 for $0 > a \ge -1$. Consequently inequality (1) is reversed for this latter domain of a. More generally,

$$\sum_{k=1}^{n} F(x_k + 1/x_k)/n \ge F\left\{\sum_{k=1}^{n} (x_k + 1/x_k)/n\right\} \ge F(n + 1/n)$$

for convex increasing F and the same domain as (1) for the outer inequality.

Comment. Inequality (1) is given and proved in D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, pp.282–283, in a longer way using Lagrange multipliers. Math. Mag., 53(1980) 49.

1089. Proposed by M. S. Klamkin, University of Alberta

Determine the highest power of 1980 which divides

$$\frac{(1980n)!}{(n!)^{1980}}$$

Math. Mag., 54(1981) 36–37.

Solution by G. A. Heuer, Concordia College & Karl Heuer, Moorhead, Minnesota. Let $V_m(x)$ be the exponent of the highest power of m which divides x. If p is prime,

$$V_p\left(\frac{(mn)!}{(n!)^n}\right) = V_p((mn)!) - mV_p(n!) = \sum_{k \ge 1} \left(\left[\frac{mn}{p^k}\right] - m\left[\frac{n}{p^k}\right] \right)$$

Thus, if m has the prime factorization $m = \prod_{i=1}^r p_i^{e_i}$

$$V_m\left(\frac{(mn)!}{(n!)^m}\right) = \min_i \left\lfloor \frac{1}{e_i} \sum_{k \ge 1} \left(\left\lfloor \frac{mn}{p_i^k} \right\rfloor - m\left\lfloor \frac{n}{p_i^k} \right\rfloor \right) \right\rfloor$$

The brackets $[\cdot]$ in all cases denote the greatest integer, and we note that the summand is the *m*-residue of $[mn/p_i^k]$. In particular, since $1980 = 2^2 5^2 \cdot 5 \cdot 11$,

$$V_{1980}\left(\frac{(1980n)!}{(n!)^{1980}}\right) = \min_{1 \le i \le 4} \left[\frac{1}{e_i} \sum_{k \ge 1} \left(\left[\frac{1980n}{p_i^k}\right] - 1980\left[\frac{n}{p_i^k}\right]\right)\right]$$

where $e_1 = e_2 = 2$, $e_3 = e_4 = 1$ and p_1 , p_2 , p_3 and p_4 are 2, 3, 5 and 11 respectively. Depending upon *n*, the minimum may occur in any of the four terms. $V_p((mn)!/(n!)^m)$ is small when *n* is a power of *p*, and for $p^{k-1} \leq n < p^k$ it grows with the sum of the digits of *n* in base *p*. If n = 8 the minimum in V_{1980} occurs when p = 2; if n = 9, 5 or 11 it occurs when p = 3, 5 or 11 respectively. Math. Mag., 53(1980) 112.

1093. Proposed by M. S. Klamkin, University of Alberta

Prove that for complex numbers u, v and w,

$$|u - v| + |u + v - 2w + |u - v|| < |u + v|$$
(1)

if and only if

$$|w - v| + |w + v - 2u + |w - v|| < |w + v|$$
(2)

Math. Mag., 54(1981) 212.

Editor's Comment. While the result is true if the numbers are real, it is not necessarily true if the numbers are complex. J. M. Stark provides an example which can be simplified to u = 3i, v = 2i and w = i for which the first inequality is true, but the second is false.

Math. Mag., 53(1980) 180.

1100. Proposed by M. S. Klamkin & M. V. Subbarao, University of Alberta

Suppose that F(x) is a power series (finite or infinite) with rational coefficients and $A_k = \int_0^1 x^k F(x) dx$ for integers $k \ge 0$.

(i) If all the A_k are rational, must F(x) be a polynomial?

(ii) Does there exist an F(x) such that all the A_k except one are rational?

(iii) Does there exist an F(x) such that all the A_k except $A_{P(k)}$, k = 0, 1, 2, ..., are rational, where P(k) is an integer valued polynomial, e.g., P(k) = 2k?

(iv)* Given a finite indexed set $A_{m(k)}$ k = 0, 1, 2, ..., n, does there exist an F(x) such that all the A_k except the $A_m(k)$ are rational?

Math. Mag., **54**(1981) 214.

Solution by Paul J. Zwier, Calvin College. Note first that the series $\sum_{i=0}^{\infty} 1/(k+i+1)(m+i+1)$, where k and m are nonnegative integers, is rational if $k \neq m$ and irrational if k = m. This follows from the fact that $k \neq m$

$$\frac{1}{(k+i+1)(m+i+1)} = \frac{1}{m-k} \left[\frac{1}{k+1+1} - \frac{1}{m+i+1} \right]$$

so that the series is telescopic and, if k = m, converges to an irrational number since the series

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

We can now answer questions (i), (ii) and (iv). If we define $c_i = 1/(m+i+1)$ and then $F_m(x) = \sum c_i x^i$

$$A_k = \sum_{i=0}^{\infty} \frac{1}{(k+i+1)(m+i+1)}$$

which is rational for $k \neq m$ and irrational for k = m. This gives a positive answer to (ii). If we are given a finite set $\{m_1, m_2, \ldots, m_n\}$ then $F_{m_1} + F_{m_2} + \cdots + F_{m_n}$ is a power series with the corresponding A_i rational except when k is one of the m_i . This answers (iv) in the affirmative.

As to part (iii), if we take $F(x) = (1 - x^2)^{-1/2}$, then F has a power series expansion with rational coefficients and

$$A_k = \int_0^1 x^k (1 - x^2)^{-1/2} \, dx$$

We find $A_0 = \pi/2$, $A_1 = 1$ and for $k \ge 2$,

$$A_k = \frac{k-1}{k} A_{k-2}$$

Thus A_{2k} is irrational and A_{2k+1} is rational.

Math. Mag., 53(1980) 300.

1107. Proposed by M. S. Klamkin, University of Alberta

Determine the maximum value of $\sin A_i \sin A_2 \cdots \sin A_n$ if $\tan A_1 \tan A_2 \cdots \tan A_n = 1$

Math. Mag., **55**(1982) 45.

Solution by Jeremy D. Primer, Columbia High School, Maplewood, New Jersey. Let P denote the product of the sines. From $\prod_{i=1}^{n} \sin A_i = \prod_{i=1}^{n} \cos A_i$ Then

$$P^{2} = \prod_{1}^{n} \sin A_{i} \cos A_{i} = 2^{-n} \prod_{1}^{n} \sin 2A_{i} \le 2^{-n}$$

Thus $|P| \leq 2^{-n/2}$ and equality is attained when $A_1 = A_2 = \cdots = A_n = \pi/4$. Therefore $2^{-n/2}$ is the maximum value.

Math. Mag., 53(1980) 301, 305.

Q 666. Submitted by M. S. Klamkin, University of Alberta

If w and z are complex numbers, prove that

$$2|w| \cdot |z| \cdot |w - z| \ge (|w| + |z|)|w|z| - z|w|$$

A 666. Let $w = re^{i\alpha}$ and $z = se^{i\beta}$. Then we have to show that $2|re^{i\alpha} - se^{i\beta}| \ge (r+s)|e^{i\alpha} - e^{i\beta}|$ or

$$4\left\{(r\cos\alpha - s\cos\beta)^2 + (r\sin\alpha - s\sin\beta)^2\right\} \ge (r+s)^2\left\{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\right\}$$

This is equivalent to

$$2(r^{2} + s^{2} - 2rs\cos(\alpha - \beta)) \ge (r^{2} + s^{2} + 2rs)(1 - \cos(\alpha - \beta))$$

or finally $(r-s)^2(1+\cos(\alpha-\beta)) \ge 0$. There is equality if r=0 or s=0 or r=s or $\alpha-\beta=\pm\pi$.

Math. Mag., 55(1982) 44.

1137. Proposed by M. S. Klamkin, University of Alberta

It is known that $\tan x + \sin x \ge 2x$ for $0 \le x < \pi/2$, which is a stronger inequality than $\tan x \ge x$. Establish the still stronger inequality

$$a^{2} \tan x (\cos x)^{1/3} + b^{2} \sin x \ge 2xab$$

for $0 \le x \le \pi/2$.

Math. Mag., 56(1983) 53–54.

Solution by Anders Bager, Hjørring, Denmark. The stated inequality is valid for all pairs $(a,b) \in \mathbb{R} \times \mathbb{R}$ and all $x \in [0, \pi/2)$, with equality if and only if either x = 0 or (a,b) = (0,0).

Proof. The quadratic form

$$a^{2}(\tan x)(\cos x)^{1/3} - 2xab + b^{2}\sin x$$

(in (a, b), for fixed x) has the discriminant

$$D = 4 \left(x^2 - (\tan x) (\cos x)^{1/3} (\sin x) \right)$$

If x = 0, then D = 0 and the quadratic form vanishes. From now on we suppose $0 < x < \pi/2$. We shall prove that D < 0, which is equivalent to

$$\cos x < \left(\frac{\sin x}{x}\right)^3 \tag{1}$$

and which imples that the quadratic form is positive, except when (a, b) = (0, 0). Since

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
 and $\frac{\sin x}{x} > 1 - \frac{x^2}{6}$ for $0 < x < \frac{\pi}{2}$

we need only prove that

$$1 - \frac{x^2}{2} + \frac{x^4}{24} < \left(1 - \frac{x^2}{6}\right)^3$$

Since this is equivalent to $x^2 < 9$, hence true, the proof is complete.

It may be remarked that (1) is true for $0 < x \le \pi$, since $\cos x \le 0$ for $\pi/2 \le x \le \pi$.

Math. Mag., 55(1982) 300, 307.

Q 679. Submitted by M. S. Klamkin, University of Alberta & M. R. Spiegel, East Hartford, Connecticut

Let S_n be the sum of the digits of 2^n . Prove or disprove that $S_{n+1} = S_n$ for some positive integer n.

A 679. Let S(k) be the sum of the digits in the base-ten representation of the positive integer k. Then $k \equiv S(k) \pmod{9}$. Hence if $S(2^{n+1}) = S(2^n)$ then

$$2^{2n+1} - 2^{n} \equiv S(2^{n+1}) - S(2^{n}) \equiv 0 \pmod{9}$$

which is impossible, since $9 \nmid 2^n$.

Math. Mag., 56(1983) 177.

1172. Proposed by M. S. Klamkin University of Alberta, Canada

Determine the number of real solutions $x \ (0 \le x \le 1)$ of the equation

$$(x^{m+1} - a^{m+1})(1 - a)^m = \left\{ (1 - a)^{m+1} - (1 - x)^{m+1} \right\} a^m$$

where $0 \le a \le 1$ and m is a positive integer.

Math. Mag., 57(1984) 180.

Solution by Vania D. Mascioni, stident, ETH Zürich, Switzerland. Let f(x) = l.h.s. - r.h.s. and note that

$$sgn f'(x) = sgn ((m+1) (x^m (1-a)^m - (1-x)^m a^m)) = sgn (x(1-a) - (1-x)a) = sgn (x-a)$$

Thus f(x) is minimal when x = a. Since f(a) = 0, x = a is the only solution to the given equation.

Math. Mag., 56(1983) 178, 182.

Q 685. Submitted by M. S. Klamkin, University of Alberta, Canada.

If two altitudes of a plane triangle are congruent, then the triangle must be isosceles. Does the same result hold for a convex spherical triangle?

A 685. No. Consider a spherical triangle cut off from a lune AA' by an arc BC through its center. By symmetry, the altitudes from B and C are congruent.



B

Math. Mag., 56(1983) 240, 244.

Q 686. Submitted by M. S. Klamkin, University of Alberta.

It is known that for any triangle of side lengths a, b and c:

 $3(bc + ca + ab) \le (a + b + c)^2 \le 4(bc + ca + ab)$

Prove more generally that if a_1, a_2, \ldots, a_n are the sides of an *n*-gon, then

$$\frac{2n}{n-1}\sum_{i$$

and determine when there is equality.

A 686. The first inequality is equivalent to $\sum_{i < j} (a_i - a_j)^2 \ge 0$, with equality if and only if the polygon is equilateral. The second inequality is equivalent to $\sum_{i=1}^{n} a_i(p - a_i) \ge \sum_{i=1}^{n} a_i^2$ where $p = \sum_{i=1}^{n} a_i$, with equality only for the degenerate *n*-gon in which n-2 of the sides are of length zero.

Math. Mag., 57(1984) 110, 115.

Q 688. Submitted by M. S. Klamkin, University of Alberta

Show that

$$\begin{split} \sqrt{a^2 + b^2 + c^2} + \sqrt{b^2 + c^2 + d^2} + \sqrt{c^2 + d^2 + a^2} + \sqrt{d^2 + a^2 + b^2} &\geq 3\sqrt{a^2 + b^2 + c^2 + d^2} \\ \mathbf{A \ 688.} \ |(a, b, c, 0)| + |(0, b, c, d)| + |(a, 0, c, d)| + |(a, b, 0, d)| \\ &\geq |(a, b, c, 0)| + |(0, b, c, d)| + |(a, 0, c, d) + (a, b, 0, d)| = 3|(a, b, c, d)| \end{split}$$

Lore generally, let A_1, \ldots, A_n be vectors with sum **S**. Then by the triangle inequality,

$$|\mathbf{S} - \mathbf{A}_1| + \dots + |\mathbf{S} - \mathbf{A}_n| \ge (n-1)|\mathbf{S}|$$

The inequality to be proved corresponds to the special case in which n = 4 and the vectors \mathbf{A}_i are mutually orthogonal. There is equality if and only if at least three of the four vectors are null.

Math. Mag., **57**(1984) 304–305.

Equal Areas and Centroid of a Triangle

November 1983

1181. Proposed by George Tsintsifas, Thessaloniki, Greece

Let $A_1A_2A_3$ be a triangle and M an interior point. The straight lines MA_1 , MA_2 , MA_3 intersect the opposite sides at the points B_1 , B_2 , B_3 respectively. Show that if the areas of triangles A_2B_1M , A_3B_2M and A_1B_3M are equal, then M coincides with the centroid of triangle $A_1A_2A_3$.

II. Solution by Murray S. Klamkin, University of Alberta. Using barycentric coordinates, let $\mathbf{M} = x_i \mathbf{A}_1 + x_2 \mathbf{A}_2 + x_3 \mathbf{A}_3$, where x_1, x_2 and x_3 are positive and $x_1 + x_2 + x_3 = 1$. Then $x_1 = [MA_2A_3]/[A_1A_2A_3]$, etc., where [PQR] denotes the area of PQR. Also

$$\mathbf{B}_1 = \frac{x_2 \mathbf{A}_2 + x_3 \mathbf{A}_3}{x_2 + x_3} \qquad \text{etc}$$

Since $\overline{A_2B_1}/\overline{A_2A_3} = x_3/(x_2+x_3)$, we have

$$\frac{[A_2MB_1]}{[A_1A_2A_3]} = \frac{x_3x_1}{x_2 + x_3} \qquad \text{etc.}$$

By hypothesis we have

$$\frac{x_3x_1}{x_2+x_3} = \frac{x_1x_2}{x_3+x_1} = \frac{x_2x_3}{x_1+x_2}$$

or

$$x_2(1-x_1) = x_3(1-x_2) = x_1(1-x_3)$$

Without loss of generality we assume that $x_1 \ge x_2 \ge x_3$. Then $x_1(1-x_3) = x_3(1-x_2) \le x_3(1-x_3)$ and so $x_1 \le x_3$. Thus $x_1 = x_2 = x_3$ and M is the centroid.

Math. Mag., 58(1985) 299, 304.

Q 702. Submitted by M. S. Klamkin, University of Alberta

Are there any integral solutions to the Diophantine equation $x^2 + y^2 + z^2 = xyz - 1$?

A 702. By parity considerations, exactly two of x, y, z must be even. Hence by the substututions x = 2a, y = 2b, z = 2c+1 the equation reduces to $4a^2+4b^2+4c^2+4c+2 = 4ab(2c+1)$. This is impossible.

Math. Mag., 59(1986) 44-45, 53.

Q 704. Submitted by M. S. Klamkin, University of Alberta

Determine the maximum value of

$$\cos^2 \angle POA + \cos^2 \angle POD + \cos^2 \angle POC + \cos^2 \angle POD$$

where ABCD is a face of a cube inscribed in a sphere with center O, and P is any point on the sphere.

A 704. We choose a rectangular coordinate system so that the direction cosines of OA, OB, OC and OD are $(\frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$. Let the direction cosines of OP be (u, v, w). Then

$$\sum \cos^2 \angle POA = \sum \left(\frac{u}{\sqrt{3}} \pm \frac{v}{\sqrt{3}} \pm \frac{w}{\sqrt{3}}\right)^2 = \frac{4}{3} \qquad \text{(constant)}.$$

he four vectors are null.

Math. Mag., **59**(1986) 114–115.

An Inequality for the Logarithm

1212. Proposed by L. Bass and R. Výborný, The University of Queensland, Australia; and V. Thomée, Chalmers Institute of Technology, Sweden

Prove that if x > 1 and 0 < u < 1 < v, then

$$\frac{v(x-1)(x^{v-1}-1)}{(v-1)(x^v-1)} < \log x < \frac{u(x-1)(1-x^{u-1})}{(1-u)(x^u-1)}$$

Solution by M. S. Klamkin, University of Alberta, Canada. The inequality can be rewritten as

$$\frac{F(v-2)}{F(v-1)} < \frac{F(-1)}{F(0)} < \frac{F(u-2)}{F(u-1)} \quad \text{for} \quad 0 < x \neq 1 \quad \text{and} \quad u < 1 < v$$

where

$$F(\lambda) = \int_{1}^{x} t^{\lambda} dt$$
 for $x > 0$ and any real λ .

Hence it suffices to show that $F(\lambda)/F(\lambda+1)$ decreases as λ increases. This follows from the continuity of F and the fact that

$$\begin{aligned} \frac{d}{d\lambda} \left(\frac{F(\lambda)}{F(\lambda+1)} \right) &= \frac{1}{(F(\lambda+1))^2} \left(\int_1^x t^{\lambda+1} dt \int_1^x t^{\lambda} (\log t) dt - \int_1^x t^{\lambda} dt \int_1^x t^{\lambda+1} (\log t) dt \right) \\ &= \frac{1}{(F(\lambda+1))^2} \int_1^x \int_1^x s^{\lambda} t^{\lambda} (\log t) (s-t) ds dt \\ &= \frac{1}{2(F(\lambda+1))^2} \int_1^x \int_1^x s^{\lambda} t^{\lambda} ((s-t) \log t + (t-s) \log s) ds dt \\ &= -\frac{1}{2(F(\lambda+1))^2} \int_1^x \int_1^x s^{\lambda} t^{\lambda} ((s-t) (\log s - \log t) ds dt < 0 \quad \text{if } \lambda \neq -1 \end{aligned}$$

March 1985

Math. Mag., **59**(1986) 173, 180.

Q 711. Submitted by M. S. Klamkin, University of Alberta, Canada

Determine the extreme values of the circumradii $R(\theta)$ of the set of triangles $T(\theta)$ whose sides are $\sin \theta$, $\cos \theta$, $\cos 2\theta$ for $0 < \theta < \pi/4$.

A 711. First one should verify that the triangles $T(\theta)$ actually exist for all θ in the given interval. This will follow from the elementary inequality $\cos 2\theta + \sin \theta > \cos \theta$. However, it will also follow from the subsequent geometry.

For $\theta = 0$ we get a degenerate triangle of sides 0, 1, 1 whose circumradius is R = 1/2. For $\theta = \pi/4$ we get a degenerate triangle of sides $1/\sqrt{2}$, $1/\sqrt{2}$, 0 with $R = 1/2\sqrt{2}$. So it may appear that $1/2\sqrt{2} < R(\theta) < 1/2$. However, we will show that $R(\theta) = 1/2$ for all θ in the open interval $(0, \pi/4)$.

Consider a triangle ABC inscribed in a circle of readius 1/2 as shown, where AB and BC subtend angles of 2θ and $\pi - 4\theta$, respectively, at the center O. Here $AB = \sin \theta$, $BC = \cos 2\theta$ and $AC = \cos \theta$.



Math. Mag., 59(1986) 240, 248.

Q 713. Submitted by M. S. Klamkin, University of Alberta

If one of the arcs joining the midpoints of the sides of a spherical triangle is 90° , show that he other two arcs are also 90° .

A 713. If A, B, C denote vectors from the center of the sphere to the vertices of the spherical triangle, we have to show equivalently that $|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}|$ and $(\mathbf{B} + \mathbf{C}) \cdot (\mathbf{C} + \mathbf{A}) = 0$ imply that $(\mathbf{B} + \mathbf{C}) \cdot (\mathbf{A} + \mathbf{B}) = 0$ and $(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{C}) = 0$. Since $(\mathbf{B} + \mathbf{C}) \cdot (\mathbf{C} + \mathbf{A}) - (\mathbf{B} + \mathbf{C}) \cdot (\mathbf{A} + \mathbf{B}) = (\mathbf{B} + \mathbf{C}) \cdot (\mathbf{C} - \mathbf{B}) = \mathbf{C}^2 - \mathbf{B}^2 = 0$, and $(\mathbf{B} + \mathbf{C}) \cdot (\mathbf{C} + \mathbf{A}) - (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{C}) = (\mathbf{C} - \mathbf{A}) \cdot (\mathbf{A} + \mathbf{C}) = 0$, the result is now immediate. Math. Mag., 60(1987) 40, 50.

Q 717. Submitted by M. S. Klamkin, University of Alberta

Are there any integral solutions to the Diophantine equation

 $x_1^{1987} + 2x_2^{1987} + 4x_3^{1987} + \dots + 2^{1986}x_{1987}^{1987} = 1986x_1x_2\cdots x_{1987}?$

(Note: This problem is an extension of a problem from the Wisconsin Talent Search.)

A 717. The only solution is the trivial one, $x_1 = x_2 = \cdots = x_{1987} = 0$. This follows by induction and infinite descent. Note that x_1 must be even. Thus $x_1 = 2y_1$ which gives the identical equation in the variables $x_2, x_3, \ldots, x_{1987}, y_1$.

Math. Mag., 60(1987) 179, 184.

Q 723. Submitted by M. S. Klamkin, University of Alberta

ABCD is a quadrilateral inscribed in a circle. Prove that the four lines, each passing through amidpoint of one of the sides of ABCD and perpendicular to the opposite side, are concurrent.

A 723. Let A, B, C, D bevecors from the center of the circle to the respective vertices A, B, C, D. The four line will intersect at the point P given by $\mathbf{P} = (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})/2$. Note that the vector from the midpoint of AB to P is $(\mathbf{C} + \mathbf{D})/2$, and this is perpendicular to CD since $|\mathbf{C}| = |\mathbf{D}|$, and similarly for the other segments BC, CD and DA. It also hold for the diagonals AC and BD, so that there are six concurrent lines.

This result was given by R. E. Lester, *Math. Gaz.*, 46(1962) 147. For other interesting properties of the point *P*, see R. A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p.252.

Math. Mag., 60(1987) 115.

1265. Proposed by M. S. Klamkin University of Alberta, Canada

Determine the maximum area F of a triangle ABC if one side is of length λ and twoof its medians intersect at right angles.

Math. Mag., **61**(1988) 125–126.

Solution by Cornelius Groenewoud, Bartow, Florida. We show that $F = 3\lambda^2/8$ or $F + 3\lambda^2/4$ depending on whether one of the perpendicular medians is drawn to the side of length λ or not.

Two cases are considered: (i) One of the perpendicular medians is drawn to the side AB of length λ ; (ii) Neither of the perpendicular medians is drawn to AB.

Place the side AB along the positive x-axis with A at the origin of a rectangular coordinate system. For the first case let M be the midpoint of AB and N the midpoint of AC.



Imposing the negative reciprocal condition between the slopes of medians BN and CM shows that the medians to AB and AC are perpendicular when C lies on a circle of radius $3\lambda/4$, centered at $P(5\lambda/4, 0)$. The area, $\lambda h/2$ will be measimum when h attains its maximum of $3\lambda/4$. For this case $F = 3\lambda^2/8$.

For the second case let L be the midpoint of BC. Imposing the negative reciprocal condition on the slopes of the medians to AL and BN shows that the medians to AC and BC are perpendicular if C lies on a circle of radius $3\lambda/2$ centered at the midpoint of AB. The area is $\lambda h/2$, but in this case the maximum value of h is $3\lambda/2$. The corresponding area is $F + 3\lambda^2/4$.
II. Solution by Thomas Jager, Calvin College, Michigan. The maximum is $3\lambda^2/4$. Label triangle ABC as shown, where the medians from B and C intersect in an angle θ .



By the law of cosines, $a^2 = 4t^2 + 4s^2 - 8st \cos \theta$, $(c/2)^2 = t^2 + 4s^2 - 4st \cos(\pi - \theta)$ and $(b/2)^2 = s^2 + 4t^2 - 4st \cos(\pi - \theta)$. These equations imply that $5a^2 - b^2 - c^2 = -8st \cos \theta$. It follows that the medians from B and C intersect in a right angle if and only if $5a^2 = b^2 + c^2$.

Suppose $\theta = \pi/2$. By the law of cosines, $a^2 = b^2 + c^2 - 2bc \cos A$, and thus,

$$F^{2} = \left(\frac{1}{2}bc\sin A\right)^{2} = \frac{1}{4}b^{2}c^{2} - \frac{1}{4}b^{2}c^{2}\cos^{2}A = \frac{1}{4}b^{2}c^{2} - \frac{1}{4}\left(\frac{b^{2}+c^{2}-a^{2}}{2}\right)^{2} = \frac{1}{4}b^{2}c^{2} - a^{4}b^{2}c^{2} - a^{4}b^{2}c^{2$$

If $a = \lambda$, $F^2 = \frac{1}{4}b^2(5\lambda^2 - b^2) - \lambda^4$ and this is minimized when $b^2 = 5\lambda^2/2$, giving $F = 3\lambda^2/4$.

If $b = \lambda$, $F^2 = \frac{1}{4}\lambda^2(5a^2 - \lambda^2) - a^4 = \frac{9}{64}\lambda^4 - (a^2 - \frac{5}{8}\lambda^2)^2$ is maximized when $a = 5\lambda^2/8$, giving $F = 3\lambda^2/8$.

Math. Mag., 60(1987) 329.

1281. Proposed by M. S. Klamkin University of Alberta

a. Determine the least number of acute dihedral angles in a tetrahedron.

b*. Deneralize the result for an *n*-dimensional simplex. Here a dihedral angle is the supplement of the angle between outward normals to two (n-1)-dimensional faces of the simplex.

Math. Mag., 61(1988) 320.

a. Solution by the proposer. There is at least one vertex of the tetrahedron such that its corresponding face angles are all acute. Otherwise, since the sum of two face angles at a vertex is greater than the third face angle, the sum of all face angles would be greater than 4π . However, since there are four faces, this sum must be equal to 4π .

We now give two lemmas obtained from the following two laws of cosines from spherical trigonometry for the face angles a, b, c and the opposite dihedral angles, respectively, of a trihedral angle:

 $\sin b \sin c \sin A = \cos a - \cos b \cos c \quad \text{etc.},\tag{1}$

$$\sin B \sin C \cos a = \cos A + \cos B \cos C \quad \text{etc.}, \tag{2}$$

LEMMA 1. If $\pi/2 > a \ge b \ge c$, then from (1), B and C are acute.

LEMMA 2. If $A \ge B \ge \pi/2 > C$, then by (2), $a, b \ge \pi/2 > c$.

In a tetrahedron PQRS, we can take P as a vertex with all acute face angles. Then by Lemma 1, we can take PQ and PS as the edges of two acute dihedral angles. We now assume that there are no more acute dihedral angles and obtain a contradiction. By Lemma 2 applied to the trihedral angles at vertices Q and S, it follows that angles PQS and PSQ are both non-acute. Since this is impossible, there are always at least three acute dihedral angles in any tetrahedron.

b. Partial solution by L. P. Pook, Glasgow, Scotland. For finite n the n(n-1)/2 dihedral angles of a regular n-dimensional simplex are acute, so this is an initial upper bound.

A lower value for the upper bound is obtained by constructing a low altitude right hyper-pyramid, vertex V, with a regular (n-1)-dimensional simplex as base. The altitude is chosen such that the dihedral angles between pyramidal faces are obtuse; the remaining n dihedral angles between the base and the pyramidal faces must be acute.

Distorting the pyramid by moving V, in a hyperplane parallel to the base, to a position such that at least one dihedral angle at the base becomes obtuse results in at least an equal number of dihedral angles between pyramidal faces becoming acute. In view of this it is conjectured that n is indeed the required minimum number of acute dihedral angles. Math. Mag., **61**(1988) 46.

1289. Proposed by M. S. Klamkin University of Alberta

Two identical beads slide on two straight wires intersecting at right angles. If the beads start from rest in any position other than the intersection point of the wires and attract each other in an arbitrary mutual fashion but also subject to a drag proportional to the speed, show that the beads will arrive at the intersection simultaneously.

Math. Mag., **62**(1989) 62–63.

Solution by Yan-loi Wong (student), University of California, Berkeley. Resolving the attractive force f(x, y) along the axes (the wires), the equations of motion for the two beads are given by

$$\begin{array}{lll} x'' & = & \frac{-xf(x,y)}{\sqrt{x^2 + y^2}} + kx' \\ y'' & = & \frac{-yf(x,y)}{\sqrt{x^2 + y^2}} + ky' \end{array}$$

where k is a positive constant. Eliminating f(x, y) we find that

$$yx'' - xy'' = k(yx' - xy')$$

 $(yx' - xy')' = k(yx' - xy')$

It follows that

$$yx' - xy' = Ce^{kt}$$

for some constant C. Since x'(0) = y'(0) = 0, we have yx' - xy' = 0. Hence, until

$$y(t) = 0$$
 $\frac{d}{dt}\left(\frac{x}{y}\right) = 0$

so that x(t) = Ay(t) for some nonzero constant A, and this gives the desired result.

Math. Mag., 61(1988) 47, 58.

Q 729. Submitted by M. S. Klamkin, University of Alberta

Determine the extreme values of

$$S = \frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{zx+z+1}$$

where xyz = 1 and $x, y, z \ge 0$. [[I don't like that '= 0' — R.]] A 729. Let A = x/(xy + x + 1), B = y/(yz + y + 1), C = z/(zx + z + 1). Then A = x/(1/z + x + 1) = zx/(zx + z + 1) = 1/(yz + y + 1)

$$\begin{array}{rcl} B &=& y/(1/x+y+1) = xy/(xy+x+1) = 1/(zx+z+1) \\ C &=& z/(1/y+z+1) = yz/(yz+y+1) = 1/(xy+x+1) \end{array}$$

Therefore 3(A + B + C) = 3 and S has the constant value 2.

An alternative and quicker solution is

$$S = \frac{x+1}{xy+x+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{zx+z+1}$$
$$= \frac{x+1}{xy+x+1} + \frac{y+1}{\frac{1}{x}+y+1} + \frac{\frac{1}{xy}+1}{\frac{1}{y}+\frac{1}{xy}+1} = 2$$

[[I guess I'm too much of a sloth to follow this! - R.]]

A related open problem is to find the extremes of S if the condition xyz = 1 is replaced by xyz = a. Math. Mag., 61(1988) 114.

1292. Proposed by M. S. Klamkin University of Alberta, Canada

Determine the maximum value of

$$x_1^2 x_2 + x_2^2 x + 3 + \dots + x_n^2 x_1$$

given that $x_1 + x_2 + \dots + x_n = 1, x_1, x_2, \dots, x_n \ge 0$ and $n \ge 3$.

Math. Mag., 61(1988) 137.

I. Solution by the 1988 Olympiad Math Team. The maximum is 4/27, achieved when the x_k comprise a cyclic permutation of $(2/3, 1/3, 0, \ldots, 0)$.

Suppose n > 3. Choose j so that $x_{j+1} \ge x_j$ (where $x_{n+1} = x_1$). The modified sequence $(\ldots, x_{j-2}, x_{j-1} + x_j, x_{j+1}), \ldots$, with only n-1 terms, gives at least as high a value for the objective function as does the original sequence, since

$$\begin{aligned} x_{j-2}^2(x_{j-1}+x_j) + (x_{j-1}+x_j)^2 x_{j+1} &\geq x_{j-2}^2 x_{j-1} + (x_{j-1}^2+x_j^2) x_{j+1} \\ &\geq x_{j-2}^2 x_{j-1} + x_{j-1}^2 x_j + x_{j-1}^2 x_j + x_j^2 x_{j+1} \end{aligned}$$

Thus it suffices to prove the bound of 4/27 only for n = 3.

So, suppose n = 3. Without loss of generality, cycle the three indices so that x_2 takes on the intermediate value. Then $(x_2 - x_1)(x_2 - x_3) \le 0 \le x_1 x_2$ so

$$x_1^2 x_2 + (x_2 - x_1)(x_2 - x_3)x_3 \le x_1^2 x_2 + x_1 x_2 x_3$$

These terms can be rearranged to give

$$\begin{aligned} x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 &\leq (x_1 + x_3)^2 x_2 = (1 - x_2)^2 x_2 \\ &\leq 4 \left(\frac{\frac{1}{2}(1 - x_2) + \frac{1}{2}(1 - x_2) + x_2}{3} \right)^3 = 4/27 \end{aligned}$$

the last by the AM-GM inequality whenever $0 \le x_2 \le 1$. [[There's also a longer solution by Eugene Lee. — R.]] Math. Mag., **61**(1988) 115, 127.

Q 733. Submitted by M. S. Klamkin, University of Alberta

If A, B, C are the angles of a triangle, determine the maximum area of a triangle whose sides are $\cos(A/2)$, $\cos(B/2)$, $\cos(C/2)$.

A 733. Comment: One might start from the expression for the square of the area of the triangle, i.e.,

$$F^{2} = s\left(s - \cos\frac{A}{2}\right)\left(s - \cos\frac{B}{2}\right)\left(s - \cos\frac{C}{2}\right) \quad \text{where} \quad 2s = \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

and then maximize subject to the constraints $A + B + C = 180^{\circ}$ and $A, B, C \ge 0$.

Solution: First, one shoud verify that triangles with the given sides exist for all triangles ABC. If $A \ge B \ge C$, it suffices to show that

$$\cos\frac{A}{2} + \frac{B}{2} > \cos\frac{C}{2}$$

or equivalently

$$\cos\frac{A-B}{4} > \sin\frac{A+B}{4}$$

which follows.

An alternative proof of the existence of a triangle follows from the fact that the three sides are $\sin \frac{\pi-A}{2}$, $\sin \frac{\pi-B}{2}$, $\sin \frac{\pi-C}{2}$, and that the angles $(\pi - A)/2$, etc., are positive and add to 180°. Since in any triangle DEF of sides d, e, f, we have $d = 2R \sin D$, etc., where R is the circumradius, it follows that $\sin \frac{\pi-A}{2}$, $\sin \frac{\pi-B}{2}$, $\sin \frac{\pi-C}{2}$ are sides of a triangle with *fixed* circumradius 1/2. It is well known and easy to prove that of all triangles inscribed in a given circle, the equilateral triangle has themaximum area. Thus, our maximum area corresponds to an equilateral triangle of sides $\cos 30^\circ$, or $3\sqrt{3}/16$.

As a bonus, by using the known result that the product of the sides of a triangle equals four times the product of its area and circumradius, we have the triangle identity

$$\cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} = 4s \left(s - \cos \frac{A}{2}\right) \left(s - \cos \frac{B}{2}\right) \left(s - \cos \frac{C}{2}\right)$$

[[Another very slow quickie ! - R.]]

Math. Mag., **59**(1986) 118–119.

Integral triangles tetrahedrons

1261. Proposed by Stanley Rabinowitz, Digital Equipment Corporation, Nashua, New Hampshire

a. What is the area of the smallest triangle with integral sides and integral area?.

b^{*}. What is the volume of the smallest tetrahedron with integral sides and integral volume?

[[I wd make that 'integer edges' - R.]]

(a) Solution by M. S. Klamkin, University of Alberta. We will show that the area of a triangle with integral sides and integral area is divisible by 6. Since a 3-4-5 triangle has area 6, the smallest area is necessarily 6.

Let T be a triangle with integral sides and integral area. We may assume that gcd(a, b, c) = 1. Let s denote the semiperimeter.

The inradius r = Area/s and thus must be rational. Since $\tan(A/2) = r/(s-a)$, $\tan(A/2)$ is rational and let it equal n/m where (m, n) = 1. Similarly, $\tan(B/2) = q/p$ with (p,q) = 1. Then

$$\sin A = \frac{2mn}{(m^2 + n^2)}$$
 $\cos A = \frac{(m^2 - n^2)}{(m^2 + n^2)}$ $\sin B = \frac{2pq}{(p^2 + q^2)}$ etc.,

and

$$\sin C = \sin(A+B) = \frac{2(mq+np)(mp-nq)}{(m^2+n^2)(p^2+q^2)}$$

The sides of any triangle are proportional to the sines of the opposite angles; specifically, $a = 2R \sin A$, etc., where R is the circumradius. Let us take $4R = (m^2 + n^2)(p^2 + q^2)$ and let \overline{T} be the triangle with sides

$$\bar{a} = mn(p^2 + q^2) \qquad \bar{b} = pq(m^2 + n^2)$$

$$\bar{c} = (mq + np)(mp - nq) = mn(p^2 - q^2) + pq(m^2 - n^2) \qquad (1)$$

We may assume that mn is relatively prime to pq, otherwise we can divide out the common factor. \overline{T} has integral area (Area $\overline{T} = tfrac12\overline{b}\overline{c}\sin A$) and \overline{T} is similar to T. Also

Area
$$\overline{T} = \frac{\overline{a}\overline{b}\overline{c}}{4R} = mnpq\left(mn(p^2 - q^2) + pq(m^2 - n^2)\right)$$
 (2)

Suppose that $\bar{a} = da$, $\bar{b} = db$, $\bar{c} = dc$ for integer $d \ge 1$. Then, Area $\bar{T} = d^2 \operatorname{Area} T$.

Suppose d is even. Then, from (1) our supposition that gcd(m, n, p, q) = 1 it must be the case that m, n, p, q are odd. In this case $\bar{a}, \bar{b}, \bar{c}$ are divisible by 2 but not by 4; i.e., d isnot divisible by 4. Also, Area \bar{T} is divisible by 8 (in (2), $p^2 - q^2 \equiv m^2 - n^2 \equiv 0 \pmod{8}$) and therefore Area T is even.

Equation (1), together with gcd(m, n, p, q) = 1, shows that 3 is not a common factor of \bar{a} , \bar{b} , \bar{c} ; that is, d is not a multiple of 3.

If one of m, n, p, q is divisible by 3, then the sides are not divisible by 3, but Area \overline{T} (and Area T) is. If none of m, n, p, q is divisible by 3, Area \overline{T} (and Area T) is because $p^2 - q^2 \equiv m^2 - n^2 \equiv 0 \pmod{3}$.

Thus, Area T is divisible by 6 and the proof is complete.

No solutions were received for Part (b). The proposer supplied computer generated evidence (a list of tetrahedra with small volumes) that suggests that the smallest volume is 6. Also, see *Crux Math.* (May 1985) 162–166 for a consideration of tetrahedra having integer-valued edge lengths, face areas, and volume.

Math. Mag., **61**(1988) 196, 203.

Q 734. Submitted by M. S. Klamkin, University of Alberta, Canada

Determine the lous of all points whose parametric representation is given by

$$x = \frac{\xi(h\xi + k\eta + l\zeta)}{(\xi^2 + \eta^2 + \zeta^2)}$$
$$y = \frac{\eta(h\xi + k\eta + l\zeta)}{(\xi^2 + \eta^2 + \zeta^2)}$$
$$z = \frac{\zeta(h\xi + k\eta + l\zeta)}{(\xi^2 + \eta^2 + \zeta^2)}$$

where the parameters ξ , η , ζ take on all values in [0,1] and h, k, l are positive constants.

A 734. Geometrically, if one wants the locus of points which are the orthogonal projections of the fixed point (h, k, l) on all lines through the origin with direction numbers (ξ, η, ζ) , one can obtain the given parametric representation. From this interpretation, it follows quickly that the locus is that portion of the sphere with the segment joining the origin to (h, k, l) as a diameter and which lies in the first orthant.

Alternatively, it follows that

$$x^2 + y^2 + z^2 = hx + ky + lz$$

Thus the surface is that part of the sphere with radius $r = \sqrt{h^2 + k^2 + l^2}/2$ and center (h/2, k/2, l/2) which lies in the first orthant.

Math. Mag., 61(1988) 193–194.

[[There's a Note by Murray: Symmetry in probability distributions.]]

Math. Mag., 61(1988) 261, 265-266.

Q 737. Submitted by M. S. Klamkin, University of Alberta, Edmonton, Canada Determine $P_n(n^2)$ where

$$P_n(x) = \frac{x-1^2}{2!^2} - \frac{(x-1^2)(x-2^2)}{3!^2} + \dots + \frac{(-1)^n(x-1^2)(x-2^2)\cdots(x-(n-1)^2)}{n!^2}$$

A 737. We show more generally by induction that if

$$Q_n(x) = 1 - \frac{x}{a_1} + \frac{x(x - a_1)}{a_1 a_2} - \dots + \frac{(-1)^n (x)(x - a_1) \cdots (x - a_{n-1})}{a_1 a_2 \cdots a_n}$$
(1)

then

$$Q_n(x) = \frac{(a_1 - x)(a_2 - x)\cdots(a_n - x)}{a_1 a_2 \cdots a_n}$$

so that

$$P_n(x) = \frac{Q_n(x) - 1 + x/a_1}{x}$$

where $a_i = i^2$. Thus, $P_n(n^2) = 1 - 1/n^2$. Assume (1) is valid for n = k. Then

$$Q_{k+1}(x) = \frac{(a_1 - x)(a_2 - x)\cdots(a_k - x)}{a_1 a_2 \cdots a_k} + \frac{(-1)^{k+1}(x)(x - a_1)\cdots(x - a_k)}{a_1 a_2 \cdots a_{k+1}}$$
$$= \frac{(a_1 - x)(a_2 - x)\cdots(a_k - x)}{a_1 a_2 \cdots a_k} \left(1 - \frac{x}{a_{k+1}}\right)$$

Thus our result is also true for n = k + 1. Since the result also holds for n = 1, it holds for all n.

Math. Mag., 61(1988) 315, 325.

Q 739. Submitted by M. S. Klamkin, University of Alberta, Edmonton, Canada Determine the largest value of the constant k such that the inequality

$$(x_1 + x_2 + \dots + x_n)^2 (x_1 x_2 + x_2 x_3 + \dots + x_n x_1) \ge (x_1^2 x_2^2 + x_2^2 x_3^2 + \dots + x_n^2 x_1^2)$$

is valid for all $x_1, x_2, \ldots, x_n \ge 0$.

A 739. Without loss of generality we can let $x_1 + x_2 + \cdots + x_n = 1$. The letting n = 2, $x_1 = x_2 = 1/2$, we must satisfy $1/4 \ge k/4^2$. Thus k must be ≤ 4 . For k = 4, the inequality becomes

$$x_1x_2(1-4x_1x_2) + x_2x_3(1-4x_2x_3) + \dots + x_mx_1(1-4x_nx_1) \ge 0$$

and this inequality holes since $\max(x_i x_j) = 1/4$. There is equality if and only if one variable equals 1 or else two successive variables equal 1/2, or trivially, if all the variables are zero.

Math. Mag., 62(1989) 58.

1315. Proposed by M. S. Klamkin and A. Liu, University of Alberta, Canada Determine all real values of λ such that the roots of

$$P(x) \equiv x^{n} + \lambda \sum_{r=1}^{n} (-1)^{r} x^{n-r} = 0 \qquad (n > 2)$$

are *all* real.

Math. Mag., **63**(1990) 61.

I. Solution by Seung-Jin Bang, Seoul, Korea. If $\lambda = 0$ then all the roots of P are real. We will show that if $\lambda \neq 0$ then P has at least one nonreal root.

Suppose $\lambda \neq 0$ and set x = -1/y. The equation becomes

$$G(y) \equiv 1/\lambda + \sum_{r=1}^{n} y^r$$

Suppose the roots r_1, r_2, \ldots, r_n are all real. Then the roots of G are $s_i \equiv 1/r_i$ $(i = 1, 2, \ldots, n)$, and

$$\sum_{i=1}^{n} s_i^2 = \left(\sum_i s_i\right)^2 - 2\sum_{i < j} s_i s_j = 1^2 - 2 = -1,$$

a contradiction.

II. Solution by David Callan, University of Bridgeport, Bridgeport, Connecticut. If $\lambda = 0$ obviously all roots are real. Suppose $\lambda \neq 0$. We cab write P(x) as Q(x)/(x+1) where $Q(x) = x^{n+1} + (1-\lambda)x^n + (-1)^n \lambda$. Since $P(0) \neq 0$ the number of real roots of P is the number of positive roots of P(x) plus the number of positive roots of P(-x). By Descartes's rule of signs, the number of positive roots of Q(x) (resp., Q(-x)) is at most the number of sign changes in the list of coefficients 1, $1-\lambda$, $(-1)^n \lambda$ (resp. 1, $\lambda-1$, $-\lambda$). The total of these sign changes is clearly ≤ 3 . Noting [[that]] P(-x) = Q(-x)/(1-x) and so x = 1 counts as positive root of Q(-x) one more time than it does as a root of P(-x), the above results for Q imply that P has at most two real roots.

Math. Mag., 62(1989) 59, 66.

Q 742. Submitted by M. S. Klamkin, University of Alberta, Edmonton, Canada Let

$$S_n = \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha} + \frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha}$$

If $S_k = 1$ for some positive integer k, show that $S_n = 1$ for n = 1, 2, ...

A 742. More generally, let

$$S_n = \sum_{i=1}^n \frac{x_i^{2n+2}}{y_i^{2n}}$$

where

$$x_1^2 + x_2^2 + \dots + x_n^2 = 1 = y_1^2 + y_2^2 + \dots + y_n^2$$

Then if $S_k = 1$ for some positive integer $k, S_n = 1$ for n = 1, 2, ...[[I don't follow – has this sentence been misplaced? — R.]] By Hölder's inequality

$$\left(\sum_{i=1}^{n} \frac{x_i^{2k+2}}{y_i^{2k}}\right)^{1/(k+1)} \left(\sum_{i=1}^{n} y_i^2\right)^{k/(k+1)} \ge \sum_{i=1}^{n} x_i^2$$

Since we have the equality case by the hypotheses, we must have $x_i^{2k+2}/y_i^{2k} = \lambda y_i^2$ or $x_i^2 = cy_i^2$ for all *i*. It then follows that c = 1 and $S_n = 1$ for n = 1, 2, ...

(Comment: The initial problem for the special case k = 1 appears in E. W. Hobson, A Treatise on Plane and Advanced Trigonometry, Dover, New York, 1957, p.96.)

Math. Mag., 62(1989) 137, 143.

Q 746. Submitted by M. S. Klamkin, University of Alberta, Canada

If V_i denotes the (n-1)-dimensional volume of the face F_i opposite the vertex A_i of the (n-1)-dimensional simplex $S_n : A_0A_1 \ldots A_n$, show that

$$V_0 + V_1 + \dots + V_n > 2V$$

for all i.

A 746. Orthogonally project the faces $F_0, F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_n$ onto the space of F_i . If A'_i (he projection of A_i) lies in the convex hull H_i of $A_0, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ then

$$V'_0 + V'_1 + \dots + V'_{i-1} + V'_{i+1} + \dots + V'_n = V_i$$

where V'_j denotes the (n-1)-dimensional volume of the orthogonal projection of face F_j . If A'_i lies outside the convex hull H_i then

$$V'_0 + V'_1 + \dots + V'_{i-1} + V'_{i+1} + \dots + V'_n > V_i$$

Finally, since $V_j > V'_j$ we are done. Note that there is equality only in the case of a degenerate simplex in which case one vertex lies in the convex hull of the remaining vertices.

Note that for n = 2 we have $a_1 + a_2 + a_3 > 2a_i$ for the sides of a triangle, and for n = 3 we have $F_1 + F_2 + F_3 + F_4 > 2F_i$ for the areas of the faces of a tetrahedron.

On pp.198–199 of *Math. Mag.*, **62**(1989) we read:

Murray Klamkin was awarded the MAA's Award for Distinguished Service to Mathematics in 1988, largely in recognition of his many contributions to problem solving. In this issue we further recognize this work: every problem posed in this issue is a proposal of Murray Klamkin's.

[[we don't really need that double genitive - R.]]

1322. An *n*-gon of consecutive sides a_1, a_2, \ldots, a_n is circumscribed about a circle of unit radius. Determine the minimum value of the product of all of its sides.

Math. Mag., 63(1990) 191–192.

Solution by the proposer. If we denote the consecutive angles of the *n*-gon by $2A_1$, $2A_2$, ..., $2A_n$ then

$$P \equiv \prod a_i = \prod (\cot A_i + \cot A_{i+1})$$

where the products and sums here and subsequently are cyclic over i and $\sum A_i = (n-2)\pi/2$ with $A_i < \pi/2$.

First, we consider n > 4. By taking two consecutive angles very close to π , we can make P arbitrarily small; that is to say, there is no minimum in this case.

Consider the case n = 4. Since $\cot x$ is convex in $(0, \pi/2)$

$$\cot x + \cot y \ge 2 \cot \frac{(x+y)}{2}$$

Hence

$$P = a_1 a_2 a_3 a_4 \ge 16 \cot \frac{A_1 + A_2}{2} \cdot \cot \frac{A_2 + A_3}{2} \cdot \cot \frac{A_3 + A_4}{2} \cdot \cot \frac{A_4 + A_1}{2}$$

Then, since

$$\cot\frac{A_3 + A_4}{2} = \frac{1}{\cot\frac{A_1 + A_2}{2}} \quad \text{and} \quad \cot\frac{A_4 + A_1}{2} = \frac{1}{\cot\frac{A_2 + A_3}{2}}$$

we get that

$$a_1 a_2 a_3 a_4 \ge 16$$

This is a stronger result than the minimum perimeter circumscribed quadrilateral is a square. [['quadrilateral's being' ? - R.]]

Now consider the case n = 3. We start with the known inequality [[known to whom?]] $(a_1a_2a_3)^2 \ge (4F/\sqrt{3})^3$ where F is the area of the triangle and equality holds if $a_1 = a_2 = a_3$. Since $a_1a_2a_3 = 4RF$, where R is the circumradius, a geometrical interpretation of this inequality is that the inscribed triangle of largest are in a circle of radius R is the equilateral one. Now $F = \cot A_1 + \cot A_2 + \cot A_3$ and it is known that the minimum area circumscribed triangle is the equilateral one. This follows easily from the convexity of $\cot x$ for x in $(0, \pi/2)$. Again there is equality only for the equilateral problem. Thus $\min(a_1a_2a_3) = 24\sqrt{3}$, or equivalently,

$$\prod (\cot A_i + \cot A_{i+1})^2 \ge \left(4\sum \cot A_i/\sqrt{3}\right)^3$$

1323. Submitted by M. S. Klamkin, University of Alberta, Canada

A parallelepiped has the property that all cross sections that are parallel to any fixed face F have the same area as F. Are there any other polyhedra with this property?

Math. Mag., **63**(1990) 191–192.

Solution by the proposer. First, the polyhedron must be convex. If not, there would be a pair of reentrant faces and the area of cross sections parallel to one of these two faces could not be the same. We now show that the polyhedron must be a parallelepiped. Consider three parallel sections whose distances from a face F are x, x_1 and x_2 where $x = w_1x_1 + w_2x_2$ $w_1 + w_2 = 1$ and $w_1, w_2 > 0$. It then follows by the Brun-Minkowski inequality (L. A. Lyusternik, *Convex Figures and Polyhedra*, Dover, New York, 1963, pp.117–118), that the areas of the three sections must satisfy

$$A(x) \ge w_1 A(x_1) + w_2 A(x_2)$$

and equality holds if and only if the region between the outer sections is a cylindrical solid. Since we have equality by hypothesis, the figure must be a prism with respect to each face and hence must be a parallelepiped.

Comment from the proposer: I set the same problem with *area* replaced by *perimeter* in the 1980 Canadian Mathematics Olympiad. In this case the figure could also be a regular octahedron. Whether or not there are any other solutions for this problem is still open.

1324. Submitted by M. S. Klamkin, University of Alberta, Canada

Determine the maximum value of

$$x_1 x_2 \cdots x_n (x_1^2 + x_2^2 + \cdots + x_n^2)$$

where $x_1 + x_2 + \dots + x_n = 1$ and $x_1, x_2, \dots, x_n \ge 0$.

Math. Mag., 63(1990) 192–193.

Solution by Eugene Lee, Boeing Commercial Airplanes, Seattle, Washington. The maximum is $(1/n)^{n+1}$. To prove this we establish inductively the proposition P_n : The function $f_n(x_1, \ldots, x_n) \equiv x_1 \cdots x_n(x_1^2 + \cdots x_n^2)$ over the simplex $\sigma_{n-1}(\lambda) \equiv \{(x_1, \ldots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = \lambda\}$, any $\lambda > 0$, attains its maximum uniquely when $x_1 = \cdots = x_n$.

One way to prove P_2 is to observe that

$$f_2(x, \lambda - x) = f_2(\lambda/2, \lambda/2) - 2(x - \lambda/2)^4$$

Now let $n \ge 3$. Fix $x_n \quad 0 < x_n < \lambda$ and rewrite f_n as

$$f_n(x_1, \dots, x_n) = x_n f_{n-1}(x_1, \dots, x_{n-1}) + x_n^3 \prod_{i=1}^{n-1} x_i$$

The induction hypothesis says that $f_{n-1}(x_1, \ldots, x_{n-1})$ has a unique maximum over $\sigma_{n-2}(\lambda - x_n)$ at $x_1 = \cdots = x_{n-1}$. But the same is true of $\prod_{i=1}^{n-1}$ (from the arithmetic mean-geometric mean inequality). Hence, for any fixed $0 < x_n < \lambda$, $f_n(x_1, \ldots, x_{n-1}, x_n)$ attains its maximum uniquely when $x_1 = \cdots = x_{n-1}$.

The role of x_n being replaceable by any x_k , we have proved P_n . (For if (x_1, \ldots, x_n) lies in the interior of $\sigma_{n-1}(\lambda)$ with $x_1 \neq x_j$ for some i, j, then, taking k different from i or j and letting $y_k = x_k$ $y_r = (\lambda - x_k)/(n-1)$ for $r \neq k$, we see that $f_n(x_1, \ldots, x_n) < f_n(y_1, \ldots, y_n)$.)

II. Solution by Professor Freidkin, University of the Witwatersrand, Johannesburg, Republic of South Africa. Let $F(x_1, x_2, ..., x_n) \equiv x_1 x_2 \cdots x_n (x_1^2 + x_2^2 + \cdots + x_n^2)$. We will show that F attains its maximum value (of $n^{-(n+1)}$) when all of the arguments are equal.

Suppose that two of the arguments of F, say x_1 and x_2 , are not equal. The we have the following.

$$F(x_1, x_2, \dots, x_n) = x_1 x_2 \prod_{i=3}^n x_i \left(x_1^2 + x_2^2 + \sum_{i=3}^n \right)$$

$$= \left(\frac{x_1 + x_2}{2} + \frac{x_1 - x_2}{2} \right) \left(\frac{x_1 + x_2}{2} - \frac{x_1 - x_2}{2} \right)$$

$$\times \prod_{i=3}^n x_i \left(\frac{1}{2} (x_1 + x_2)^2 + \frac{1}{2} (x_1 - x_2)^2 + \sum_{i=3}^n x_i^2 \right)$$

$$= \frac{1}{8} \left((x_1 + x_2)^4 - (x_1 - x_2)^4 \right) \prod_{i=3}^n x_i$$

$$+ \frac{1}{4} \prod_{i=3}^n x_i \left((x_1 + x_2)^2 - (x_1 - x_2)^2 \right) \sum_{i=3}^n x_i^2$$

$$< \frac{1}{8} (x_1 + x_2)^4 \prod_{i=3}^n x_i + \frac{1}{4} \prod_{i=3}^n x_i (x_1 + x_2)^2 \sum_{i=3}^n x_i^2$$

$$\equiv F \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3 \dots, x_n \right)$$

Thus the maximum must occur when all the x_i are equal.

III. Solution by the proposer. We will show that

$$\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{n+2} \ge x_1 x_2 \cdots x_n \left(\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}\right) \tag{1}$$

so that the desired maximum value is $n^{-(n+1)}$ and is taken on when all the x_i are equal. Our proof is by induction. First, (1) is valid for n = 2, since it reduces to $(x_1 - x_2)^4 \ge 0$. We now assume (1) is valid for n = k and we will prove it valid for n = k + 1. Let $A = (x_1 + x_2 + \cdots + x_k)/k$ and $P + x_1 x_2 \cdots x_k$. Then from (1) with n = k,

$$Px(x_1^2 + x_2^2 + \dots + x_k^2) + Px^3 \le kxA^{k+2} + Px^3 \le kxA^{k+2} + x^3A^k$$

It now suffices to show that

$$(k+1)\left(\frac{x_1+x_2+\dots+x_k+x}{k+1}\right)^{k+3} = (k+1)\left(\frac{kA+x}{k+1}\right)^{k+3} \ge kxA^{k+2} + x^3A^k$$

or equivalently, that $kxA^{k+2} + x^3A^k \le k+1$, where, without loss of generality, we have assumed that kA + x = k + 1.

Using the standard calculus technique, we differentiate with respect to ${\cal A}$ and set it to zero,

$$D_A(kxA^{k+2} + x^3A^k) = k(k+2)xA^{k+1} + kx^3A^{k-1} + (kA^{k+2} + 3x^2A^k)(-k) = 0$$

or

$$(kt^{3} - (k+2)t^{2} + 3t - 1)t^{k-1} = 0$$

where t = A/x. The cubic factors into $(t-1)(kt^2 - 2t + 1)$. The only real roots are 0 and 1. The maximum occurs for A = x = 1 and we have proved the case for n = k + 1. Thus the result is valid for all $n = 2, 3, 4, \ldots$

1325. Submitted by M. S. Klamkin, University of Alberta, Canada

a. Determine the *minimum* value of

$$\max_{0 \le x_i \le 1} |F_1(x_1) + F_2(x_2) + \dots + F_n(x_n) - x_1 x_2 \cdots x_n|$$

over all possible real-valued functions $F_i(t)$, $0 \le t \le 1, 1 \le i \le n$. **b.** Determine the *minimum* value of

$$\max_{0 \le x_i \le 1} |F_1(x_1)F_2(x_2)\cdots F_n(x_n) - (x_1 + x_2 + \dots + x_n)|$$

over all possible real-valued functions $F_i(t)$, $0 \le t \le 1, 1 \le i \le n$. Math. Mag., **63**(1990) 194–196.

Solution by the proposer.

a. We will show that the minimum is (n-1)/2n. The first part of the proof is indirect. Assume that for any F_i and all x_i that

$$|F_1(x_1) + F_2(x_2) + \dots + F_n(x_n) - x_1 x_2 \cdots x_n| < a \le \frac{n-1}{2n}$$

Let

$$S_0 = F_1(0) + F_2(0) + \dots + F_n(0)$$

$$S_1 = F_1(1) + F_2(1) + \dots + F_n(1)$$

We have

$$|S_0| < a$$
 $|1 - S_1| < a$

and for j = 1, 2, ..., n,

$$|S_1 - F_j(1) + F_j(0)| < a$$

Thus,

$$a + (n-1)a + na > |S_0| + (n-1)|1 - S_1| + \sum_{j=1}^n |S_1 - F_j(1) + F_j(0)|$$

and therefore

$$2na > |-S_0 + (n-1)(1-S_1) + (n-1)S_1 + S_0| = n - 1$$

Hence

$$T_1 \equiv |F_1(x_1) + F_2(x_2) + \dots + F_n(x_n) - x_1 x_2 \cdots x_n| \ge \frac{n-1}{2n}$$

for some choice of the x_i . That the minimum is (n-1)/2n will follow by the choice of functions

$$F_i(x_i) \equiv \frac{x_i}{n} - \frac{n-1}{2n^2}$$

for all i. Here

$$T_1 = \left| \sum_{i=1}^n \left(\frac{x_i}{n} - \frac{n-1}{2n^2} \right) - x_1 x_2 \cdots x_n \right|$$

and all we need now show is that $T_{\text{max}} = (n-1)/2n$. Since T is linear in the x_i its maximum will be taken on by each variable being 0 or 1. For all ones, T = (n-1)/2n. For r(< n) ones and n-r zeros,

$$T = \left| \frac{r}{n} - \frac{n-1}{2n} \right| \le \frac{n-1}{2n}$$

and with equality only for r = n - 1.

Comment: The special case of showing that the minimum is greater than or equal to 1/4 for n = 2 wasa 1959 Putnam problem (A. M. Gleason, R. E. Greenwood, L. M. Kelly, *The William Lowell Putnam Mathematical Competition, Problems and Solutions, 1938–1964*, MAA, Washington, D.C., 1980, p.499).

b. First consider the even case; that is, replace n by 2n. We will show that the minimum is n/4 The first part of the proof is indirect. Assume that for any F_i and all x_i that

$$|F_1(x_1)F_2(x_2)\cdots F_{2n}(x_{2n}) - x_1 - x_2 - \cdots - x_{2n}| < a \le n/4$$

Let

$$P_0 = |F_1(0)F_2(0)\cdots F_{2n}(0)|$$

$$P_1 = |F_1(1)F_2(1)\cdots F_{2n}(1)|$$

We then have

$$-a < P_0 < a \qquad 2n - a < P_1 < 2n + a \qquad P_o P_1 < a(2n + a)$$
$$n - a < |F_1(0)F_2(0)\cdots F_n(0)F_{n+1}(1)F_{n+2}(1)\cdots F_{2n}(1)| < n + a$$

We now take all combinations similar to the last inequality with n zeros and n ones and multiply them to give

$$(n-a)^{\alpha} < (P_0P_1)^{\alpha/2} < (n+a)^{\alpha}$$

where $\alpha = \binom{2n}{n}$. Equivalently,

$$(n-a)^2 < P_0 P_1 < (n+a)^2$$

Also

$$(n-a)^2 < P_0 P_1 < a(2n+a)$$

and from this we get a > n/4. Hence

$$T_2 \equiv |F_1(x_1)F_2(x_2)\cdots F_{2n}(x_{2n}) - x_1 - x_2 - \cdots - x_{2n}| \ge n/4$$

for some choice of the x_i . That the minimum is n/4 will follow by the choice of functions $F_i(x_i) \equiv ax_i + b$ for all *i* where *a*, *b* are taken to satisfy

$$(a+b)^{2n} = 2n + n/4$$
 and $a^{2n} = n/4$

Here

$$T_2 = \left| \prod_{i=1}^{2n} (ax_i + b) - x_1 - x_2 - \dots - x_{2n} \right|$$
$$(a+b)^{2n-1} = \frac{n(n-1)}{2(2n-1)} \quad \text{and} \quad a^{2n-1} = \frac{n(n-1)}{2(2n-1)}$$

Since T_2 is linear in the x_i it takes on its extreme values when all the x_i are 0 or 1. Thus it now remains to show that

$$(a+b)^r a^{2n-1-r} < \frac{n(n-1)}{2(2n-1)} + r$$

or, equivalently, that

$$(9n^2 - 9r + 2)^r (n^2 - n)^{2n - 1 - r} \le (n^2 + n(4r - 1) - 2r)^{2n - 1}$$

for $r = 0, 1, \ldots, 2n-1$. The proof goes through as before using concavity.

1326. Submitted by M. S. Klamkin, University of Alberta, Canada

A particle is projected vertically upwards in a uniform gravitational field and subject to a drag force mv^2/c . The particle in its ascent and descent has equal speeds at two points whose respective heights above the point of projection are x and y. It has been shown by Newton that if a denotes the maximum height of the particle, then x and yare related by

$$e^{2(a-x)/c} + e^{-2(a-y)/c} = 2 \tag{1}$$

Consider the same problem except that the drag force is now $F(mv^2/c)$ where F is a smooth function. Show that if (1) still holds for all possible y values, then F(u) = u.

Math. Mag., 63(1990) 197–198.

Solution by Michael Golomb, Purdue University, West Lafayette, Indiana. We may assume [[that]] the unit of mass is chosen so that m = 1. The equation of motion during ascent is

$$v dv/dx = -g - F(v^2/c), \quad v(0) = v_0 > 0$$

At the maximum height a the velocity is 0, thus

$$\int_{0}^{v} \frac{v \, dv}{g + F(v^{2}/c)} = -\int_{a}^{x} dx = a - x$$

We set $v^2/c = u$ and obtain

$$\int_{0}^{v^{2}/c} \frac{du}{g + F(u)} = \frac{2}{c}(a - x)$$

In the same way, we obtain for the descent

$$\int_{0}^{v^{2}/c} \frac{du}{g - F(u)} = \frac{2}{c}(a - y)$$

Thus, if (1) is to hold then

$$\exp\left(\int_0^{v^2/c} \frac{du}{g+F(u)}\right) + \exp\left(\int_0^{v^2/c} \frac{du}{-g+F(u)}\right) = 2$$
(2)

Set

$$w = v^2/c$$
 $P(w) = \frac{1}{g + F(w)}$ $Q(w) = \frac{1}{-g + F(w)}$

Then differentiating (2) twice with respect to w, we obtain

$$(P' + P^2)e^{\int_0^w P} + (Q' + Q^2)e^{\int_0^w Q} = 0$$
(3)

But $P' = -F'P^2$, $Q' = -F'Q^2$, hence (3) becomes

$$(1 - F')(P^2 e^{\int_0^w P} + Q^2 e^{\int_0^w P}) = 0$$

Since $P^2 e^{\int P} + Q^2 e^{\int Q} > 0$ we conclude [[that]] F' = 1, i.e., F(w) = w + k, where k is a constant. With this F, (2) becomes

$$\frac{g + v^2/c + k}{g + k} + \frac{-g + v^2/c + k}{-g + k} = 2$$

It is readily seen that this equation implies k = 0, thus F(w) = w.

Q 748. Submitted by M. S. Klamkin, University of Alberta, Canada

Determine the maximum value of

$$F = \frac{(x^{2n} - a^{2n})(b^{2n} - x^{2n})}{(x^{2n} + a^{2n})(b^{2n} + x^{2n})}$$

over all real x.

A 748. On dividing,

$$F = -1 + \frac{2(a^{2n} + b^{2n})x^{2n}}{x^{4n} + (a^{2n} + b^{2n})x^{2n} + a^{2n}b^{2n}}$$

Now by the artithmetic mean-geomtric mean inequality,

$$x_{2n} + \frac{a^{2n}b^{2n}}{x^{2n}} + (a^{2n} + b^{2n})$$

takes onits minimum value when $x^2 = ab$ (we can assume that $a, b \ge 0$). Finally,

$$F_{\max} = \frac{(b^n - a^n)^2}{(b^n + a^n)^2}$$

Q 749. Submitted by M. S. Klamkin, University of Alberta, Canada Prove that

$$\frac{x^{\lambda+1}}{y_{\lambda}} + \frac{y^{\lambda+1}}{z_{\lambda}} + \frac{z^{\lambda+1}}{x_{\lambda}} \ge x+y+z$$

where $x, y, z, \lambda > 0$.

A 749. Expanding out, we have to show that

$$z^{\lambda}x^{\lambda}(x^{\lambda+1} - y^{\lambda+1}) + x^{\lambda}y^{\lambda}(y^{\lambda+1} - z^{\lambda+1}) + y^{\lambda}z^{\lambda}(z^{\lambda+1} - x^{\lambda+1}) \ge 0$$

Since the inequality is cyclic, we can assume without loss of generality that (i) $x \ge y \ge z$ or else (ii) $x \ge z \ge y$.

For (i) we can rewrite the inequality in the obvious form

$$z^{\lambda}(x^{\lambda}-y^{\lambda})(x^{\lambda+1}-y^{\lambda+1})+y^{\lambda}(x^{\lambda}-z^{\lambda})(y^{\lambda+1}-z^{\lambda+1}) \ge 0$$

For(ii) we rewrite the inequality in the form

$$z^{\lambda}(x^{\lambda}-y^{\lambda})(x^{\lambda+1}-z^{\lambda+1})+x^{\lambda}(z^{\lambda}-y^{\lambda})(z^{\lambda+1}-y^{\lambda+1}) \ge 0$$

More generally, the given inequality is the soecial case a = y, b = z, c = x of the inequality

$$\frac{x^{\lambda+1}}{a_{\lambda}} + \frac{y^{\lambda+1}}{b_{\lambda}} + \frac{z^{\lambda+1}}{c_{\lambda}} \ge \frac{(x+y+z)^{\lambda+1}}{(a+b+c)^{\lambda}}$$

where $\lambda = (x + y + z)/(a + b + c)$ and x, y, z, a, b, c > 0.

To prove this, let

$$F(\lambda) \equiv \left(\frac{a(x/a)^{\lambda+1} + b(y/b)^{\lambda+1} + c(z/c)^{\lambda+1}}{a+b+c}\right)^{1/(\lambda+1)}$$

Then by the power mean inequality

$$F(\lambda) \ge F(0)$$

which gives the desired result. It is to be noted that λ may be 0 and the inequality can be extended to

$$\sum_{i} \frac{x_i^{\lambda+1}}{a_i^{\lambda}} \ge \frac{\left(\sum_{i} x_i\right)^{\lambda+1}}{\left(\sum_{i} a_i\right)^{\lambda}}$$

Another proof of

$$\frac{x_1^{\lambda+1}}{x_2^{\lambda}} + \frac{x_2^{\lambda+1}}{x_3^{\lambda}} + \dots + \frac{x_n^{\lambda+1}}{x_1^{\lambda}} \ge x_1 + x_2 + \dots + x_n$$

follows immediately from the rearrangement inequality; i.e., if $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, $b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$, and the c_i are a permutation of the b_i then $a_1c_1 + a_2c_2 + \cdots + a_nc_n \ge a_ib_n + a_2b_{n-1} + \cdots + a_nb_1$.

[[Another not-so-quick quickie. - R.]]

Q 750. Submitted by M. S. Klamkin, University of Alberta, Canada

Ptolemy's inequality states that $ac + bd \ge ef$, where a, b, c, d are consecutive sides of a quadrilateral (it need not be planar), and e, f are its diagonals. There is equality if and only if the quadrilateral is cyclic (has a circumcircle). Determine a corresponding inequality for a spherical quadrilateral.

A 750. Let a, b, c, d and e, f denote the sides and diagonals of a spherical quadrilateral. Then the chord lengths of the spherical arcs of the sides and diagonals are given by $a' = 2R\sin(a/2), b' = 2R\sin(b/2)$, etc., where R is the radius of the sphere. Then by Ptolemy's inequality above

$$\sin(a/2) \cdot \sin(c/2) + \sin(b/2) \cdot \sin(d/2) \ge \sin(e/2) \cdot \sin(f/2)$$

Again there is equality if and only if the quadrilateral is cyclic.

Q 751. Submitted by M. S. Klamkin, University of Alberta, Canada

If z_1, z_2, \ldots, z_5 are complex numbers such that

$$|z_{i+1} + z_{i+2}| = |z_{i+3} + z_{i+4} + z_{i+5}|$$

for $i = 1, 2, \ldots, 5$ and $z_{i+5} + z_i$ prove that $z_1 + z_2 + \cdots + z_5 = 0$.

A 751. We will show more generally that if A_1, A_2, \ldots, A_n are vectors in \mathbf{E}^n such that

$$|\mathbf{A}_{i+1} + \mathbf{A}_{i+2} + \cdots + \mathbf{A}_{i+r}| = |\mathbf{A}_{i+r+1} + \mathbf{A}_{i+r+2} + \cdots + \mathbf{A}_{i+n}|$$

for 2r < n, i = 0, 1, ..., n-1 and $A_{i+n} = A_i$, then

$$\mathbf{A}_1 + \mathbf{A}_2 + \dots + \mathbf{A}_n = 0$$

Proof. Let $\mathbf{S}_i = \mathbf{A}_{i+1} + \mathbf{A}_{i+2} + \cdots + \mathbf{A}_{i+r}$ and $\mathbf{S} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n$. The given relations become $|\mathbf{S}_i| = |\mathbf{S} - \mathbf{S}_i|$ which on squaring becomes $\mathbf{S}^2 = 2\mathbf{S} \cdot \mathbf{S}_i$. On summing over *i*, we get $n\mathbf{S}^2 = 2r\mathbf{S}^2$. Hence $\mathbf{S} = 0$.

Math. Mag., 63(1990) 57, 63.

Q 758. Submitted by M. S. Klamkin, University of Alberta, Alberta, Canada Evaluate

$$I = \int_0^a x^n (2a - x)^n \, dx \div \int_0^a x^n (a - x)^n \, dx$$

A 758. Letting x = 2t in the first integral, we get

$$I = 2^{2n+1} \int_0^{a/2} t^n (a-t)^n \, dt \div \int_0^a t^n (a-t)^n \, dt$$

Since

$$\int_0^a t^n (a-t)^n \, dt = \int_0^{a/2} t^n (a-t)^n \, dt + \int_{a/2}^a t^n (a-t)^n \, dt = 2 \int_0^{a/2} t^n (a-t)^n \, dt,$$

 $I = 2^{2n}$.

This problem is due to E. B. Elliott and appears in *Mathematical Problems from the Educational Times*, where each integral is evaluated separately.

Math. Mag., 63(1990) 126, 133.

Q 761. Submitted by M. S. Klamkin, University of Alberta, Edmonton, Canada If $a_1, a_2, \ldots, a_{n+1} > 0$, prove that

$$a_1 a_2 \cdots a_{n+1} (a_1^{-n} + a_2^{-n} + \dots + a_{n+1}^{-n}) \ge a_1 + a_2 + \dots + a_{n+1}$$

A 761. Letting $P = \prod_{i=1}^{n+1} a_i$, $S = \sum_{i=1}^{n+1} a_i^{-n}$, the inequality can be written as

$$P\left[(S - a_1^{-n}) + (S - a_2^{-n}) + \dots + (S - a_{n+1}^{-n})\right] \ge n \sum_{i=1}^{n+1} a_i$$

Since by the arithmetic mean - geometric mean inequality,

$$S - a_i^{-n} \ge na_i/P$$

we get the desired inequality. There is equality if and only if all the a_i are equal.

Alternative solution with generalization. Letting $a_k = 1/x_k$ we then have to show equivalently that

$$x_1^n + x_2^n + \dots + x_{n+1}^n \ge x_1 x_2 \dots x_n + x_2 x_3 \dots x_{n+1} + \dots + x_n x_1 \dots x_{n-1}$$

for $x_k > 0$. More generally, we have

$$x_1^p + x_2^p + \dots + x_{n+1}^p \ge x_1 x_2 \dots x_p + x_2 x_3 \dots x_{p+1} + \dots + x_n x_1 \dots x_{p-1}$$

for all positive integers p and n. A proof follows immediately by applying Hölder's inequality to

$$(x_1^p + x_2^p + \dots + x_n^p)^{1/p} (x_2^p + x_3^p + \dots + x_1^p)^{1/p} \cdots (x_p^p + x_{p+1}^p + \dots + x_{p+n-1}^p)^{1/p}$$

where $x_{n+k} = x_k$.

Math. Mag., 64(1991) 65–66.

Comment

Q 761. In Vol.63, No.2, April 1990, p.133 *Murray Klamkin* offered the following generalization to his Quickie (**Q761**): "For positive integers n and p and numbers $x_1, \ldots, x_n > 0$,

$$x_1^p + x_2^p + \dots + x_n^p \ge x_1 x_2 \cdots x_p + x_2 x_3 \cdots x_{p+1} + \dots + x_n x_1 \cdots x_{p-1}$$

where each subscript on the right is understood to be reduced nodulo n to one of $1, 2, \ldots, n$." Peter D. Johnson, Jr., Auburn University, offers the following generalization.

We start with the well-known rearrangement inequality (e.g., see *Inequalities*, Hardy, Littlewood & Pólya, Chapter 10): If $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$ and π is a permutation of $\{1, 2, \ldots, n\}$, then $\sum_i a_i b_i \geq \sum_i a_i b_{\pi(i)}$.

For non-negative sequences this generalizes, by induction on p, to the following.

If $p \ge 2$ and n are positive integers, and $\mathbf{A} = (a_{ij})$ is a $p \times n$ matrix of non-negative numbers with non-decreasing rows, then for any permutations $\pi_2, \pi_3, \ldots, \pi_p$ of $\{1, 2, \ldots, n\}$,

$$\sum_{j=1}^{n} a_{1j} a_{2j} \cdots a_{pj} \ge \sum_{j=1}^{n} a_{1j} a_{2,\pi_2(j)} \cdots a_{p,\pi_p(j)}$$

Klamkin's inequality is obtainable from this by taking the rows of **A** to be equal (to a non-decreasing rearrangement of x_1, \ldots, x_n) and the permutations π_2, \ldots, π_p to be successive powers of a certain cycle (π_2) .

Math. Mag., **63**(1990) 190, 198.

Q 763. Submitted by M. S. Klamkin, University of Alberta, Edmonton, Canada

Determine all real solutions of the simultaneous equations

$$2x(1 + y + y^{2}) = 3(1 + y^{4})$$

$$2y(1 + z + z^{2}) = 3(1 + z^{4})$$

$$2z(1 + x + x^{2}) = 3(1 + x^{4})$$

A 763. Since $1 + x + x^2 > 0$, etc., it follows that x, y, z areall positive. Without loss of generality we may assume $x \ge y \ge z$. Then

$$2x(1 + x + x^2) \ge 3(1 + x^4)$$

so that

$$0 \ge (x-1)^2(3x^2+4x+3)$$

Thus x = 1 giving the one real solution x = y = z = 1.

Math. Mag., 63(1990) 274.

1355. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the extreme values of

$$F \equiv x_1 x_2 \cdots x_n - (x_1 + x_2 + \cdots + x_n)$$

where $b \ge x_i \ge a \ge 0$ for all *i*.

Math. Mag., 64(1991) 278.

Solution by Christos Athanasiadis, student, Massachusetts Institute of Technology, Cambridge, Massachusetts. We note that as a function of x_j alone, F is linear, so it attains its extreme values at $x_j = a$ and $x_j = b$. It easily follows that the extreme values of F are attained at n-tuples (x_1, x_2, \ldots, x_n) for which $x_i \in \{a, b\}$ for all i. Thus, by the symmetry of F, the extreme values of F are among the numbers c_0, c_1, \ldots, c_n defined by

$$c_k = a^{n-k}b^k - (n-k)a - kb$$

Note that $c_{k+1} - c_k = (a^{n-k-1}b^k - 1)(b-a)$ for $k = 0, 1, \ldots, n-1$. We now distinguish three cases.

Case 1. $a \ge 1$. Then $c_{k+1} \ge c_k$ for all k, so the minimum of F is $c_0 = a^n - na$ and the maximum $c_n = b^n - nb$.

Case 2. $b \leq 1$. Then $c_{k+1} \leq c_k$ for all k so the minimum is c_n and the maximum is c_0 . Case 3. a < 1 < b. In this case, $c_{k+1}|geqc_k$ if and only if

$$k \ge \frac{(n-1)\log(1/a)}{\log(b/a)} \tag{(*)}$$

Thus, if k is the smallest integer for which (*) holds, then the minimum of F is c_k , while the maximum is $\max\{c_0, c_n\}$.

Math. Mag., 63(1990) 274, 280.

Q 768. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Show that

$$(a+b+c)^n(a^{2n}+b^{2n}+c^{2n}) \ge (a^n+b^n+c^n)(a^2+b^2+c^2)^n$$

where $a, b, c \ge 0$ and $n \ge 1$.

A 768. By the power mean inequality,

$$\frac{\sum a^{2n}}{\sum a^n} = \frac{\sum a^n \cdot a^n}{\sum a^n} \ge \left(\frac{\sum a^n \cdot a}{\sum a^n}\right)^n$$

where the sums here and subsequently are symmetric over a, b, c. Then

$$\frac{\sum a^{n+1}}{\sum a^n} \geq \frac{\sum a^2}{\sum a}$$

since it is equivalent to

$$\sum ab(a^{n-1} - b^{n-1})(a - b) \ge 0$$

The inequality can be easily extended for a more general combination of exponents and to any number of variables. Math. Mag., 63(1990) 279.

Comments

1311. Murray Klamkin notes that the result of this problem is equivalent to a generalization he gave to a USA Olympiad Problem (M. S. Klamkin, USA Mathematical Olympiads, 1972–1976, MAA, Wasington, D.C., 1988, p.84). Moreover, the result in this reference is valid for both the odd and even cases.

[[The problem referred to is

Math. Mag., **61**(1988) 315.

1311. Proposed by Mihály Bencze, Braşon, Romania

Let $0 < m \leq x_1, x_2, \ldots, x_{2n+1} \leq M$. Prove that

$$(M-m)^2 + 4Mm\left(\sum_{k=1}^{2n+1} x_k\right) \left(\sum_{k=1}^{2n+1} \frac{1}{x_k}\right) \le (2n+1)^2(M+m)^2 \qquad]]$$

Math. Mag., 63(1990) 350–351.

1362. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

If (a_i, b_i, c_i) are the sides, R_i the circumradii, r_i the inradii, and s_i the semi-perimeters of n triangles (i = 1, 2, 3, ..., n) respectively, show that

$$3\left(\prod a_{i}^{-1/n} + \prod b_{i}^{-1/n} + \prod c_{i}^{-1/n}\right) \leq \prod \left(\frac{s_{i}}{r_{i}R_{i}}\right)^{1/n} \\ \leq 2^{n} \prod \left(a_{i}^{-1/n} + b_{i}^{-1/n} + c_{i}^{-1/n}\right)$$

where the sums and products are over i = 1 to n.

Math. Mag., **64**(1991) 355.

(We regret that the statement of the problem inadvertently contained an extra summation sign [[removed from the above]] which rendered the problem meaningless, and omitted the necessary condition that n > 1.)

Solution by Jiro Fukuta, Motosu-gun, Gifu-ken, Japan. From Hölder's inequality,

$$3\left(\prod a_i^{-1/n} + \prod b_i^{-1/n} + \prod c_i^{-1/n}\right) \le 3\prod \left(a_i^{-1/n} + b_i^{-1/n} + c_i^{-1/n}\right)^{1/n}$$

Now $(a_i + b_i + c_i)^2 = a_i^2 + B_i^2 + c_i^2 + 2(b_ic_i + c_ia_i + a_ib_i) \ge 3(b_ic_i + c_ia_i + a_ib_i)$ and therefore $3(a_i^{-1} + b_i^{-1} + c_i^{-1}) \le (a_i + b_i + c_i)^2/(a_ib_ic_i) = s_i/(r_iR_i)$. It follows that

$$3\left(\prod a_{i}^{-1/n} + \prod b_{i}^{-1/n} + \prod c_{i}^{-1/n}\right) \leq \prod \left(\frac{s_{i}}{r_{i}R_{i}}\right)^{1/n}$$

For the other inequality, it is enough to prove that if a, b, c are the lengths of the three sides of a triangle,

$$\left(\frac{(a+b+c)^2}{abc}\right)^{1/n} \le 2(a^{-1/n}+b^{-1/n}+c^{-1/n})$$

or

$$(a+b+c)^2 \le \left(2(bc)^{1/n} + 2(ca)^{1/n} + 2(ab)^{1/n}\right)^n$$

Let $A = a^{1/n}$, $B = b^{1/n}$ and $C = c^{1/n}$. Then the preceding equation can be rewritten as

$$(A^{n} + B^{n} + C^{n})^{2} \le (2BC + 2CA + 2AB)^{n}$$

But

$$(2BC + 2CA + 2AB)^n = (A(B + C - A) + B(C + A - B) + C(A + B - C) + A^2 + B^2 + C^2)^n \\ \ge (A^2 + B^2 + C^2)^n$$

because $(B+C)^n = (b^{1/n} + c^{1/n})^n \ge b + c \ge a = A^n$ and thus $B+C \ge A$ and so on. Applying Jensen's inequality, we have

$$(A^{n} + B^{n} + C^{n})^{2} \le (A^{2} + B^{2} + C^{2})^{n}$$

for n > 0. The result follows.

Math. Maq., 63(1990) 351, 357.

Q 770. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Canada Determine the minimum value of $x^2 + y^2 + z^2$ given that

xyz - x - y - z = 2 and $x, y, z \ge 0$.

A 770. The constraint condition can be rewritten as

$$1/(1+x) + 1/(1+y) + 1/(1+z) = 1$$

Then by Jensen's inequality for convex functions,

$$1/(1+x) + 1/(1+y) + 1/(1+z) \ge 3/(1+A)$$

where A + (x + y + z)/3. Thus, $A \ge 2$. Then by the power mean inequality,

$$(x^2 + y^2 + z^2)/3 \ge A^2$$

so that $x^2 + y^2 + z^2 \ge 12$ and with equality if and only if x = y = z = 2. More generally, if $\sum 1/(1+x_i) = 1$ and $x_i \ge 0$ for i = 1, 2, ..., n, then $\sum x_i^p \ge n(n-1)^p$ for $p \ge 1$ and with equality if and only if $x_i = n - 1$.

Math. Mag., 63(1990) 328–329.

[[There's a Note:]]

A Single Inequality Condition for the Existence of Many r-gons

Murray S. Klamkin & Zrzysztof Witczynski

Math. Mag., 63(1990) 356–357.

Comments

Q759. In this problem, proposed by Norman Schaumberger, Bronx Community College, a, b, c and d are the lengths of the sides of a quadrilateral and P is its perimeter. Then

$$abc/d^2 + bcd/a_c^2 da/b^2 + dab/c^2 > P$$

unless a = b = c = d.

Murray Klamkin, University of Alberta, makes the following comments.

First, one can obtain the given inequality in one step by an application of Hölder's inequality, i.e.,

$$(a^{3}b^{3}c^{3} + b^{3}c^{3}d^{3} + c^{3}d^{3}a^{3} + d^{3}a^{3}b^{3})^{1/3} \times (d^{3}a^{3}b^{3} + a^{3}b^{3}c^{3} + b^{3}c^{3}d^{3} + c^{3}d^{3}a^{3})^{1/3} \times (c^{3}d^{3}a^{3} + d^{3}a^{3}b^{3} + a^{3}b^{3}c^{3} + b^{3}c^{3}d^{3})^{1/3} \ge a^{3}b^{2}c^{2}d^{2} + b^{3}c^{2}d^{2}a^{2} + c^{3}d^{2}a^{2}b^{2} + d^{3}a^{2}b^{2}c^{2}$$

For a generalization of this, one can start with the *m*-th root of the cyclic sum (with respect to the a_i) whose first term is $a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n}$ and multiplying by (m-1) *m*-th roots of successive permutations of this sum as above $(1 \le m \le n)$.

The given inequality is also a special case of Muirhead's inequality [Hardy, Littlewood & Pólya, *Inequalities*, Cambridge University Press, London, 1934, pp.44–48]:

Let

$$[\alpha_1, \alpha_2, \dots, \alpha_n] = (1/n!) \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$$

where the sum is over the n! terms obtained from $a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n}$ by all possible permutations of the a_i and $a_i > 0$, $\alpha_i > 0$. If

$$\{\alpha_1, \alpha_2, \dots, \alpha_n\}$$
 majorizes $\{\beta_1, \beta_2, \dots, \beta_n\}$ $\beta_i \ge 0$

that is,

$$\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n \qquad \beta_1 \ge \beta_2 \ge \dots \ge \beta_n \\ \alpha_1 + \alpha_2 + \dots + \alpha_k \ge \beta_1 + \beta_2 + \dots + \beta_k \qquad n > k \ge 1 \\ \alpha_1 + \alpha_2 + \dots + \alpha_n = \beta_1 + \beta_2 + \dots + \beta_n$$

then

$$[\alpha_1, \alpha_2, \dots, \alpha_n] \preceq [\beta_1, \beta_2, \dots, \beta_n]$$

Since $\{3, 3, 3, 0\}$ majorizes $\{3, 2, 2, 2\}$, we obtain the original given inequality. It also follows that

$$[9,0,0,0] \leq [6,3,0,0] \leq [3,3,3,0] \leq [3,3,2,1] \leq [3,2,2,2],$$
etc.

Math. Mag., 64(1991) 61, 67.

Q 774. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Canada

If A, B, C, D are distinct coplanar vectors with equal lengths such that $\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{C}$, show that $\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{D} = \mathbf{0}$.

A 774. If *O* is the origin of the vectors, then their endpoints *A*, *B*, *C*, *D*, respectively, lie on a circle centered at *O*. Since $(\mathbf{A} - \mathbf{C}) \cdot (\mathbf{B} - \mathbf{D}) = 0$, $AC \perp BD$. Hence, *A*, *B*, *C*, *D* are in consecutive order on the circle and do not lie on any semicircle.



D

Referring to the preceding figure, we have to show that $\cos \alpha + \cos \delta = 0$. Since $90^{\circ} = \angle APB = \frac{1}{2}(\operatorname{arc} AB + \operatorname{arc} CD), \alpha + \delta = 180^{\circ}$ and we are done.

Math. Mag., 64(1991) 356–357.

Comments

Q774. In the Quickie Solution (Vol.64, No.1, pp.61,67) to this problem, the author states that "A, B, C,D are in consecutive order on the circle and do not lie on any semicircle." The following example, pointed out by *Mike Schramm (student)* and *Kevin Farrell, Lyndon State College*, shows that this conclusion is false. Let $\mathbf{A}=(5,14)$, $\mathbf{B}=(14,-5)$, $\mathbf{C}=(-5,14)$ and $\mathbf{D}=(14,5)$. These vectors satisfy the hypotheses of the theorem, namely, they are distinct coplanar vectors with equal lengths and $\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{C}$. They do not satisfy the conclusion stated above. They are not in consective order and do lie on a semicircle.

Murray Klamkin, University of Alberta comments: In the solution, it was assumed that AC and BD, which must be perpendicular chords of a circle, intersect at a point P lying within or on a circle. The example given above shows that P might lie outside the circle. Here is a simpler solution that takes care of both cases.

Choose a rectangular coordinate system with origin at the center O of the circle and with axes parallel to AC and BD. Then the four vectors have the representations

 $\mathbf{A} = (\alpha, \beta), \mathbf{C} = (-\alpha, \beta) \mathbf{B} = (\gamma, \delta)$ and $\mathbf{D} = (\gamma, -\delta)$. (Note that α need not be positive, etc.) Finally,

$$\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{D} = \alpha \gamma + \beta \delta - \alpha \gamma - \beta \delta = 0$$

Since also

$$A^{2} + B^{2} + C^{2} + D^{2} - (A - B)^{2} - (C - D)^{2} = 2(A \cdot B + C \cdot D) = 0$$

we have equivalently that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4R^2$$
 (*R* = radius of the circle)

The latter corresponds to the known result (*Crux Mathematicorum*, **15**(1989) 293, #1) that the sum of the areas of the four circles whose diameters are AP, BP, CP and DP is equal to the area of the given circle. In this result it is assumed that P lies within the circle. But the above proof shows that it is valid if P is outside the circle. This four-circle result apparently has been generalized (*Crux Mathematicorum*, **16**(1990) p.109, #1535) to a result concerning two intersecting chords in an ellipse. However, the ellipse result can be shown to follow from the circle result by an affine transformation.

Math. Mag., 64(1991) 132, 137–138.

Q 777. Submitted by Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Alberta, Canada

 T_1 and T_2 are two acute triangles inscribed in the same circle. If the perimeter of T_1 is greater than the perimeter of T_2 , must the area of T_1 also be treater than the area of T_2 ?

A 777. By considering two triangles with angles $(80^\circ, 50^\circ, 50^\circ)$ and $(70^\circ, 70^\circ, 40^\circ)$ the answer is in the negative.

The result would be valid for two general triangles if the angles of T_2 majorized those of T_1 , that is, if $A_1 \ge B_1 \ge C_1$, $A_2 \ge B_2 \ge C_2$, then $A_2 \ge A_1$ and $A_2 + B_2 \ge A_1 + B_1$. Then by the majorization inequality,

$$F(A_1) + F(B_1) + F(C_1) \ge F(A_2) + F(B_2) + F(C_2)$$
(1)

for concave functions F. The rest follows since the perimeter and area of a triangle ABC is [[are?]] given by $2R(\sin A + \sin B + \sin C)$ and $2R^2 \sin A \sin B \sin C$ respectively, and $\sin x$ and $\ln \sin x$ are concave on $(0, \pi)$.

Comments. For the special case when T_1 and T_2 have a common angle (or equivalently a common side), then the angles of T_2 majorize those of T_1 . It would be of interest to give an elementary geometric proof of (1) for $F(x) = \sin x$ or $\ln \sin x$.
Math. Mag., 64(1991) 198, 206.

Q 779. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the best upper and lower bounds for the sum

$$\frac{a}{f+a+b} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{c+d+e} + \frac{e}{d+e+f} + \frac{f}{e+f+a}$$

where a, b, c, d, e and f are nonnegative and no denominator is zero.

A 779. Let S be the given sum and T = a + b + c + d + e + f. Then

$$S > a/T + b/T + c/T + d/T + e/T + f/T = 1$$

That 1 is the best possible bound follows by choosing

$$a = 1/\epsilon^6$$
 $b = 1/\epsilon^5$ $c = 1/\epsilon^4$ $d = 1/\epsilon^3$ $e = 1/\epsilon^2$ and $f = 1/\epsilon$

where $\epsilon \ll 1$.

[[The last 'f' was misprinted as 'e']]

The least upper bound is 3 and follows from

$$\frac{a}{f+a+b} + \frac{b}{a+b+c} \le \frac{a+b}{a+b} = 1$$
$$\frac{c}{b+c+d} + \frac{d}{c+d+e} \le \frac{c+d}{c+d} = 1$$
$$\frac{e}{d+e+f} + \frac{f}{e+f+a} \le \frac{e+f}{e+f} = 1$$

The 3 bound is achievable by either setting a = c = e = 0 or else by setting b = d = f = 0.

Math. Mag., 64(1991) 275, 281.

Q 783. Submitted by Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Canada

If all the vertices of a regular *n*-gon are lattice points in a plane tessellated by equilateral triangles, then n = 3 or n = 6.

A 783. Suppose $n \neq 3$ and $n \neq 6$ but there exists a regular lattice *n*-gon $A_1A_2 \ldots A_n$. Construct equilateral triangles $A_1A_2B_1$, $A_2A_3B_2$, ..., $A_nA_1B_n$ inside $A_1A_2 \ldots A_n$. It is easy to see that $B_1B_2 \ldots B_n$ is also a regular lattice *n*-gon, and smaller than $A_1A_2 \ldots A_n$. This construction can be repeated to generate a sequence of lattice *n*-gons converging to a single point. This is clearly impossible. On the other hand, there certainly exist regular lattice *n*-gons for n = 3 and n = 6. For n = 3, $B_1B_2B_3 = A_1A_2A_3$. For n = 6, $B_1 = B_2 = B_3 = B_4 = B_5 = B_6$ and the construction cannot be repeated.

Math. Mag., 64(1991) 277.

Quadrilateral subdivision

1354. Proposed by Frank Schmidt, Bryn Maur College, Bryn Maur, Pennsylvania, and Rodica Simion, George Washington University, Washington, DC.

Let ABCD be a convex quadrilateral in the plane with trisection points joined as in the figure to form nine smaller quadrilaterals.

a. Show that the area of A'B'C'D' is one-ninth the area of ABCD.

b. Give necessary and sufficient conditions so that all nine quadrilaterals have equal area.



II. Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. Results generalizing part a, and easily implying part b, are given in Donald Batman & Murray Klamkin, Solution to Problem E2423 (1974, Amer. Math. Monthly, pp.666–668). Part a also appeared previously as Problem E1548 (1962, Amer. Math. Monthly).

Math. Mag., 65(1992) 57, 65.

Q 786. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Evaluate the absolute value of the $n \times n$ determinant of the matrix (a_{rs}) , where $a_{rs} = \omega^{rs}$ (r, s = 1, 2, ..., n) and ω is a primitive root of $x^n = 1$.

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A 786. The *rs*-th term of the matrix $(a_{rs})(a_{rs})$ is given by

$$\omega^{r+s} + \omega^{2(r+s)} + \dots + \omega^{n(r+s)}$$

The latter is 0 unless r + s = n or 2n, in which case it is equal to n. Thus,

$$\det(a_{rs})(a_{rs}) = \det\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & n & 0\\ 0 & 0 & 0 & \cdots & n & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots\\ 0 & 0 & n & \cdots & 0 & 0 & 0\\ 0 & n & 0 & \cdots & 0 & 0 & 0\\ n & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix} = \pm n^n$$

and therefore $|\det(a_{rs})| = n^{n/2}$

Math. Mag., 65(1992) 195, 200.

Q 792. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Canada Determine all positive integer triples (x, y, z) satisfying the Diophantine equation

$$x^4 + y^4 + z^4 = 2y^2z^2 + 2z^2x^2 + 2x^2y^2 - 3z^2z^2 + 2z^2y^2 - 3z^2 + 2z^2y^2 - 3z^2z^2 + 2z^2y^2 - 3z^2 + 2z^2 +$$

A 792. The equation is equivalent to

$$(x + y + z)(y + z - x)(z + x - y)(x + y - z) = 3$$

Hence x + y + z = 3 and then x = y = z = 1.

A more interesting problem is to find integers w such that

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z) = 3w^4$$

has solutions other than x = y = z = w. Geometrically, this problem is equivalent to finding integer triangles having the same area as an equilateral triangle of side w.

[[is this an unsolved problem ? — R.]]

Math. Mag., **65**(1992) 266.

1407. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the maximum value of the sum

$$x_1^p + x_2^p + \dots + x_n^p - x_1^q x_2^r - x_2^q x_3^r - \dots - x_n^q x_1^r$$

where p, q, r are given numbers with $p \ge q \ge r > 0$ and $0 \le x_i \le 1$ for all i.

Math. Mag., 66(1993) 271.

Solution by David Jonathan Barrett, New York, New York. We show that $\lfloor n/2 \rfloor$ is the maximum value.

Let $f_{p,q,r}(x_1,\ldots,x_n)$ denote the given expression. Then for the x_i as given

$$f_{p,p,p}(x_1,\ldots,x_n) \ge f_{p,q,r}(x_1,\ldots,x_n) \tag{1}$$

We show that the left side reaches its maximum at some point where equality holds. For n = 2 it is easy to check that $f_{p,p,p}(x_1, x_2)$ reaches its maximum of 1 at (1,0) and (0,1).

Assume $n \geq 3$. For fixed p, x_3, x_n let

$$g(x_1, x_2) = x_1^p + x_2^p - x_1^p x_2^p - x_2^p x_3^p - x_n^p x_i^p$$

Then $f_{p,p,p}(x_1, \ldots, x_n)$ is the sum of $g(x_1, x_2)$ and some function independent of x_1 and x_2 . To maximize $f_{p,p,p}(x_1, \ldots, x_n)$ with respect to x_1, x_2 we need only to maximize $g(x_1, x_2)$. Since the latter has no relative extremum in the interior of the unit square, it must reach its maximum on the boundary, that is, where at least one of x_1, x_2 is 0 or 1. Examination reveals that either (1,0) or (0,1) must be a maximal point.

But the same argument goes through for any two adjacent variables in the function, so that some *n*-tuple of 0s and 1s (with nver more than two consecutive 0s and 1s) must be a maximal point for $f_{p,p,p}(x_1, \ldots, x_n)$. In fact, by starting with $x_1 = 1$ and alternating 0s and 1s, we get the desired maximum at a point where equality in (1) holds. Math. Mag., 65(1992) 266, 272–273.

Q 794. Submitted by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

The general problem of Appolonius [[sic]] is to draw a circle tangent to three given circles. Special cases ensue when all or some of the circles are replaced by points or lines. Solve the problem in the case of two points O, Q and a circle C, where O is the center of C and Q is an interior point of C.

A 794. The center(s) of the desired tangent circle(s) is (are) the midpoint(s) of the hypotenuse(s) of the right triangle(s) drawn as in the figure.



[[Note: only one solutions was given. The above has been written with a plural alternative, and the diagram includes the second solution. - R.]]

Math. Mag., 65(1992) 349, 354.

Q 797. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and dedicated to the late Joseph Konhauser, Macalester College, St. Paul, Minnesota

Determine the maximum and minimum values of

$$S = \sin^2 x + \sin^2 y + 2k \sin x \sin y$$

where $x, y \ge 0, x + y = \alpha$ and α, k are given constants with $0 \le \alpha \le \pi$.

A 797. We rewrite S in the form

$$S = \sin^2 x + \sin^2 y + 2\cos\alpha \sin x \sin y + 2(k - \cos\alpha) \sin x \sin y$$

and it then follows easily that

$$S = \sin^2 \alpha + 2(k - \cos \alpha) \sin x \sin y$$

= $\sin^2 \alpha + (k - \cos \alpha) (\cos(x - y) - \cos(x + y))$

Case 1. $k \ge \cos \alpha$.

$$\max S = \sin^2 \alpha + (k - \cos \alpha)(1 - \cos \alpha) = (1 + k)(1 - \cos \alpha)$$
$$\min S = \sin^2 \alpha$$

Case 2. $k < \cos \alpha$.

$$\min S = (1+k)(1-\cos\alpha)$$
$$\max S = \sin^2 \alpha$$

This problem was suggested by the following Quickie problem by the late Joe Konhauser (private communication).

$$B$$
 c A b c

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From a point A on the circular arc (radius R) of a sector of a nonobtuse angle α , drop perpendiculars to the sides. If the two feet are B and C, determine the extreme values of BC. Since OBAC is cyclic, the circumradius of ABC is the same as that of OAB, which is OA/2 since $\angle OBA$ is a right angle. Then since the product of the sides of triangle $BAC = 4 \times$ its circumradius \times its area,

$$bc \cdot BC = 4(R/2)(bc\sin(\pi - \alpha))/2$$

or $BC = R \sin \alpha = \text{constant}$.

Math. Mag., 66(1993) 65.

Q791. Murray Klamkin, University of Alberta, Canada, points out that this problem, generalizations, and the continuous analogues, have appeared in the following notes

1. Murray S. Klamkin, A probability of more heads, this MAGAZINE, 44(1971) 146-149.

2. Murray S. Klamkin, Symmetry in probability distributions, this MAGAZINE, 61(1988) 193-194.

[[**Q791** was:

Math. Mag., 65(1992) 195, 200.

Q791. Proposed by Barry Cipra, Northfield, Minnesota

Suppose you have n coins and your opponent has n + 1. You each toss allyour coins and count the number of heads. You lose if you have fewer heads, otherwise you win (i.e., you win all ties). Assuming that the coins are fair, is this game fair?

A791. For each combination of coin tosses C, let C' be the combination produced by reversing every coin. It is clear that this association yields an involution on the set of all combinations of coin tosses. But it is also easy to see that it reverses the win-loss outcome of any combination. Therefore the number of winning combinations equals thenumber of losing combinations, so the game is fair.]]

Math. Mag., 65(1992) 127, 132.

Q 802. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Prove that if one altitude intersects two other altitudes of a tetrahedron, then all four altitudes of the tetrahedron are concurrent.

[[I think that this occurs somewhere else as well. — with hindsight, it must E2226 in the MONTHLY: see below — R.]]

A 802. Let the vertices of the tetrahedron be A_0 , A_1 , A_2 , A_3 and assume the altitude from A_0 intersects the altitudes from A_1 and A_2 . We now choose a vector origin to be the point P that is the intersection of the altitude from A_0 with the altitude from A_1 . The vector \mathbf{A}_i 1 = 1, 2, 3, 4 will denote the vector from P to A_i . Since A_0 is orthogonal to the face opposie A_0 , we immediately have

$$\mathbf{A}_0 \cdot \mathbf{A}_1 = \mathbf{A}_0 \cdot \mathbf{A}_2 = \mathbf{A}_0 \cdot \mathbf{A}_3 \tag{1}$$

and similarly

$$\mathbf{A}_1 \cdot \mathbf{A}_0 = \mathbf{A}_1 \cdot \mathbf{A}_2 = \mathbf{A}_1 \cdot \mathbf{A}_3 \tag{2}$$

The intersection of the altitudes from A_0 and A_2 is $k\mathbf{A}_0$ for some k. Then as above, we also have

$$(\mathbf{A}_2 - k\mathbf{A}_0) \cdot \mathbf{A}_0 = (\mathbf{A}_2 - k\mathbf{A}_0) \cdot \mathbf{A}_1 = (\mathbf{A}_2 - k\mathbf{A}_0) \cdot \mathbf{A}_3$$

Then using (1) and (2), we have that all the $\mathbf{A}_i \cdot \mathbf{A}_j$ $(i \neq j)$ are equal. Hence P is the orthocenter of the tetrahedron.

Proceeding in the same way, it follows that if one altitude of an *n*-dimensional simplex intersects n-1 other altitudes, all n+1 altitudes are concurrent.

This problem first appeared as Problem E2226 in the *Amer. Math. Monthly.* This solution is much simpler than the one published there (February 1971, Vol.78, No.2, p.200).

Math. Mag., 66(1993) 193, 201.

Q 805. Submitted by Murray S. Klamkin and A. H. Rhemtulla, University of Alberta, Edmonton, Alberta, Canada

Let S be a semigroup.

(i) Given that $x^r y^r = y^r x^r$ for all $x, y \in S$ and for r = 2, 3, ..., must S be commutative?

(ii) Given that $x^2 = x^3 = x^4 = \cdots$ for all $x \in S$, must S be commutative?

A 805. Let S be the multiplicative semigroup with generators

$$A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} C = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with a, b, c nonzero real numbers. Then for any matrix D in S, $D^r = 0$ for r = 2, 3, ...However

$$AB = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = BA.$$

Math. Mag., 66(1993) 266.

1428. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the remainder when $(x^2 - 1)(x^3 - 1) \cdots (x^{16} - 1)(x^{18} - 1)$ is divided by $1 + x + x^2 + \cdots + x^{16}$

Math. Mag., 67(1994) 305.

Solution by F. J. Flanigan, San Jose State University, San Jose, California.

The remainder is 17. More generally, for p an odd prime, the remainder when $F_p(x) = (x^2 - 1)(x^3 - 1)\cdots(x^{p-1} - 1)(x^{p+1} - 1)$ is divided by $\Phi(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is the constant p.

To see this, write $F_p(x) = Q(x)\Phi_p(x) + R(x)$ where R(x) is a polynomial of degree at most p-2. Let ζ be a root of $\Phi_p(x)$, that is, a primitive p th root of unity. Clearly $F_p(\zeta) = R(\zeta)$. But

$$F_p(\zeta) = (\zeta^2 - 1) \cdots (\zeta^{p-1})(\zeta^{p+1} - 1) = (1 - \zeta)(1 - \zeta^2) \cdots (1 - \zeta^{p-1})$$

= $\Phi_p(1)$

where we have used $\zeta^{p+1} = \zeta$ and the standard factorization $\Phi(x) = (x - \zeta)(x - \zeta^2) \cdots (x - \zeta^{p-1})$, as well as the fact that p-1 is even (to reverse each factor).

From this we learn that $R(\zeta) = F_p(\zeta) = \Phi_p(1) = 1 + 1 + \dots + 1 = p$, the given prime. But this is true for each of the p-1 primitive p th roots of unity. Since the polynomial R(x) has degree no larger than p-2, $R(x) \equiv p$ for all x, proving the claim.

Note: For a positive integer n, recall that a complete positive reduced residue system modulo n is a set $\kappa = \{k_1, k_2, \ldots, k_{\phi(n)}\}$ of positive integers, no two of which are congruent modulo n and each of which is coprime to n. For n and κ as above, define $E_{\kappa}(x) = \prod_{k \in \kappa} (x^k - 1)$. Then, for $n \geq 3$, the remainder when $E_{\kappa}(x)$ is divided by then th cyclotomic polynomial $\Phi_n(x)$ is the integer $\Phi_n(1)$. The proof given above, mutatis mutandis, works here as well. Math. Mag., 66(1993) 267, 272.

Q 809. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

If the four altitudes of a tetrahedron are concurrent, prove that the six midpoints of the edges of the tetrahedron are cospherical.

A 809. Let **A**, **B**, **C**, **D** denote vectors from the orthocenter to the vertices A, B, C, D, respectively, of the tetrahedron. Because **A** is orthogonal to **B** - **C** and **C** - **D**, etc., it follows that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{D} = \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{D} = \mathbf{C} \cdot \mathbf{D} = \lambda$$

We now show that the six midpoints of the edges $(\mathbf{A} + \mathbf{B})/2$, $(\mathbf{A} + \mathbf{C})/2$, etc., are all equidistant from the centroid $(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})/4$. All we need to show is that 16 times the square of one of the six distances, say

$$16|(\mathbf{A} + \mathbf{B})/2 - (\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D})/4|^2 = |\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D}|^2, \dots$$

is symmetric with respect to A, B, C, D. Expanding out and using the above identities, we get

$$|\mathbf{A}|^{2} + |\mathbf{B}|^{2} + |\mathbf{C}|^{2} + |\mathbf{D}|^{2} - 4\lambda.$$

Math. Mag., 66(1993) 339, 345.

Q 812. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the maximum value of

$$(x+y+z)\left(\sqrt{a^2-x^2}+\sqrt{b^2-y^2}+\sqrt{c^2-z^2}\right)$$

A 812. The problem can be done in several ways using the multivariate calculus. Here is a generalization by elementary means.

We determine the maximum value of

$$P = \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} \sqrt{a_i^2 - x_i^2}$$

Let $x_i = a_i \sin \theta_i$, $-\pi/2 \le \theta_i \le \pi/2$, $a_i > 0$, so that

$$P = \sum_{i=1}^{n} a_i \sin \theta_i \cdot \sum_{i=1}^{n} a_i \cos \theta_i$$
$$= \frac{1}{2} \left(\sum_{i=1}^{n} a_i^2 \sin 2\theta_i + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i a_j \sin(\theta_1 + \theta_j) \right)$$

Clearly the maximum is taken on when all $\theta_i = \pi/4$ so that

$$\max P = \frac{1}{2} \left(\sum_{i=1}^{n} a_i \right)^2$$

Math. Mag., 66(1993) 67, 74.

Q 815. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

ABCD is a convex quadrilateral inscribed in the base of a right circular cone of vertex P. Show that for the pyramid PABCD, the sum of the dihedral angles with edges PA and PC equals the sum of the dihedral angles with edges PB and PD.

A 815. Equivalently, we want to show that for a cyclic spherical quadrilateral, the sum of onepair of opposite angles equals the sum of the other pair of opposite angles. Let O be the pole of the circumcircle of ABCD. Since triangles AOB, BOC, COD and DOA are isosceles, the result follows. Note that it does not matter if O lies in the interior of ABCD or not.

Math. Mag., 67(1994) 146.

1446. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Canada Determine the least number of times the graph of

$$y = \frac{a^2}{x^2 - 1} + \frac{b^2}{x^2 - 4} + \frac{c^2}{x^2 - 9} - 1$$

intersects the x-axis (a, b, c are nonzero real constants).

Math. Mag., 68(1995) 151.

Solution by Jerrold W.Grossman, Oakland University, Rochester, Michigan. The graph always intersects the x-axis six times. By symmetry, it suffices to consider $x \ge 0$. It is clear that y < 0 when x = 0, that $\lim_{x\to n^-} y = -\infty$ for n = 1, 2, 3, that $\lim_{x\to n^+} y = \infty$ for n = 1, 2, 3, and that $\lim_{x\to\infty} y = -1$. Furthermore

$$\frac{dy}{dx} = -2x\left(\frac{a^2}{(x^2-1)^2} + \frac{b^2}{(x^2-4)^2} + \frac{c^2}{(x^2-9)^2}\right)$$

so y is decreasing on (0,1), (1,2), (2,3) and $(3,\infty)$. Since y is also continuous on these intervals, it follows from all these statements that there is precisely one x-intercept in each of (1,2), (2,3) and $(3,\infty)$ and no x-intercept in (0,1).

Math. Mag., **67**(1994) 225.

1453^{*}. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Let $E^{n+1} = x^{E_n}$ where $E_0 = 1, n = 0, 1, ...,$ and $x \ge 0$ $(E_1 = x, E_2 = x^x, \text{ etc.})$ It is easy to show that for $x \ge 1$, E_n (n > 1) is a strictly increasing convex function.

[[Not true for x = 1? – Later: the analysis below sets this straight – R.]]

Prove or disprove each of the following.

(i) E_{2n} is a unimodal convex function for n > 1 and all $x \ge 0$.

(ii) E_{2n+1} is an increasing function for $x \ge 0$, and is concave in a small enough interval $[0, \epsilon(n)]$.

Math. Mag., 68(1995) 230–231.

Composite analysis by Richard Holzsager, The American University, Washington, DC, and George Gilbert, Texas Christian University, Fort Worth, Texas. Neither statement gives the correct concavity. For instance, one can set up a recursion and let the computer calculate that $E'_{10}(0.02) \approx -15.5$. Also, a straightforward calculation shows that $\lim_{x\to 0^+} E''_3(x) = \infty$, so that E_3 cannot be concave in any interval containing 0.

In fact, the function E_n is convex in some interval of the form $[0, \epsilon_n)$ for n > 1. To see this, call a function f "small of order r" near 0 if, for any $\delta > 0$, $f(x)/x^{r-\delta} \to 0$ as $x \to 0^+$. Note that $x^r \ln^k x$ is small of order r for all k. Let κ_r stand for any function that is small of order r. The convexity follows once we establish that, for $n \ge 1$

$$E_{2n} = 1 + x \ln x + \kappa_2 \qquad E'_{2n} = \ln x + 1 + \kappa_1 \qquad E_{2n} = 1/x + \kappa_0 \tag{1}$$

 $E_{2n+1} = x + x^2 \ln^2 x + \kappa_3 E'_{2n+1} = 1 + 2x(\ln^2 x + \ln x) + \kappa_2 E''_{2n+1} = 2(\ln^2 x + 3\ln x + 1) + \kappa_1 \quad (2)$ From $E_{n+1} = x^{E_n}$ we find that

$$E'_{n+1} = E_{n+1} \left(\frac{E_n}{x} + E'_n \ln x\right)$$

and

$$E_{n+1}'' = E_{n+1} \left(\frac{E_n^2}{x^2} + \frac{2E_n E_n' \ln x}{x} + E_n'^2 \ln^2 x - \frac{E_n}{x^2} + E_n'' \ln x \right)$$

It is easy to show that these formulas hold for n = 1. Furthermore, one finds that $(1)_n$ implies $(2)_n$ and $(2)_n$ implies $(1)_{n+1}$ so, by induction, (1) and (2) hold for all $n \ge 1$. This completes the proof of our claim.

Finally, we show that E_{2n+1} is increasing for $x \ge 0$. We need only consider 0 < x < 1. For such x, a < b implies $x^b < x^a$ and applying this repeatedly we see that, for 0 < x < 1

$$x = E_1 < E_3 < \dots < E_{2n+1} < \dots < E_{2n} < \dots < E_2 < E_0 = 1$$

Repeated application of the recurrence $E'_n = E_n(E_{n-1}/x + E'_{n-1}\ln x)$ yields

$$E'_{n} = \frac{E_{n}E_{n-1}}{x} (1 + E_{n-2}\ln x + E_{n-2}E_{n-3}\ln^{2} x + \dots + E_{n-2}E_{n-3}\dots E_{0}\ln^{n-1} x)$$

For a given x in (0,1)

$$E'_{2n+1} = \frac{E_{2n+1}E_{2n}}{x} \left[(1 + E_{2n-1}\ln x) + E_{2n-1}E_{2n-2}\ln^2 x (1 + E_{2n-3}\ln x) + \cdots + E_{2n-1}\cdots E_2\ln^{2n-2} x (1 + E_1\ln x) + E_{2n-1}\cdots E_0\ln^{2n} x \right]$$

is positive unless $1 + E_{2j+1} \ln x < 0$ for some j. However, in this case $1 + E_{2k} \ln x < 0$ for all k, hence, rearranging

$$E'_{2n+1} = \frac{E_{2n+1}E_{2n}}{x} \left[1 + E_{2n-1}\ln x(1 + E_{2n-2}\ln x) + \dots + E_{2n-1}\dots E_1\ln^{2n-1}x(1 + E_0\ln x)\right]$$

we see that each summand is positive, proving that E_{2n+1} is increasing.

The second step of iterating the recurrence yields

$$E'_{2n+2} = E_{2n+2}E_{2n+1}\left[\frac{1+E_{2n}\ln x}{x} + E'_{2n}\ln^2 x\right]$$

Note that $x^x > 1/e$ and that $x^{x^{1/e}}$ decreases on $(0, e^{-e})$. If $E_{2n} > 1/e$ on $(0, e^{-e})$, then also

$$E_{2n+2} > x^{x^{E_{2n}}} > x^{x^{1/e}} > (e^{-e})^{(e_{-e})^{1/e}} = \frac{1}{e}$$

[[shd that first '>' be an '=' ? - R.]]

It follows that $1 + E_{2n} \ln x < 0$ on $(0, e^{-e})$, hence that $E'_{2n} < 0$ on $(0, e^{-e})$ for all $n \ge 1$. If E_{2n} were convex for $x \ge e^{-e}$, unimodality would be established. One final observation: Because $1 + E_{2n} \ln x$ increases with n, if $E'_{2n}(x) > 0$ for a given x, then $E'_{2n+2k}(x) > 0$ for all integers k > 0.

Math. Mag., 67(1994) 305, 310.

Q 824. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine all positive rational solutions of $x^x = y^y$ with $x \ge y > 0$.

A 824. Obviously, one set of solutions is x = y. To obtain all the others, we let x = 1/u and y = 1/v to give $u^v = v^u$ where v > u. It is known that all the solutions of the latter equation are given by

$$v = (1 + 1/m)^{m+1}$$
 $u = (1 + 1/m)^m$ $m = 1, 2, ...$

(See W. Sierpiński, *Elementary Theory of Numbers*, Hafner, New York, 1964, 106–107.) Hence

$$x = (1 + 1/m)^{-m}$$
 $y - (1 + 1/m)^{-1-m}$

Math. Mag., 67(1994) 385, 390.

Q 826. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Which of the two integrals

$$\int_0^1 \left(\sqrt{1 + x^{2r} \tan^2 \alpha} - x^r \sec \alpha\right)^{1/s} \, dx \qquad \int_0^1 \left(\sqrt{1 + x^{2s} \tan^2 \alpha} - x^s \sec \alpha\right)^{1/r} \, dx$$

is larger, given that r > s > 0 and $\pi/2 > \alpha > 0$?

A 826. The first integral equals the area in the first quadrant bounded by the x and y axes and the curve

$$(y^s + x^r \sec \alpha)^2 = 1 + x^{2r} \tan^2 \alpha$$

which simplifies to

$$y^{2s} + 2y^s x^r \sec \alpha + x^{2r} = 1$$

Since the mirror image of the latter curve across the x = y line is

$$y^{2r} + 2y^r x^s \sec \alpha + x^{2s} = 1$$

the two integrals are equal in value.

Math. Mag., 67(1994) 69, 74.

Q 830. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine $\int_0^1 (1-x^m)^n dx \div \int_0^1 (1-x^m)^{n-1} dx \ m, n > 0$ without using beta function integrals.

A 830. Integrating by parts,

$$F_n \equiv \int_0^1 (1 - x^m)^n \, dx = mn \int_0^1 x^m (1 - x^m)^{n-1} \, dx$$
$$= mn \int_0^1 \left((x^m - 1)(1 - x^m)^{n-1} + (1 - x^m)^{n-1} \right) \, dx$$

or, $F_n(1+mn) = mnF_{n-1}$. Hence

$$\frac{F_n}{F_{n-1}} = \frac{mn}{1+mn}$$

Math. Mag., 68(1995) 228–229.

Concurrent Lines in a Triangle

1452. Proposed by John Frohliger and Adam Zeuske (student), St. Norbert College, DePere, Wisconsin

Let ABC be a given triangle and θ an angle between -90° and 90° . Let A', B', C' be points on the perpendicular bisectors of BC, CA, AB, respectively, so that $\angle BCA'$, $\angle CAB'$, $\angle ABC'$ all have measure θ . Prove that the lines AA', BB', CC' are concurrent, provided that points A', B', C' are not equal to A, B, C, respectively.

II. Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. Since $BA' = CA' = \frac{1}{2} \sec \theta$, it follows that the areal coordinates of A' are

$$\frac{1}{4} \left(a^2 \tan \theta, ab \sec \theta \sin(C - \theta), ac \sec \theta \sin(B - \theta) \right)$$

If **V** denotes a vector from a given origin to a point V, then the vector representation of A' is given by

$$\mathbf{A}' = \frac{1}{4} (\mathbf{A}a^2 \tan \theta + \mathbf{B}ab \sec \theta \sin(C - \theta) + \mathbf{C}ac \sec \theta \sin(B - \theta))$$

It now follows that the line AA' intersects BC in a point A'' such that

$$\frac{BA''}{A''C} = \frac{c\sin(B-\theta)}{b\sin(C-\theta)}$$

and similarly for the other two lines (by cyclic interchange). Then since

$$\frac{c\sin(B-\theta)}{b\sin(C-\theta)} \frac{a\sin(C-\theta)}{c\sin(A-\theta)} \frac{b\sin(A-\theta)}{a\sin(B-\theta)} = 1$$

it follows by Ceva's theorem that the lines AA', BB', CC' are concurrent, provided that the points A', B', C' are not equal to A, B, C, respectively.

The two angles of θ that must be excluded are -90° and 90° .

Math. Mag., **69**(1996) 74.

Comments

S1452. Concurrent Lines in a Triangle

Peter Yff writes that "the locus of the point of concurrence is a conic known as Kiepert's hyperbola. This is a rectangular hyperbola passing through A, B, C, the centroid, the orthocenter, the Spieker center, the isogonic centers and the Napoleon points." He refers to page 223 of R. A. Johnson's Advanced Euclidean Geometry.

June 1994

June 1995.

Math. Mag., 68(1995) 307.

1481. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

It is known that if a point moves on a straight line with constant acceleration and s_1 , s_2 , s_3 are its positions at times t_1 , t_2 , t_3 respectively, then the constant acceleration is given by

$$2\left(\frac{(s_2-s_3)t_1+(s_3-s_1)t_2+(s_1-s_2)t_3}{(t_2-t_3)(t_3-t_1)(t_1-t_2)}\right)$$

Show that this property characterizes uniformly accelerated motion; that is, if a particle moves on a straight line and s_1 , s_2 , s_3 are its positions at any times t_1 , t_2 , t_3 respectively, then if

$$\frac{(s_2 - s_3)t_1 + (s_3 - s_1)t_2 + (s_1 - s_2)t_3}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)} = \text{ constant}$$

the motion is one of constant acceleration.

Math. Mag., **69**(1996) 308.

Solution by Victor Kutsenok, St. Francis College, Fort Wayne, Indiana. Fix $t_2 \neq t_3$. Then

$$\frac{(s_2 - s_3)t + (s_3 - s)t_2 + (s - s_2)t_3}{(t_2 - t_3)(t_3 - t)(t - t_2)} = \frac{a}{2}$$

for some real number a and $t \neq t_2, t_3$, where s is the position corresponding to time t. Then $(s_2 - s_3)t + (s_3 - s)t_2 + (s - s_2)t_3 = (a/2)(t_2 - t_3)(t_3 - t)(t - t_2)$ for all t. Solving for s yields a quadratic for in t, so the given motion is one of constant acceleration with s'' = a.

Math. Mag., 67(1994) 308, 318.

Q 840. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

If $A_1A_2A_3A_4$ is a plane cyclic quadrilateral and H_i is the orthocenter of triangle $A_{i+1}A_{i+2}A_{i+3}$ where $A_{i+4} = A_i$ for i = 1, 2, 3, 4, prove that

(i) $\operatorname{Area}(A_1A_2A_3A_4) = \operatorname{Area}(H_1H_2H_3H_4)$ and

(ii) the lines A_1H_1 , A_2H_2 , A_3H_3 , A_4H_4 are concurrent.

A 840. A vector proof is particularly apt here since if \mathbf{A}_i is a vector from the circumcenter of $A_1A_2A_3A_4$ to the vertex A_i , the orthocenter of $A_{i+1}A_{i+2}A_{i+3}$ is given simply by $\mathbf{H}_i = \mathbf{S} - \mathbf{A}_i$ i = 1, 2, 3, 4, where $\mathbf{S} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4$ (note for example that $(\mathbf{H}_1 - \mathbf{A}_2) \cdot (\mathbf{A}_3 - \mathbf{A}_4) = \mathbf{A}_3^2 - \mathbf{A}_4^2 = 0$). It follows that $H_1H_2H_3H_4$ is congruent to $A_1A_2A_3A_4$, which establishes (i).

For (ii), the lines $H_i A_i$ are given by $\mathbf{A}_i + \lambda_i (\mathbf{H} - \mathbf{A}_i)$, where λ_i are scalar parameters. Letting $\lambda_i = 1/2$, the four lines are concurrent at the point given by $\mathbf{S}/2$. Math. Mag., 68(1995) 310–312.

Recurrence in which positivity implies uniqueness

October 1994

1456. Proposed by Howard Morris, Chatsworth, California

Show that the only sequence of numbers (α_i) that satisfies the conditions

(i)
$$\alpha_i > 0$$
 for all $i \ge 1$, and

(ii)
$$\alpha_{i-1} = \frac{i\alpha_i+1}{\alpha_i+i}$$
 for all $i > 0$.

III. Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. If we let $\alpha_i = x_n + 1$, (ii) becomes

$$\frac{n+1}{x_n} - \frac{n-1}{x_{n-1}} = -1$$

Then, with $x_n = n(n+1)F_n$ we get the telescoping difference equation

$$\frac{1}{F_n} - \frac{1}{F_{n-1}} = -n$$

Hence,

$$F_n = \frac{2F_1}{2 - (n+2)(n-1)F_1}$$

and thus

$$\alpha_n = \frac{2 + (n^2 + n + 2)F_1}{2 - (n^2 + n - 2)F_1}$$

Since $2F_1 = \alpha_1 - 1$, (i) implies that $F_1 > -1/2$. Since α_n is positive for n = 2, 3, ..., it must be the case that $F_1 = 0$, or equivalently, $\alpha_n = 1$ for n = 1, 2, ... Finally, from (ii) with i = 1, we get $\alpha_0 = 1$.

Math. Mag., 68(1995) 400, 406.

Q 841. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and Stanley Rabinowitz, MathPro Press, Westford, Massachusetts

Prove that the sequence $u_n = 1/n$, n = 1, 2, ..., cannot be the solution of a nonhomogeneous linear finite-order difference equation with constant coefficients.

[[N.B. On p.391 of *Math. Mag.*, **69**(1996) there is a correction from "nonhomogeneous" to "homogeneous" ! — R.]]

A 841. Assume to the contrary that it is possible. Then there exist constants a_i , not all zero, such that

$$a_0 + \frac{a_1}{n} + \frac{a_2}{n+1} + \dots + \frac{a_r}{n+r-1} = 0$$
 (1)

for n = 1, 2, ... It then follows that the left-hand side of (1), which is a rational function of *n*,must identically vanish for all *n*. Letting $n \to 0$, it follows that $a_1 = 0$. Then letting $n \to -1$, it follows that $a_2 = 0$, and similarly, all the a_i are zero, and this is a contradiction.

In a similar way, it follows that no strictly rational function can be the solution of a linear finite-order difference equation with constant coefficients.

Math. Mag., 67(1994) 400, 406.

Q 843. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Evaluate the following two $n \times n$ determinants:

(i) $D_1 = \det(a_{rs})_{r,s=1,\ldots,n}$ where $a_{rr} = a_r, r = 2, \ldots, n$ and all the remaining elements are 1.

(ii) $D_2 = \det(b_{rs})_{r,s=1,\ldots,n}$ where $b_{rr} = b_r$, $r = 1, 2, \ldots, n$ and all remaining elements are 1.

A 843. (i) On setting any $a_r = 1$, D_1 vanishes. Hence $(a_2 - 1)(a_3 - 1)\cdots(a_n - 1)$ is a factor of D_1 . The remaining factor can only be the constant 1 since D is a polynomial in the a_i with leading term $a_2a_3\cdots a_n$.

(ii) If $b_r = 1$ for some r, the determinant reduces to the case considered in (i), so we will assume that none of the b_r is equal to 1. Replace b_r by $x/x_r + 1$ so that

$$x_1 x_2 \cdots x_n D_2 = D'_2 = \det(c_{rs})$$
 where $c_{rr} = x + x_r$ and $c_{rs} = x_r$ for $r \neq s$

By setting x = 0 in D'_2 we get *n* rows that are proportional. Hence x^{n-1} is a factor of D'_2 . The other factor must be linear in *x* having the form $x + \lambda$ since the the coefficient of x^n in D'_2 must be 1. It is clear that $\lambda = \sum x_1$ since the coefficient of x^{n-1} can only come in from the main diagonal. Finally

$$D'_2 = \left(x + \sum_{i=1}^n\right) x^{n-1}$$
 and $D_2 = \left(1 + \sum_{i=1}^n \frac{1}{b_i - 1}\right) \prod_{i=1}^n (b_1 - 1)$

Alternatively, one can split the first row as $(1, 1, ..., 1) + (b_1 - 1, 0, ..., 0)$ and then use the linearity of the determinant in a row, part (i), and induction.

Math. Mag., 69(1996) 143.

1496. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Find a solution to the differential equation $d^2y/dx^2 = -kx/y^4$, k > 0, other than one of the form $y = ax^{3/5}$.

Math. Mag., 69(1996) 145-147.

I. Solution by Hongwei Chen, Christopher Newport University, Newport News, Virginia. The given differential equation is a special case of the Emden-Fowler equation $d^2y/dx^2 = Ax^ny^m$. All possible solvable cases are given in A. D. Polyanin & V. F. Zaitsev, Handbook of Exact Solutions for Ordinary Differential Equations, CRC Press, 1995, 241–250.

We claim that the general solution to the differential equation is given in the parametric form

$$x = \left(C_2 \pm \int \left(\frac{2k}{3}t^{-3} + C_1\right)^{-1/2} dt\right)^{-1}$$
$$y = t \left(C_2 \pm \int \left(\frac{2k}{3}t^{-3} + C_1\right)^{-1/2} dt\right)^{-1}$$

where t is a parameter, C_1 and C_2 are arbitrary constants.

The transformation x = 1/s, y = t(s)/s changes the equation into

$$\frac{d^2t}{ds^2} = \frac{1}{s^3}\frac{d^2y}{dx^2} = -kt^{-4}$$

By using the substitution w(t) = dt/ds this equation is reduced to the first order equation

$$\frac{dw}{dt} = \frac{dw/ds}{dt/ds} = \frac{d^2t/ds^2}{w} = -\frac{kt^{-4}}{w}$$

Integrate to obtain

$$w^2 = \frac{2k}{3}t^{-3} + C_1$$

where C_1 is a constant. Thus,

$$\frac{dt}{ds} = \pm \left(\frac{2k}{3}t^{-3}\right)^{1/2}$$

so that

$$\pm \int \left(\frac{2k}{3}t^{-3}\right)^{-1/2} dt = \int ds$$

and therefore

$$s = C_2 \pm \int \left(\frac{2k}{3}t^{-3}\right)^{-1/2} dt$$

where C_2 is an additional constant. Hence, the general solution of the original equation is given by

$$x = \left(C_2 \pm \int \left(\frac{2k}{3}t^{-3} + C_1\right)^{-1/2} dt\right)^{-1}$$
$$y = t \left(C_2 \pm \int \left(\frac{2k}{3}t^{-3} + C_1\right)^{-1/2} dt\right)^{-1}$$

Setting $C_1 = 0$ leads to

$$x = \left(C_2 \pm \sqrt{\frac{6}{25k}} t^{5/2}\right)^{-1}$$

so that

$$t = \left(C + \sqrt{\frac{25k}{6}} \, x^{-1}\right)^{2/5}$$

and

$$y = x \left(C + \sqrt{\frac{25k}{6}} x^{-1} \right)^{2/5}$$

II. Solution by the proposer. Setting y = xt(x) we get

$$x^4 \frac{d^2t}{dx^2} + 2x^3 \frac{dt}{dx} = -\frac{k}{t^4}$$

Multiplying by the integrating factor 2 dt/dx we get

$$\frac{d}{dx}\left(x^4\left(\frac{dt}{dx}\right)^2\right) = \frac{d}{dx}\frac{2k}{3t^3}$$

Integrating and taking square roots yields

$$\frac{dt}{dx} = \pm \frac{\sqrt{\frac{2k}{3t^3} + C_1}}{x^2}$$

As in the first solution above, separation of variables leads to the parametric solution, and setting $C_1 = 0$ allows us to perform the integral to obtain the analytic solution. Math. Mag., 68(1995) 225, 230.

Q 851. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Prove that among all parallelepipeds of given edge lengths, the rectangular one has the greatest sum of the lengths of the four body diagonals.

A 851. Let a, b and c be the given edge lengths. Let vectors A, B and C denote three corresponding vectors along three coterminal edges of the parallelepiped. We want to maximize

$$S = |A + B + C| + |A + B - C| + |A - B + C| + |-A + B + C|$$

We may write

$$\mathbf{B} \cdot \mathbf{C} = bc \cos \alpha$$
 $\mathbf{C} \cdot \mathbf{A} = ca \cos \beta$ $\mathbf{A} \cdot \mathbf{B} = ab \cos \gamma$

where α , β and γ denote the angles between the pairs of vectors. Hence

$$|\pm \mathbf{A} \pm \mathbf{B} \pm \mathbf{C}| = (a^2 + b^2 + c^2 \pm 2bc\cos\alpha \pm 2ca\cos\beta \pm 2ab\cos\gamma)^{1/2}$$

where the appropriate \pm signs are chosen. Since \sqrt{x} is concave for $x \ge 0$

$$S \le 4(a^2 + b^2 + c^2)^{1/2}$$

with equality if and only if the four body diagonals are equal, or, equivalently, if the parallelepiped is rectangular.

Math. Mag., **69**(1996) 304.

1505. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and Cecil C. Rousseau, The University of Memphis, Memphis, Tennessee

Let a and b be positive numbers satisfying $a + b \ge (a - b)^2$. Prove that

$$x^{a}(1-x)^{b} + x^{b}(1-x)^{a} \le \frac{1}{2^{a+b-1}}$$

for $0 \le x \le 1$ with equality if and only if x = 1/2.

Math. Mag., **70**(1997) 301–302.

I. Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana. With the change of variables $x = \frac{1}{2} - \frac{1}{2}y$ the given inequality becomes

$$(1-y)^a (1+y)^b + (1-y)^b (1+y)^a \le 2$$
(1)

. for $|y| \leq 1$. Let g(y) be the left-hand side of (1).

If a = b then (1) becomes $(1 - y^2)^a \leq 1$, which is clearly true for $|y| \leq 1$.

We assume in the following that b > a. Note that the maximum of g on the interval [-1, 1] must occur at some interior point of the interval. A routine calculation shows that g'(y) = 0 if and only if

$$\left(\frac{1+y}{1-y}\right)^{b-a} = \frac{1+\frac{b+a}{b-a}y}{1-\frac{b+a}{b-a}y}$$
(2)

Observe that the left-hand side of (2) is always positive on (-1, 1), while the right-hand side is positive only for |y| < (b-a)/(b+a). Using the series representation

$$\ln\left(\frac{1+y}{1-y}\right) = 2\sum_{n=0}^{\infty} \frac{1}{2n+1}y^{2n+1} \qquad |y| < 1$$

to expand each side of (2), we seek |y| < (b-a)/(b+a) such that

$$y\sum_{n=0}^{\infty} \frac{1}{2n+1} \left[\left(\frac{b+a}{b-a} \right)^{2n+1} - (b-a) \right] y^{2n} = 0$$
(3)

From the conditions imposed on a and b, it follows that

$$\left(\frac{b+a}{b-a}\right)^{2n+1} - (b-a) \ge \left(\frac{b+a}{b-a}\right) - (b-a) \ge 0$$

where the first inequality is strict for n > 0. This means [[that]] (3), and hence the equation g'(y) = 0, has the unique solution y = 0 in the interval (-1, 1). It follows easily that $g(y) \le g(0) = 2$ on [-1, 1], with strict inequality for $y \ne 0$.

II. Solution by Joseph G. Gaskin, SUNY College at Oswego, Oswego, New York. Let $f(x) = x^a(1-x)^b + x^b(1-x)^a$ where $0 \le x \le 1$. If a = b, then $f(x) \le f(1/2) = 2(1/4)^a$, with equality if and only if x = 1/2. So, supposing that $a \ne b$, we may assume a < b.

Since f(0) = f(1) = 0 and since f is differentiable and positive on (0,1), it follows that f is maximized at a critical point in (0,1). From

$$f'(x) = x^{a-1}(1-x)^{a-1} \left[(a - (a+b)x)(1-x)^{b-a} + (b - (a+b)x)x^{b-a} \right]$$

we see that f'(x) = 0 for $x \in (0, 1)$ if and only if

$$g(x) = \frac{(a+b)x - a}{b - (a+b)x} \left(\frac{1-x}{x}\right)^{b-a} = 1$$

Note that g(1/2) = 1 and that if g(x) = 1 > 0 on (0,1), then a/(a+b) < x < b/(a+b). After a bit of algebra we find that

$$g'(x) = \frac{b-a}{(b-(a+b)x)^2} \left(\frac{1-x}{x}\right)^{b-a} \frac{\left[(a+b)-(a+b)^2\right](x-x^2)+ab}{x(1-x)}$$

Every factor is clearly positive on (a/(a+b), b/(a+b)) except possibly for

$$[(a+b) - (a+b)^2](x-x^2) + ab$$

This term is clearly positive if $(a + b) - (a + b)^2 \ge 0$. Otherwise, the hypothesis $(a + b) \ge (a - b)^2$ implies

$$\left[(a+b) - (a+b)^2 \right] (x-x^2) + ab \ge \left[(a+b) - (a+b)^2 \right] / 4 + ab \\ \ge \left[(a-b)^2 - (a+b)^2 \right] / 4 + ab = 0$$

with strict inequality for $x \neq 1/2$. We conclude that g is strictly increasing on (a/(a+b), b/(a+b)), thereby proving that the only critical point of f(x) is when x = 1/2. The inequality $f(x) \leq f(1/2) = 2^{1-a-b}$ follows immediately.

Math. Mag., 68(1995) 385, 391.

Q 858. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Show that the Diophantine equation

$$x^{2} + 27y^{2}z^{2}(y+z)^{2} = 4(y^{2} + yz + z^{2})^{3}$$

has an infinite number of integral solutions (x, y, z) with x, y, z relatively prime and $xyz \neq 0$.

A 858. I. Since $4(y^2 + yz + z^2)^3 - 27y^2z^2(y+z)^2 = (y-z)^2(y+2z)^2(z+2y)^2$ the general solution is immediate.

II. Provided by the Editors. The roots of $f(t) = t^3 - (y^2 + yz + z^2)t + yz(y+z)$ arey, z and -y-z. Therefore the discriminant of the cubic satisfies

$$4(y^{2} + yz + z^{2})^{3} - 27y^{2}z^{2}(y+z)^{2} = (y-z)^{2}(y+2z)^{2}(z+2y)^{2}$$

Thus every integral pair (y, z) with $yz(y - z)(y + 2z)(2y + z) \neq 0$ gives rise to two integral solutions (x, y, z) to the given equation with $xyz \neq 0$. In particular, we may take y and z to be distinct, relatively prime, positive integers.

Math. Mag., **70**(1997) 142, 150.

Q 863. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Prove that

$$n\left[\sum_{i=1}^{n} a_i b_i + \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{1/2}\right] \ge 2\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$$

where the a_i and b_i are real. Determine when equality holds.

A 863. Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ and \mathbf{c} denote *n*-dimensional vectors. The given inequality will follow from the more general $|\mathbf{c}|^2(\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}| |\mathbf{b}|) \ge 2(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c})$ by setting $\mathbf{c} = (1, \ldots, 1)$. Let α , β , γ denote the angles between \mathbf{a} and \mathbf{c} , between \mathbf{b} and \mathbf{c} , and between \mathbf{a} and \mathbf{b} , respectively. The generalizy inequality is now equivalent to $|\mathbf{a}| |\mathbf{b}| \cos \gamma + |\mathbf{a}| |\mathbf{b}| \ge 2|\mathbf{a}| |\mathbf{b}| \cos \alpha \cos \beta$, or $\cos \gamma + 1 \ge 2 \cos \alpha \cos \beta$. Since in the trihedral angle $\alpha + \beta \ge \gamma$ and $2\pi - (\alpha + \beta) \ge \gamma$, it suffices to show that

$$1 + \cos(\alpha + \beta) \ge 2\cos\alpha\cos\beta$$
 or $1 \ge \cos(\alpha - \beta)$

Equality holds if and only if $\alpha = \beta$ and either $\alpha + \beta = \gamma$ or $\alpha + \beta = 2\pi - \gamma$. In particular, **a**, **b** and **c** must be linearly dependent if equality holds.

Math. Mag., **70**(1997) 382.

1538. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and George T. Gilbert, Texas Christian University, Fort Wirth, Texas

Find all integr solutions to $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.

Math. Mag., 71(1998) 395-396.

I. Solution by Brian D. Beasley, Presbyterian College, Clinton, South Carolina. We show that the given equation holds for integers x and y if and only if x + y + 1 = 0.

The given equation is true if and only if

$$2(x^{5} + y^{5} + 1) - 5xy(x^{2} + y^{2} + 1) = (x + y + 1)f(x, y) = 0$$

where

$$\begin{aligned} f(x,y) &= & 2x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + 2y^4 - 2x^3 - x^2y - xy^2 - 2y^3 \\ &+ & 2x^2 - xy + 2y^2 - 2x - 2y + 2 \end{aligned}$$

Thus we need only show that $f(x, y) \neq 0$ for all integers x and y. Observe that in any solution of the original equation x and y must have opposite parity. By symmetry, we may assume without loss of generality that x is even and y is odd. Then

$$f(x,y) \equiv 2y^4 - xy^2 - 2y^3 - xy + 2y^2 - 2y + 2 \pmod{4}$$

However, each of the expressions $2y^4 - 2y^3 = 2y^3(y-1)$, $-xy^2 - xy = -xy(y+1)$ and $2y^2 - 2y = 2y(y-1)$ is divisible by 4 for x even, leaving $f(x,y) \equiv 2 \pmod{4}$.

II. Solution by Lenny Jones and students Karen Blount, Dennis Reigle and Beth Stockslager, Shippensburg University, Shippensburg, Pennsylvania. The only solutions are ordered pairs of integers (x, y) with x + y + 1 = 0.

To see this, factor $2(x^5 + y^5 + 1) - 5xy(x^2 + y^2 + 1)$ as (x + y + 1)f(x, y) where

$$f(x,y) = [2x^{3}(x-y-1)] + [x(2y^{2}-y+2)(x-y-1)]$$
$$[2y^{4}-2y^{3}] + [2y^{2}-2y] + 2$$

If y = x, then $f(x, y) = 2x^4 - 6x^3 + 3x^2 - 4x + 2$, which has no integer roots by the rational root theorem. Note that x and y cannot both be negative. By symmetry, it suffices to show that $f(x, y) \neq 0$ for $x \geq y + 1$ with $x \geq 0$. In this case, observe that each of the bracketted terms in f(x, y) is nonnegative, so that f(x, y) > 0.

[[On p.389 of *Math. Mag.*, **70**(1997) we read

Acknowledgements. The editors would like to thank Murray S. Klamkin, Loren C. Larson, Efton Park, Daniel H. Ullman and Peter Yff for their help in reviewing problem proposals over the last two years.]]

[[On p.230 of *Math. Mag.*, **71**(1998) we read, in connexion with

1528. Proposed by Florin S. Pirvănescu, Slatina, Romania

Let M be a point in the interior of convex polygon $A_1A_2...A_n$. If d_k is the distance from M to A_kA_{k+1} $(A_{n+1} = A_1)$, show that

$$(d_1 + d_2)(d_2 + d_3) \cdots (d_n + d_1) \le 2^n \cos^n \frac{\pi}{n} \cdot MA_1 \cdot MA_2 \cdots MA_n$$

and determine when equality holds.

the following

Comment. Murray Klamkin observed that the result follows from the stronger inequality with d_k redefined to be the length of the angle bisector of $\angle A_k M A_{k+1}$, referring to D. S. Mitrinović, J. E. Pecarić & V. Volenec, *Recent Advances in Geometric Inequalities*, p.423.]] Math. Mag., **71**(1998) 316, 322.

Q 883. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Given N rays in \mathbb{R}^n forming a non-degenerate *n*-hedral angle with vertex O and a point P in the interior of this angle, find points on the rays minimizing the volume of the simplex formed by the points and O under the restriction that P is in the hyperplane formed by the points.

(This generalizes Q847 from the April 1996 issue of this MAGAZINE.)

A 883. Choosing the origin to be at O, let \mathbf{v}_i denote the unique vector from O along the *i*th ray such that $\mathbf{v}_1 + \cdots + \mathbf{v}_n = \mathbf{P}$. If the chosen points are $x_i \mathbf{v}_i$ then the restriction implies that $1/x_1 + \cdots + 1/x_n = 1$. The volume of the simplex is

$$x_1 \cdots x_n \det(\mathbf{v}_1 \dots \mathbf{v}_n)/n!$$

The arithmetic-geometric mean inequality implies [[that]] the volume is minimized when

$$x_1 = \dots = x_n = n$$

so that P is the centroid of the (n-1)-simplex formed by the n chosen points.

[[On p.396 of *Math. Mag.*, **71**(1998) we read

Acknowledgements. The editors would like to thank Murray S. Klamkin, Loren C. Larson, Harvey Schmidt and Daniel H. Ullman for their help in reviewing problem proposals over the last year.]]

Math. Mag., **72**(1999) 69–70.

Extrema of Volumes of Truncated Simplexes

February 1998

1456. Proposed by Michael Golomb, Purdue University, West Lafayette, Indiana

Let S be a given n-dimensional simplex with centroid C. A hyperplane through C divides the simplex into two regions, one or both of which are simplexes. Find the extrema of the volumes of those regions which are simplexes.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada. We will show that the maximum volume is $\operatorname{vol}(S)/2$ and the minimum volume is $[n/(n+1)]^n \operatorname{vol}(S)$ where $\operatorname{vol}(S)$ denotes the volume of S.

Let $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$ and \mathbf{C} denote vectors from one vertex to all other vertices of S and C, respectively. Let a hyperplane through C cut these vectors at points given by $x_1\mathbf{V}_1$, $x_2\mathbf{V}_2, \ldots, x_n\mathbf{V}_n$ respectively, where the x_i lie in [0,1]. The, since $\mathbf{C} = \sum_{i=1}^n \mathbf{V}_i/(n+1)$ lies in the hyperplane, thereare nonnegative weights w_1, w_2, \ldots, w_n with sum 1, such that $\sum_{i=1}^n w_i x_i \mathbf{V}_i = \sum_{i=1}^n \mathbf{V}_i/(n+1)$. Since the \mathbf{V}_i are linearly independent, $w_i x_i = 1/(n+1)$, so that $\sum_{i=1}^n 1/x_i = n+1$. Since the volume cut off the simplex is

$$\operatorname{vol}(S) \prod_{i=1}^{n} x_i \ge \operatorname{vol}(S) \left/ \left(\frac{1}{n} \sum_{i=1}^{n} 1/x_i \right)^n = \operatorname{vol}(S) [n/(n+1)]^n \right|$$

so that the minimum volume occurs when $x_i = n/(n+1)$ for all *i*.

To obtain the maximum volume, we let $1/x_i = 1 + y_i$ so that we now want to minimize $\prod_{i=1}^{n} (1+y_i)$ subject to $\sum_{i=1}^{n} y_i = 1$. Expanding the product, we see that

$$\prod_{i=1}^{n} (1+y_i) \ge 1 + \sum_{i=1}^{n} y_i = 2$$

with equality if and only if one y_i is 1 and the rest are 0. This yields a maximum volume of vol (S)/2.

Math. Mag., **73**(2000) 156.

1596. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

From the vertices A_0, A_1, \ldots, A_n of a simplex S, parallel lines are drawn intersecting the hyperplanes containing the opposite faces in the corresponding points B_0, B_1, \ldots, B_n . Determine the ratio of the volume of the simplex determined by B_0, B_1, \ldots, B_n to the volume of S.

Math. Mag., **74**(2001) 157–158.

Solution by L. R. King, Davidson College, Davidson, NC. The ratio for n-dimensional simplexes is n. Without loss of generality we may consider the standard simplex S_0 with one vertex at the origin and the others at e_1, e_2, \ldots, e_n , where e_1, \ldots, e_n denotes the usual basis for \mathbb{R}^n . (The translate $S - A_0$ is the image $L(S_0)$ for some linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Because L has rank n, ratios of volumes are invariant under L.)

Let $B_0 = (s_1, s_2, \ldots, s_n)$ be the intersection point of the line from the origin with the face opposite origin, so $s_1 + s_2 + \cdots + s_n = 1$. We then find $B_k = e_k - \frac{1}{s_k}B_0$ and $B_k - B_0 = e_k - t_k B_0$ where $t_k = \left(1 + \frac{1}{s_k}\right)$. (Note that $s_k \neq 0$ because otherwise the line through e_k and parallel to B_0 would be parallel to the face opposite e_k .) The volume of the simplex with vertices B_0, \ldots, B_n is

$$\frac{1}{n!} |\det(B_1 - B_0, \dots, B_k - B_0, \dots, B_n - B_0)|$$

$$= \frac{1}{n!} |\det(e_1 - t_1 B_0, \dots, e_k - t_k B_0, \dots, e_n - t_n B_0)|$$

$$= \frac{1}{n!} |\det(e_1, \dots, e_n) - t_1 \det(B_0, e_2, \dots, e_n) - t_2 \det(e_1, B_0, \dots, e_n) - \dots - t_n \det(e_1, \dots, e_{n-1}, B_0)|$$

$$= \frac{1}{n!} |1 - (t_1 s_1 + t_2 s_2 + \dots + t_n s_n)|$$

$$= \frac{1}{n!} \left|1 - n - \sum_{k=1}^n s_k\right| = \frac{1}{n!} n$$

Because the standard simplex has volume 1/n! the ratio of the volumes is n.

Comment: Leon Gerber notes that this problem has a long history and that it appeared, with his solution, as a problem in *Amer. Math. Monthly*, **80**(1973) 1145–1146.

Math. Mag., **73**(2000) 403, 410.

Q 905. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Determine the maximum volume of a tetrahedron given the lengths of three of its medians.

A 905. Let ABCD be the tetrahedron, m_A , m_B , m_C and m_D the median lengths, the last three given, and G the centroid. The medians are concurrent and are such that $AG = 3m_A/4$, $BG = 3m_B/4$, $CG = 3m_C/4$ and $DG = 3m_D/4$. Thus the volume of ABCD is four times the volume of GBCD. Furthermore the latter volume will be a maximum when BG, CG and DG are mutually orthogonal. Hence the maximum volume is $4(BG \cdot CG \cdot DG/6) = 9m_Bm_Cm_D/32$.

This easily generalizes to determining the maximum volume of an n-dimensional simplex given the lengths of n of its medians.

Math. Mag., 74(2001) 155, 161.

Q 910. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Given [[a]] positive integer k, it is easy to find two base 10 numbers whose product has more yhan k digits andhas all digits the same. As an example, take 9 and $(10^n - 1)/9$ with n > k. Give examples for which the two numbers have the same number of digits.

A 910. First note that $10^{3+6n} + 1 \equiv 0 \pmod{7}$ and that $(10^{3+6n} - 1)/9$ and $(10^{3+6n} + 1)/7$ have the same number of digits. We then have

$$[(10^{3+6n})/7][7(10^{3+6n})/9] = (10^{6+12n} - 1)/9$$

Math. Mag., 74(2001) 239.

1624. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

An ellipsoid is tangent to each of the six edges of a tetrahedron. Prove that the three segments joining the points of tangency of the opposite edges are concurrent.

Math. Mag., 75(2002) 229–230.

Solution by Michel Bataille, Rouen, France. Under a suitable affine transformation, the ellipsoid becomes a sphere, and concurrency and tangency are preserved. Thus we need only consider the case in which the ellipsoid is a sphere that is tangent at points R, S, T, U, V and W to sides [[edges?]] BC, CA, AB, DA, DB and DC, respectively, of tetrahedron ABCD. Because all segments of tangents from a vertex to the point of tangency on the sphere have the same length, we can set x = AS = AT = AU, y = BT = BR = BV, z = CR = CS = CW and t = DU = DV = DW. Denoting by **M** the vector from the point M, let I be the point determined by

$$m\mathbf{I} = zt(y\mathbf{A} + x\mathbf{B}) + xy(t\mathbf{C} + z\mathbf{D}) = zt(y+x)\mathbf{T} + xy(t+z)\mathbf{W}$$

Because zt(y + x) and xy(t + z) are positive and sum to m, it follows that I lies on segment TW. Similarly,

$$m\mathbf{I} = yz(t\mathbf{A} + x\mathbf{D}) + tx(z\mathbf{B} + y\mathbf{C}) = yz(t+x)\mathbf{U} + tx(z+y)\mathbf{R}$$

and

$$m\mathbf{I} = ty(z\mathbf{A} + x\mathbf{C}) + zx(t\mathbf{B} + y\mathbf{D}) = ty(z + x)\mathbf{S} + zx(t + y)\mathbf{V}$$

showing that I lies on segments UR and SV as well. Thus the three segments joining points of tangency of opposite edges are concurrent at I.

Math. Mag., 74(2001) 240, 246.

Q 911. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Two points P and Q are on opposite sides of a given plane in \mathbb{R}^3 . Describe how to determine a point R in the plane so that |PR - QR| is maximal.

A 911. Let Q' be the reflection across the plane of Q, so QR = Q'R. By the triangle inequality, $|PR - Q'R| \leq PQ'$. The maximal value PQ' is achieved when R is the intersection of line PQ' with the plane. In the event that PQ' is parallel to the plane, the value PQ' is approached as R approaches the point at infinity in the plane that is in the direction of line PQ'.

Math. Mag., 74(2001) 325, 330.

Q 913. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Let F be a function that has a continuous third derivative on [0,1]. If F(0) = F'(0) = F''(0) = F''(0) = F''(1) = F''(1) = 0 and F(1) = 1, prove that $F'''(x) \ge 24$ for some x in [0,1].

A 913. Consider the Taylor series expansions about the points x = 0 and x = 1,

$$F(x) = F(0) + F'(0)x + \frac{F''(0)}{2}x^2 + \frac{F'''(c_1)}{6}x^3$$

$$F(x) = F(1) + F'(1)(x-1) + \frac{F''(1)}{2}(x-1)^2 + \frac{F'''(c_2)}{6}(x-1)^3$$

where $0 \le c_1 \le x$ and $x \le c_2 \le 1$. These reduce to

$$F(x) = \frac{F'''(c_1)}{6}x^3$$
 and $F(x) = 1 + \frac{F'''(c_2)}{6}(x-1)^3$

Setting x = 1/2 we find that there are c_1 and c_2 with $F'''(c_1) + F'''(c_2) = 48$. Thus at least one of $F'''(c_1)$ and $F'''(c_2)$ is greater than or equal to 24.

Math. Mag., 74(2001) 404, 409.

Q 916. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Do there exist any integers k such that there are an infinite number of relatively prime positive integer triples (x, y, z) satisfying the Diophantine equation

$$x^{2}y^{2} = k^{2}(x+y+z)(y+z-x)(z+x-y)(x+y-z)?$$

A 916. In order for the right-hand side of the equation to be positive, x, y, z must be the lengths of the sides of a triangle. If A is the area of the triangle and α is the angle between the sides of lengths x and y, then

$$\frac{4A^2}{\sin^2\alpha} = 16k^2A^2$$

so $\sin \alpha = \frac{1}{2k}$. By the Law of Cosines,

$$z^{2} = x^{2} + y^{2} - 2xy\cos\alpha = x^{2} + y^{2} \pm \frac{xy}{k}\sqrt{4k^{2} - 1}$$

Because the left-hand side is an integer and the right-hand side is irrational, there are no solutions.
Math. Mag., **75**(2002) 63–64.

1641. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Show that if the midpoints of the six edges of a tetrahedron lie on a sphere, then the tetrahedron has an orthocenter.

Math. Mag., 75(2002) ??395–396.

Solution by Daniele Donini, Bertinoro, Italy. Let A_1 , A_2 , A_3 and A_4 be the vertices of the tetrahedron, and let O and r be, respectively, the center and radius of the sphere. Fix an orthogonal coordinate system with origin O. Given a point P, let \mathbf{P} denote the vector from O to P. By hypothesis, $\|\mathbf{A}_i + \mathbf{A}_j\| = r$ for each pair of distinct indices i, j. The condition can be rewritten as

$$4r^{2} = \|\mathbf{A}_{i} + \mathbf{A}_{j}\|^{2} = \mathbf{A}_{i} \cdot \mathbf{A}_{i} + 2\mathbf{A}_{i} \cdot \mathbf{A}_{j} + \mathbf{A}_{j} \cdot \mathbf{A}_{j}$$

or equivalently

$$\mathbf{A}_i \cdot \mathbf{A}_j = 2r^2 - \frac{1}{2}(\mathbf{A}_i \cdot \mathbf{A}_i + \mathbf{A}_j \cdot \mathbf{A}_j) \tag{1}$$

The orthocenter of $A_1A_2A_3A_4$ is a point H such that the vectors $\mathbf{A}_i - \mathbf{A}_j$ and $\mathbf{A}_k - \mathbf{H}$ are orthogal, that is, such that

$$() \cdot () = 0$$
 for any triple of distinct indices i, j, k

By (1), this condition is equivalent to

$$(\mathbf{A}_i - \mathbf{A}_j) \cdot \mathbf{H} = \frac{1}{2} (\mathbf{A}_i \cdot \mathbf{A}_i + \mathbf{A}_j \cdot \mathbf{A}_j)$$
 for any triple of distinct indices i, j, k

which is in turn equivalent to the conditions

$$(\mathbf{A}_i - \mathbf{A}_1) \cdot \mathbf{H} = \frac{1}{2} (\mathbf{A}_1 \cdot \mathbf{A}_1 + \mathbf{A}_i \cdot \mathbf{A}_i) \quad \text{for any index } i = 2, 3, 4$$
(2)

Note that the reverse implication follows from

$$\begin{aligned} (\mathbf{A}_i - \mathbf{A}_j) \cdot \mathbf{H} &= (\mathbf{A}_i - \mathbf{A}_1) \cdot \mathbf{H} - (\mathbf{A}_j - \mathbf{A}_1) \cdot \mathbf{H} \\ &= \frac{1}{2} (\mathbf{A}_1 \cdot \mathbf{A}_1 - \mathbf{A}_i \cdot \mathbf{A}_i) - \frac{1}{2} (\mathbf{A}_1 \cdot \mathbf{A}_1 - \mathbf{A}_j \cdot \mathbf{A}_j) \\ &= \frac{1}{2} (\mathbf{A}_j \cdot \mathbf{A}_j - \mathbf{A}_i \cdot \mathbf{A}_i) \end{aligned}$$

Now consider (2) as a system of three linear equations in three unknowns; the unknowns are the three coordinates of H. Because the three vectors $\mathbf{A}_2 - \mathbf{A}_1$, $\mathbf{A}_3 - \mathbf{A}_1$ and $\mathbf{A}_4 - \mathbf{A}_1$ are linearly independent, the system has exactly one solution. The solution gives the coordinates of the orthocenter H.

Math. Mag., **75**(2002) 146, 151.

Q 920. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is inscribed in a parallelogram. Determine the area of the parallelogram if two of the points of tangency are $(a \cos u, b \sin u)$ and $(a \cos v, b \sin v)$, with $0 \le u < v < \pi$.

A 920. Under the transformation $(x, y) \longrightarrow (bx/a, y)$ the parallelogram is transformed into another parallelogram and the ellipse is transformed into an inscribed circle of radius b. The circle is tangent to the image parallelogram at the points $(b \cos u, b \sin u)$ and $(b \cos v, b \sin v)$. Draw radii to the four points of tangency. The result is two quadrilaterals of area $b^2 \tan(\frac{v-u}{2})$ and two of area $b^2 \tan(\frac{\pi-v+u}{2})$, for a total area of $4b^2 \csc(v-u)$. Because ratios of areas are preserved under the transformation, the desired area is $4ab \csc(v-u)$.

Math. Mag., **75**(2002) 228, 233.

Q 922. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Two directly homothetic triangles are such that the incircle of one of them is the circumcircle of the other. If the ratio of their areas is 4, prove that the triangles are equilateral.

A 922. Let the sides, area, circumradius and inradius of the larger triangle be a, b, c, F, R and r respectively, and let the corresponding sides and area of the smaller triangle be a', b', c' and F'. We then have

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = 2 \qquad 4FR = abc \qquad \text{and} \qquad 4F'r = a'b'c'$$

It follows that

$$\frac{FR}{F'r} = \frac{abc}{a'b'c'} = 8$$

and hence that R = 2r. However, it is known that $R \ge 2r$ with equality if and only if the triangle is equilateral.

Math. Mag., 75(2002) 318, 323.

Q 923. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Let x_k , y_k , $(1 \le k \le n)$ be positive real numbers and let r, s be real numbers with $\sum_{k=1}^n x_k^r = \sum_{k=1}^n y_k^s = 1$. Prove that if $\sum_{k=1}^n (x_k^{r(m+1)}/y_k^{sm}) = 1$ for some positive integer m, then $\sum_{k=1}^n (x_k^{r(m+1)}/y_k^{sm}) = 1$ for every positive integer m.

A 923. Applying Hölder's inequality for the particular value of m we find

$$1 = \sum_{k=1}^{n} x_{k}^{r} = \sum_{k=1}^{n} \frac{x_{k}^{r}}{y_{k}^{sm/(m+1)}} y_{k}^{sm/(m+1)}$$
$$\leq \left(\sum_{k=1}^{n} \frac{x_{k}^{r(m+1)}}{y_{k}^{sm}}\right)^{1/(m+1)} \left(\sum_{k=1}^{n} k_{k}^{s}\right)^{m/(m+1)} = 1$$

Because we have equality and $\sum x_k^r = \sum y_k^s$ it follows that $x_k^r = y_k^s$ $1 \le k \le n$. The desired result follows immediately.

Math. Mag., **75**(2002) 320–321.

A Centroidal Equality

October 2001

1630. Proposed by Geoffrey A. Kandall, Hamden, CT

Let P be in the interior of $\triangle ABC$, and let lines AP, BP and CP intersect sides BC, CA and ABin L, M and N respectively. Prove that if

$$\frac{AP}{PL} + \frac{BP}{PM} + \frac{CP}{PN} = 6$$

then P is the centroid of $\triangle ABC$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada. We prove the following generalization:

Let P be a point in the interior of the n-dimensional simplex $A_0A_1A_2...A_n$ and for $0 \le k \le n$ let the cevian from A_k through P intersect the opposite face in B_k . If

$$\sum_{k=0}^{n} \frac{A_k P}{PB_k} = n(n+1)$$

then P is the centroid of the simplex.

We use barycentric coordinates. Let \mathbf{P} , \mathbf{A}_k and \mathbf{B}_k denote vectors from a common origin to the points P, A_k and B_k respectively. Then $\mathbf{P} = \sum_{k=0}^n x_k \mathbf{A}_k$ where $\sum_{k=0}^n x_k = 1$ and each $x_k > 0$, and $\mathbf{B}_k = (\mathbf{P} - x_k \mathbf{A}_k)/(1 - x_k)$. We then have

$$A_k P = \|\mathbf{A}_k - \mathbf{P}\|$$

$$A_k B_k = \|\mathbf{A}_k - (\mathbf{P} - x_k \mathbf{A}_k)/(1 - x_k)\| = \|\mathbf{A}_k - \mathbf{P}\|/(1 - x_k)$$

and

$$B_k P = A_k B_k - A_k P = x_k \|\mathbf{A}_k - \mathbf{P}\|/(1 - x_k)$$

It follows that

$$n(n+1) = \sum_{k=0}^{n} \frac{A_k P}{P B_k} = \sum_{k=0}^{n} \left(\frac{1}{x_k} - 1\right)$$

so $\sum_{k=0}^{n} 1/x_k = (n+1)^2$. However, by the Cauchy-Schwarz inequality

$$\sum_{k=0}^{n} \frac{1}{x_k} = \left(\sum_{k=0}^{n} x_k\right) \left(\frac{1}{x_k}\right) \ge (n+1)^2$$

with equality if and only if $x_k = 1/(n+1)$, $0 \le k \le n$. It follows that P is the centroid of the simplex.

This completes the solution of the problem. However, other similar results also hold. Indeed, it also follows that if any of the following three equations holds, then P is the centroid of the simplex:

$$\sum_{k=0}^{n} \frac{PB_k}{A_k P} = \frac{n+1}{n} \qquad \sum_{k=0}^{n} \frac{A_k B_k}{A_k P} = \frac{(n+1)^2}{n} \qquad \sum_{k=0}^{n} \frac{A_k B_k}{PB_k} = (n+1)^2$$

If thefirst of these equations is true, then

$$\frac{n+1}{n} = \sum_{k=0}^{n} \frac{PB_k}{A_k P} = \sum_{k=0}^{n} \frac{x_k}{1-x_k} = -(n+1) + \sum_{k=0}^{n} \frac{1}{1-x_k}$$

so $\sum_{k=0}^{n} 1/(1-x_k) = (n+1)^2/n$. By the Cauchy-Schwarz inequality

$$\sum_{k=0}^{n} 1/(1-x_k) \ge (n+1)^2 \left/ \sum_{k=0}^{n} (1-x_k) = (n+1)^2/n \right|$$

with equality if and only if $x_k = 1/(n+1)$, $0 \le k \le n$. Similar arguments can be used to show that if either of the other two equations is true, then P is the centroid of the simplex.

Note: Miguel Amengual Covas of Spain points out that the triangle version of this problem appeared on a Romanian Mathematics Competition. See *Revista de Matematica din Timisora*, Annul II (seria a 4-a), nr.1-1997, pp.16–17. The triangle version of this problem also appeared as problem E 1043 in the *Amer. Math. Monthly*, **59**(1952) p.697, with solution in **60**(1953) p.421.

Math. Mag., **75**(2002) 400, 404.

Q 925. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Evaluate the determinant

$$\begin{array}{c|cccc} \underline{y+z} & \underline{x} & \underline{y+z} & \underline{x} \\ \underline{y} & \underline{y+z} & \underline{y+z} & \underline{y+z} \\ \underline{y} & \underline{z+x} & \underline{y} & \underline{z+x} \\ \underline{z} & \underline{z} & \underline{z} & \underline{x+y} \\ x+y & \underline{x+y} & \underline{z} \end{array}$$

A 925. Let D be the value of the determinant. Clearing fractions we have

$$xyz(y+z)(z+x)(x+y)D = \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$
(*)

Because the resulting determinant vanishes when x = 0 or y = 0 or z = 0, it has xyz as a factor. Next note that if x + y + z = 0, then the determinant has three proportional rows. Hence the determinant in (*) also has $(x + y + z)^2$ as a factor. Thus the determinant in (*) has the form

$$Pxyz(x+y+z)^2$$

Because this determinant is a symmetric, homogeneous polynomial of degree 6, it follows that P = k(x + y + z) for some constant k. To determine k, set x = y = z = 1. The determinant in (*) then has value 54 and it follows that k = 2. We then find

$$D = \frac{2(x+y+z)^3}{(y+z)(z+x)(x+y)}$$

Math. Mag., **75**(2002) 400, 404–405.

Q 926. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Find the maximum value of

$$\sin^{2}(2A) + \sin^{2}(2B) + \sin^{2}(2C) + 2\cos(2A)\sin(2B)\sin(2C) + 2\cos(2B)\sin(2C)\sin(2A) + 2\cos(2C)\sin(2A)\sin(2B)$$

where A, B, C are the angles of a triangle ABC.

A 926. Let the vertices of $\triangle ABC$ be given in counter-clockwise order, let *D* be a point in the plane of the triangle, and let **A**, **B** and **C**, respectively, be the vectors from *D* to *A*, *B* and *C*. It is known that

$$\mathbf{A}[DBC] + \mathbf{B}[DCA] + \mathbf{C}[DAB] = \mathbf{0} \tag{(*)}$$

where [XYZ] denotes the directed area of $\triangle XYZ$. Now let D be the circumcenter of $\triangle ABC$. Then

$$[DBC] = \frac{1}{2} \|\mathbf{B}\| \|\mathbf{C}\| \sin(\angle BDC) = \frac{1}{2} \|\mathbf{B}\| \|\mathbf{C}\| \sin(2A)$$

with similar expressions for [DCA] and [DAB]. Substitute these results into (*), then calculate the length of the resulting expression. Noting that $\mathbf{B} \cdot \mathbf{C} = \|\mathbf{B}\| \|\mathbf{C}\| \cos(2A)$, with similar expressions for $\mathbf{C} \cdot \mathbf{A}$ and $\mathbf{A} \cdot \mathbf{B}$, we have

$$0 = (\mathbf{A}[DBC] + \mathbf{B}[DCA] + \mathbf{C}[DAB]) \cdot (\mathbf{A}[DBC] + \mathbf{B}[DCA] + \mathbf{C}[DAB])$$

=
$$\frac{\|\mathbf{A}\|^2 \|\mathbf{B}\|^2 \|\mathbf{C}\|^2}{4} \left(\sin^2(2A) + \sin^2(2B) + \sin^2(2C) + 2\cos(2A)\sin(2B)\sin(2C) + 2\cos(2A)\sin(2B)\sin(2C)\sin(2A) + 2\cos(2C)\sin(2A)\sin(2B)\right)$$

Thus the expression in the problem statement is identically 0.

Math. Mag., 76(2003) 151.

1667. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada Let a, b and c be nonnegative constants. Determine the maximum and minimum values of

$$\sqrt{a^2x^2 + b^2y^2 + c^2z^2} + \sqrt{a^2y^2 + b^2z^2 + c^2x^2} + \sqrt{a^2z^2 + b^2x^2 + c^2y^2}$$

subject to $x^2 + y^2 + z^2 = 1$.

Math. Mag., 77(2004) 157–158.

I. Solution by Michael Andreoli, Miami-Dade Community College, North Campus, Miami, Fl. Let $S = \sqrt{a^2x^2 + b^2y^2 + c^2z^2} + \sqrt{a^2y^2 + b^2z^2 + c^2x^2} + \sqrt{a^2z^2 + b^2x^2 + c^2y^2}$. We show that if $x^2 + y^2 + z^2 = 1$, then the maximum and minimum values of S are $\sqrt{3(a^2 + b^2 + c^2)}$ and a + b + c, respectively.

Because the square root function is concave, Jensen's inequality implies

$$\frac{1}{3}S \leq \sqrt{\frac{1}{3}\left((a^2x^2 + b^2y^2 + c^2z^2) + (a^2y^2 + b^2z^2 + c^2x^2) + (a^2z^2 + b^2x^2 + c^2y^2)\right)} = \sqrt{\frac{1}{3}(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} = \sqrt{\frac{1}{3}(a^2 + b^2 + c^2)}$$

Thus $S \leq \sqrt{3(a^2 + b^2 + c^2)}$. Because equality holds if $x = y = z = 1/\sqrt{3}$, this value of S is the minimum value.

Because $x^2 + y^2 + z^2 = 1$, it also follows from Jensen's inequality that

$$\sqrt{a^2x^2 + b^2y^2 + c^2z^2} \ge \sqrt{a^2x^2} + \sqrt{b^2y^2} + \sqrt{c^2z^2} = ax^2 + by^2 + cz^2$$

and similar inequalities hold for $\sqrt{a^2y^2 + b^2z^2 + c^2x^2}$ and $\sqrt{a^2z^2 + b^2x^2 + c^2y^2}$. It follows that

$$S \ge (a+b+c)(x^2+y^2+z^2) = a+b+c$$

Because equality occurs when x = 1 and y = z = 0, this value is the minimum value for S.

II. Solution by Michel Bataille, Rouen, France. Let

$$u = \sqrt{a^2x^2 + b^2y^2 + c^2z^2} \quad v = \sqrt{a^2y^2 + b^2z^2 + c^2x^2} \quad w = \sqrt{a^2z^2 + b^2x^2 + c^2y^2}$$

and S = u + v + w, and observe that $u^2 + v^2 + w^2 = a^2 + b^2 + c^2$ when $x^2 + y^2 + z^2 = 1$. Hence, by the Cauchy-Schwarz inequality,

$$S = u + v + w \le \sqrt{1^2 + 1^2 + 1^2} \sqrt{u^2 + v^2 + w^2} = \sqrt{3(a^2 + b^2 + c^2)}$$

with equality when $x = y = z = 1/\sqrt{3}$.

Again by the Cauchy-Schwarz inequality,

$$uv = \sqrt{(ax)^2 + (by)^2 + (cz)^2}\sqrt{(cx)^2 + (ay)^2 + (bz)^2} \ge cax^2 + aby^2 + bcz^2$$

with analogous inequalities for vw and wu. Summing we find

$$uv + vw + wu \ge (ab + bc + ca)(x^2 + y^2 + z^2) = ab + bc + ca$$

Thus

$$S^{2} = a^{2} + b^{2} + c^{2} + 2(uv + vw + wu) \ge a^{2} + b^{2} + c^{2} + 2(ab + bc + ca)$$
$$= (a + b + c)^{2}$$

so $S \ge a + b + c$. Equality holds when x = 1 and y = z = 0.

Math. Mag., **76**(2003) 319, 325.

Q 934. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

A tetrahedron has base with sides [[edges?]] of length a, b, c and an altitude of length h to this base. Determine the minimum possible surface area for the tetrahedron.

A 934. Let the distances from the foot of the altitude to the sides of length a, b, c be x, y, z respectively. Then the sum of the areas of the three lateral faces is

$$\frac{1}{2}a\sqrt{x^2+h^2} + \frac{1}{2}b\sqrt{y^2+h^2} + \frac{1}{2}c\sqrt{z^2+h^2}$$

By Minkowski's inequality, this quantity is greater than or equal to

$$\frac{1}{2}\sqrt{h^2(a+b+c)^2 + (ax+by+cz)^2} = \frac{1}{2}\sqrt{h^2(a+b+c)^2 + 4F^2}$$

where F is the area of the base. Equality occurs if and only if x = y = z = r, where r is the radius of the incircle of the base; that is, if and only if the incentre of the base is the foot of the given altitude. Thus the minimal surface area is $\frac{1}{2}(a+b+c)(r+\sqrt{h^2+r^2})$

Math. Mag., 76(2003) 399.

1683. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

For integer $n \ge 2$ and nonnegative real numbers x_1, x_2, \ldots, x_n define

$$A_n = (x_1^2 + x_2^2)(x_2^2 + x_3^2) \cdots (x_n^2 + x_1^2)/2^n$$

$$B_n = (x_1x_2 + x_2x_3 + \cdots + x_nx_1)^n/n^n$$

(a) Determine all n, if any, such that $A_n \ge B_n$ for all choices of x_k .

(b) Determine all n, if any, such that $B_n \ge A_n$ for all choices of x_k .

Math. Mag., 77(2004) 399-400.

Solution by Roy Barbara, American University of Beirut, Beirut, Lebanon.

a. We show that $A_n \ge B_n$ always holds for n = 2, 3, but need not hold for $n \ge 4$. For n = 2 the inequality follows immediately from $(x_1 - x_2)^2 \ge 0$. For the case n = 3, let $S_1 = x_1 + x_2 + x_3$, $S_2 = x_1x_2 + x_2x_3 + x_3x_1$ and $S_3 = x_1x_2x_3$. We may assume that $S_2 = 1$. Then the n = 3 case of the inequality becomes

$$\frac{1}{8}(x_1^2 + x_2^2)(x_2^2 + x_3^2)(x_3^2 + x_1^2) \ge \frac{1}{27}$$

To prove this, first observe that

$$S_1^2 = x_1^2 + x_2^2 + x_3^2 + 2S_2 \ge 3S_2 = 3$$
, so $S_1 \ge \sqrt{3}$

and by the arithmetic-geometric mean inequality

$$\frac{1}{3} = \frac{S_2}{3} \ge S_3^{2/3}$$
, so $S_3 \le \frac{\sqrt{3}}{9}$

Thus

$$\frac{1}{8}(x_1^2 + x_2^2)(x_2^2 + x_3^2)(x_3^2 + x_1^2) \geq \left(\frac{x_1 + x_2}{2}\right)^2 \left(\frac{x_2 + x_3}{2}\right)^2 \left(\frac{x_3 + x_1}{2}\right)^2 \\
= \frac{1}{64}(S_1 - x_3)^2(S_1 - x_1)^2(S_1 - x_2)^2 \\
= \frac{1}{64}(S_1S_2 - S_3^2) \geq \frac{1}{64}\left(\sqrt{3} - \frac{\sqrt{3}}{9}\right)^2 = \frac{1}{27}$$

as desired.

Now let $n \ge 4$. Setting $x_1 = x_2 = 0$ and $x_3 = x_4 = \cdots = x_n = 1$, we get $A_n = 0$ and $B_n > 0$, showing that $A_n < B_n$ is possible.

b. For $n \ge 2$, the inequality $B_n \ge A_n$ does not hold for all real x_k . Set $x_k = 1$ for k odd and $x_k = 0$ for k even. Then for even $n \ge 2$,

$$B_n = 0 < \frac{1}{2^n} = A_n$$

and for odd $n \ge 2$,

$$B_n = \frac{1}{n^n} < \frac{1}{2^{n-1}} = A_n$$

Math. Mag., 77(2004) 156

1691. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada

Let p, r and n be integers with 1 < r < n and let k be a positive constant. Determine the maximum and minimum values of

$$\sum_{j=1}^{n} \frac{t_j^p}{1+kt_j}$$

where $t_j = x_j + x_{j+1} + \dots + x_{j+r-1}$ with $x_i \ge 0$ $(1 \le i \le n)$, $x_1 + x_2 + \dots + x_n = 1$ and $x_{i+n} = x_i$.

Math. Mag., 78(2005) 159–160.

Solution by the proposer. The second derivative of $F(t) = t^p/(1+kt)$ is

$$F''(t) = \frac{t^{p-2}(p(p-1) + 2kp(p-2)t + k^2(p-1)(p-2)t^2)}{(1+kt)^3}$$

Thus F is concave for p = 1 and convex for $p \ge 2$ and $p \le 0$.

Noting that for any choice of the t_j we have $t_1+t_2+\cdots+t_n=r$, we apply a majorization result due to Hardy, Littlewood & Pólya (see A. W. Marshall & I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, NY, 1979).

Given a vector $\mathbf{y} = (y_1, \ldots, y_n)$, let $y_{[1]}, y_{[2]}, \ldots, y_{[n]}$ be the components of \mathbf{y} in decreasing order. For vectors \mathbf{y} and \mathbf{z} , write $\mathbf{y} \prec \mathbf{z}$ if $\sum_{j=1}^{k} y_{[j]} \leq \sum_{j=1}^{k} z_{[j]}$ for $1 \leq k < n$ and $\sum_{j=1}^{n} y_{[j]} = \sum_{j=1}^{n} z_{[j]}$. If g is convex on $[a, b], \mathbf{y}, \mathbf{z} \in [a, b]^n$, and $\mathbf{y} \prec \mathbf{z}$, then $\sum_{j=1}^{n} g(y_j) \leq \sum_{j=1}^{n} g(z_j)$.

For all choices of the x_j we have

$$\left(\frac{r}{n}, \frac{r}{n}, \dots, \frac{r}{n}, \right) \prec (t_1, t_2, \dots, t_n) \prec (1, 1, \dots, 1, 0, \dots, 0)$$

where the last *n*-tuple consists of *r* ones followed by n-1 zeroes. Thus, if p = 1 (so that *F* is concave), we have

$$\sum_{i=1}^{n} F(t_i) \le nF\left(\frac{r}{n}\right) = \frac{nr}{n+kr} \quad \text{and} \quad \sum_{i=1}^{n} F(t_i) \ge rF(1) + (n-r)F(0) = \frac{r}{1+k}$$

If $p \ge 2$ or $p \le 0$ (so that F is convex), then

$$\sum_{i=1}^{n} F(t_i) \ge nF\left(\frac{r}{n}\right) = \frac{r^p}{n^{p-2}(n+kr)}$$

and for $p \geq 2$ or p = 0,

$$\sum_{i=1}^{n} F(t_i) \le rF(1) + (n-r)F(0) = \frac{r}{1+k}$$

If p < 0, the sum is not bounded above.

Math. Mag., 77(2004) 320.

1701. Proposed by Murray S. Klamkin, University of Alberta, Edmunton [[sic]], AB Prove that for all positive real numbers a, b, c, d,

$$a^{4}b + b^{4}c + c^{4}d + d^{4}a \ge abcd(a + b + c + d)$$

Math. Mag., **78**(2005) 324–325.

I. Solution by Zuming Feng, Philips Exeter Academy, Exeter, NH.

By the arithmetic-geometric mean inequality,

$$a^4b + abc^2d + abcd^2 \ge 3a^2bcd$$

Similarly,

$$b^{4}c + abcd^{2} + a^{2}bcd \ge 3ab^{2}cd$$

$$c^{4}d + a^{2}bcd + ab^{2}cd \ge 3abc^{2}d$$

$$d^{4}a + ab^{2}cd + abc^{2}d \ge 3abcd^{2}$$

Adding the four inequalities leads to the desired result.

II. Solution by Chip Curtis, Missouri Southern State University, Joplin MO. Note that

$$(a^4b)^{23/51}(b^4c)^{7/51}(c^4d)^{1/51}(d^4a)^{10/51} = a^2bcd$$

so by the weighted arithmetic-geometric mean inequality,

$$\frac{23}{51}a^4b + \frac{7}{51}b^4c + \frac{1}{51}c^4d + \frac{10}{51}d^4a \ge a^2bcd$$

Adding this to the analogous results for $b^2 c da$, $c^2 dab$, $d^2 a b c$ gives the desired inequality. Note. A few readers proved that if a_1, a_2, \ldots, a_n are nonnegative real numbers, then

$$a_1^n a_2 + a_2^n a_3 + \dots + a_n^n a_1 \ge a_1 a_2 \dots + a_n (a_1 + a_2 + \dots + a_n).$$

[The following were published posthumously.]]

Math. Mag., 77(2004) 397, 403.

Q945. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada Determine the maximum value of

$$\prod_{k=1}^{n+1} (1 + \tanh x_k) / \prod_{k=1}^{n+1} (1 - \tanh x_k)$$

for real numbers $x_1, x_2, \ldots, x_{n+1}$ with $\sum_{k=1}^{n+1} x_k = 0$.

A945. Because

$$\tanh(a+b) = \frac{\tanh a + |tanhb|}{1 + \tanh a \tanh b}$$

it follows that

$$\tanh(x_1 + x_2 + \dots + x_n) = \frac{T_1 + T_3 + T_5 + \dots}{1 + T_2 + T_4 + \dots}$$

where

$$T_k = \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} \tanh x_{i_1} \tanh x_{i_2} \cdots \tanh x_{i_k}$$

is the symmetric sum of the products of the $tanh x_j$ taken k at a time. Hence

$$1 + \tanh x_{n+1} = 1 - \tanh(x_1 + x_2 + \dots + x_n) = \frac{1 - T_1 + T_2 - T_3 + \dots}{1 + T_2 + T_4 + T_6 + \dots}$$

and

$$1 - \tanh x_{n+1} = \frac{1 + T_1 + T_2 + T_3 + \cdots}{1 + T_2 + T_4 + T_6 + \cdots}$$

We then have

$$\frac{\prod_{k=1}^{n+1}(1+\tanh x_k)}{\prod_{k=1}^{n+1}(1-\tanh x_k)} = \frac{1+T_1+T_2+T_3+\cdots}{1-T_1+T_2-T_3+\cdots} \cdot \frac{1+\tanh x_{n+1}}{1-\tanh x_{n+1}} = 1$$

showing that the expression is identically 1.

Q946. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada

Let ABCD be a tetrahedron. Let ℓ_A denote the line through the centroid of face BCDand perpendicular to the face, and let ℓ_B , ℓ_C and ℓ_D be defined in a similar way. Prove that ℓ_A , ℓ_B , ℓ_C and ℓ_D are concurrent if and only if the four altitudes of the tetrahedron are concurrent.

A946. Let **X** denote the vector from the origin to the point X. First assume that ℓ_A , ℓ_B , ℓ_C and ℓ_D are concurrent at P. Then ℓ_A is parallel to the vector $\frac{1}{3}(\mathbf{B}+\mathbf{C}+\mathbf{D})-\mathbf{P}$. The altitude from A is parallel to ℓ_A so has vector equation

$$\mathbf{R}_A(t) = \mathbf{A} + t \left(\frac{1}{3} (\mathbf{B} + \mathbf{C} + \mathbf{D}) - \mathbf{P} \right)$$

Setting t = 3 we obtain $\mathbf{R}_A(3) = 4\mathbf{G} - 3\mathbf{P}$, where G is the centroid of ABCD. By a similar argument, each of the other three altitudes also contains this point. Hence the four altitudes are concurrent,

Next assume that the altitudes are concurrent at H. Then the vector equation of ℓ_A is

$$\mathbf{L}_A(t) = \frac{1}{3}(\mathbf{B} + \mathbf{C} + \mathbf{D}) + t(\mathbf{A} - \mathbf{H})$$

Setting $t = \frac{1}{3}$ we obtain $\mathbf{L}(\frac{1}{3}) = \frac{4}{3}\mathbf{G} - \frac{1}{3}\mathbf{H}$. By a similar argument, ℓ_B , ℓ_C and ℓ_D also contain this point, so the four lines are concurrent.

Math. Mag., 78(2005) 322.

40 Years ago in the MAGAZINE

Readers who enjoyed Kung's noye, "A Butterfly Theorem for Quadrilaterals", might enjoy recalling Klamkin's "An Extension of the Butterfly Problem". **38**(1965) 206–208. His main result can be summarized as follows:

Let AB be an arbitrary chord of a circle with midpoint P, and let chords JH and GI intersect AB at M and N respectively. If MP = PN, the if AB intersects JI at C and GH at D we have CP = PD. Furthermore, if line segments AB, GJ and IH are extended so that AB intersects GJ and IH at E and F respectively, the EO + OF,

$$E \qquad A \qquad M P \qquad D B \qquad F$$

$$C \qquad N$$

G

H

Ι

Murray Klamkin, Mathematical Gazette

Richard K. Guy

June 22, 2006

This edition started on 2005-10-27

This is the (lost count!) of a number of files listing problems, solutions and other writings of Murray Klamkin.

The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

Math. Gaz., 52(1968) 156–157

3202. Inequalities Concerning the Arithmetic, Geometric and Harmonic Means

In mathematical note 5168 [*Math. Gaz.*, 50(1966) 310], Mitrinović establishes by a usual calculus approach that

$$\left\{\frac{G_n}{A_n}\right\}^n \le \left\{\frac{G_{n-1}}{A_{n-1}}\right\}^{n-1} \le \dots \le \left\{\frac{G_2}{A_2}\right\}^n \le \frac{G_1}{A_1} = 1$$

where

$$A_n = \frac{1}{n} \sum_{r=1}^n a_r, \qquad G_n = \prod_{r=1}^n a_r^{1/n} \qquad (n = 1, 2, \ldots)$$

and a_1, a_2, \ldots are arbitrary positive numbers. Here we give a more elementary proof, starting with the well known inequality $A_n \geq G_n$ which can also be derived in an elementary fashion without calculus. In addition, we give some analogous inequalities involving the harmonic mean

$$H_n = n \left/ \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \right.$$

By the A.M.–G.M. inequality,

$$\left[n\left\{\frac{a_1+a_2+\dots+a_n}{n}\right\}+a_{n+1}\right]/(n+1) \ge \sqrt[(n+1)]{\left\{\frac{a_1+a_2+\dots+a_n}{n}\right\}^n a_{n+1}\right]}$$

or, equivalently,

$$\left\{\frac{a_1 + a_2 + \dots + a_{n+1}}{n+1}\right\} / (a_1 a_2 \cdots a_{n+1}) \ge \left\{\frac{a_1 + a_2 + \dots + a_n}{n}\right\}^n / (a_1 a_2 \cdots a_n)$$

which implies the desired result.

It also follows immediately that

$$\left\{\frac{G_{n+1}}{H_{n+1}}\right\}^{n+1} \ge \left\{\frac{G_n}{H_n}\right\}^n \qquad (n = 1, 2, \ldots)$$

Using the A.M.–G.M. inequality for n = 2, one can easily show that the minimum value (over x) of

$$(A+x)(B+1/x) = (AB+1+Bx+A/x), \quad (A,B,x \ge 0)$$

occurs when $x = \sqrt{(A/B)}$. Thus,

$$(A+x)(B+1/x) \ge \{\sqrt{(AB)}+1\}^2$$

Now letting

$$A = a_1 + a_2 + \dots + a_n$$

$$B = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

$$x = a_{n+1}$$

we obtain

$$(n+1)\cdot\sqrt{\left(\frac{A_{n+1}}{H_{n+1}}\right)-n\cdot\sqrt{\left(\frac{A_n}{H_n}\right)}} \ge 1 \quad (n=1,2,\ldots)$$

Note that by successive addition of the last relation for n = 1, 2, ..., we obtain $A_n \ge H_n$ with equality if and only if $a_1 = a_2 = \cdots = a_n$.

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MURRAY S. KLAMKIN

Math. Gaz., 52(1968) 156–157

3249. On the roots of a certain determinantal equation

In a recent note in this *Gazette*, **51**(1967), Vermeulen showed that the roots of the determinantal equation $A_n = 0$ are all real and negative where

$$A_{n} = \begin{vmatrix} \lambda + a_{1} + e_{1} & a_{1} & \dots & a_{1} \\ a_{2} & \lambda + a_{2} + e_{2} & \dots & a_{2} \\ \vdots & \vdots & & \vdots \\ a_{n} & a_{n} & \dots & \lambda + a_{n} + e_{n} \end{vmatrix}$$

and the a_i and e_i are strictly positive.

In this note we establish the result more elementarily and in addition give simple bounds for the roots.

It follows easily, as was done by Vermeulen, that

$$A_n = (\lambda + e_n)A_{n-1} + a_n \prod_{1}^{n-1} (\lambda + e_i)$$

This difference equation is solved simply by letting

$$A_n = \phi_n \prod_{1}^n (\lambda + e_i)$$

whence we find

$$A_n = \left\{ 1 + \frac{a_1}{\lambda + e_1} + \frac{a_2}{\lambda + e_2} + \dots + \frac{a_n}{\lambda + e_n} \right\} \prod_1^n (\lambda + e_i) \tag{1}$$

Without loss of generality we can assume that $0 < e_i < E_{i+1}$. Now consider the graph of $a_1 \qquad a_2 \qquad a_n$

$$y = 1 + \frac{a_1}{x + e_1} + \frac{a_2}{x + e_2} + \dots + \frac{a_n}{x + e_n}$$

where at first the e_i are assumed to be distinct. The graph is continuous except at the points $x = e_1, e_2, \ldots e_n$ which correspond to vertical asymptotes. It follows by continuity that there are n real negative roots $-\lambda_i$ such that

$$e_1 < \lambda_1 < e_2 < \lambda_2 < e_3 < \dots < e_{n-1} < \lambda_{n-1} < e_n < \lambda_n$$

A typical graph is shown in the figure for n = 3.



If some of the e_i coincide, it follows from (1) that there will be roots at the $-e_i$ which coincide and the others are located as before. For example, if n = 6 and the only equalities are $e_1 = e_2 = e_3$ and $e_5 = e_6$, then three roots are $-e_1$, $-e_1$, $-e_5$ and the other three roots occurin the intervals $(-\infty, -e_5)$, $(-e_5, -e_4)$ and $(-e_4, -e_3)$.

The special case corresponding to $e_1 = e_2 = \cdots = e_n = 0$ is well known. Here the roots are $-\sum a_i, 0, 0, \ldots, 0$.

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Math. Gaz., 66(1982) 156–157

66.27 An algebraic theorem related to the theory of relativity

In a previous note with the same title (*Gazette*, December 1972) M. D. Dampier establishes the following theorem which has an application to the special theory of relativity.

THEOREM. If K(x, y, z, t) is a homogeneous quadratic function of x, y, z, t with the property that K(x, y, z, t) = 0 whenever x, y, z, t are real numbers such that $x^2 + y^2 + z^2 - t^2 = 0$ (1)

then there exists a constant a such that

$$K(x, y, z, t) = a(x^{2} + y^{2} + z^{2} - t^{2}).$$

Here we give extensions with a more transparent proof.

THEOREM. If K(x, y, z, t) is a polynomial in x, y, z, t with the property that K(x, y, z, t) = 0 whenever x, y, z, t are real numbers such that

$$x^2 + y^2 + z^2 - t^2 = 0$$

then K is divisible by $x^2 + y^2 + z^2 - t^2$).

PROOF. It follows by division of K by $t^2 - x^2 - y^2 - z^2$ that

$$K(x, y, z, t) = F(x, y, z, t)(t^{2} - x^{2} - y^{2} - z^{2}) + A(x, y, z)t + B(x, y, z)$$

where F, A and B are polynomials. Since K vanishes for all real values $t = \pm \sqrt{x^2 + y^2 + z^2}$

$$\begin{array}{rcl} A(x,y,z)\sqrt{x^2+y^2+z^2}+B(x.y.z) &=& 0\\ -A(x,y,z)\sqrt{x^2+y^2+z^2}+B(x.y.z) &=& 0 \end{array}$$

for all real x, y, z. Thus A and B must vanish identically.

The previous result can easily be extended to the case where equation (1) is replaced by

$$\sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{n} x_i^2 = 0$$

and K is a polynomial in the n variables.

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The *Mathematical Gazette* started a Problems Corner in 1980. I've skimmed through it, and may have missed some items, but the first mention of Marray that I noticed was in the list of solvers of

Math. Gaz, 80(1996) 426–427

79.I (C.F. Parry) ABC is a scalene triangle with sides a, b and c. If

$$c^2(b^2-c^2)^2 = a^2(c^2-a^2)^2 = b^2(a^2-b^2)^2$$

show that

$$\frac{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)} = 13$$

Solutions were usually published editorially, with little or no attribution to particular individuals. [Though later this improved somewhat as is evidenced by some of the quotes below.]

The next mention that I found was in connexion with

Math. Gaz., 84(2000) 327 83.K (C. F. Parry)

 $A_1B_1C_1$ and $A_2B_2C_2$ are two arbitrary triangles. Show that

 $\cot A_1(\cot B_2 + \cot C_2) + \cot B_1(\cot C_2 + \cot A_2) + \cot C_1(\cot A_2 + \cot B_2) \ge 2$

with equality only in the equiangular case, i.e., when $A_1 = A_2$, $B_1 = B_2$ and $C_1 = C_2$. Math. Gaz., 85(2001) 525–526

Addendum. Murray S. Klamkin has submitted the following commentary upon Problem 83.K (C. F. Parry) July 2000, pp.327–328.

The inequality

$$\cot A_1(\cot B_2 + \cot C_2) + \cot B_1(\cot C_2 + \cot A_2) + \cot C_1(\cot A_2 + \cot B_2) \ge 2$$
$$["\ge 2" \text{ is missing here}]$$

is another representation of the known Neuberg-Pedoe inequality

$$a_1^2(b_2^2 + c_2^2 - a_2^2) + b_1^2(c_2^2 + a_2^2 - b_2^2) + c_1^2(a_2^2 + b_2^2 - c_2^2) \ge 16F_1F_2$$

where a, b, c and F are the sides and area of a triangle. A number of proofs and generalizations are given in [1, pp.354–364]. The equivalence is obtained by noting that

$$\cot B + \cot C = \frac{\sin A}{\sin B \sin C} = \frac{2Ra}{bc} = \frac{a^2}{2F}$$

and

$$\cot A = \frac{R(b^2 + c^2 - a^2)}{abc} = \frac{b^2 + c^2 - a^2}{4F}$$

The following simple geometric interpretation is due to Pedoe:

Let $\triangle A'_2 B'_2 C'_2$ be similar to $\triangle A_2 B_2 C_2$ with sides $a_1, b_2 a_1/a_2, c_2 a_1/a_2$ and superimpose $\triangle A'_2 B'_2 C'_2$ on $\triangle A_1 B_1 C_1$ by letting $B'_2 C'_2$ coincide with $B_1 C_1$ and A_1, A'_2 beon the same side of $B_1 C_1$. Then the Neuberg-Pedoe inequality is simply $(A_1 A'_2)^2 \ge 0$ (just use the law of cosines in $\triangle A_1 B_1 A'_2$ and expand out $\cos(B_1 - B_2)$).

Reference

1. D. S. Mitrinović, J. E. Pecarić & V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

Murray features in

Math. Gaz., 86(2002) 149–151

85.G (Lee Ho-Joo)

From a point inside a triangle ABC perpenduculars OP, OQ, OR are drawn to its sides BC, CA, AB resolutively. Prove that

 $OA.OB + OB.OC + OC.OA \ge 2(OA.OP + OB.OQ + OC.OR)$

From the few responses to this problem we have chosen Michel Bataille's solution.

A



B P C By the sine rule, $\frac{QR}{\sin A} = OA$, $\frac{PR}{\sin B} = OB$ and $\frac{PQ}{\sin C} = OC$. Now, using the cosine rule,

$$QR^{2} = OR^{2} + OQ^{2} + 2OR.OQ\cos A = OR^{2} + OQ^{2} - 2OR.OQ\cos(B+C)$$

= $(OR\cos B - OQ\cos C)^{2} + (OR\sin B + OQ\sin C)^{2}$
 $\geq (OR\sin B + OQ\sin C)^{2}$

so that $QR \ge OR \sin B + OQ \sin C$.

In the same way, we obtain $PR \ge OP \sin C + OR \sin A$ and $PQ \ge OQ \sin A + OP \sin B$. It follows that for any positive real numbers r, s, tr OA + s OB + t OC

$$\geq r \frac{OR\sin B + OQ\sin C}{\sin A} + s \frac{OP\sin C + OQ\sin A}{\sin B} + t \frac{OQ\sin A + OQ\sin B}{\sin C}$$
$$= OP\left(s \frac{\sin C}{\sin B} + t \frac{\sin B}{\sin C}\right) + OQ\left(t \frac{\sin A}{\sin C} + r \frac{\sin C}{\sin A}\right) + OR\left(r \frac{\sin B}{\sin A} + s \frac{\sin A}{\sin B}\right)$$
$$\geq 2(\sqrt{st} OP + \sqrt{tr} OQ + \sqrt{rs} OR)$$

[the latter because of the general inequality $a + b \ge 2\sqrt{ab}$ for positive a, b]. Thus $rOA + sOB + tOC \ge 2(\sqrt{st}OP + \sqrt{tr}OQ + \sqrt{rs}OR)$.

Taking $r = (OB.OC)^2$, $s = (OC.OA)^2$, $t = (OA.OB)^2$ and dividing out by OA.OB.OC now yields the desired result.

Remark. Taking r = s = t = 1 instead yields the well-known Erdős-Mordell inequality $OA + OB + OC \ge 2(OP + OQ + OR).$

W. Janous and M. S. Klamkin cite a 1961 paper on this problem [1], and the latter has sent the following commentary.

With OP, OQ, OR denoted by r_1 , r_2 , r_3 respectively and OA, OB, OC denoted by R_1 , R_2 , R_3 respectively, Oppenheim showed that for any homogeneous inequality $I(R_1, R_2, R_3, r_1, r_2, r_3) \ge 0$ we also have two further dual inequalities, i.e.,

$$I(R_1, R_2, R_3, r_1, r_2, r_3) \ge 0 \Leftrightarrow I(1/r_1, 1/r_2, 1/r_3, 1/R_1, 1/R_2, 1/R_3) \ge 0$$
$$\Leftrightarrow I(R_2R_3, R_3R_1, R_1R_2, r_1R_1, r_2R_2, r_3R_3) \ge 0.$$

Applying these duality transformation relations repeatedly to the Erdős-Mordell inequality

$$R_1 + R_2 + R_3 \ge 2(r_1 + r_2 + r_3) \tag{1}$$

he gets the following set of equivalent inequalities.

$$P(1) \qquad \qquad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \ge 2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$Q(1) R_2 R_3 + R_3 R_1 + R_1 R_2 \ge 2(r_1 R_1 + r_2 R_2 + r_3 R_3)$$

where here P(1) and Q(1) denote the first and second transformations on (1) respectively. Then we have

$$QP(1) \qquad \qquad \frac{R_2 R_3}{r_1} + \frac{R_3 R_1}{r_2} + \frac{R_1 R_2}{r_3} \ge 2(R_1 + R_2 + R_3)$$

$$PQ(1) \qquad \qquad \frac{1}{r_2r_3} + \frac{1}{r_3r_1} + \frac{1}{r_1r_2} \ge 2\left(\frac{1}{r_1R_1} + \frac{1}{r_2R_2} + \frac{1}{r_3R_3}\right)$$

$$PQP(1) r_1R_1 + r_2R_2 + r_3R_3 \ge 2(r_2r_3 + r_3r_1 + r_1r_2)$$

$$QPQ(1) r_1R_1 + r_2R_2 + r_3R_3 \ge 2(r_2r_3 + r_3r_1 + r_1r_2)$$

Finally, L. J. Mordell proves the result, $R_1R_2R_3 \ge (r_2 + r_3)(r_3 + r_1)(r_1 + r_2)$, in his October 1962 *Gazette* article "On geometric problems of Erdős and Oppenheim" pp.213–215.

Reference

1. A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly*, **68**(1961) 226–230; addendum 349; *MR* **23** #A1254 [a review in Italian].

Murray is acknowledged on Math. Gaz, 86(2002) 152 as a solver of

85.H (N. Lord)

Let L_n , S_n denote the arc length and surface area of revolution (about the x-axis) of the astroid-like curve $x = \cos^n t$, $y = \sin^n t$ ($0 \le t \le \frac{\pi}{2}$). Prove or disprove the geometrically plausible assertions: $L_n \to 2$ and $S_n \to \pi$ as $n \to \infty$.

Murray features in

Math. Gaz., 87(2003) 591–592

87.C (Nick Lord)

Find the smallest value of α for which

$$\frac{1}{27} - xyz \le \alpha \left[\frac{1}{3} - (yz + zx + xy)\right]$$

holds for all non-negative x, y, z satisfying x + y + z = 1. (That $\alpha = \frac{7}{9}$ works is the substance of British Mathematical Olympiad **2**(1999) qn.3.)

This was a popular problem which attracted a wide variety of different methods of solution ranging from ingenious algebra and roots of cubics to polar coordinates and calculus of one or more variables. Li Zhou's solution is striking in its brevity:

By symmetry, we may assume that $x \leq y \leq z$. Then $x \leq \frac{1}{3}$. Substituting x = 0 and $y = z = \frac{1}{2}$ into the given inequality, we get $\frac{1}{27} \leq \alpha(\frac{1}{3} - \frac{1}{4})$, thus $\alpha \geq \frac{4}{9}$. When $\alpha = \frac{4}{9}$ we use $yz \leq \left(\frac{y+z}{2}\right)^2$ and y + z = 1 - x to obtain

$$\begin{aligned} \alpha[\frac{1}{3} - (yz + zx + xy)] - \frac{1}{27} + xyz &= -(\frac{4}{9} - x)yz - \frac{4}{9}x(y + z) + \frac{1}{9} \\ &\geq -(\frac{4}{9} - x)(\frac{1-x}{2})^2 - \frac{4}{9}x(1-x) + \frac{1}{9} \\ &= \frac{1}{4}x(x - \frac{1}{3})^2 \ge 0. \end{aligned}$$

Hence the smallest value of α is $\frac{4}{9}$.

Mario Catalani's solution typified those which used calculus.

We can rewrite the desired inequality as

$$\frac{\frac{1}{27} - xyz}{\frac{1}{3} - (yz + zx + xy)} \le \alpha$$

Let g(x, y, z) be th LHS. Then

$$g(x, y, z) = \frac{1}{9} \times \frac{1 - 27xyz}{1 - 3(yz + zx + xy)}$$

= $\frac{1}{9} \times \frac{1 - 27xy(1 - x - y)}{1 - 3(xy + x(1 - x - y) + y(1 - x - y))}$
 $\leq \frac{1}{9} \times \frac{1 - 27xy(1 - x - y)}{1 - 3(1 - x)x}$
 $\leq \frac{1}{9} \times \frac{1}{1 - 3(1 - x)x}$

where we used the restrictions z = 1 - x - y, $0 \le y \le 1 - x$ and $x \ge 0$. Now consider the function

$$f(x) = \frac{1}{9} \times \frac{1}{1 - 3(1 - x)x}$$

Either by calculus or completing the square, we readily see that f(x) attains its maximum at x = 0.5 and $f(0.5) = \frac{4}{9}$. It follows

$$f(x) \le \frac{4}{9}$$

Then

$$g(x, y, z) \le \frac{4}{9}$$

Now it is easy to compute

$$g(0.5, 0.5, 0) = \frac{4}{9}$$

This shows that $\alpha = \frac{4}{9}$.

Michel Bataille and Murray Klamkin point out that, when $\alpha = \frac{4}{9}$, the inequality rearranges to

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \ge 0$$

which is a case of Schur's inequality discussed, for example, in P. Ivády's June 1983 *Gazette* article (pp.126–127).

Murray Klamkin and Walther Janous further observe that, writing $T_1 = \sum x$, $T_2 = \sum yz$, $T_3 = xyz$, when $\alpha = \frac{4}{9}$ the given inequality is the same as $T_1^3 \ge 4T_1T_2 - 9T_3$ which is the best inequality of a homogeneous symmetric inequality of three non-negative variables x, y, z with equality if x = y = z. For if $T_1^3 \ge aT_1T_2 + bT_3$, the latter condition forces 9a + b = 27. But then

$$4T_1T_2 - 9T_3 \ge aT_1T_2 + (27 - 9a)T_3$$

since $T_1T_2 \ge 9T_3$.

Perhaps Murray's last contribution to Math. Gaz. appears at

Math. Gaz., 88(2004) 324-325

87.I (Michel Bataille)

Let A, B, C and D be distinct points on a circle with radius r. Show that

$$AB^{2} + BC^{2} + CD^{2} + DA^{2} + AC^{2} + BD^{2} \le 16r^{2}$$

When does equality occur?

This extremely popular problem attracted a wide range of solutions. Notable for its algebraic elegance was that of I. G. Macdonald.

We may assume that r = 1 and that A, B, C, D are represented by four complex numbers $t_1, \ldots t_4$ of modulus 1 (so that $\bar{t}_i = t_i^{-1}$). Then

$$AB^{2} = |t_{1} - t_{2}|^{2} = (t_{1} - t_{2})(t_{1}^{-1} - t_{2}^{-1}) = 2 - t_{1}t_{2}^{-1} - t_{2}t_{1}^{-1}$$

and so on, so that

$$S = AB^{2} + BC^{2} + \dots + BD^{2} = \sum_{i < j} (2 - t_{i}t_{j}^{-1} - t_{j}t_{i}^{-1})$$
$$= 12 - \sum_{i \neq j} t_{i}t_{j}^{-1}$$
$$= 16 - e_{1}\bar{e_{1}}$$
$$= 16 - |e_{1}|^{2}$$

where $e_1 = t_1 + t_2 + t_3 + t_4$. Hence $S \leq 16$ and S = 16 if and only if $e_1 = 0$. This means that the quartic equation whose roots are $t_1, \ldots t_4$ is of the form $x^4 + e_2x^2 + e_4 = 0$ (because the coefficient of x^3 is $-e_1 = 0$ and that of x is $-\bar{e_1}e_4 = 0$). Hence S = 16 if and only if *ABCD* is a rectangle.

John Rigby observed that the inequality may be extended to n points on a circle. Several solvers used vectors: these gave rise to the shortest proofs. Moreover, Murray S. Klamkin (quoted) and Li Zhou noted that the proof extends (unchanged) to more than 2 dimensions to give the definitive generalisation: if P_1, \ldots, P_n are n points (distinct or otherwise) all distance r from O in \mathbb{R}^d , then

$$\sum_{i < j} |P_i P_j|^2 \le n^2 r^2$$

To see this, write $\mathbf{p}_i = \overrightarrow{OP_i}$ and observe that $\sum_{i < j} |\mathbf{p}_i - \mathbf{p}_j|^2 \le n^2 r^2$ in expanded-out form is equivalent to $n^2 r^2 + 2 \sum_{i < j} \mathbf{p}_i \cdot \mathbf{p}_j \ge 0$. This now follows since the LHS is also the expansion of $(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n)^2$. There is equality if and only if $\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n = \mathbf{0}$, i.e., the centroid of P_1, P_2, \ldots, P_n coincides with O.

It is interesting to note that, in the 3-dimensional extension of 87.I with four points, equality holds for inscribed isosceles tetrahedra (which includes rectanglesin a circle of radius r as a degenerate case). And equality also holds for the Platonic solids—a pretty observation!

Finally, Christopher Bradley noted that a proof of the n = 3 case was the key to Problem 3 of the 1996 Lithuanian Mathematical Olympiad (*Crux Mathematicorum*, **26** 275–277). Math. Gaz,, 89(2005) 114

Readers will be saddened to learn of the passing of Murray Klamkin who died aged 83 in August 2004. He was universally admired as the doyen of problemists worldwide with an encyclopaedic knowledge which he was delighted to share. His problems and solutions graced the columns of every mathematics journal which a problems section; in particular, he responded regularly to the problems in the *Gazette*. His last contribution (July 2004, pp.324–325) typified his elegant, incisive style.

Nick Lord

Excerpts from Murray Klamkin, SIAM Rev. 1959–98

Richard K. Guy

June 22, 2006

File updated 2006-05-22.

This is the first of a number of files listing problems, solutions and other writings of Murray Klamkin. [I haven't accessed Vols. 1 and 2 of *SIAM Rev.* Murray may have been Problems Editor from the start? YES! He was – see page 4 below.]

The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

SEE PAGES 24–25 FOR FURTHER REMARKS ABOUT EDITING.

The rest may be of use to others who are wishing to make collections of Murray's work.

To reduce confusion, my own remarks from now on will be in double brackets:

[[Perhaps Bruce S. and Andy L. (and presumably we wouls welcome input from others?) will give me some detail as to the format. Not just a catalog of Murray problems?

Is there any record of any of Murray's excellent problem-solving talks? E.g., those he gave at training sessions for IMO teams. (The appendix to) the following article gives a glimpse of what I'd like to see in the way of general introduction.]]

M. S. Klamkin & D. J. Newman, The philosophy and applications of transform theory, SIAM Rev., **3**(1961) 10–36; MR **22** #9810.

[[There is an Appendix (pp.35–36) to this paper which is intended to illustrate the problem solving dictum: TRANSFORM, SOLVE, INVERT.]]

APPENDIX

1. Solve

$$y'(x^2 + 1) = xy + \sqrt{x^3 + x - y^2}$$

2. Given four mutually external circles C_1 , C_2 , C_3 , C_4 , with radii r_1 , r_2 , r_3 , r_4 , respectively, find the relation among the circles if they have a common tangent circle (Casey's relation).

3. *ABCD* is a parallelogram circumscribing a conic. Show that its diagonals are conjugate diameters.

4. If the side-lines of a skew quadrilateral touch a sphere, the points of contact are coplanar.

5. Through a point P inside an ellipse, draw a line cutting off a minimum area from the ellipse.

6. Show that it is possible to "weight" a pair of dice, such that ten of the eleven sums which can come up have equal probability.

7. Is it possible to "weight" N (N > 2) dice such that all possible sums have equal probability of coming up?

8. Construct a linkage which will convert circular motion into straight-line motion.

9. Determine the condition on two circles such that a closed Steiner chain can be formed.

10. Determine the leading term in the asymptotic ecorpansion of A_n , where

$$A_{n+1} = A_n + A_n^{-1} \qquad (A_0 = 1)$$

11. Determine the number of ways of inserting parentheses to form products r + 1 at a time in $(a_1 a_2 a_3 \cdots a_n)$.

12. What is the analogue of problem 11 as a dissection of a convex n-gon?

13. The points of contact of the sides of triangles of minimum area circumscribing an ellipse are the mid-points of the sides (also true if the ellipse is replaced by a convex oval).

14. Show that

$$\int_0^1 \frac{dt}{[1-t^{2n}]^{1/n}} \div \int_0^1 fracdt [1+t^{2n}]^{1/n} = \sec\frac{\pi}{2n} \qquad n = 2, 3, \dots$$

without integrating.

15. Construct the mid-point of a line segment with compass alone.

16. Construct the straight line joining two given points with a straightedge alone whose length is too small to span the two points.

17. Determine the shape of a smooth curve such that the time of descent (to the bottom) of a particle sliding down is independent of the initial position of the particle.

18. Sum

$$S_n = \sum_{r=1}^{\infty} \frac{a(r,n)}{r}$$

where *n* is an integer >! and $a(r, n) = \begin{cases} 1 & \text{if } n \nmid r \\ 1 - n & \text{if } n \mid r \end{cases}$ 19. Show that the theorems of Menelaus and Ceva are dual.

20. Show that one cannot decompose the integers into a finite number of arithmetic progressions such that all the common differences are distinct.

21. Determine a four parameter solution of the Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = (a_1^2 + a_2^2 + a_3^2 + a_4^2)z^n$$
 (n > 1, a_r given)

[[Let me backtrack. I've now got access to the first two volumes of SIAM Review, and there's some important stuff, Murray-wise:]]

SIAM Rev., 1(1959) 68-70.

PROBLEMS

EDITED BY MURRAY S. KLAMKIN, AVCO Manufacturing Corporation All problems should be sent to Murray S. Klamkin, R-6 Physics, AVCO Research and Advanced Development Division, Wilmington, Massachusetts.

The editors of the SIAM Review desire the development and continuance of a Problems Section. To this end, Murray S. Klamkin has been appointed an editor of the Problem Section, and he seeks contributed problems, with or without solution, which would be of interest to the SIAM membership. Each problem should be cast in its applied or industrial setting, complete with references and indications of known approaches; solutions to problems must be self-contained. Discussions should, where applicable, call attention to areas where there is a need for the development of mathematical techniques. Problems (as well as solutions) should be submitted in accordance with the instructions given on the inside front cover. Solutions will be oublished, and the editors will list annually the problems yet to be solved. An asterisk placed beside a problem number indicates that the problem was submitted without solution.

Problem 59-1^{*}, *The Ballot Problem*, by Mary Johnson (American Institute of Physics) and M. S. Klamkin.

A society is preparing 1560 ballots for an election for three offices for which there are 3, 4 and 5 candidates, respectively. In order to eliminate the effect of the ordering of the candidates on the ballot, there is a rule that each candidate must occur an equal number of times in each position as any other candidate for the same office. what is the least number of different ballots necessary?

It is immediately obvious that 60 different ballots would suffice. However, the following table gives a solution for 9 different ballots:

No. of ballots	312	78	130	234	182	104	208	286	26
Office									
1	A	А	Α	В	В	В	С	С	C
2	D	D	Ε	Е	F	G	F	G	Е
3	Н	Ι	Κ	Ι	Κ	J	J	L	L

Another solution (by C. Berndtson) is given by

No. of ballot	s 260	182	78	234	52	130	104	312	208
Office									
1	Α	A	A	В	В	В	В	C	C
2	D	F	E	G	G	D	G	E	F
3	H	I	J	J	Η	Ι	K	L	K

The above tables just give the distribution for the first position on the ballot for each office. The distributions for the other positions are obtained by cyclic permutations.

We now show that 9 is the least possible number of ballots. Let us consider the distribution for office 3 using only 8 different ballots. We must have the following (for simplicity we consider a total of 60 ballots):

No. of ballots	x	12-x	y	12 - y	z	12 - z	12	12
Office								
3	Η	Н	Ι	Ι	J	J	Κ	L

Now to get a total of 15 representations for each position for office 2, we must have x = y = 3, z = 6. But this does not satisfy the requirements for office 1. Similarly no number of ballots fewer than 8 will suffice.

It would be of interest to solve this problem in general. The problem is to determine a distribution of the candidates such that the system of linear equations for the number of each type of ballot, which contains more equations than unknowns, is solvable in positive integers.

A trick solution to the problem can be obtained using 5 different ballots: add two fictitious names to the group of 3 and one to the group of 4. We then have 3 offices for which there are 5 "candidates" for each. This would also provide a survey on the effect of ordering of the candidates on the ballot.

Problem 59-2^{*}, *N*-dimensional Volume, by Maurice Eisenstein (AVCO Manufacturing Corporation) and M. S. Klamkin.

Determine the volume in N-space bounded by the region

$$0 \le a_1 x_1 + a_2 x_2 + \dots + a_N x_n < 1$$
 $(a_r \ge 0)$
 $b_r \ge x_r \ge c_r$ $(r = 1, 2, \dots, N)$

This problem has arisen from the following physical situation: a series-parallel circuit of N resistances is given where each of the resistances R_i are not known exactly but are uniformly distributed in the range $R_i \pm \epsilon_i R_i$ ($\epsilon_i \ll 1$). We wish to determine the distribution function for the circuit resistance

$$R = F(R_1, R_2, \ldots, R_N)$$

To first order terms

$$\triangle R = \sum_{i=1}^{N} \frac{\partial F}{\partial R_i} dR_i \qquad (|dR_i| \le \epsilon_i R_i)$$

The probability that the circuit resistance lies between R and $R + \Delta R$ will be proportional to the volume bounded by the region

$$0 \le \sum_{i=1}^{N} \frac{\partial F}{\partial R_i} x_i \le \Delta R \qquad (-\epsilon_i R_i \le x_i \le \epsilon_i R_i)$$

Special cases of the problem arise in the two following examples:

(A) A sequence of independent random variables with a uniform distribution is chosen from the interval (0,1). The process is continued until the sum of the chosen numbers exceeds L. What is the expected number of such choices? The expected number E will be given by

$$E = 1 + F_1 + F_2 + F_3 + \cdots$$

where F_i is the probability of failure up to and including the *i* th trial. Geometrically, F_i will be given by the volume enclosed by

$$x_1 + x_2 + \dots + x_i \le L$$

$$0 \le x_r \le 1 \qquad (r = 1, 2, \dots, i)$$

For the case L = 1

$$E = \sum_{m=0}^{\infty} \frac{1}{m!} = e$$
(D. J. Newman & M. S. Klamkin, Expectations for sums of powers, Avco Research and Development Division, RAD-U-58-13; to be published shortly in the American Mathematical Monthly.)

[[D. J. Newman & M. S.Klamkin, Expectations for sums of powers, *Amer. Math. Monthly*, **66**(1959) 50–51; *MR* **21** #390.]]

(B) What is the probability that N points picked at random in a plane form a convex polygon?

If we denote the interior angles by θ_i , the probability that the polygon will be convex will be proportional to the volume of the region given by

$$\theta_1 + \theta_2 + \theta_3 + \dots + \theta_N = (N-2)\pi$$

 $0 \le \theta_r < 2\pi$

(we are assuming that the angles are uniformly distributed).

SIAM Rev., 2(1960) 41–45

Solution by I. J. Schoenberg, University of Pennsylvania.

Let B_{ω} denote the volume of the *n*-dimensional polyhedron

$$0 \le x_i \le a_i \qquad i = 1, 2, \dots, n \tag{1}$$
$$0 \le \lambda_1 x_1 + \lambda_2 x_2 + \dots \le \omega \tag{2}$$

where $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = 1$ and $a_r, \lambda_r, \omega \ge 0$. Also, let $b_i = a_i \lambda_i$. If F(u) is a function of one variable u, we define the operator L_n by

$$L_n F(u) = \sum_{(\alpha_1, \alpha_2, \dots, \alpha_i)} (-1)^{n-i} F(b_{\alpha_1 + b_{\alpha_2} + \dots + b_{\alpha_i}})$$
(3)

where $(\alpha_1, \alpha_2, \ldots, \alpha_i)$ runs through all the 2^n combinations of the *n* quantities b_1, b_2, \ldots, b_n . For example

$$L_1F(u) = F(b_1) - F(0)$$

$$L_2F(u) = F(b_1 + b_2) - F(b_1) - F(b_2) + F(0)$$

It follows that

$$L_n F(u) = L_{n-1} F(u+b_n) - L_{n-1} F(u)$$

If F(u) is sufficiently smooth,

$$\int \cdots \int F^{(n)}(\lambda_1 x_1 + \dots + \lambda_n x_n) \, dx_1 \cdots dx_n = \prod r = 1^n \lambda_r^{-1} L_n F(u) \tag{4}$$

where B denotes the box defined by (1). To establish (4) we assume it holds for n = 1, 2, ..., n - 1. Then

$$\int_{0}^{a_{n}} dx_{n} \int \dots \int F^{(n)}(\lambda_{1}x_{1} + \dots + \lambda_{n}x_{n}) dx_{1} \dots dx_{n-1}$$

$$= \frac{1}{\lambda_{1} \dots \lambda_{n-1}} \int_{0}^{a_{n}} L_{n-1}F'(u + \lambda_{n}x_{n}) dx_{n}$$

$$= \frac{1}{\lambda_{1} \dots \lambda_{n-1}} \{L_{n-1}F(u + b_{n}) - L_{n-1}F(u)\}$$

$$= \prod \lambda_{r}^{-1}L_{n}F(u)$$

Since (4) is valid for n = 1 it is valid for all n by induction. One consequence of (4) is that $L_n F(u) = 0$ whenever F(u) is a polynomial of degree less than n. By a known theorem of Peano we can write

$$L_n F(u_{\pm} \int_{-\infty}^{\infty} \Phi_n(x) F^{(n)}(x) \, dx \tag{5}$$

where the kernal Φ_n may be described as follows: If we define the truncated power function x_+^k by

$$x_{+}^{k} = \begin{cases} x^{k} & \text{if } x \ge 0\\ 0 & \text{if } x < 0 \end{cases} \qquad \qquad k = 0, 1, 2, \dots$$
 (6)

then

$$\Phi_n(x) = L_n \frac{(u-x)_+^{n-1}}{(n-1)!} \tag{7}$$

where the right side x is treated as a parameter and L_n operates on the variable u. Since $\Phi_n(x) = 0$ if x < 0 or $x > \sum_{1}^{n} b_r = b$,

$$\int \dots \int F^{(n)}(\lambda_1 x_1 + \dots + \lambda_n x_n) \, dx_1 \dots dx_n = \prod \lambda_r^{-1} \int_0^b \Phi_n(x) F^{(n)}(x) \, dx \qquad (8)$$

Equation (8) shows that $\prod \lambda_r^{-1} \Phi_n(x)$ is the area of the intersection of the box B with the hyperplane $\lambda_1 x_1 + \cdots + \lambda_n x_n = x$ (x fixed). To see this more clearly, we choose F(x) in (8) such that

$$F^{(n)}(x) = \begin{cases} 1 & \text{if } x \le \omega \\ 0 & \text{if } x > \omega \end{cases}$$

i.e.

$$F(x) = (-1)^n \frac{(\omega - x)_+^n}{n!}$$

Equation (8) now reduces to

$$B_{\omega} = \prod \lambda_r^{-1} \int_0^{\omega} \Phi_n(x) \, dx \tag{9}$$

Since the operator L_n commutes with the integration

$$\prod \lambda_r B_{\omega} = L_n \int_0^{\omega} \frac{(u-x)_+^{n-1}}{(n-1)!} dx = -L_n \left\{ \frac{(u-x)_+^n}{n!} \right\}_{x=0}^{x=\omega}$$
$$= \frac{1}{n!} L_n u_+^n - \frac{1}{n!} L_n (u-\omega)_+^n$$

Writing $B = a_1 a_2 \cdots a_n$ and observing that if $\omega \ge b$ then $L_n (u - \omega)_+^n = 0$ and $B_\omega = B$. We may now write our final result as

$$B_{\omega} = B - \frac{\prod \lambda_r^{-1}}{n!} L_n (u - \omega)_+^n \tag{10}$$

As an example, let us consider the hypercube when $a_r = 1$ and $\lambda_r = n^{-1/2}$, $r = 1, 2, \ldots, n$. Then also $b_r = n^{-1/2}$ and (10) gives

$$B_{\omega} = 1 - \frac{n^{n/2}}{n!} \Delta^n (u - \omega)^n_+|_{u=0}$$
(11)

where \triangle^n is the ordinary n^{th} order advancing difference operator of step $h = n^{-1/2}$. Now, if $\omega = 0$ then $B_{\omega} = 0$ and (11) gives

$$\Delta^{n} u_{+}^{n}|_{u=0} = \Delta^{n} u^{n}|_{u=0} = n^{-n/2} n!$$

which is a known relation. If $\omega = n^{-1/2}$ then again for the ordinary power function

$$\Delta^{n}(u-\omega)^{n}|_{u=0} = n^{-n/2}n!$$
 (12)

Passing to the truncated power function only one term of the left side of (12) drops out so that

$$\Delta^n (u - \omega)^n_+|_{u=0} = n^{-n/2} \{ n! - 1 \}.$$

Finally (11) gives for $\omega = n^{-1/2}$ the value

$$B_{n^{-1/2}} = \frac{1}{n!}$$

which is also known.

The expression (7) shows that $\Phi_n(x)$ is what has been called elsewhere [Bull. Amer. Math. Soc., **64**(1958) 352–357] a spline curve of degree n - 1, i.e. a composite of different polynomials of degree n - 1 having n - 2 continuous derivatives while $\Phi_n^{(n-1)}$

has jumps at the "knots" $x = b_{\alpha_1} + \cdots + B_{\alpha_i}$. The Laplace transform of $\Phi_n(x)$, however, has the simple form

$$\int_{-\infty}^{\infty} e^{-sx} \Phi_n(x) \, dx = \prod_{r=1}^n \frac{1 - e^{-sb_r}}{s} \tag{13}$$

This transform is particularly useful if we wish to discuss the limit properties of the distribution $\Phi_n(x)$ for large *n*. *Remark*: No originality is claimed for the matters presented here. The operator L_n was studied by M. Frechet, T. Popoviciu and others. Laplace transforms of the kind obtained here were already derived by Laplace himself. Finally, G. Pólya's Hungarian doctoral dissertation [*Mathematikai es Physikai Lapok* **XXII**] is devoted to an intensive study of the transforms (13). As a matter of fact, Pólya starts from the problem of determining the volume B_{ω} and also stresses the relations with probability theory which are obtained if *n* is allowed to tend to infinity.

Also solved by Larry Shepp who shows that the probability that an n+1 sided polyon be convex (the angles of which are assumed uniformly distributed) is

$$P_{n+1} = \frac{2^n - n - 1}{(n-1)^n - \binom{n+1}{1}(n-3)^n + \dots + (-1)^{[n/2]+1}\binom{n+1}{[n/2]-1}(n-2[n/2]+1)^n}$$

This generalizes the result of H. Demir for the case n = 3 (*Pi Mu Epsilon J.*, Spring 1958).

Editorial Note: E. G. Olds in "A note on the convolution of uniform distributions", *Ann. Math. Statist.*, **23**(1952) 282–285, gives a derivation for the probability density function for a sum of independent rectangularly distributed random variables.

[[Here's the review of the BAMS ref – Schoenberg's introduction of the idea of "spline":

MR0100746 (20 #7174) Schoenberg, I. J., Spline functions, convex curves and mechanical quadrature. Bull. Amer. Math. Soc. 64 1958 352–357.

By a spline function of degree n-1 is meant a function of the form

$$S_{n-1,k}(x) = P_{n-1}(x) + \sum_{\nu=1}^{k} C_{\nu}(x-\xi_{\nu})_{+}^{n-1},$$

where $P_{n-1}(x)$ is a polynomial of degree $\leq n-1$ and $x_+^{n-1} = x^{n-1}$ for $x \geq 0$ and 0 if x < 0. In this research announcement a fundamental theorem of algebra is given for spline functions and applications are indicated to mechanical quadrature formulas of Gauss and Radau type. The determination of the knots (ξ_{ν}) of a spline function with given zeros is made to depend upon a refinement of a theorem of Carathéodory on convex hulls.

Reviewed by P. J. Davis]]

[[I've now got myself out of chronological order, since next is Problem 60-11, which is below.]]

SIAM Rev., 1(1959) 172.

Problem 59-6^{*}, *The Smallest Escape Asteroid*, by M. S. KLAMKIN (AVCO Research and Advanced Development Division).

A problem which was solved in the American Mathematical Monthly (May, 1953, p.332) was to determine the largest asteroid that one could jump "clear" oo (escape). A more interesting and more difficult problem would be to determine the smallest asteroid that one could jump "clear" off. The difficulty arises in the reaction of the asteroid. For a large one the reaction is negligible. But this is not true for a small one. [[I don't think that a solution was ever offered.]]

SIAM Rev., **2**(1960) 41.

Problem 60-3, *A Center of Gravity Perturbation*, by M. S. KLAMKIN (AVCO Research and Advanced Development Division).

Determine a vector $\mathbf{Z} = (z_1, z_2, \dots, z_n)$ which maximizes $(\mathbf{A} \cdot \mathbf{Z})^2 + (\mathbf{B} \cdot \mathbf{Z})^2$ where \mathbf{A} and \mathbf{B} are given vectors and $|z_r| \leq 1, r = 1, 2, \dots, n$. This problem arises from the following physical situation:

A composite body consists of n component masses $\{m_r\}$ with individual cC.G.s at (x_r, y_r) . The masses will not be known exactly but can vary within a tolerance of $\pm \epsilon_r m_r$ ($\epsilon_r \ll 1$). What is the greatest distance the C.G.can be from the C.G. which is calculated by using the nominal masses?

If the origin of our coordinate system is taken at the nominal C.G., then to first order terms the perturbation in the position of the C.G. due to perturbations in the masses will be given by

$$\Delta x = \frac{\sum x_r \Delta m_r}{\sum m_r}$$
$$\Delta y = \frac{\sum y_r \Delta m_r}{\sum m_r}$$

Then

$$\triangle x^2 + \triangle y^2 = (\mathbf{A} \cdot \mathbf{Z})^2 + (\mathbf{B} \cdot \mathbf{Z})^2$$

$$\mathbf{A} = \left\{ \frac{\epsilon_r m_r x_r}{\sum m_r} \right\} \qquad r = 1, 2, \dots, n$$
$$\mathbf{B} = \left\{ \frac{\epsilon_r m_r y_r}{\sum m_r} \right\} \qquad r = 1, 2, \dots, n$$
$$\mathbf{Z} = \{z_r\} \qquad |z_r| \le 1, \qquad r = 1, 2, \dots, n$$

It follows that the maximizing vector \mathbf{Z} (emanating from the origin) will terminate on one of the vertices of the hypercube $(\pm 1, \pm 1, \ldots, \pm 1)$. The difficulty in the probem resides in the fact that in the actual problem involved, n = 43 and thus the number of vertices is 2^{43} which is much too large to check each one. Crude upper and lower bounds can be immediately obtained by considering \mathbf{Z} to terminate on the circumscribed and inscribed hyperspheres, respectively. In these cases the maximizing vector will lie in the plane of \mathbf{A} and \mathbf{B} and is easily determined. The ratio of these bounds is \sqrt{n} . The lower bound can be improved by choosing the "closest" vextex vector to the latter maximizing vector.

[[No solution appeared until:]]

SIAM Rev., **34**(1992) 651–652.

Solution by JOHN QUINN (St. Francis Xavier University, Nova Scotia, Canada).

We let

$$\Omega = \{ (\mathbf{A} \cdot \mathbf{Z}, \mathbf{B} \cdot \mathbf{Z}) : |z_i| \le 1, \quad i = 1, \dots, n \}$$

Our problem is to find a point (α, β) in Ω , whose distance r from the origin is a maximum. If $(\bar{\alpha}, \bar{\beta}), \bar{r}$ is optimal, then the vector $(\bar{\alpha}, \bar{\beta})$ is an outward normal to the disk of radius \bar{r} about the origin, and to a line of support to the centro-symmetric convexpolygon Ω at the vertex $(\bar{\alpha}, \bar{\beta})$. This line has equation

$$\alpha \bar{\alpha} + \beta \bar{\beta} = \bar{r}^2$$

and intersects Ω only at $(\bar{\alpha}, \bar{\beta})$. It follows that the corresponding ${}^{-}\mathbf{Z}$ is uniquely determined by the condition that

$$\bar{\alpha}\sum_{i=1}^n a_i z_i + \bar{\beta}\sum_{i=1}^n b_i z_i = \sum_{i=1}^n (\bar{\alpha}a_i + \bar{\beta}b_i) z_i$$

is maximum, where $\mathbf{A} = (a_i)$, $\mathbf{B} = (b_i)$. Thus \mathbf{Z} is given by

$$z_i = \begin{cases} +1 & \text{if } \bar{\alpha}a_i + \bar{\beta}b_i > 0\\ -1 & \text{if } \bar{\alpha}a_i + \bar{\beta}b_i < 0 \end{cases}$$
(1)

and, rather than maximizing $(\mathbf{A} \cdot \mathbf{Z})^2 + (\mathbf{B} \cdot \mathbf{Z})^2$ over all \mathbf{Z} with $|z_i| \leq 1$, we need only consider \mathbf{Z} given by (1) as $(\bar{\alpha}, \bar{\beta})$ ranges over all nonzero vectors in \mathbb{R}^2 . But (1) simply states that $z_i = 1$ or -1, according to whether (a_i, b_i) lies above or below the line

$$\ell : \bar{\alpha}\alpha + \beta\beta = 0.$$

If ℓ is rotated, the corresponding ${}^{-}\mathbf{Z}$ changes only when ℓ intersects one of the points (a_i, b_i) . Since there are only n such points, there are at most n distinct ${}^{-}\mathbf{Z}$ generated by (1). We shall refer to these \mathbf{Z} as the *suspects*.

A systematic way of generating the suspects is to use lines having normal vectors $(-b_j, a_j)$. We then defin $\mathbf{Z}^{(j)}$ for j = 1, 2, ..., n by

$$z_i^{(j)} = \begin{cases} +1 & \text{if } -b_j a_i + a_j b_i \ge 0\\ -1 & \text{if otherwise} \end{cases}$$

and determine j so that

$$(\mathbf{A} \cdot \mathbf{Z}^{(j)})^2 + (\mathbf{B} \cdot \mathbf{Z}^{(j)})^2$$

is a maximum.

Note. The above method generalizes to, for example, the problem of finding

$$\max\{\mathbf{A}\cdot\mathbf{Z}+\mathbf{B}\cdot\mathbf{Z}+\mathbf{C}\cdot\mathbf{Z}\}.$$

The argument involving a family of lines is replaced by one involving a family of planes through the origin, and the *n* suspects $\mathbf{Z}^{(j)}$ for a maximum are replaced by $\mathbf{Z}^{(j,k)}$, j = 1, 2, ..., n; k = 1, 2, ..., n; $k \neq j$, where

$$z_i^{(j,k)} = \begin{cases} +1 & \text{if } \langle a_i, b_i, c_i \rangle N^{(j,k)} \ge 0\\ -1 & \text{if otherwise} \end{cases}$$

and

$$N^{(j,k)} = \langle a_j, b_j, c_j \rangle \times \langle a_k, b_k, c_k \rangle.$$

Editorial comment. It is to be noted that for the solution given here, the number of calculations necessary to determine the optimum \mathbf{Z} is $O(n^2)$. This is far better than trying all vectors $\mathbf{Z} = (\pm 1, \pm 1, \dots, \pm 1)$, which requires $O(2^n)$ calculations. An open problem is to determine whether or not one can do better than $O(n^2)$. [M.S.K.]

[[Now I'm back on track, but out of order. Chronologically, but not logically, the reader should step back to the (Appendix to the) article by MSK & D.J.Newman.]]

SIAM Rev., **3**(1961) 72.

Problem 61-4, *Flight in an Irrotational Wind Field*, by M. S. KLAMKIN (AVCO) and D. J. NEWMAN (Yeshiva University)

If an aircraft travels at a constant air speed, and traverses a closed curve in a horizontal plane (with respect to the ground), the time taken is always less when there is no wind, than when there is any constant wind. Showthat this result is also valid for any irrotational wind field and any closed curve (the constant wind case is due to T. H. Matthews, *Amer. Math. Monthly*, Dec. 1945, Problem 4132).

SIAM Rev., 4(1962) 155–156.

Solution by the proposers.

If we let $\mathbf{W} = \text{wind velocity}$, $\mathbf{V} = \text{actual plane velocity}$ (which is tangential to the path of flight), then $|\mathbf{V} - \mathbf{W}|$ is the constant air speed of the airplane (without wind) and will be taken as unity for convenience.

We now have to show that

$$\oint \frac{ds}{\mathbf{V}} \ge \oint \frac{ds}{1} \tag{1}$$

By the Schwarz inequality

$$\oint |\mathbf{V}| \, ds \cdot \oint \frac{ds}{|\mathbf{V}|} \ge \left\{ \oint ds \right\}^2 \tag{2}$$

Since

$$\oint |\mathbf{V}| \, ds = \oint \mathbf{V} \cdot d\mathbf{R} = \oint (\mathbf{V} - \mathbf{W}) \cdot d\mathbf{R} + \oint \mathbf{W} \cdot d\mathbf{R}$$

and

$$\oint \mathbf{W} \cdot d\mathbf{R} = 0 \qquad (\mathbf{W} \text{ is irrotational})$$
$$\oint |\mathbf{V}| \, ds \leq \oint |\mathbf{V} - \mathbf{W}| \, |dR| = \oint ds. \tag{3}$$

(1) now follows from (2) and (3).

r

[[Articles:]]

Murray S. Klamkin, On cooking a roast, SIAM Rev., 3(1961) 167–169; MR 23 #B238.

Murray S. Klamkin, On the transformation of a class of boundary problems into initial value problems for ordinary differential equations, SIAM Rev., 4(1962) 43–47; MR 27 #5950.

SIAM Rev., 4(1962) 257.

[[The following is out of order, having been taken from the SOLUTIONS. The problem was first published at SIAM Rev., 2(1960) 219.]]

Problem 60-11 *A Parking Problem*, by M. S. KLAMKIN (AVCO), D. J. NEWMAN (Yeshiva University) and L. SHEPP (University of California, Berkeley).

Let E(x) denote the expected number of cars of length 1 which can be parked on a block of length x if cars park randomly (with a uniform distribution in the available space). Show that $E(x) \sim cx$ and determine the constant c.

[[On p.258, after a solution by ALAN G. KONHEIM and LEOPOLD FLATTO is the following *Editorial Note*: [Murray was Problems Editor as well as one of the setters.]]]

This problem was obtained third-hand by the proposers and attempts were made to track down the origin of the problem. These efforts were unsuccessful until after the problem was published. Subsequently, H. Robbins, Stanford University, has informed me that he had gotten the problem from C. Derman and M. Klein of Columbia University in 1957 and that in 1958 he had proven jointly with A. Dvoretzky that

$$E(x) = cx - (1 - c) + O(x^{-n})$$
 $n \ge 1$

plus other results like asypmtotic normality of x etc. They had intended to publish their results but did not when they found that A. Rényi had published a paper proving (8) [i.e., the above displayed formula] in 1958, i.e., "On a One-Dimensional Problem Concerning Random Place Filling," Mag. Tud. Akad. Kut. Mat. Intézet. Közleményei, pp. 109–127. Also (8) is proven by P. Ney in his Ph.D. thesis at Columbia.

A reference to the Rényi paper was also sent in by T. Dalenius (University of California, Berkeley).

An abstract of the Rényi paper was sent in anonymously from the National Bureau of Standards. The abstract appears in the International Journal of Abstracts: Statistical Theory and Method, Vol.I, No.1, July 1959, Abstract No.18. According to the abstract, there is a remark due to N. G. DeBruijn in the Rényi paper stating that a practical application of the Rényi result is in the parking problem that was proposed here. In addition, the constant c has been evaluated to be 0.748.

SIAM Rev., 4(1962) 396.

Problem 62-15, *A Property of Harmonic Functions*, by M. S. KLAMKIN (University of Buffalo).

A. For what functions F do there exist harmonic functions ϕ satisfying

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = F(\phi)$$

B. Give a physical interpretation for (A).

[[No solution appeared until]]

SIAM Rev. 40(1998) 981–983.

[The section heading of this issue bears the dedication:

This last issue is dedicated with respect, gratitude and affection to Murray S. Klamkin, founder of the Problems and Solutions section and its editor for over thity years.

The affiliation 'U of Buffalo' has a footnote added:

Now Professor Emeritus, University of Alberta.

See also the final paragraph of the solution.]]

Solution by O. G. RUEHR (Graduate Student, University of Michigan (1962)—Professor Emeritus, Michigan Technological University (1998)).

The following generalization and solution for part (A) is taken entirely from [2], with motivation in [1]. We treat the Helmholtz equation $\nabla^2 \phi + k^2 \phi = 0$ in *n* dimensions, *k* is constant. To illustrate the procedure we will work out the cases n = 1, 2. For n = 1, we differentiate $F(\phi) = \phi_x^2$ with respect to *x* and divide by ϕ_x to get $F'(\phi) = 3\phi_{xx}$. From the Helmholtz equation we find that $F'(\phi) + 2k^2\phi = 0$. For n = 2, we differentiate $F(\phi) = \phi_x^2 + \phi_y^2$ twice with respect to *x* and with respect to *y* to get

$$F'\phi_x = 2\phi_x\phi_{xx} + 2\phi_y\phi_{yx}$$

$$F'\phi_y = 2\phi_x\phi_{xy} + 2\phi_y\phi_{yy}$$

$$F''\phi_x^2 + F'\phi_{xx} = 2\phi_{xx}^2 + 2\phi_x\phi_{xxx} + 2\phi_{xy}^2 + 2\phi_y\phi_{yxx}$$

$$F''\phi_y^2 + F'\phi_{yy} = 2\phi_{xy}^2 + 2\phi_x\phi_{xyy} + 2\phi_{yy}^2 + 2\phi_y\phi_{yyy}$$

Adding the last two equations and using the Helmholtz equation and the definition of F yields

$$FF'' - k^2 \phi F' = 4\phi_{xy}^2 + 2(\phi_{xx}^2 + \phi_{yy}^2) - 2k^2 F$$

From the first two equations we obtain

$$(F' - 2\phi_{xx})(F' - 2\phi_{yy}) = 4\phi xy^2$$

Combining the last two equations we have

$$FF'' - k^2 \phi F' = F'^2 + 2k^2 \phi F' - 2k^2 F + 4\phi_{xx}\phi_{yy} + 2(\phi_{xx}^2 + \phi_{yy}^2)$$

Since the last two terms are just twice the square of the Helmholtz equation, we obtain finally the differential equation for F as follows:

$$FF'' = F'^2 + 3k^2\phi F' - 2k^2F + 2k^4\phi^2$$

From now on let us agree to write f(x) for $F(\phi)$. The function f satisfies the differential equation $\beta_n = 0$, where the differential polynomials β are defined by

$$\beta_1 = \frac{df}{dx} + 2k^2x$$
$$\beta_2 = 2f\frac{d\beta_{i-1}}{dx} - \left(i\frac{df}{dx} + 2k^2x\right)\beta_{i-1} \qquad i = 2, 3, \dots, n$$

These relationships are found formally by repeated differentiation and algebraic simplification. For n = 1 we have $f' + 2k^2x = 0$, for n = 2 we have $f(f'' + 2k^2) - (f' + k^2x)(f' + 2k^2x) = 0$, in agreement with the cases worked out above. For a solution of the differential equation for f when n = 2 and k = 1, see [3]. It is to be emphasized that these are necessary conditions that f must satisfy, and we say nothing about the existence of the solutions of the partial differential equations themselves. Similar results are found in [2] for the heat equation in n dimensions and it is shown there that such restrictions cannot exist for the scalar wave equation (because of d'Alembert's solution).

Again, omitting details, which are in [2], we find a parametric solution for the differential equation $\beta_n = 0$ for arbitrary n as follows:

$$f = \left(\frac{dx}{dt}\right)^2$$

and

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} \left(\frac{1}{(t-c_1)} + \frac{1}{(t-c_2)} + \dots + \frac{1}{(t-c_{n-1})} \right) + k^2 x = 0$$

Verification of this solution is facilitated by writing $f = \dot{x}^2$, $\ddot{x} + \dot{s}\dot{x}/s + k^2x = 0$ and finding by induction that $\beta_i = -2^i \dot{x}^i s^{(i)}/s$. Since $\beta_n = 0$, s is an arbitrary polynomial of degree n - 1 and the result follows as indicated.

Returning to the case n = 3 and harmonic functions (k = 0), we obtain the differential equation

$$4f^2f''' - 10ff'f'' + 6f'^3 = 0$$

which has solutions $f(x) = Be^{Ax}$, $A \sinh^4(Bx + C)$ and $A(x - B)^4$, where A, B, C are arbutrary constants and the results come from eliminating the parameter t from the general solution given above.

The solver cannot resist remarking on the irony that the solution had been accomplished before the problem was published. I was not aware of the problem nor was Murray Klamkin aware of my solution during the more than twenty years we worked together.

REFERENCES

[1] R. K. RITT, The modeling of physical systems, *IRE Trans. Antennae and Propagation*, **AP-**4(1956) 216–218.

[2] O. G. RUEHR, Studies in nonlinear modeling V; Nonlinear modeling functions of a special type, *Univ. Mich. Radiation Lab. Report*, **2648-5-T**(1962).

[3] O. G. RUEHR, Problem 73-12, A nonlinear differential equation, SIAM Rev., 16(1974) 261–262.

SIAM Rev., 5(1963) 275–276.

Problem 63-9^{*}, An optimal search, by RICHARD BELLMAN (The Rand Corporation)

Suppose that we know that a particle is located in the interval (x, x + dx), somewhere along the real line $-\infty < x < \infty$ with a probability density function g(x). We start at some initial point x_0 and can move in either direction. What policy minimizes the expected time required to find the particle, assuming a uniform velocity and

(a) assuming that the particle will be recognized when we pass x, or

(b) assuming that there is a probability p > 0 of missing the particle as go past it?

Also, what would be the optimum starting point x_0 ?

Editorial note. A related class of two dimensional search problems are the following "swimming in a fog" problems. A person has been shipwrecked in a fog and wishes to determine the optimal path of swimming to get to shore (in the least expected time—assuming a uniform rate of swimming). The boundary conditions can be any of the following:

- 1. The ocean is a helf-plane,
- 2. Condition (1) plus the knowledge that the initial distance to shore is $\leq D$ (with a uniform distribution),
- 3. The ocean boundary is a given closed curve, i.e., a circle, rectangle, or possibly not closed (a parabola),
- 4. Condition (2) and (3), etc.

SIAM Rev., **28**(1986) 234.

Comment by WALLACE FRANCK (University of Missouri–Columbia)

I had given a necessary and sufficient condition for this problem in this Journal (7(1965) 503–512). In the comment by Anatole Beck on this problem in the September 1985 Problem Section, p.447, he had given a councerexample to the necessity condition. The purpose of this comment is to alert the reader to my erratum (8(1966) 524) which gives the correct form for this condition. I have corresponded with Professor Beck and he agrees that the corrected condition is indeed correct.

SIAM Rev., 5(1963) 275–276.

Problem $63 - 13^*$, An Infinite Permutation, by M. S. KLAMKI+N (State University of New York at Buffalo).

Consider the infinite permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n & \cdots \\ 1 & 3 & 2 & 5 & 7 & 4 & \cdots & f(n) & \cdots \end{pmatrix}$$

where

$$f(3n-2) = 4n-3 f(3n-1) = 4n-1 f(3n) = 2n$$

We now write P as a product of cycles:

$$P = (1)(2,3)(4,5,7,9,6)(8,11,15,\ldots)\cdots$$

It is conjectured that the cycle (8, 11, 15, ...) is infinite. Other problems concerning P are

(a) Can P be expressed as a product of a finite number of cycles?

(b) Are there any other finite cycles other than those indicated?

Editorial Note: For a similar problem where there are cycles of every length, see Problem 5109, Amer. Math. Monthly, May, 1963. A major difference between the two permutations is that in the latter case, the ratio of odd to even numbers of f(n) approaches 1, whereas in the former case, the ratio approaches 2.

[[Here is the MONTHLY problem:]] AMM 70(1963) 572–573.

5109. Proposed by M. S. Klamkin, State University of New York at Buffalo, and A. L. Tritter, Data Processing, Inc.

Given the infinite permutation

 $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ 1 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & 10 & 12 & \cdots \end{pmatrix}$

where the second row is formed by taking in order from the natural numbers, 1 odd, 2 even, 3 odd, ..., 2n even, 2n + 1 odd, What is the cycle structure of this permutation?

[[Here's the solution. Note that Alan Tritter's name has disappeared from the setters.]] AMM **71**(1964) 569–570.

5109. Proposed by M. S. Klamkin, State University of New York at Buffalo

Given the infinite permutation

 $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ 1 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & 10 & 12 & \cdots \end{pmatrix}$

where the second row is formed by taking in order from the natural numbers, 1 odd, 2 even, 3 odd, ..., 2n even, 2n + 1 odd, What is the cycle structure of this permutation?

Solution by George Bergman, Harvard University. Let I_n designate the set of integers $\{i|\frac{1}{2}n(n-1) < i \leq \frac{1}{2}n(n+1)\}$. Examination of the given permutation shows that it acts on I_n by the law: $i \to 2i-u_n$ where $u_n = \frac{1}{2}n^2$ if n is even, $u_n = \frac{1}{2}(n^2+1)$ if n is odd. The "pivot" of this action is u_n ; u_n is fixed, numbers of I_n less than u_n are decreased, numbers of I_n greater than u_n are increased. But we see that even the greatest integer of I_n is not increased as far as u_{n+1} , and even the least integer of I_{n+1} is not decreased as far as u_n ; hence the interval $J_n = \{i|u_n \leq i < u_{n+1}\}$ is sent into itself. This J_n contains 2[n/2] + 1 elements. Let us represent them by the integers 0 through 2[n/2], writing j for $u_n + j$. Then the action of our permutation is: $j \to 2j$ for $j \leq [n/2]$, $j \to 2j - 2[n/2] - 1$ otherwise. In other words the elements of J_n are permuted exactly as the residue classes (mod 2[n/2] + 1) are permuted under multiplication by 2.

The nature of the permutation is as follows: for each divisor d of 2[n/2]+1 the elements $i = u_n + j$ of J_n such that (2[n/2] + 1, j) = d, form a cycle of order f((2[n/2] + 1)/d), where f(k) is the least m such that $k \mid 2^m - 1$. This numbertheoretic function is described in the standard texts. For example, let n = 15, $J_n = \{i|113 \le i < 128\}$, represented by $\{j|0 \le j < 15\}$. The permutation for these integers is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{pmatrix}$$

The cycles are given by:

 $d = 1: (1 \ 2 \ 4 \ 8), (7 \ 14 \ 13 \ 11) \sim (114 \ 115 \ 117 \ 121), (120 \ 127 \ 126 \ 124)$ $d = 3: (3 \ 6 \ 12 \ 9) \sim (116 \ 119 \ 125 \ 122)$ $d = 5: (5 \ 19) \sim (118 \ 123)$ $d = 15: (0) \sim (113), \text{ fixed.}$

f takes on every integral value (for $f(2^m - 1) = m$); therefore all cycles are finite, and there are infinitely many cycles of every finite order.

Also solved by L. Carlitz, Donald Liss, P. Catherine Varga and Oswald Wyler.

[I continue from SIAM Rev. The following comment from Dan Shanks, (and another from Oliver Atkin) deserves quoting, especially as it helps to establish the provenance of connecting Collatz's name with the 3x + 1 problem. Collatz mentioned the problem to me, later, so it's nice to have this earlier corroboration. — RKG]]

SIAM Rev., 7(1965) 285–286.

Comment by DANIEL SHANKS (David Taylor Model Basin).

These problems date back, at least, to 1950 when L. Collatz mentioned them in personal conversations during the International Congress at Harvard. In 1955, on one of the first 650's, the writer found that the cycle (8, 11, ...) contained members $> 10^{10}$ and would not close under that limit. Further, the cycle (14, 19, ...) behaved similarly; it could not be made either to close or to join the cycle (8, 11, ...). Many other such open and nonjoining cycles (with the limit 10^{10}) were found.

However, the cycle $(44, 59, \ldots, 66)$ closes with a period of 12. The known finite cycles therefore have periods 1, 2, 5 and 12. These periods are consistent with the following approximate theory.

Let *m* be a member of a finite cycle, and let there be *a*, *b* and *c* transformations respectively of the form f(3n+1) = 4n+1, f(3n-1) = 4n-1 and f(3n) = 2n before the cycle returns to *m*. Then we have approximately

$$m\left(\frac{4}{3}\right)^a \left(\frac{4}{3}\right)^b \left(\frac{2}{3}\right)^c \approx m$$

Thus

$$2^{2a+2b+c} \approx 3^{a+b+c}$$

or

$$\log_2 3 \approx \frac{2a+2b+c}{a+b+c}$$

The most likely periods for a finite cycle, which are here given by the sum a + b + c, are therefore the *denominators* in the best rational approximations of $\log_2 3$. Since the

continued fraction

$$\log_2 3 = 1 + \frac{1}{1+1} \frac{1}{1+2} \frac{1}{2+2} \frac{1}{2+3} \frac{1}{3+1} \frac{1}{1+} \cdots$$

has the convergents

$$\frac{1}{1}, \quad \frac{2}{1}, \quad \frac{3}{2}, \quad \frac{8}{5}, \quad \frac{19}{12}, \quad \frac{65}{41}, \cdots,$$

we do find denominators of 1, 2, 5, 12.

However, it has not been proven [[sic]] that these denominators constitute the only allowable periods. Nor has a finite cycle of period 41 been discovered. Nor have any other finite cycles been so far discovered.

SIAM Rev., 8(1966) 234–236.

Comment by A. O. L. ATKIN (Atlas Computer Laboratory, Chilton, England).

[[edited to avoid repetition of what's in Shanks's comment.]]

We give a method applicable in principle to the problem of finding all cycles (if any) of given period p, although the computation required becomes formidable if q/p is a good approximation to $\log_2 3$. The method utizes a quantitative refinement of the "approximate theory" given by Shanks. We show that there are no new cycles of period less than 200; in particular, there are none of periods 41 and 53 which are denominators of convergents to $\log_2 3$.

Suppose that there is a cycle (a_r) of period p, and that h is its least term. If there are p-k transformations of the form f(3n) = 2n and k transformations of the other two kinds, then

$$1 = \left(\frac{2}{3}\right)^{p-k} \left(\frac{4}{3}\right)^k \prod_{r=1}^k \left(\frac{3f(a_r)}{4a_r}\right)$$

reordering the a_r if necessary.

Also for all $r, 1 \leq r \leq k$, we have

$$|1 - 3f(a_r)/4a_r| \le 1/4m$$

Hence

$$(1 - 1/4m)^k \le 3^p/2^{p+k} \le (1 + 1/4m)^k$$

Now for 0 < x < 1 we have

$$log(1+x) < xlog(1-x) > -x - x^2/2(1-x)$$

so that

$$-63k/248m \le p\log(3/2) - k\log 2 \le k/4m$$

Thus for a given p we must have

$$m \le \frac{63}{248} / \min_{k} \left| \frac{p}{k} \log(1.5) - \log 2 \right| = g(p) , \text{ say.}$$

A program was run on the I.C.T. Atlas computer of the Science Research Council at Chilton, to show that all cycles, other than the known ones, with least terms less than 5000 have at least 342 terms in their periods. Next, for $p \leq 341$, a tabulation of g(p) showed that g(p) < 5000 except when p = 200, 253, 306. Hence the only possible periods of new cycles are p = 200, 253, 306 and p > 341.

A similar run was performed for the permutations obtained by permuting the values of f. For instance, with f(3n) = 4n + 3, f(3n + 1) = 2n, f(3n + 2) = 4n + 1, there is a cycle of period 94, least term 140. While 149/94 is a good approximation to $\log_2 3$ it is not a convergent. As the referee points out, however, there is more chance of such a (nondenominator) period here since, for example, (4n + 3)/(3n + 2). [[I wasn't the referee, but note that $\frac{149}{94} = \frac{65+84}{41+53}$ is a mediant. – RKG]]

My general conjecture, on a probability basis, is that for any "congruence" permutation of this kind, the number of finite cycles is finite, since (here) the "expected" value of f(t)/t is about $(\frac{2}{3} \frac{4}{3} \frac{4}{3})^{1/3}$ and that of $f^{-1}(t)/t$ is $(\frac{3}{2} \frac{2}{3} \frac{3}{4} \frac{3}{4})^{1/4}$ so that most cycles tend to infinity in both directions. Dr. D. A. Burgess of Nottingham University has given an elegant proof that these expected ratios cannot be unity for any congruence permutation.

Editorial note. R. EDDY (David Taylor Model Basin) notes that that there are "near" closures at periods 41, 53 and at "counterexample" periods 17 and 29 for the original permutation. Here

$36 \rightarrow 37$:	17 transforms,	
$46 \rightarrow 47$:	17 transforms,	
$78 \rightarrow 77$:	29 transforms,	
$50 \rightarrow 49$:	41 transforms,	
$554 \rightarrow 553$: 53 transforms.	

[[Article:]]

Murray S. Klamkin, A duality relation in differential equations and some associated functional equations, *SIAM Rev.*, 8(1966) 494–500; *MR* **35** #3154.

[[RKG speaking. This seems a good place (Later: perhaps now too late, but better late than never) to interpose an essay on some points of editing:

- 1. What about further editorial comments by the editors of the volume under construction?
- 2. What is Canadian English?
- 3. How far must one be precise in making quotations?
- 4. Other typographical considerations.

1. The last problem is a good example of where we should step in with additional comments. (Note the asterisk, denoting absence of a submitted solution.) This is an analog of the notorious 3x + 1 problem, and the subject of the classic paper

J. H. Conway, Unpredictable iterations, in *Proc. Number Theory Conf.*, Boulder CO, 1972, 49–52; *MR* **52** #13717.

It is discussed in UPINT **E17** where there a couple of other references.

I have yet to discover what more appeared in *SIAM Rev.* Later: quite a bit, which I've appended.

2. What is Canadian English? No doubt the simplest example is 'centre' (I hope!) which is the same in both Canadian languages and consistent with its etymology, Greek 'kentrum', a sharp point. [The other spelling has been copied many times in going through SIAM Rev.] I hope that it [i.e., 'Canadian English'] includes 'proved' when used in mathematical contexts; 'proven' being the p.p. of the obsolete verb *preve*, meaning 'test'.

3. In regard to quotations, these should be [sic]! But I'm strongly tempted to remove one of the 'others' from part (b) of the last problem of Murray above. Also I'd like to give Rényi his accent five times in the Editorial Note quoted above. [Later: I did so!] But presumably I'm not allowed to change 'proven', even if we agree on 'proved' in our own writings. There's another example below in which 'steady-state' is hyphenated in one sentence and not in the next. [there are occasions on which one would want to use both forms, but this isn't one of them.] And in the next sentence, does one really need commas round the y ?

4. Other typographical matters. E.g., will references normally be in MR style, with journal titles in ital., vol. nos. in bold followed by year nos. in paren.? But in this document I have often copied the typographical conventions (fonts, capitalization) of the original. I hope soon to send this file to interested parties to ensure that I'm not involving them in extra work, and that I'm not unnecessarily using my own time. LATER: I've taken it upon myself to do some minor editing as I took stuff from

periodicals, e.g., references put more or less into MR style, and adding MR references occasionally.

I should also warn other editors of some of my idiosyncrasies, so that they can do a query-replace if they object to any of them.

(a) I do not automatically put a comma before 'and' or 'or'.

(b) I prefer equation numbers to be on the right. Because the numbers can't conveniently run consecutively throughout this document, I've cooked their appearance, and they're not flush right, as they should be.

(c) I object to the use of the genitive in place of a plural. E.g., there's no need for apostrophe esses in 'Here the S_i and the c_i are given'.

(d) But I do like an apostrophe ess (and I do like it to be pronounced) in "Pythagoras's", "Lucas's", "Shanks's", etc., else we tend to wind up with "Shank's", etc.

(e) I prefer not to punctuate displayed formulas. Punctuation is to get the reader to pause. Good writing minimizes the number of occasions on which we have to warn the reader to take a breath. A displayed formula always gives pause, and should never be punctuated if it ends with a subscript:

$$X = \sum_{i=1}^{n} x_i, \qquad X = \sum_{i'}^{n} x_{i'}, \quad ???]]$$

[[As to what we include is concerned, there's an interesting test case at SIAM Rev., 5(1963) 157–158. Five solutions to W. L. BADE's Problem 61-9* (note the asterisk) are carefully edited by Murray, but there's no indication that any of them are due to him in the first instance. If we're short on material, which I'm sure we're not, this would be an excellent example of Murray's very competent editing.]]

SIAM Rev., 5(1963) 76–77.

[[This is the published solution for the problem proposed in SIAM Rev., 3(1961) 263. The prime on the solution number is due to a misnumbering in which the same numbers were used for different problems. Note the asterisk.]]

Problem 61-7^{*}, *One-Dimensional Steady-State Ablation*, by TOM MUNSON (AVCO Research and Advanced Development Division).

The following second order boundary value problem arises in a consideration of steadystate ablation in thermally decomposing, non-charring plastic materials. The problem results from the mathematical model when the steady state ablation is combined with the assumption that the specific heat of the decomposition products is negligible with respect to that for the undecomposed material. The coordinate, y, is measured from the receding surface and the boundary conditions correspond to a specification of the initial temperature of the material and the effective heat flux at the receding surface.

$${D^2 + (a + be^{-\lambda y})D + c}T = d, \quad T(\infty) = k, \quad \left[\frac{dT}{dy}\right]_{y=0} = Q$$

Solution by M. S. KLAMKIN (State University of New York at Buffalo). Let $mx = be^{-\lambda y}$. This transforms the equation into

$$\left\{x^2D^2 + \left[\left(1 - \frac{a}{\lambda}\right)x - \frac{mx^2}{\lambda}\right] + \frac{c}{\lambda^2}\right\}T = \frac{d}{\lambda^2}$$

The solution of the equation

$$\{x^{2}D^{2} + (\alpha x + 2\beta)xD + [\beta(\beta - 1) + (\alpha + \beta)(1 - \alpha - \beta)]\}T = 0$$

is given by

$$T = x^{1-\alpha-2\beta}e^{-\alpha x} \left\{ C_1 + C_2 \int x^{2\alpha+2\beta-2}e^{\alpha x} dx \right\}$$

(KAMKE, E., *Differentialgleichungen*, Chelsea, New York, 1948, p.451). We now have to identify the constants, i.e.,

$$1 - \frac{a}{\lambda} = 2\beta$$

$$-\frac{m}{\lambda} = \alpha$$

$$\frac{c}{\lambda^2} = \beta(\beta - 1) + (\alpha + \beta)(1 - \alpha - \beta)$$

These 3 equations suffice to determine m. α and β in terms of a, c and λ . The solution will be complicated for the case α and m complex. To obtain the general solution, the particular solution $T_p = d/c$ is added to the complementary one. The two arbitrary constants C_1 and C_2 suffice in general to satisfy the two boundary conditions.

Also solved by BERNARD G. GRUNEBAUM [[but we're presumably not interested in his solution]].

SIAM Rev., 6(1964) 61.

Problem 64-5^{*}, *A Physical Characterization of a Sphere*, by M. S. KLAMKIN (University of Buffalo).

Consider the heat conduction problem for a solid:

$$\frac{\partial T}{\partial t} = \nabla^2 T$$

Initially, T = 0. On the boundary, T = 1.

The solution to this problem is well known for a sphere and, as to be expected, it is radially symmetric. Consequently, the equipotential (isothermal) surfaces do not vary with the time (the temperature on them, of course, varies). It is conjectured for the boundary value problem above, that the sphere is the only bounded solid having the property of invariant equipotential surfaces. If we allow unbounded solids, then another solution is the infinite right circular cylinder which corresponds to the spherical solution in two dimensions.

[[Interpolation by RKG: Also the infinite halfspace, corresponding to the 1-dimensional solution. — Later: I'm now up to 1986 and haven't found any solution to this unsolved problem.]]

SIAM Rev., 6(1964) 177–178.

Problem 64-10, *A Boundary Value Problem*, by M. S. KLAMKIN (University of Buffalo).

The Thomas-Fermi equation

$$\frac{d^2y}{dx^2} = \sqrt{y^3/x} \tag{1}$$

subject to the boundary conditions y(0) = 1, $y(\infty) = 0$ arises in the problem of determining the effective nuclear charge in heavy atoms (H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, U. S. Atomic Energy Commission, 1960, pp.405–407).

Transform this boundary value problem into an initial value one.

SIAM Rev., 7(1965) 566–567. Solution by the proposer.

If we let x = 1/t, then (1) is transformed into

$$\{t^4D^2 + 2t^3D\}y = \sqrt{y^3t}$$

subject to the boundary conditions y(0) = 0, $y(\infty) = 1$. Now let $y'(0) = \lambda$ (to be determined) and (see M. S. KLAMKIN, On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations, SIAM Rev., 4(1962) 43–47; MR 27 #5950)

$$y(t) = \lambda^{3/2} F(\lambda^{-1/2} t) \tag{2}$$

It follows that F(x) satisfies the initial value problem

$${x^4D^2 + 2x^3D}F = \sqrt{F^3t}, \quad F(0) = 0, \quad F'(0) = 1$$

Then by letting $t \to \infty$ in (2) we have $\lambda = F(\infty)^{-2/3}$. Consequently, the initial boundary value problem has been transformed into two similar initial-value problems. This avoids interpolation techniques for numerically determining λ .

SIAM Rev., 6(1964) 178.

[[It is stated that 'the solutions [to the problem stated below] by E. DEUTSCH (Institute of Mathematics, Bucharest, Rumania), THOMAS ROGGE (Iowa State University), J. ERNEST WILKINS JR. (General Dynamics Corporation) and M. S. KLAMKIN (University of Buffalo) were essentially the same and are given by [what follows the statement of the problem below].' Since Murray was the Editor, he modestly puts his name last, but as the original submission was as an unsolved problem, and as Murray extended the problem and its solution [see ref at end], it may qualify as a Murray original. — RKG]]

SIAM Rev., 4(1962) 148.

Problem 62-1^{*}, *A Steady-State Temperature*, by ALAN L. TRITTER (Data Processing Inc.) and A. I. MLAVSKY (Tyco, Inc.)

Consider the steady-state temperature (T(r, z)) distribution boundary-value problem for an infinite solid bounded by two parallel planes:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \quad 0 < z < H, \quad r \ge 0, \qquad (1)$$
$$\begin{cases} -k \frac{\partial T}{\partial z} = \begin{array}{c} Q, \quad r < R\\ 0, \quad r > R \end{array} \end{cases}_{z=0}$$

 ${T = 0}_{z=H}$ and |T| < M (boundedness condition), (all the parameters involved are constants). Determine the temperature at the point r = z = 0.

[[Solution by MSK et al.]] Letting

$$\phi(\lambda, z) = \int_0^{infty} r J_0(\lambda r) T(r, z) \, dr$$

it follows by integration by parts that the Hankel transform of Eq. (1) is

$$\{D^2 - \lambda^2\}\phi = 0$$

subject to the boundary conditions

$$k \left. \frac{\partial \phi}{\partial z} \right]_{z=0} = \int_0^R Qr J_0(\lambda r) \, dr = \frac{QR J_1(\lambda R)}{\lambda}, \quad \{\phi = 0\}_{z=H}$$

Consequently

$$\phi(\lambda, z) = \frac{QR}{k} \frac{J_1(\lambda R)}{\lambda^2} \frac{\sinh \lambda (H - z)}{\cosh H}$$

Inverting the latter transform:

$$T(r,z) = \frac{QR}{k} \int_0^\infty \frac{\sinh\lambda(H-z)}{\lambda\cosh\lambda H} J_0(\lambda r) J_1(\lambda R) \, d\lambda$$

On letting $H \to \infty$ we obtain

$$\lim_{H \to \infty} T(r, z) = \frac{QR}{k} \int_0^\infty E^{-\lambda z} J_0(\lambda r) J_1(\lambda R) \frac{d\lambda}{\lambda}$$

which corresponds to a result given in Carslaw and Jaeger, Conduction of Heat in Solids, Oxford University Press, London, 1959, p.215.

In particular, the temperature at r = 0, z = 0 is given by

$$T(0,0) = \frac{QR}{k} \int_0^\infty \lambda^{-1} J_1(\lambda R) \tanh \lambda H \, d\lambda$$

The series expansion

$$T(0,0) = \frac{QR}{k} \left\{ 1 - \frac{R}{H} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m + \sqrt{m^2 + R^2/4H^2}} \right\}$$

is obtained by expanding $\tanh \lambda H$ into the exponential series

$$\tanh \lambda H = 1 - 2\sum_{m=1}^{\infty} (-1)^{m+1} e^{-2m\lambda H}$$

and employing the integral

$$\int_0^\infty \lambda^{-1} e^{-a\lambda} J_1(\lambda R) \, d\lambda = (\sqrt{a^2 + R^2} - a)/R$$

(Watson, Theory of Bessel Functions, Cambridge University Press, London, 1952, p.386).

[[There follows further detail from Deutsch, Wilkins, D. E. Amos & J. E. Warren, about which we needn't bother, but we should add:]]

For extensions of this problem to the unsteady-state in finite or infinite cylinders see Unsteady Heat Transfer into a Cylinder Subject to a Space- and Time-Varying Surface Flux, by M. S. Klamkin, TR-2-58-5, AVCO Research and Advanced Development Division, May, 1958.

SIAM Rev., 6(1964) 311.

Problem 64-15, *On a Probability of Overlap*, by M. S. KLAMKIN (University of Minnesota).

Prove directly or by an immediate application of a theorem in statistics that the conjecture in the following abstract from Mathematical Reviews, March 1964, p.589 is valid:

"Oleskiewicz, M.

The probability that three independent phenomenon [sic] of equal duration will overlap. (Polish, Russian and English summaries) Prace Mat. 4(1960) 1–7.

The value P_3 of the probability that 3 stochastically independent phenomenon of equal duration t_0 which all occur during the time $t + t_0$ will overlap is shown by geometrical methods to be equal to $(3tt_0^2 - 2t_0^3)/t^3$. The author makes the conjecture that a similar formula holds for n independent events, namely $P_n = (ntt_0^{n-1} - (n-1)t_0^n)/t^n$."

SIAM Rev., 8(1966) 112–113.

Solution by P. C. HEMMER (Norges Tekniske Høgskole, Trondheim, Norway).

We note that an overlap has to start simultaneously with the onset of one of the n events. The probability of overlap when this event is assumed to be number 1 gives 1/n of the desired probability P_n . Denote the whole time by $(0, t + t_0)$ and let event number 1 occur in $(\lambda, \lambda + t_0)$, where λ is uniformly distributed in (0, t). An overlap exists if and only if the other n - 1 events occur at time λ . Independence and uniform distribution guarantee the following probability for this to happen:

$$P(\lambda) = \begin{cases} (\lambda/t)^{n-1} & if \quad 0 \le \lambda \le t_0\\ (t_0/t)^{n-1} & if \quad t_0 \le \lambda \le t \end{cases}$$

Thus

$$\frac{P_n}{n} = \int_0^t \frac{P(\lambda) \, d\lambda}{t} = \left(\frac{t_0}{t}\right)^n \left[\frac{t - t_0}{t_0} + \frac{1}{n}\right]$$

in agreement with the conjecture.

[[Solutions also given by A. J. BOSCH, W. OETTLI. Also solved by J. RICHMAN and the proposer. There is also an editorial note:]]

Editorial Note. Hemmer also gives two related problems:

1. Determine the probability of overlap of n independent events of durations t_1, t_2, \ldots, t_n which all occur during the time t.

2. Determine the distribution of the duration of the overlap.

[[Solutions of these follow – can send in the event they're considered worth including. — Later! In *SIAM Rev.*, 10(1968) 112 there's a further editorial note. Here it is. — R.]]

Editorial Note. A published solution to this problem appeared prior to the appearance of the problem and solution in this Review as indicated in the following abstract from Mathematical Reviews, November, 1966, p.1115:

ZUBRZYCKI, S. A problem concerning simultaneous duration of several phenomena. (Polish, Russian and English summaries) Prace Mat. 7(1962) 7–9.

The author proves a conjecture formulated by M. Oleszkiewicz [same Prace 4(1960) 1–7; MR 27 #3006]. The theorem proved by the author reads as follows: Let n independent events start at moments randomly chosen in 0 < t < T under the assumption of a uniform probability distribution. Let each event last t_0 time units ($t_0 < T$). Then the probability P_n that the n events will have a common interval of duration is given by

$$P_n = \frac{nTt_0^{n-1} - (n-1)t_0^n}{T^n}$$

SIAM Rev., 8(1966) 107.

Problem 66-5, *Two Integrals*, by A. D. BRAILSFORD and M. S. KLAMKIN (Ford Scientific Laboratory).

Evaluate, in closed form, the following two integrals occurring in the calculation of the elastic strain energy of a rectangular dislocation loop [1].

$$I_1 = \int_0^1 \frac{1 - J_0(\lambda x)}{x\sqrt{1 - x^2}} dx$$
$$I_2 = \int_0^1 \frac{J_2(\lambda x)}{x\sqrt{1 - x^2}} dx$$
REFERENCE

[1] A. D. BRAILSFORD, The strain energy of a kink, Tech. Rept. SL 65-72, Ford Scientific Laboratory.

[[In SIAM Rev., 9(1967) 124–127, solutions are given by D. P. THOMAS, LESTER RUBENFELD, YUDELL LUKE, A. J. STRECOK and J. L. BROWN, JR. – also solved by 16 others and the proposers. I just give the first.]]

Solution by D. P. THOMAS (Queen's College, Dundee, Scotland).

Using the result [2, p.45]

$$1 - J_0(\lambda x) = x \int_0^\lambda J_1(yx) \, dy$$

and making the substitution $x = \sin \theta$, we find that

$$I_1 = \int_0^{\pi/2} \int_0^{\lambda} J_1(y\sin\theta) \, dy \, d\theta = \int_0^{\lambda} \int_0^{\pi/2} J_1(y\sin\theta) \, d\theta \, dy$$
$$I_2 = \int_0^{\pi/2} J_2(\lambda\sin\theta) \csc\theta \, d\theta$$

The identity [2, p.374]

$$\int_0^{\pi/2} J_{\mu}(z\sin\theta)(\sin\theta)^{1-\mu} \, d\theta = \left(\frac{\pi}{2z}\right)^{1/2} \mathbf{H}_{\mu-1/2}(z)$$

where μ is unrestructed and \mathbf{H}_{ν} is the Struve function of order ν [2, p.328], enables us to evaluate the integrals with respect to θ . Hence

$$I_{1} = \int_{0}^{\lambda} \left(\frac{\pi}{2x}\right)^{1/2} \mathbf{H}_{1/2}(x) \, dx = \int_{0}^{\lambda} \frac{1 - \cos x}{x} \, dx$$
$$I_{2} = \left(\frac{\pi}{2x}\right)^{1/2} \mathbf{H}_{3/2}(\lambda)$$

Tables exist for I_1 and $2I_2$ [3]. Note also that

$$I_1 = \log \lambda + \gamma - \operatorname{Ci}(\lambda)$$

$$I_2 = \frac{1}{2} - \frac{\sin \lambda}{\lambda} + \frac{1 - \cos \lambda}{\lambda^2}$$

where γ is Euler's constant and

$$\operatorname{Ci}(\lambda) = \int_{\infty}^{\lambda} \frac{\cos x}{x} \, dx$$

REFERENCES

[2] G. N. WATSON, *Theory of Bessel Functions*, 2nd ed. Cambridge University Press, Cambridge, 1944.

[3] A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD and L. J. COMRIE, An Index of Mathematical Tables, 2nd ed. Blackwell Scientific Publications, Oxford, 1962.

[[Article:]]

Murray S. Klamkin, A duality relation in differential equations and some associated functional equations, *SIAM Rev.*, 8(1966) 494–500; *MR* **35** #3154.

[[Short Note:]]

Murray S. Klamkin & Donald J. Newman, Extended reducibility of some differential operators, *SIAM Rev.*, **9**(1967) 577–580; *MR* **36** #445.

[[See also MR0104852 ($\mathbf{21} \# 3603$) Klamkin, Murray S.; Newman, Donald J. On the reducibility of some linear differential operators, *Amer. Math. Monthly*, $\mathbf{66}(1959)$ 293–295.

The authors prove $x^n D^{2n} = [xD^2 - (n-1)D]^n$ and $x^{2n}D^n = [x^2D - (n-1)x]^n$, where D = d/dx (and point out that the second of these relations is equivalent to a result of Glaisher [Nouvelle Corr. Math., 2(1876) 240–243, 349–350]); and they use these relations to solve the differential equations $x^n D^{2n}y = y$, $x^{2n}D^n y = y$, and $D^2x^4D^2u = x^2u$. They also give some generalizations of their identities and solve some further differential equations.

Reviewed by A. Erdélyi]]

SIAM Rev., **10**(1968) 225.

Problem 68-10, *Rank and Eigenvalues of a Matrix*, by SYLVAN KATZ (Aeronutronic Division, Philco-Ford Corporation) and M. S. KLAMKIN (Ford Scientific Laboratory).

Determine the rank and eigenvalues of the $n \times n$ $(n \geq 3)$ matrix $||A_{r,s}||$ where $A_{r,s} = \cos(r-s)\theta$ and $\theta = 2\pi/n$. This problem arose in a study of electromagnetic wave propagation.

SIAM Rev., **11**(1969) 628–630.

[[Solutions of 68-10 by G. J. FOSCHINI, CARLENE ARTHUR & CECIL ROUSSEAU, S. H. EISMAN; a generalization by HARRY APPLEGATE and an editorial note. Here are the first solution, the generalization and the note. — R.]]

Solution by G. J. FOSCHINI (Bell Telephone Laboratories, Holmdel, New Jersey).

From elementary complex algebra it follows that if k is an integer then

$$\sum_{j=1}^{n} e^{i\theta kj} = \begin{cases} 0, & k \not\equiv 0 \pmod{n} \\ n, & k \equiv 0 \pmod{n} \end{cases}$$
(1)

Using (1) we see that the Vandermonde $||n^{-1/2}e^{i\theta rs}||$ has inverse $||n^{-1/2}e^{-i\theta rs}||$ and furthermore that

$$\|n^{-1/2}e^{i\theta rs}\| \cdot \|A_{r,s}\| \cdot \|n^{-1/2}e^{-i\theta rs}\| = \|n^{-1/2}e^{i\theta rs}\| \cdot \|\frac{e^{i\theta(r-s)}}{2} + \frac{e^{-i\theta(r-s)}}{2}\| \cdot \|n^{-1/2}e^{-i\theta rs}\|$$

has zero entries except in the (1,1) and (n-1, n-1) positions, where 1/2 appears. Thus the rank of $||A_{r,s}||$ is 2 and its eigenvalues are 1/2 and 0 with multiplicities 2 and n-2 respectively.

Additionally, the same similarity transformation of $\|\sin(r-s)\theta\|$ yields a matrix with zero entries except in the (1, n-1) and (n-1, 1) positions, where 1/2i appears. Thus the rank of $\|\sin(r-s)\theta\|$ is 2 and its eigenvalues are 0 (multiplicity n-2), i/2 and -i/2.

Generalization by HARRY APPLEGATE (City College of New York).

Let t_1, t_2, \ldots, t_n be n real numbers $(n \ge 2)$ such that at least one difference $t_i - t_j$ is not a multiple of π . Then the matrix A with entries $a_{ij} = \cos(t_i - t_j)$ has rank 2.

Proof. Define vectors

$$c = \begin{pmatrix} \cos t_1 \\ \cos t_2 \\ \vdots \\ \cos t_n \end{pmatrix} \quad s = \begin{pmatrix} \sin t_1 \\ \sin t_2 \\ \vdots \\ \sin t_n \end{pmatrix}$$

It is easy to see that $A = cc^T + ss^T$ where T means transpose. The condition that some difference $t_i - t_j$ is not a multiple of π shows that c and s are linearly independent. If

x is an arbitrary vector, Ax = (c, x)c + (s, x)s where (,) is the usual scalar product. Hence Ax = 0 if and only if (c, x) = (s, x) = 0. This shows that the kernel of A is the orthogonal complement of the 2-dimensional subspace generated by c and s. Hence $\dim(\ker A) = n - 2$ which implies rank A = 2.

If $n \geq 3$ and $t_i = 2\pi i/n$ we get the problem as stated.

Remark. We get a similar result if $a_{ij} = \sin(t_i - t_j)$.

Editorial note. Some re; ated results are in the paper

L. Carlitz, Some cyclotomic matrices, Acta Arith., 5(1959) 293–308; MR 22 #2578.

SIAM Rev., 11(1969) 74–75.

Problem 69-5, *A Quartic*, by H. HOLLOWAY and M. S. KLAMKIN (Ford Scientific Laboratory).

Solve the quartic equation

$$x^{4} + (1-b)x^{3} + (1-3a-b+3a^{2}+3ab+b^{2})x^{2} + b(2-3a-2b+b^{2}) = 0$$

Also show that if 0 < a + b < 1, a, b > 0, then all the roots have absolute value less than unity.

The quartic arises from an analysis of x-ray diffraction by faulted cubic close-packed crystals. The crystal lattice may be regarded as an array of close-packed $\{1, 1, 1\}$ planes which can occupy three sets of stacking positions. In a faulted crystal the stacking has randomness and the diffraction problem requires specification of the probable stacking relationships between pairs of layers. This can be done by using genealogical tables of stacking arrangements to generate difference equations which give the probability for a given stacking relationship as a function of the fault probability and of the separation between the layers. Growth faulting and intrinsic faulting both generate second order difference equations [1,2]. A more complex model for growth faulting [3] gives a fourth order difference equation, and the quartic which results was solved numerically. Analysis of diffraction by crystals with both intrinsic and extrinsic faulting also yields a fourth order difference equation whose solution involves solution of the above quartic, where *a* and *b* are the intrinsic and extrinsic fault probability.

REFERENCES

[1] A. J. C. WILSON, Imperfections in the structure of cobalt II. Mathematical treatment of proposed structure, *Proc. Roy. Soc. London Sect. A*, **180**(1942) 277–285.

[2] M. S. PATERSON, X-ray diffraction by face-centered cubic crystals with deformation faults, J. Appl. Phys., 23(1952) 805–811.

 [3] H. JAGODZINSKI, Eindimensionale Fehlordnung in Kristallen und ihr Einfluss auf die Röntgeninterferenzen I, Acta Cryst., 2(1949) 201–207.

SIAM Rev., **12**(1970) 472–473.

Solution by the proposers.

The given quartic can be factored into the following complex form:

$$\left\{ x^2 + \left[\omega(1-b) - (\omega+\omega^2)a \right] x + \omega b \right\} \left\{ x^2 + \left[-\omega^2(1-b) + (\omega+\omega^2)a \right] x - \omega^2 b \right\} = 0$$

where $\omega = (1 + i\sqrt{3})/2$ (a cube root of -1). Thus the four roots are

$$\frac{1}{2} \left\{ \omega^2 (1-b) - (\omega + \omega^2) a \pm \left[-\omega (1-b)^2 - 3a^2 + 2(1+\omega)a(1-b) + 4\omega^2 b \right]^{1/2} \right\}$$
$$\frac{1}{2} \left\{ -\omega (1-b) + (\omega + \omega^2) a \pm \left[\omega^2 (1-b)^2 - 3a^2 + 2(1-\omega^2)a(1-b) - 4\omega b \right]^{1/2} \right\}$$

To show that the absolute values of the roots are less than one, we use the following theorem of Cauchy [M. MARDEN, *Geometry of Polynomials*, American Mathematical-Society, Providence RI, 1966, p.122]:

All the roots of

$$a_0 + a_1 z + \dots + a_n z^n = 0, \qquad \qquad a_n \neq 0$$

lie in the circle $|z| \leq r$, where r is the positive root of the equation

$$|a_0| + |a_1|z + \dots + |a_{n-1}|z^{n-1} - |a_n|z^n = 0$$

Taking one of the quadratic factors above, r is the positive root of

$$b + |\omega(1-b) - (\omega + \omega^2)a|z - z^2 = 0$$

or

$$z^{2} = z \left\{ \left(\frac{1-b}{2}\right)^{2} + 3\left(\frac{1-b}{2} - a\right) \right\}^{1/2} + b$$

and r will be less than one if

$$(1-b)^2 > \left(\frac{1-b}{2}\right)^2 + 3\left(\frac{1-b}{2}-a\right)^2$$

or 3a(1-b-a) > 0 (and similarly for the other quadratic term).

For applications of the solution, see the authors' paper *Diffraction by fcc crystals with intrinsic and extrinsic faults*, J. Appl. Phys., **40**(1969) 1681–1689.

SIAM Rev., **36**(1994) 491–493.

Comment by STANLEY RABINOWITZ (Westford, Massachusetts).

In the published solution [1], the quartic was factored as the product of two quadratic polynomials having complex coefficients. Since its zeros occur in complex conjugate pairs, the quartic must also factor as the product of two quadratics with real coefficients. The editor [M.S.K.] requested that I find these two factors.

The four roots are known. In [1], Holloway and Klamkin pointed out that the equation can be written $(x^2 + \alpha x + b\omega)(x^2 + \bar{\alpha} + b\bar{\omega}) = 0$, where $\omega = (1 + i\sqrt{3})/2$, and

$$\alpha = (1-b)\omega - ia\sqrt{3} = \frac{(1-b) + i(1-2a-b)\sqrt{3}}{2}$$

Thus the roots of the equation are z_1 , $\bar{z_1}$, z_2 , $\bar{z_2}$, where

$$z_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4b\omega}}{2}$$
 and $z_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4b\omega}}{2}$

Note that $\alpha^2 - 4b\omega = (p + iq\sqrt{3})/2$, where

$$p = (1-b)^2 + 6a(1-b) - 6a^2 - 4b$$
 and $q = (1-b)^2 - 2a(1-b) - 4b$

We need to determine $z_1 + \bar{z_1}$, $z_2 + \bar{z_2}$, $z_1 \bar{z_1}$ and $z_2 \bar{z_2}$ in terms of a and b. The required complex square root is

$$\sqrt{\frac{p + iq\sqrt{3}}{2}} = \frac{1}{2} \left[\sqrt{\sqrt{p^2 + 3q^2} + p} + i\operatorname{sgn} q\sqrt{\sqrt{p^2 + 3q^2} - p} \right]$$

(See, for example, [2, p.95].) Thus

$$z_1 + \bar{z_1} = \frac{1}{2} \left(b - 1 + \sqrt{\sqrt{p^2 + 3q^2} + p} \right) \quad \text{and} \quad z_2 + \bar{z_2} = \frac{1}{2} \left(b - 1 - \sqrt{\sqrt{p^2 + 3q^2} + p} \right)$$

Also

$$z_1 \bar{z_1} = \frac{1}{16} \left[\left(b - 1 + \sqrt{\sqrt{p^2 + 3q^2} + p} \right)^2 + \left((2a + b - 1)\sqrt{3} + \operatorname{sgn} q \sqrt{\sqrt{p^2 + 3q^2} - p} \right)^2 \right]$$

and

$$z_2 \bar{z_2} = \frac{1}{16} \left[\left(b - 1 - \sqrt{\sqrt{p^2 + 3q^2} + p} \right)^2 + \left((2a + b - 1)\sqrt{3} - \operatorname{sgn} q \sqrt{\sqrt{p^2 + 3q^2} - p} \right)^2 \right]$$

Thus the desired factors are

$$x^{2} - \frac{1}{2}\left(b - 1 + \sqrt{\sqrt{p^{2} + 3q^{2}} + p}\right)x + (P_{1} + P_{2} + P_{3})$$

and

$$x^{2} - \frac{1}{2}\left(b - 1 - \sqrt{\sqrt{p^{2} + 3q^{2}} + p}\right)x + (P_{1} - P_{2} - P_{3})$$

where

$$P_{1} = \frac{(b-1)^{2} + 3(2a+b-1)^{2} + 2\sqrt{p^{2}+3q^{2}}}{16}$$

$$P_{2} = \frac{(b-1)\sqrt{\sqrt{p^{2}+3q^{2}}+p}}{8}$$

$$P_{3} = \operatorname{sgn} q \frac{(2a+b-1)\sqrt{3\left(\sqrt{p^{2}+3q^{2}}-p\right)}}{8}$$

[1] H. HOLLOWAY & M. S.KLAMKIN, Solution of Problem 69-5, *SIAM Rev.*, **12**(1970) 471–473; reprinted in M.S.KLAMKIN, Problems in Applied Mathematics, *Selections from SIAM Review*, Society for Industrial and Applied Mathematics, Philadelphia PA, 1990 529–530.

[2] A. MOSTOWSKI and M. STARK, Introduction to Higher Algebra, Pergamon Press, New York, 1964.

SIAM Rev., **12**(1970) 289–290.

Problem 70-14^{*}, Conductors of Unit Resistance, by M. S. KLAMKIN (Ford Scientific Laboratory).

Aknown result due to Rayleigh [1,2] is that conjugate conductors have reciprocal resistances.



Here the conductor is a two-dimensional simply connected region R with boundary arcs a and b as terminals. The complementary part of the boundary consists of two arcs c and d which are insulated (see Fig. 1). The conjugate conductor consists of a region congruent to R but now the arcs c and d are the terminals and the arcs a and b are insulated.

A self-conjugate conductor is one in which the region R is a reflection of itself in the straight line connecting the initial point A of terminal a with the initial point C of terminal b (see Fig. 2). It follows immediately by Rayleigh's result and symmetry that a self-conjugate conductor has unit resistance.



If a given region has unit resistance for arbitrary chords BD which are perpendicular to a given chord AC, it is then conjectured that the conductor is self-conjugate (i.e., \overline{AC} is an axis of symmetry) (see Fig. 3).

REFERENCES

[1] J. W. S. RAYLEIGH, On the approximate solution of certain problems related to the potential, *Proc. London Math. Soc.*, **7**(1876) 70–75; Scientific Papers, Vol.1, No.39.

[2] R. J. DUFFIN, Distributed and lumped networks, J. Math. Mech., 8(1959) 816-819; MR 21 #4766.

SIAM Rev., 12(1970) 581; corrected (-n inserted) 13(1971) 388.

Problem 70-25^{*}, On the Diagonals of a Polygon, by M. S. KLAMKIN (Ford Motor Company).

If D_i , D_e , D_m denote the number of diagonals which, except for their endpoints, lie in the interior, the exterior, or neither in the interior nor exterior, respectively, of a simple n-gon P, then

$$D_i + D_e + D_m = \binom{n}{2} - n.$$

It is obvious and well known that $\max D_i = \binom{n}{2} - n$ occurring when P is convex.

The determination of $\max D_e$ is given by Problem E2214 (Amer. Math. Monthly, **77**(1970) 79). To complete this classification, determine $\max D_m$. Also consider the corresponding problem for higher-dimensional polytopes.

[[The problem referred to, and its solution, are inserted here. -R.]]

Amer. Math. Monthly, **77**(1970) 79.

E2214. Proposed by M. S. Klamkin, Ford Scientific Laboratory and B. Ross Taylor, York High School

It is intuitive that every simple n-gon (n > 3) possesses at least one interior diagonal. For a simple n-gon what is the least number of diagonals which, except for their endpoints, lie wholly in its interior?

Amer. Math. Monthly, **77**(1970) 1111–1112.

Solution by Anders Bager, Hjørring, Denmark. The two tangents from a point P outside a circle Γ touch Γ in points A and B. Connect A and B with a broken line consisting of n-2 chords succeeding each other along the smaller arc from A to B. Join P to A and B to obtain a simple *n*-gon with exactly n-3 inner diagonals (all issuing from P).

The number n-3 is minimal. This is trivially so if n = 3. Suppose it tru for some n and consider an arbitrary simple (n + 1)-gon. From this cut off a triangle such that two sides are sides of the (n + 1)-gon, and the third side an inner diagonal. This is always possible and leaves a simple n-gon which, by assumption, has at least n - 3 inner diagonals. Hence the (n + 1)-gon has at least (n - 3) + 1 = (n + 1) - 3 inner diagonals. Thus the assertion of the problem is true by induction.

[Also solved by ten others, including the proposers and ...]]

R. B. Eggleton establishes the result that a simple n-gon has precisely n-3 inner diagonals if and only if no two of its diagonals intersect.
SIAM Rev., **13**(1971) 248.

Problem 71-14, An Expected Value, by M. S. KLAMKIN (Ford Motor Company).

N numbers are chosen independently at random, one from each of the N intervals $[0, L_i]$ (i = 1, 2, ..., N). If the distribution of each random number is uniform with respect to length in the interval it is chosen from, determine the expected value of the smallest of the N numbers chosen.

SIAM Rev., 14(1972) 378–391.

Solution by O. G. RUEHR (Michigan Technological University).

If the random variable x_i be associated with the interval $[0, L_i]$ and if $X = \min(x_i)$, $L = \min(L_i)$, then

$$F(x) = \Pr\{X \le x\} = 1 - \Pr\{X > x\} = 1 - \prod(1 - x/L_i)$$

The expected value of X is then

$$E(X) = \int_0^L x \, dF(x) = xF(x) \Big]_0^L - \int_0^L F(x) \, dx = \int_0^L \{1 - F(x)\} \, dx$$
$$\frac{1}{S_n} \left\{ LS_n - \frac{L^2}{2}S_{n-1} + \frac{l^3}{3}S_{n-2} - \dots (-1)^n \frac{L^{n+1}}{n+1} \right\}$$

where the S_i are the elementary symmetric functions of the L_i . e.g., $S_1 = \sum L_i$, $S_2 = \sum_{i \neq j} L_i L_j$

[[In connexion with the following, note that the Editor, the Proposer, and a member of the Putnam Questions Committee, were all MSK — R.]]

Comment by the proposer. The special case n = 3 occurs in the 31st William Lowell Putnam Mathematical Competition (Amer. Math. Monthly, Sept. 1971 – p.765, Problem A-6.). A more involved set of problems is to determine $E\{M_i(x_1, x_2, \ldots, x_n)\}$ where M_i denotes the *i*th smallest of the x_i . For i > 1 the above method is apparently not applicable. Here, we indicate how to solve this general class of problems by determining the result for the two cases i = 2 and *n*. Letting

$$\Phi(r,n) = \int_0^{L_n} \cdots \int_0^{L_n} \int_0^{L_{n-1}} \cdots \int_0^{L_1} \max_{i=1}^{n+r-1} (x_i) \prod_{i=1}^{n+r-1} dx_i$$

we obtain symbolically that

$$\Phi(r,n) = \left\{ \int_{L_{n-1}}^{L_n} + \int_0^{L_{n-1}} \right\}^r \int_0^{L_{n-1}} \cdots \int_0^{L_1} \max(x_i) \prod dx_i$$

= $\Phi(r+1, n-1) + F(r, n)$

where

$$F(r,n) = \sum_{j=0}^{r-1} \binom{r}{j} \left\{ \int_{L_{n-1}}^{L_n} \right\}^{r-j} \left\{ \int_0^{L_{n-1}} \right\}^{j+1} \int_0^{L_{n-1}} \cdots \int_0^{L_{n-2}} \cdots \int_0^{L_1} \max(x_i) \prod dx_i$$

It is to be noted that $\Phi(r, n)$ and F(r, n) are also functions of L_1, L_2, \ldots, L_n but we have left them out for convenience. The summand for F equals

$$\binom{r}{j}L_1L_2\cdots L_{n-2}L_{n-1}^{j+1}\int_{L_{n-1}}^{L_n}\cdots \int_{L_{n-1}}^{L_n}\max_{k=1}^{r-j}\prod_{k=1}^{r-j}dx_k$$

In a manner similar to the above or otherwise, we obtain

$$\int_0^a \dots \int_0^a M_r(x_i) \, dx_1 \, dx_2 \, \dots \, dx_m = \frac{ra^{m+1}}{m+1}$$

Thus

$$F(r,n) = \sum_{j=0}^{r-1} \binom{r}{j} L_1 L_2 \cdots L_{n-2} L_{n-1}^{j+1} (L_n - L_{n-1})^{r-j} \frac{(r-j)L_n + L_{n-1}}{r-j+1}$$

It follows from the recurrence equations that

$$\Phi(r,n) = \Phi(r+n-1,1) + F(r+n-2,2) + F(r+n-3,3) + \dots + F(r,n)$$

where

$$\Phi(r,1) = rL_1^{r+1}/(r+1)$$

Finally

$$E\{\max(x_1, x_2, \dots, x_n)\} = \Phi(1, n)/L_1L_2\cdots L_n$$

To obtain $E\{M_2(x_1, x_2, \ldots, x_n)\}$ we first consider

$$\psi(r,n) = \int_0^{L_n} \int_0^{L_{n-1}} \cdots \int_0^{L_2} \cdots \int_0^{L_2} \int_0^{L_1} M_2(x_i)_{i=1}^{n+r-1} \prod_{i=1}^{n+r-1} dx_i$$

Then using $\int_0^{L_n} = \int_0^{L_2} + \int_{L_2}^{L_n}$ we obtain

$$\psi(r,n) = \psi(r+1,n-1) + (L_n - L_2)\psi(r,n-1)$$

Whence

$$\psi(r,n) = \psi(r+n-2,2) + S_1\psi(r+n-3,2) + S_2\psi(r+n-4,2) + \dots + S_{n-2}\psi(r,2)$$

where the S_i are the elementary symmetric functions of

$$L_3 - L_2, L_4 - L_2, \ldots, L_n - L_2$$

It now remains to determine

$$\psi(r,2) = \int_0^{L_2} \cdots \int_0^{L_2} \int_0^{L_1} M_2(x_i) \, dx_1 \, dx_2 \, \cdots \, dx_{r+1}$$

Symbolically,

$$\psi(r,2) = \sum_{j=0}^{R} {\binom{r}{j}} \left\{ \int_{L_1}^{L_2} \right\}^{r-j} \left\{ \int_0^{L_1} \right\}^{j+1} M_2(x_i) \, dx_1 \, dx_2 \, \cdots \, dx_{r+1}$$

or

$$\psi(r,2) = L_1 \int_{L_1}^{L_2} \cdots \int_{L_1}^{L_2} \min(x_i) \, dx_1 \, dx_2 \cdots \, dx_r$$
$$+ \sum_{j=1}^r \binom{r}{j} (L_2 - L_1)^{r-j} \int_0^{L_1} \cdots \int_0^{L_1} M_2(x_i) \, dx_1 \, dx_2 \cdots \, dx_{j+1}$$
$$= \frac{L_1 (L_2 - L_1)^r (rL_1 + L_2)}{r+1} + \sum_{j=1}^r \binom{r}{j} \frac{2L_1^{j+2} (L_2 - L_1)^{r-j}}{j+2}$$

Then

$$E\{M_2(x_1, x_2, \dots, x_n)\} = \psi(1, n)/L_1L_2\cdots L_n$$

The previous problems become even more involved if we replace the intervals $[0, L_i]$ by $[K_i, L_i]$.

[[also solved by a dozen others as well as the proposer]]

SIAM Rev., **13**(1971) 572.

Problem 71-28, *An Inequality*, by M. S. KLAMKIN (Ford Motor Company) and D. J. NEW-MAN (Yeshiva University).

Determine the largest value of the constant k such that

$$a^{3} + b^{3} + c^{3} \ge 3abc + K(a - b)(b - c)(c - a)$$

for all nonnegative a, b, c.

SIAM Rev., 14(1972) 656–657.

Solution by C. C. ROUSSEAU (Memphis State University).

The arithmetico-geometric mean inequality gives

$$a^3 + b^3 + c^3 \ge 3abc$$

for all nonnegative a, b, c. Hence, taking k to be positive, we need only consider the case where (a - b)(b - c)(c - a) > 0. Without loss of generality, we can require that a < b < c. The desired value of k is given by

$$k = \min_{\substack{a,b,c \ge 0 \\ a < b < c}} fraca^3 + b^3 + c^3 - 3abc(a-b)(b-c)(c-a)$$

Using the identity

$$a^{3} + b^{3} + c^{3} - 3abc = \frac{1}{2}(a+b+c)[(a-b)^{2} + (b-c)^{2} + (c-a)^{2}]$$

we write

$$k = \min_{\substack{a \ge 0\\s>t>0}} \frac{(3a+s+t)[s^2+t^2+(s-t)^2]}{2st(s-t)}$$

where s = c - a and t = b - a. From the above equation it is clear that a = 0 for the minimum. Hence, letting c = bx, we have simply

$$k = \min_{x>1} \frac{x^3 + 1}{x(x-1)}$$

By elementary calculs the desired value of x is the real root, greater than 1, of the quartic equation

$$x^4 - 2x^3 - 2x + 1 = 0$$

The quartic equation can be factored to yield

$$[x^{2} + (\sqrt{3} - 1)x + 1][x^{2} - (\sqrt{3} + 1)x + 1] = 0$$

It follows that the desired value of k is given by

$$k = \frac{\alpha^3 + 1}{\alpha(\alpha - 1)}$$

where $\alpha = \frac{1}{2}(1 + \sqrt{3} + \sqrt[4]{12})$. An approximate numerical value is $k \doteq 4.403669475$ [[solved by a dozen others and the proposers. Followed by the following MSK note:]] *Editorial note.* A simpler expression for k, as was noted by the proposers and several of the solvers, is $k = \sqrt{9 + 6\sqrt{3}}$. D. Shanks and the proposers also noted that it follows symmetrically that the smallest permissible value for the constant is $-\sqrt{9 + 6\sqrt{3}}$.

If a, b, c denote the sides of an arbitrary triangle, then the largest and smallest permissible values of k are $\pm \sqrt{9 + 6\sqrt{3}}$. This follows from a duality relation established by the first proposer (*Duality in triangle inequalities*, Ford Motor Company Preprint, July 1971. Also see Notices Amer. Math. Soc., August 1971, p.782); i.e., if a, b, c are the sides of a triangle, then there exist three nonnegative numbers x, y, z such that

$$x = s - a, \quad y = s - b, \quad z = s - c, \qquad 2s = a + b + c$$

and, conversely, for any three numbers x, y, z there exist sides of a triangle a, b, c and here

$$a = y + z$$
, $b = z + x$, $c = x + y$

Then, corresponding to any inequality in x, y, z we have a corresponding inequality in a, b, c and conversely, i.e.,

$$F(x, y, z) \ge 0 \Rightarrow F(s - a, s - b, s - c) \ge 0$$
$$G(a, b, c) \ge 0 \Rightarrow G(y + z, z + x, x + y) \ge 0$$

SIAM Rev., **14**(1972) 169.

[[An MSK editorial note which is worth repeating, since it bears on problem solving. – R.]]

Editorial note. Gould also notes that he has never seen anything "essentially" new obtained by lattice-point enumerations which could not be obtained from [the generalized Vandermonde convolution]. This is analogous to the view that any integral which is obtained by contour integration can also be obtained without it. However, it is always advantageous to have more than one method for solving a class of problems. [M.S.K.]

SIAM Rev., **15**(1973) 220.

Problem 73-8, *A Polynomial Diophantine Equation*, by M. S. KLAMKIN (Ford Motor Company).

Determine all real soltuions of the polynomial Diophantine equation

$$P(x)^{2} - P(x^{2}) = x\{Q(x)^{2} - Q(x^{2})\}$$
(1)

SIAM Rev., 16(1974) 99–100.

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands). From the given equation it follows that

$$P(x^{4}) - x^{2}Q(x^{4}) = P^{2}(x^{2}) - x^{2}Q^{2}(x^{2})$$

= { $P(x^{2}) - xQ(x^{2})$ } { $p(x^{2}) + xQ(x^{2})$ }

Letting $F(x) = P(x^2) - xQ(x^2)$ we have

$$F(x^2) = F(x)F(-x)$$
(2)

Conversely, any solution of (1) may be obtained from a solution of (2) by taking

$$P(x) = \frac{1}{2} \left\{ F(\sqrt{x}) + F(-\sqrt{x}) \right\}$$
$$Q(x) = \frac{1}{2x} \left\{ -F(\sqrt{x}) + F(-\sqrt{x}) \right\}$$

Polynomial solutions of (2) may be written in the form

$$F(x) = C(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$
 (C is a constant)

Then

$$F(-x) = (-1)^n C(x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_n)$$

so that

$$F(x)F(-x) = (-1)^{n}C^{2}(x-\alpha_{1})(x+\alpha_{1})(x-\alpha_{2})(x+\alpha_{2})\cdots(x-\alpha_{n})(x+\alpha_{n})$$

On the other hand, taking β_i such that $\beta_i^2 = \alpha_i \ (i = 1, ..., n)$, we find

$$F(x^2) = C(x-\beta_1)(x+\beta_1)(x-\beta_2)(x+\beta_2)\cdots(x-\beta_n)(x+\beta_n)$$

Therefore, in view of (2), excluding the trivial case C = 0, we obtain $C = (-1)^n$ and $(\alpha_i)_{i=1}^n$ is a permutation of $(\beta_i)_{i=1}^n$.

Finite, squaring-invariant subsets of the complex plane can only contain 0 and roots of unity of odd order. The irreducible polynomials corresponding to these roots are

$$\lambda_0(x) = x, \quad \lambda_k(x) = \prod_{(2k-1,l)=1} [x - \exp\{2\pi i l/(2k-1)\}], \quad k = 2, 3, \dots$$

(the cyclotomic polynomials). Since for all k = 1, 2, 3, ... the set

$$\left[\exp\{2\pi i l/(2k-1)\}\right]_{(l,2k-1)=1}$$

is squaring-invariant and the set of solutions of (2) is closed under multiplication, the general polynomial solution of (2) is

$$F(x) = (-1)^{\deg F} \prod_{k=0}^{\infty} (\lambda_k(x))^{n_k}$$

the n_k being non-negative integers, $n_k \neq 0$ for a finite number of indices k. These polynomials all have integer coefficients.

Also solved by the proposer who notes that one can give extensions by considering higher order roots of unity. For example, letting $\omega^3 = 1$, consider $F(x^3) = F(x)F(\omega x)F(\omega^2 x)$, where $F(x) = P(x^3) + \omega x Q(x^3) + \omega^2 x^2 R(x^3)$.

[[Article:]]

M. S. Klamkin, Moving axes and the Principle of the Complementary Function, SIAM Rev., **16**(1974) 295–302; MR **50** #9094.

SIAM Rev., **16**(1974) 94.

Problem 74-5^{*}, *On the norm of a matrix exponential*, by J. C. CAVENDISH (General Motors Research Laboratories).

The following problem arose in a study of discrete approximations to heat flow problems.

(A) Let A denote any $n \times n$ diagonally dominant matrix such that $a_{ii} \ge 0$. Show that $\|e^{-tA}\|_{\infty} \le 1$ for all $t \ge 0$, where e^{-tA} is the matrix exponential and $\|\cdot\|_{\infty}$ is the usual maximum row sum norm.

(B) Conjecture. Let A denote any real $n \times n$ matrix whose eigenvalues all have nonnegative real parts. Then there exists a constant K, independent of n, such that $\|e^{-tA}\|_{\infty} \leq K$ for all $t \geq 0$.

SIAM Rev., **17**(1975) 175.

Editorial comment by MSK. The most commonly used reference for proofs of

$$||e^{tA}|| \le e^{t\mu[A]} \quad \text{for all } t \ge 0$$

where

$$\mu[A] = \lim_{h \to 0+} \frac{\|I + hA\| - 1}{h}$$

is the logarithmic form associated with $\|\cdot\|$, and of

$$\mu_{\infty}[-A] = -\min_{t} \left\{ \operatorname{Re} a_{ii} - \sum_{\substack{j=1\\j \neq i}}^{n} |a_{ij}| \right\}$$

were W. A. Coppel, Stability and asymptotic behavior of Differential Equations, Heath, Boston, 1965, pages 59 and 41. Several solvers pointed out that if $A \in \mathbb{C}^{n,n}$ is a matrix whose eigenvalues have nonnegative real parts and whose pure imaginary eigenvalues are simple roots of the minimum polynomial of A, then $||e^{-tA}||_{\infty} \leq K(A)$ for all $t \geq 0$, where K(A) is independent of t but dependent on A and hence on n. However, part (B) was meant to be an extension of part (A), in which the proposer was seeking a characterization of the class T_K of all real $n \times n$ matrices for which $||e^{-tA}||_{\infty} \leq K$ for all $t \geq 0$ and $A \in T_K$. A complete characterization of T_1 was provided by J. C. WILLEMS (University of Groningen, Groningen, The Netherlands) and W. W. MEYER (General Motors Research Laboratories), both of whom proved that $||e^{-tA}||_{\infty} \leq 1$ for all $t \geq 0$ if and only if $a_{ii} \geq \sum_{j \neq i}$, $1 \leq i \leq n$. It should be pointed out that the proofs given by Willems and Meyer were not valid for $\Lambda \in \mathbb{C}^{n,n}$. SIAM Rev., **16**(1974) 258.

Problem 74-9, *Bounds for the zero of a polynomial*, by V. E. HOGGATT (San Jose State University).

Determine upper and lower bounds for the positive zero of the polynomial equation

$$x^{r+1} - (kx)^r - () - \dots - 1 = 0$$

 $(k > 1, r \ge 1)$ whose difference $\rightarrow 0$ as $r \rightarrow \infty$.

SIAM Rev., **17**(1975) 691.

Editorial comment by MSK. The Lossers solution is more general then the others in that k > 0; it also has the best lower bound

$$k^{r} + \frac{1}{k} - \{k^{r+2}(k^{r^{2}-1} + k^{-r-1})\}^{-1}$$

The problem appears in a different form in A. M. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966, pp. 102–103. However, the corresponding difference of the upper and lower bounds do not $\rightarrow 0$ as $r \rightarrow \infty$.

SIAM Rev., 16(1974) 547.

Problem 74-23^{*}, *Bounds for the zero of a polynomial, An Optimal Strategy*, by M. S. KLAMKIN (University of Waterloo, Ontario, Canada).

A father, mother and son decide to hold a certain type of board game family tournament. The game is a two-person one with no ties. Since the father is the weakest player, he is given the choice of deciding the two players of the first game. The winner of any game is then to play the person who did not play in that game, and so on. The first player to win N games wins the tournament. If the son is the strongest player, it is intuitive that the father will maximize his probability of winning the tournament if he chooses to play in the first game and with his wife. Prove either that this strategy is, indeed, optimal or that it is not. It is assumed that any player's probability of winning an individual game from another player does not change throughout the tournament.

The special case corresponding to N = 2 was set as a problem in the Third U.S.A. Mathematical Olympiad, May 1974.

For another related problem, prove either that the previous strategy is still optimal or it is not, if now the tournament is won by the first player who wins N consecutive games. This latter problem is a generalization of a variant of a chess problem due to the late Leo Moser (see M. Gardner, *The Unexpected Hanging*, Simon and Schuster, New York, 1969, pp.171–172).

COMBINATORICS

Counting problems: paths

SIAM Rev., **16**(1974) 547.

Problem 75-1, by R. W. ALLEN

An optical fiber carries power in two modes represented by 0 and 1. The path of one photon Is represented by an N-bit binary number. The sequence 0 1 or 1 0 is counted as one transition. Thus the path 1 0 0 0 1 1 1 contains two transitions and three zeros. Determine the number of paths S(N, T, M) that contain T transitions and M zeros. Prove whether or not the following formula is valid for all N:

$$S(N,T,M) = 2H(N,T)\binom{M-1}{U}\binom{N-M-1}{U}$$

where

$$H(2N,T) = \frac{\binom{N-1}{V}}{\binom{N-1}{U}}$$
$$H(2N+1,T) = \frac{\binom{2N}{2V}}{\binom{2N}{T-1}}$$

and

$$U = \left\lfloor \frac{T-1}{2} \right\rfloor \qquad V = \left\lfloor \frac{T}{2} \right\rfloor$$

SIAM Rev., 18(1976) 300. Editorial note by M.S.K. Kleitman notes that the solution to the analogous problem for any number of symbols where the number of transitions from each symbol to each other one is given, can be obtained by similar reasoning to the combinatorial solution along with the so-called B.E.S.T. theorem that relates the number of Eulerian circuits in a graph to a minor of the determinant of a version of its adjacency matrix.

A number of solvers noted that the problem is a known one in the theory of runs and more general results exist. In particular, Bailar, Carlitz, Mallows and Marimont, respectively, refer to:

M. FISZ. *Probability Theory and Mathematical Statistics*, John Wiley, New York, 1963; Eq. 11.3.12;

Ann. Math. Statist., **11**(1940) 147–167; 367–392;

Biometrika, **36**(1949) 305–316; and

W. FELLER, Introduction to Probability Theory and its Applications, John Wiley, New York, 1950 pp. 56–58.

ALGEBRA

Finite sums: exponentials

SIAM Rev., **17**(1975) 68.

Problem 75-3, A power series expansion, by U. G. HAUSSMANN (University of British Columbis)

Last year, a former engineering student of ours wrote to the mathematics department concerning a problem encountered in the electrical design for the appurtinent structures of the Mica Dam on the Columbia River in British Columbia. These structures include a spillway, low level outlets, intermediate level outlets, auxiliary service buildings and a power intake structure.

The engineers obtained a function

$$f(u) = \frac{\exp(u + nu) + \exp(-nu)}{1 + \exp u}$$

where

$$\cosh u = 1 + \frac{x}{2}$$

and where x is a ratio of resistances. Moreover, they suspected that if y = f[u(x)], then

$$y(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^{k}$$

Show that this is valid.

SIAM Rev., **18**(1976) 303.

Editorial comment by MSK. Lossers gives the esplicit sum

$$\sum_{k=0}^{n} \binom{n+k}{2k} x^{k} = \frac{2(2+x-t)^{n}}{4+x+t} + \frac{2(2+x+t)^{n}}{4+x-t}$$

where $t^2 = x^2 + 4x$. Byrd notes that the latter sum occurs in his paper, Expansion of analytic functions in polynomials associated with the Fibonacci numbers, *Fibonacci Quart.*, 1(1963) 17, and that for x = 1, the sum reduces to the odd [ranking] Fibonacci numbers F_{2n+1} . Carlitz shows more generally that if

$$f(\lambda, u) = [e^{(\lambda+1)u} - e^{-\lambda u}]/[e^u + 1]$$

where λ is an arbitrary complex number and $2 \cosh u = 2 + t^2$, then

$$f(\lambda, u) = \sum_{k=0}^{\infty} \binom{\lambda+k}{2k} t^{2k}$$

in some region about t = 0.

The other solutions used induction, generating functions, differential equations and the known expansions of

$$\frac{\cos(2n+1)\theta}{\cos\theta} \quad \text{or} \quad \frac{\sin n\theta}{\sin\theta}$$

NUMBER THEORY

Series: binomial coefficients

SIAM Rev., **17**(1975) 168.

Problem 75-4^{*}, A combinatorial identity, by P. BARRUCAND (Université Paris IV, France)

Let

$$A(n) = \sum_{i+j+k=n} \frac{n!^2}{i!^2 j!^2 k!^2}$$

where i, j, k are integers ≥ 0 , and let

$$B(n) = \sum_{m=0}^{n} \binom{n}{m}^{3}$$

so that A(n) is the sum of the squares of the trinomial coefficients of rank n and B(n) is the sum of the cubes of the binomial coefficients of rank n ($A(n) = 1, 3, 15, 93, 639, \ldots$; $B(n) = 1, 2, 10, 56, 346, \ldots$).

Prove that

$$A(n) = \sum_{m=0}^{n} \binom{n}{m} B(m)$$

SIAM Rev., **17**(1975) 303–304.

Editorial note by MSK. Equivalently, one has to prove that

$$\sum_{m=0}^{n} \sum_{r=0}^{m} \binom{n}{m} \binom{m}{r}^{3} = \sum_{m=0}^{n} \binom{2m}{m} \binom{n}{m}^{2}$$

For other properties, e.g., recurrences, integral representations, etc., the proposer refers to his papers in *Comptes Rendus Acad. Sci. Paris*, **258**(1964) 5318–5320 and **260**(1965) 5439–5441 [not 5541]. He also notes that his solution is a tedious indirect one.

By equating the constant term in the binomial expansion on both sides of the identity

$$\{1 + (1+x)(1+y/x)(1+1/y)\}^n = \left\{1 + \frac{1+x}{y}\right\}^n \left\{1 + y\left(1 + \frac{1}{x}\right)\right\}^n$$

or an equivalent identity, the desired result was obtained by D. R. BREACH (University of Canterbury, Christchurch, New Zealand), D. MCCARTHY (University of Waterloo), D. MONK (University of Edinburgh, Scotland) and P. E. O'NEIL (University of Massachusetts, Boston).

With a little more work, one can obtain a more general identity. Expanding the above identity yields

$$\sum_{r=0}^{n} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{k=0}^{r} \binom{n}{r} \binom{r}{i} \binom{r}{j} \binom{r}{k} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{s} \binom{r}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{n} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} \binom{s}{j} x^{i-j} y^{j-k} = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{s} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{i-j} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{i-j} y^{s-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{j-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{j-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{j-r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \binom{n}{i} x^{j-r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{j=0}^{r}$$

Equating the coefficients of $x^u y^v$ on both sides gives

$$\sum_{r=0}^{n} \sum_{k=0}^{r} \binom{n}{r} \binom{r}{k} \binom{n}{r} \binom{r}{k} \binom{r}{k} \binom{r}{v+k} \binom{r}{v+u+k} =$$
$$\sum_{r=0}^{n} \binom{n}{r} \binom{n}{r+v} \sum_{j=0}^{r+v} \binom{r}{u+j} \binom{r+v}{j} = \sum_{r=0}^{n} \binom{n}{r} \binom{n}{r+v} \binom{2r+v}{r-u}$$

Equivalent but more complicated versions of the latter identity were obtained by L. CARLITZ (Duke University) and D. J. KLEITMAN and Class 18.325 (Massachusetts Institute of Technology). Also solved, using hypergeometric functions, by G. E. AN-DREWS (Pennsylvania State University), M. E. H. ISMAIL (University of Wisconsin), O. G. RUEHR (Michigan Technological University); using probability by C. L. MAL-LOWS (Bell Laboratoried, Murray Hill); and, using differential equations, the proposer in a second solution establishes the equivalent identity

$$\sum \frac{x^n A(n)}{n!} = e^x \sum \frac{x^n B(n)}{n!}$$

ALGEBRA

Inequalities: finite sums

SIAM Rev., **17**(1975) 169.

Problem 75-5^{*}, A nonnegative form, by M. M. GUPTA (Papua New Guinea University of Technology, Lae, Papua New Guinea).

Suppose p and q are positive integers, p > q, and Z_1, \ldots, Z_p are arbitrary real numbers. Define

$$\alpha = p^{-2}q^{-2}(p-q)^{-1}$$
$$\beta_p = (Z_2 - 2Z_1)^2 + \sum_{i=2}^{p-1} (Z_{i-1} - 2Z_i + Z_{i+1})^2$$

and

$$I_{p,q} = -2q^{3}\alpha Z_{1}Z_{p} + 2p^{3}\alpha Z_{1}Z_{q} - 2\alpha Z_{1}^{2} + ()\beta_{p}$$

Show that $I_{p,q} \ge 0$.

This problem arose in deriving discretization error estimates for the first biharmonic boundary value problem in a rectangular region.

SIAM Rev., **18**(1976) 764.

Editorial note by MSK. The proposer states that the result is easy to verify if q = 1 and that it also holds for q = 2, 3 and 4.

hypergeometric functions

SIAM Rev., **17**(1975) 685.

Problem 75-17^{*}, A series of hypergeometric functions, by H. M. SRIVASTAVA (University of Victoria, B.C., Canada).

Let

$$F\begin{bmatrix}a:b,b',\dots;c,c'\dots;\\d,d',\dots;e,e',\dots;x,y,z\end{bmatrix}$$

= $\sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b)_{l+m}(b')_{l+m}\dots(c)_{l+n}(c')_{l+n}\dots}{(d)_{l+m}(d')_{l+m}\dots(e)_{l+n}(e')_{l+n}\dots} \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!}$

and

$$\phi(x, y, z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n \prod_{j=1}^p (a_j)_n \prod_{j=1}^r (\alpha_j)_n}{n! \prod_{j=1}^q (b_j)_n \prod_{j=1}^s (\beta_j)_n} \left[\frac{xyz}{(1-z)^2} \right]^n \cdot G$$

where

$$G = {}_{p+1}F_q \left[\begin{array}{c} \lambda + n, a_1 + n, \cdots, a_p + n; \\ b_1 + n, \cdots, b_q + n; \end{array} \frac{xz}{z-1} \right] \times {}_{r+1}F_s \left[\begin{array}{c} \lambda + n, \alpha_1 + n, \cdots, \alpha_r + n; \\ \beta_1 + n, \cdots, \beta_s + n; \end{array} \frac{yz}{z-1} \right]$$

Prove or disprove that

$$\phi(x, y, z) = F \begin{bmatrix} \lambda : a_1, \cdots, a_p; \alpha_1, \cdots, \alpha_r; & xyz & xz \\ b_1, \cdots, b_q; \beta_1, \cdots, \beta_s; & a-z & z-1 & z-1 \end{bmatrix}$$

SIAM Rev., **18**(1976) 768.

Editorial note by MSK. Carlitz obtained the desired result as a special case of the following more general identity:

Let a(m), $\alpha(m)$, b(m), $\beta(m)$ denote arbitrary sequences of (complex) numbers. Put

$$\phi(x, y, z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n a(n)\alpha(n)}{n! b(n)\beta(n)} \left(\frac{xyz}{(1-z)^2}\right)^n \cdot G_n$$

where

$$G_n = \sum_{i=0}^{\infty} \frac{(\lambda+n)_i a(n+i)/a(n)}{i!b(n+i)/b(n)} \left(\frac{xz}{z-1}\right)^i \times \sum_{j=0}^{\infty} \frac{(\lambda+n)_j \alpha(n+j)/\alpha(n)}{j!\beta(n+j)/\beta(n)} \left(\frac{yz}{z-1}\right)^j$$

Also put

$$F(x,y,z) = \sum_{k,m,n=0}^{\infty} \frac{(\lambda)_{k+m+n}a(k+m)\alpha(k+n)}{k!m!n!b(k+m)\beta(k+n)} \left(\frac{xyz}{1-z}\right)^k \left(\frac{xz}{z-1}\right)^m \left(\frac{yz}{z-1}\right)^n$$

Then, by taking

$$a(n) = \prod_{i=1}^{p} (a_i)_n \qquad \alpha(n) = \prod_{j=1}^{p} (\alpha_j)_n$$
$$b(n) = \prod_{i=1}^{q} (b_i)_n \qquad \beta(n) = \prod_{j=1}^{s} (\beta_j)_n$$

it is easily verified that

$$\phi(x, y, z) = F(x, y, z)$$

reduces to the stated identity.

ANALYSIS

Integrals: evaluations

SIAM Rev., 17(1975) 566.

Problem 75-12, An average distance, by H. J. OSER (National Bureau of Standards). Evaluate the 4-fold integral

$$F = \int_0^1 \int_{-1}^0 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left\{ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right\}^{1/2} dx_1 dx_2 dy_1 dy_2$$

which gives the average distance between points in two adjacent unit squares.

SIAM Rev., **18**(1976) 497.

Editorial note by MSK. The proposer notes that the problem was suggested by C. R. Johnson and that the result should be of interest to workers in transportation modeling and similar fields.

SIAM Rev., 17(1975) 567–568.

Problem 75-15, An eigenvalue problem, by E. WASSERSTROM (Israel Institute of Technology, Haifa, Israel).

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \qquad T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

where d_1 , d_2 and d_3 are positive and $d_3 \leq d_1$. Show that if $d_3 < d_1/3$, then there are two other positive diagonal matrices D_1 and D_2 such that D, D_1 and D_2 are distinct but DT, D_1T and D_2T have the same eigenvalues. Show also that if $d_3 > d_1/3$ and D_1 is a positive diagonal matrix distinct from D, then DT and D_1T must have different eigenvalues. Illustration. For the three matrices D = diag(5.5596, 1.4147, 1.5257),

 $D_1 = \text{diag}(5.1030, 2.4288, 0.9682)$ and $D_2 = \text{diag}(2.9782, 4.6565, 0.8653)$, correct to the given figures, the eigenvalues of DT, D_1T and D_2T are the same, i.e., $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 12$. On the other hand, with D = diag(3.4530, 1.4584, 1.5887), the eigenvalues of DT are $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 8$, and there is no other positive diagonal matrix D_1 such that the eigenvalues of D_1T are the same.

Remark. This problem arises from the discretization of the inverse eigenvalue problem $d^2y/dx^2 = \lambda \rho(x)y$, y(0) = y(1) = 0. For a given spectrum, λ , one is then required to find the density function $\rho(x)$. (See B. M. Levitan & M. G. Gasymov, Determination of a differential equation by means of two spectra, *Uspehi Mat. Nauk.*, **19**(1964) 3–63.)

SIAM Rev., **18**(1976) 502–503.

Editorial note by MSK. The proposer's solution is essentially a numerical one. It would be desirable to give an analytic solution.

[Such a solution, by O. P. Lossers, follows.]

Bessel functions

SIAM Rev., **17**(1975) 686.

Problem 75-20, Limit of an integral, by M. L. GLASSER (University of Waterloo, Ontario, Canada).

Show that

$$\lim_{n \to \infty} n \int_0^\infty I_n(x) J_n(x) K_n(x) \, dx = 5^{-1/2}$$

where, as usual, I_n , J_n , K_n are Bessel functions.

SIAM Rev., **18**(1976) 772.

Editorial note by MSK. Also solved by D. E. AMOS (Sandia Laboratories) and A. G. GIBBS (Battelle Memorial Institute) who both showed that

$$\lim_{n \to \infty} \int_0^\infty I_n(x) J_n(ax) K_n(bx) \, dx = \frac{1}{2} (1+a^2)^{-1/2}$$

Procedures for obtaining the asymptotic behavior of this integral and more general ones appeared in

M. L. Glasser & F. W. J. Olver, Asymptotic behaviour of integrals of Bessel functions of high order, *Utilitas Math.*, **12**(1977) 225–239; *MR* **57** #16737.

Die Autoren beschäftigen sich mit zwei Methoden zur Beurteilung des asymptotischen Verhaltens der uneigentlichen Integrale

$$U = \int_0^\infty J_{\nu}(ax) I_{\nu}(bx) K_{\nu}(bx) x^{\rho-1} \, dx$$

und

$$V = \int_0^\infty J_\nu(ax) Y_\nu(ax) I_\nu(bx) K_\nu(bx) x^{\sigma-1} dx$$

für $\nu \to \infty$. $J_{\nu}, Y_{\nu}, I_{\nu}, K_{\nu}$ sind wie üblich Bessel-Funktionene reeller Ordnung $\nu \ge 0$; ρ und σ (reell oder komplex) und a > 0, b > 0 sind Konstanten. Bei der ersten Variante sie hat Bezug zu Untersuchungen von J. E. Kilpatrick, S. Katsura und Y. Inoue, *Math. Comput.*, **21**(1967) 407–412; *Zbl* **154** 415—werden *U* bzw. *V* in Zusammenhang gebracht mit Integraldarstellungen vom Barnesschen Typ für die Meijersche *G*-Funktion. Diese Transformation wurde gemacht, um die bekannte Asymptotik der hier vorkommenden Gammafunktionen ausnutzen zu können. Die zweite Methode stützt sich auf asymptotische Entwicklungen der Zylinderfunktionen für grosse Werte ν nach Termen von elementaren oder Airy-Funktionen. [Reviewed by F. Gotze]

GEOMETRY

N-dimensional geometry: curves

SIAM Rev., **17**(1975) 687.

Problem 75-21, *n*-dimensional simple harmonic motion, by I. J. SCHOENBERG (University of Wisconsin).

In R_n we consider the curve

$$\Gamma :: x_i = \cos(\lambda_i t + a_i) \qquad i = 1, \dots, n \qquad -\infty < t < \infty$$

which represents an *n*-dimensional simple harmonic motion entirely contained within the cube $U : -1 \le x_i \le 1$, i = 1, ..., n. We want Γ to be truly *n*-dimensional and will therefore assume without loss of generality that $\lambda_i > 0$ for all *i*. We consider the open sphere

$$S : \sum_{i=1}^{n} x_i^2 < r^2$$

and want the motion Γ to take place entirely outside of S, hence contained in the closed set U-S. What is the largest sphere S such that there exist motions Γ entirely contained in U-S? Show that the largest such sphere S_0 has the radius $r_0 = \sqrt{n/2}$ and that the only motions Γ within $U-S_0$ lie entirely on the boundary $\sum x_i^2 = r_0^2$ of S_0 .

SIAM Rev., 18(1976) 773.

Comment by MSK. If, as usual, we consider the 2n-dimensional motion where the velocities are the other n coordinates, then the [given] arguments show that the minimum of

$$g(t) = \sum_{i=1}^{n} [\cos^2(\lambda_i t + a_i) + \lambda_i^2 \sin^2(\lambda_i t + a_i)] \le \frac{1}{2} \left(n - \sum_{i=1}^{n} \lambda_i^2 \right) < r_0^2$$

Thus the 2*n*-dimensional problem has the same solution in the sense that all spheres with radii $< r_0$ have a Γ outside them, but the sphere with radius r_0 does not. The motions also need not be truly *n*-dimensional unless the λ_i are independent over the rationals.

NUMBER THEORY

Series: unit fractions

SIAM Rev., **18**(1976) 118.

Problem 76-5^{*}, An arithmetic conjecture, by D. J. NEWMAN (Yeshiva University) To determine positive integers a_1, a_2, \ldots, a_n such that

$$S_n = \sum_{i=1}^n \frac{1}{a_i} < 1$$

and S_n is a maximum, it is conjectured that at each choice one picks the smallest integer still satisfying the inequality constraint. Is this conjecture true?

SIAM Rev., **19**(1977) 155.

... for example, for n = 4 one would choose

$$\frac{1}{2}\frac{1}{3}\frac{1}{7}\frac{1}{43}$$

Editorial note by MSK. P. Erdős notes that this problem was raised by Kellog in 1921 and solved by Curtiss (On Kellog's Diophantine equation, Amer. Math. Monthly, **29**(1922) 380–387). Curtiss shows that if $\{u_n\}$ is defined by $u_1 = 1$, $u_{k+1} = u_k(u_k + 1)$ (giving rise to the sequence 1, 2, 6, 42, 1806, ...) and if $1/F_n = 1 - S_n$, then the maximum finite value of F_{n-1} , for all positive values of $a_1, a_2, \ldots, a_{n-1}$ is u_n and also there is but one set of the a_i which give this maximum value, namely $a_k = u_{k+1}, k = 1, 2, \ldots, n-1$.

It would be of interest to solve the following extension of the problem: we wish the stated conjecture to still be valid if the a_i are further restricted to be members of a given infinite sequence $\{b_n\}$ with $\sum 1/b_k = \infty$. Characterize all such sequences $\{b_k\}$. In particular, is the conjecture valid for $b_k = 2k$; for $b_k = 2k + 1$, $k = 1, 2, 3, \ldots$?

Differential equations: order n

SIAM Rev., **18**(1976) 118.

Problem 76-6, An *n*-th Order Linear Differential Equation, by M. S. KLAMKIN (University of Waterloo).

Solve the differential equation

$$[x^{2n}(D - a/x)^n - k^n]y = 0.$$

SIAM Rev., **19**(1977) 155.

Editorial note by MSK. Most of the solutions reduced the equation simply to $[x^{2n}D^n - k^n]u = 0$ and then referred to Kamke's Differentialgleichungen. Ortner also solved the dual equation $[x^n(D-a/x)^{2n}-k^n]y = 0$ in a similar fashion. More generally, it is just as easy to solve the pair of equations

$$[x^{2n}(D+\phi'(x))^n - k^n]y = 0 \qquad [x^n(D+\phi'(x))^{2n} - k^n]y = 0$$

for by the exponential shift theorem, they reduce to

$$[x^{2n}D^n - k^n]ye^{\phi} = 0 \qquad [x^nD^{2n} - k^n]ye^{\phi} = 0$$

The latter pair can be solved in terms of solutions of first order equations by using the known dual operational identities

$$x^{2n}D^n \equiv [x^2D + (1-n)x]^n \qquad x^nD^{2n} \equiv [xD^2 + (1-n)D]^n$$

LINEAR ALGEBRA

Matrices: Hermitian matrices

SIAM Rev., **18**(1976) 295.

Problem 76-8, A matrix inequality, by W. ANDERSON and G. TRAPP (West Virginia University).

Let A and B be Hermitian positive definite (HD) matrices. Write $A \ge B$ if A - B is HD. Show that

$$A^{-1} + B^{-1} \ge 4(A+B)^{-1}$$

SIAM Rev., **19**(1977) 331.

Editorial note by MSK. The theorem quoted by Moore follows from that of Lieb by setting k = 2, $C_1 = \lambda A$, $C_2 = (1 - \lambda)B$, $D_1 = \lambda I$ and $D_2 = (1 - \lambda)I$. Several solvers noted that for the statement of the problem to be correct, the relation $A \ge B$ should be defined to mean that A - B is Hermitian positive semi-definite (HSD).

[[Classroom Notes in Applied Mathematics.]]

M. S. Klamkin, On extreme length flight paths, SIAM Rev., 18(1976) 486–488; MR 57 #7291.

[[References include Problem 61-4 above;

the papers: M. S. Klamkin & D. J. Newman, Flying in a wind field, I, II, *Amer. Math. Monthly*, **76**(1969) 16–23, 1013–1019; *MR* **38** #5496, **40** #3951;

and the following:]]

Problem 926, Math. Mag., 48(1975) 51.

[[missing from our Research Collection, but presumably by Murray — Later: No! it's by Melvin F. Gardner, U of Toronto, so we don't need to bother? — R.]]

COMBINATORICS

Permutations

SIAM Rev., **18**(1976) 491.

Problem 76-17^{*}, *A Reverse Card Shuffle*, by DAVID BERMAN and M. S. KLAMKIN (University of Waterloo).

The following problem, originating somewhere in England, was brought to our attention by G. Cross.

A deck of n cards is numbered 1 to n in random order. Perform the following operations on the deck. Whatever the number on the top card is, count down that many in the deck and turn the whole block over on top of the remaining cards. Then, whatever the number of the (new) top card, count down that many cards in the deck and turn this whole block over on top of the remaining cards. Repeat the process. Show that the number 1 will eventually reach the top.

Consider the following set of related and more difficult problems:

I. Determine the number N(k) of initial card permutations, so that the 1 first appears on top after k steps of the process. In particular, show that N(0) = N(1) = N(2) = (n-1)! and that

$$N(3) = \begin{cases} (n-1)! - \frac{1}{2}(n-1)(n-3)(n-4)! & n \text{ odd} \\ (n-1)! - \frac{1}{2}(n-2)^2(n-4)! & n \text{ even} \end{cases}$$

(The method of the authors is apparently too unwieldy to determine N(k) for k > 3). [['k > 2' was printed – R.]]

2. Estimate the maximum number of steps it takes to get the 1 to the top.

3. For what n is there a unique permutation giving the maximum number of steps?

4. It is conjectured that the last step of a maximum step permutation leaves the cards in order (i.e., $1, 2, \ldots, n$).

Computer calculations give the following partial results:

	Maximum	Number of
n	number of	maximum
	steps	permutations
1	0	1
2	1	1
3	2	2
4	4	2
5	7	1
6	10	5
7	16	2
8	22	1
9	30	1

The first four steps of the maximum step permutation for n = 9 are:

Step	Permutation							
0	615	97	28	34				
1	279	51	68	34				
2	729	51	68	34				
3	861	59	27	34				

SIAM Rev., **19**(1977) 740–741.

Editorial note. D. E. KNUTH (Stanford University) notes that the card shuffle game here was shown to him in 1973 by J. H. CONWAY (Cambridge University) who proposed it and named it "topswaps". In the next year Knuth included part 2 on a take-home examination in the following form (also included is his solution):

Problem 3. Let $\pi = \pi[1]\pi[2]\cdots\pi[n]$ be a permutation of $\{1, 2, \ldots, n\}$ and consider the following algorithm:

```
begin integer array A[1:n]; integer k
      (A[1],\ldots,A[n]) \leftarrow (\pi[1],\ldots,\pi n);
           print (A[1], ..., A[n]);
loop:
            k \leftarrow A[1];
            if k = 1 then go to finish;
            (A[1],\ldots,A[n]) \leftarrow (A[k],\ldots,A1);
            go to loop;
```

finish: end

For example, when n = 9 and $\pi = 314562687$, the algorithm will print

3	1	4	5	9	2	6	8	7
4	1	3	5	9	2	6	8	7
5	3	1	4	9	2	6	8	7
9	4	1	3	5	2	6	8	7
7	8	6	2	5	3	1	4	9
1	3	5	2	6	8	7	4	9

and then it will stop.

Let $m = m(\pi)$ be the total number of permutations printed by the above algorithm. Prove that m [[mis?printed n in the original – RKG]] never exceeds the Fibonacci number F^{n+1} . (In particular, the algorithm always halts.)

Extra credit problem. Let $M_n = \max\{m(\pi) \mid \pi \text{ a permutation of } \{1, \ldots, n\}\}$. Find the best upper and lower bounds on M_n that you can.

Problem 3 solution. If array element A[1] takes on k distinct values during the (possibly infinite) execution of the algorithm, we will show that $m \leq F_{k+1}$ (hence m is finite). This is obvious for k = 1, since k = 1 can occur only when $\pi[1] = 1$.

If $k \ge 2$, let the distinct values assumed by A[1] be $d_1 < d_2 < \cdots < d_k$. Suppose that $A[1] = d_k$ occurs first on the *r* th permutation, and let $t = \pi[d_k]$. Then the (r + 1)st permutation will have A[1] = t and $A[d_k] = d_k$. All subsequent permutations will also have $A[d_k] = d_k$ (they leave A[j] untouched for $j \ge d_k$), hence at most k - 1 values are assumed by A[1] after the *r* th permutation has been passed. By induction, $m - r \le F_k$, so *m* is finite and $d_1 = 1$.

Interchanging d_k with 1 in π produces a permutation π' such that $m(\pi') = r$, and for which the values d_k and t never appear in position A[1] unless t = 1. If t = 1we have $r \leq F_k$, since A[1] assumes at most k - 1 values when processing π' , hence $m = r + 1 \leq F^{k+1}$. If t > 1 we have $r \leq F_{k-1}$ since A[1] assumes at most k - 2 values when processing π' (note that $t = d_j$ for j < k) hence $m \leq F_k + r \leq F_{k+1}$.

Three hours of further concentration on this problem lead to the hypothesis that it is difficult either to prove or to disprove the conjecture $M_n = O(n)$; the upper bound F_{n+1} is exact only for $n \leq 5$.

[The upper bound applies more generally to any algorithm that sets $(A[1], \ldots, A[k]) \leftarrow (A[k], A[p_2], \ldots, A[p_{k-1}], A[1])$ when $p_2 \ldots p_{k-1}$ is an arbitrary permutation of $\{2, \ldots, k-1\}$.

Computer calculations show that $M_6 = 11$, $M_7 = 17$, $M_8 = 23$, $M_9 = 31$, so $M_{n+1} - M_n$ may possibly increase without limit. This search is speeded up slightly by restricting consideration to permutations without fixed points.

The long-winded permutations on 7, 8, 9 elements are 3146752, 4762153; 61578324; 615972834.

When $n \ge 3$ and $1 \le k \le 3$, exactly (n-1)! permutations π satisfy $m(\pi) = k$. It is conjectured that exactly (n-1)! permutations π will satisfy "A[1] = n at some stage."]

[[The sequence 0, 1, 2, 4, 7, 10, 16, 22, 30 is A000375 in OEIS and continues 38, 51, 65, 80, 101, 113, 139 — RKG]]

SIAM Rev., **20**(1978) 856.

B. J. HOLLINGSWORTH (Pennzoil Company) notes that the conjecture in part 4 (see Oct. 1977, p. 739) is false, as may be seen by the counterexample 416523 for n = 6. After ten steps, one ends up with 143256 which is not in increasing order. The four remaining "long-winded" permutations for n = 6, namely 365142, 456213,564132 and 415263, do end up as 123456.

Integrals: gamma function

SIAM Rev., **19**(1977) 146.

Problem 77-1^{*}, Percentiles for the gamma distribution, by R. A. WALLER and M. S. WA-TERMAN (Los Alamos Scientific Laboratory).

If $0 < \xi_1 < \xi_2 < 1$ and 1 < b are fixed, consider solutions (λ, ϕ) of the system

$$f(\lambda,\phi) = \int_0^\lambda \frac{e^{-y}y^{\phi-1}}{\Gamma(\phi)} dy = \xi_1$$
$$g(\lambda,\phi) = \int_0^{b\lambda} \frac{e^{-y}y^{\phi-1}}{\Gamma(\phi)} dy = \xi_2$$

where $0 < \lambda$ and $0 < \phi$. Does this system always have a solution? If a solution exists, is it unique? This problem arises in a procedure for determining gamma priors in Bayesian reliability analysis. In this context, some pairs, (ξ_1, ξ_2) , of interest are (0.01, 0.50), (0.01, 0.95), (0.05, 0.50), (0.05, 0.95), (0.50, 0.95) and (0.50, 0.99).

SIAM Rev., **20**(1978) 856.

Editorial note by MSK. Only an existence proof for this problem was given in the Jan. 1978 issue. The following comment by I. W. SAUNDERS (CSIRO Division of Mathematics and Statistics, Canberra, Australia) establishes uniqueness:

Write $\zeta(\phi;\xi)$ for the ξ -quartile of the gamma distribution with parameter ϕ , so that

$$\int_0^\zeta e^{-y} y^{\phi-1} \, dy / \Gamma(\phi) = \xi$$

Then we want to show that, when $1 > \xi_2 > \xi_1 > 0$ and b > 1, the equation

$$\eta(\phi;\xi_2)/\zeta(\phi;\xi_1) = b$$

has a unique solution ϕ .

Saunders & Moran 1 show that $r(\phi) = \eta(\phi;\xi_2)/\zeta(\phi;\xi_1)$ is decreasing with ϕ . Since $\zeta^{\phi}e^{-\zeta} < \xi\Gamma(\phi+1) < \zeta^{\phi}$ it is easily shown that $r(\phi) \to \infty$ as $\phi \to 0$. Also, using the central limit theorem, noting that the gamma distribution is the convolution of unit exponential distributions, $\zeta(n;\xi) = n + O(\sqrt{n})$, so that $r(n) \to 1$ as $n \to \infty$ for n an integer. Hence, since $r(\phi)$ is decreasing, $r(\phi) \to 1$ as $\phi \to \infty$.

Thus the equation has a unique solution $\phi(b)$ for any $b \in (1, \infty)$.

REFERENCE

[1] I. W. Saunders & P. A. P. Moran, On the quantiles of the gamma and F distributions, J. Appl. Probability, **15**(1978) 426–432; MR **58** #3153.

Integrals: evaluations

SIAM Rev., **19**(1977) 147.

Problem 77-3, A definite integral of N. Bohr, by P. J. SCHWEITZER (IBM Research Center).

N. Bohr [1] investigated the integral

$$K = \int_0^\infty F(x)(F'(x) - \ln x) \, dx$$

where

$$F(x) = \int_{-\infty}^{\infty} \frac{\cos xy \, dy}{(1+y^2)^{3/2}}$$

is related to a modified Bessel funcyion [2] and he numerically obtained the rough approximate result K = -0.540. Find an exact expression for K.

REFERENCES

 Niels Bohr, Collected Works, Vol. 1, L. Rosenfeld, ed., North-Holland, Amsterdam, 1972, pp. 554–557.

[2] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions*, Applied Mathematics Series 55, U.S. Government Printing Office, National Bureau of Standards, 1964, Eq. 9.6.25.

SIAM Rev., **20**(1978) 189–190.

Editorial note by MSK. Bohr's comments on the evaluation of K are contained in a letter which he wrote to his brother, Harald Bohr. Bohr first derives a series expansion for F based on the fact that F satisfies the differential equation F'' - (1/x)F' - F = 0. He also derives an asymptotic expansion for F. The, evidently, Bohr employs two series representations in appropriate intervals and uses numerical integration techniques to evaluate K. After what he describes as "some days of numerical drudgery" he obtains $K \approx -0.540$. The exact source of error in Bohr's result is, perhaps, a subject for historical speculation. Amos notes the interesting numerical fact that $(4/\pi)G'(1) \approx$ -0.54073, which fosters speculation to the effect that the numerical value quoted by Bohr referes to only part of the integral defining K. In any case, it is worth noting that Bohr's basic approach is viable enough. With the aid of a computer, it is a relatively easy matter to implement Bohr's program and so obtain $K \approx -1.15063$.

Integrals: evaluations

SIAM Rev., **19**(1977) 329.

Problem 77-8, A Definite Integral, by M. L. GLASSER (University of Waterloo).

Prove that

$$\int_0^\infty \frac{\log |J_0(x)|}{x^2} \, dx = -\frac{\pi}{2}$$

Editorial note. A class of such integrals has been treated by B. Berndt and the author by complex integration.

Bruce C. Berndt & M. L. Glasser, A new class of Bessel function integrals, Aequationes Math., 16(1977) 183–186; MR 57 #740.

Under appropriate conditions on a rational function f(x), the authors use the calculus of residues to evaluate the principal value (PV) of $\int_{-\infty}^{\infty} \{J_{\nu+1}(x)/J_{\nu}(x)\}f(x) dx$, where $J_{\nu}(x)$ is the Bessel function of the first kind of order ν . An interesting special case is PV $\int_{0}^{\infty} \{J_{\nu+1}(x)/x J_{\nu}(x)\} dx = \pi/2$. In the latter, if $\nu = -\frac{1}{2}$, the integrand is $(\tan x)/x$. Also, PV $\int_{0}^{\infty} \{J_{\nu+1}(x)/x (x^2 + a^2) J_{\nu}(x)\} dx = \pi I_{\nu+1}(a)/2a^2 I_{\nu}(a)$.

SIAM Rev., 20(1978) 595.

comment by MSK

GEOMETRY

Triangle inequalities: circumradius

SIAM Rev., **19**(1977) 329.

Problem 77-9, A Triangle Inequality, by I. J. SCHOENBERG (University of Wisconsin).

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3, x_1 < x_2 < x_3$, be points in the Cartesian (x, y)-plane and let R denote the radius of the circumcircle Γ of the triangle $P_1P_2P_3$ ($R = \infty$ if the triangle is degenerate). Show that

$$\frac{1}{R} < 2 \left| \frac{y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{y_2}{(x_2 - x_3)(x_2 - x_1)} + \frac{y_3}{(x_3 - x_1)(x_3 - x_2)} \right|$$

unless both sides vanish, and that 2 is the best constant.

SIAM Rev., **20**(1978) 400.

Editorial note by MSK. The proposer notes that the problem had arisen in determining conditions that would ensure that an $F(x) \in C[0, 1]$ is a linear function. He then lets $F(x_1, x_2, x_3)$ be the 2nd order divided difference of F(x) where $0 \le x_1 < x_2 < x_3 \le 1$. If $F(x_1, x_2, x_3) \to 0$ whenever x_1, x_2, x_3 converge to a common limit l in [0,1], for all l, then the inequality shows that the plane arc y = F(x) has Menger curvature zero at all of its points. It then follows by a theorem of Menger that the arc is straight. Also, the proposer in his proof establishes the equivalent interesting result:

If P_1 , P_2 , P_3 are three distinct points on a parabola, then their circumcircle is larger than the circle of curvature at its vertex.

GEOMETRY

Triangle inequalities: sides

SIAM Rev., **19**(1977) 329.

Problem 77-10, *A Two Point Triangle Inequality*, by M. S. KLAMKIN (University of Alberta).

Let P and P' denote two arbitrary points and let $A^1 A^2 A^3$ denote an arbitrary triangle of sides a_1, a_2, a_3 . If $R_i = PA_i$ and $R'_i = P'A_i$, prove that

$$a_1 R_1 R_1' + a_2 R_2 R_2' + a_3 R_3 R_3' \ge a^1 a^2 a^3 \tag{1}$$

and determine the conditions for equality. It is to be noted that when P' coincides with P, we obtain a known polar moment of inertia inequality.

SIAM Rev., **20**(1978) 400–401.

Solution by the proposer.

We start with the known identity for five arbitrary complex numbers

$$\frac{z-z_1}{z_1-z_2} \cdot \frac{z'-z_1}{z_1-z_3} + \frac{z-z_2}{z_2-z_3} \cdot \frac{z'-z_2}{z_2-z_1} + \frac{z-z_3}{z_3-z_1} \cdot \frac{z'-z_1}{z_3-z_2} = 1$$
(2)

It now follows by the triangle inequality that

$$|z - z_1| |z' - z_1| |z_2 - z_3| + |z - z_2| |z' - z_2| |z_3 - z_1| + |z - z_3| |z' - z_3| |z_1 - z_2|$$

$$\ge |z_1 - z_2| |z_2 - z_3| |z_3 - z_1|$$
(3)

with equality if and only if each of the three terms on the left hand side of (2) are real. Let z_1, z_2, z_3, z, z' be the complex numbers corresponding to the points A_1, A_2, A_3, P, P' respectively, then (3) is equivalent to (1). The equality condition requires that

$$\angle A_2 A_1 P = \angle P' A_1 A_3, \quad \angle A_1 A_2 P = \angle P' A_2 A_3, \quad \angle A_2 A_3 P = \angle P' A_3 A_1$$

[[This is as printed, but I'm suspicious about it. If we use it, would some hero check if the following version is the correct one:

$$\angle A_2 A_1 P = \angle P' A_1 A_3, \quad \angle A_3 A_2 P = \angle P' A_2 A_1, \quad \angle A_1 A_3 P = \angle P' A_3 A_2]]$$

Thus the two points P and P' must be isogonal conjugates with respect to the given triangle. If P is the center of the inscribed circle of the triangle, then P' coincides with P. If P is the center of the circumcircle, then P' is the orthocenter.

It is a known result (1) that if P and P' are foci of an ellipse inscribed in the triangle $A_1A_2A_3$, then we have the equality condition of inequality (1). The proof given was

geometric. Fujiwara [2] using identity (2) easily establishes the equality condition of inequality (1) that P and P' must be isogonal conjugates. The ellipse result of [1] also follows easily from (2) by using the known general angle property of an ellipse that two tangents to an ellipse from a given point make equal angles with the focal radii to the given point.

Many other triangle inequalities can be obtained in a similar fashion. These are given in a paper *Triangle inequalities from the triangle inequality*, submitted for publication.

[1] W. J. MILLER, Mathematical Questions and Solutions from the Educational Times, 7, F. Hodgson, London, 1876. Problem 210, p.43.

M. FUJIWARA, On the deduction of geometrical theorems from algebraic identities, Tôhoku Math.
 J., 4(1913-1914) 75–77.

[The paper mentioned in the last para. is more precisely:

M. S. Klamkin, Triangle inequalities from the triangle inequality, *Elem. Math.*, **34**(1979) 49–55; *MR* **80m**:51013

The author obtains about 30 inequalities for the parts of a triangle by applying the basic inequality $|u| + |v| \ge |u + v|$ to complex number identities arising by considering the triangle in the complex plane, with an arbitrary point as origin. As observed, some of these are new, others have appeared elsewhere [see O. Bottema, R. Ž. Dordević, R. R. Janić, D. S. Mitrinović and P. M. Vasić, Geometric inequalities, Wolters-Noordhoff, Groningen, 1969; *MR* **41** #7537)], and elsewhere.

Reviewed by H. T. Croft]]

SIAM Rev., **21**(1979) 257

comment by MSK..

ALGEBRA

Inequalities: degree 3 [and 4]

SIAM Rev., **19**(1977) 563.

Problem 77-12^{*}, *Conjectured inequalities*, by PETER FLOR (University of Cologne, West Germany).

Establish or disprove the following inequalities where all the variables are positive:

$$a^{3} + b^{3} + c^{3} + 3abc \ge a^{2}(b+c) + b^{2}(c+a) + c^{2}(a+b)$$

 $\begin{array}{l} 39a^3 + 15a(b^2 + c^2) + 20ad^2 + 5bc(b + c + d) \geq 10a^2(b + c) + 43a^2d + 39abc + ad(b + c) \\ 5(a^4 + b^4 + c^4 + d^4) + 6(a^2c^2 + b^2d^2) + 12(a^2 + c^2)bd + 12(b^2 + d^2)ac \geq \end{array}$

 $2(a^{3} + b^{3} + c^{3} + d^{3})(a + b + c + d) + 4(a + c)(b + d)(ac + bd) + 2(a^{2} + c^{2})(b^{2} + d^{2}) + 8abcd$

Note that, obviously, the first two do not hold for all *real* values of the variables. The situation for the third is the same: consider a = b = 1, c = d = -1. Further note that on equating the variables, equality is obtained in all cases.

All three are particular cases of a general inequality which I published as a conjecture ten years ago and on which no progress has been reported so far (see *Bull. Amer. Math. Soc.*, 72(1966), Research Problem 1, p. 30). Their proof might indicate a method for attacking this old conjecture.

Editorial note. The proposer does have a proof of the first inequality.

Here is the BAMS Research Problem:

Peter Flor: *Matrix theory*

For any square matrix A, let per(A) denote the permanent of A and s(A) the sum of the elements of A.

Prove or disprove the following statement: "If M is any $n \times n$ matrix of real nonnegative numbers, and if k is any integer, $1 \le k \le n$, then

$$\sum (\operatorname{per}(B) - \operatorname{per}(C))(s(B) - s(C)) \ge 0$$

where B and C range independently over the $k \times k$ submatrices of M."

For the case of M being *doubly-stochastic* the statement reduces to a conjecture of Holens (see [1]) which in turn would imply the affirmative solution of van der Waerden's famous problem on permanents (see e.g. [2]).

References

1. F. Holens, Two aspects of doubly stochastic matrices and the minimum of the permanent conjecture, *Canad. Math. Bull.*, **7**(1964) 507–510.

2. M. Marcus & M. Newman, On the minimum of the permanent of a doubly stochastic matrix, *Duke Math. J.*, **26**(1959) 61–72.

SIAM Rev., 20(1978) 599.

Editorial note by MSK. The first inequality is a special case (n = 1) of an inequality of Schur, i.e.,

$$a^{n}(a-b)(a-c) + b^{n}(b-c)(b-a) + c^{n}(c-a)(c-b) \ge 0$$

COMBINATORICS

Graph theory: trees

2

SIAM Rev., **20**(1978) 329.

Problem 77-15*, A Conjectured Minimum Valuation Tree, by I. CAHIT (Turkish Telecommunications, Nicosia, Cyprus).

[[Also published as E 2671^{*} in *Amer. Math. Monthly*, **84**(1977) 651, with a solution, similar to Fan Chung's below, by G. W. Peck (MIT) at **85**(1978) 827, and a reference to *SIAM Rev.* It is closely related to the notorious Kotzig-Ringel tree-labelling problem—R.K.G.]]

Let T denote a tree on n vertices. Each vertex of the tree is labelled with distinct integers from the set 1, 2, ..., n. The weight of an edge of T is defined as the absolute value of the difference between the vertex numbers at its endpoints. If S denotes the sum of all the edge weights of T with respect to a given labelling, it is conjectured that for a k-level complete binary tree, the minimum sum is given by

$$S_{\min}^{(k)} = \min_{\text{all labellings}} \sum_{(i,j)\in T} |i-j| = (k-1)2^{(k-1)} \qquad (k>1)$$

Examples of minimum valuation trees for k = 2, 3, 4 are given by

6

.

10

6

4

2

4

14

12

2

Editorial note by Murray. A. Meir suggested the related problem of determining $\min \sum (i-j)^2$. More generally, one can consider max and $\min \sum |i-j|^m$.

SIAM Rev., **20**(1978) 601–604.

Solution by F. R. K. CHUNG (Bell Laboratories). The conjecture is not true for k > 4. The following labelling for the 5-level complete binary tree shows that $S_{\min}^{(5)} \leq 60 < 4 \cdot 2^4$



We let $S_k = S_{\min}^{(k)}$. It can be shown that

$$S_k = 2^k (k/3 + 5/18) + (-1)^k (2/9) - 2$$

Proof. The optimal labelling L_k for the k-level binary tree T_k satisfies the following properties. The proof can be found in [1] or [2] or can be easily verified.

Property 1. The vertex labelled by 1 or $n = 2^k - 1$ in L_k is a leaf (a *leaf* is a vertex of valence 1).

Property 2. Let P denote the path connecting the two vertices labelled by 1 and n in L_k . Let P have vertices v_0, \ldots, v_t . Then the labelling of the vertices of P is monotone, i.e.,

$$L(v_i) < L(v_{i+1})$$
 for $i = 0, \ldots, t-1$

or

$$L(v_i) > L(v_{i+1})$$
 for $i = 0, \ldots, t-1$

Property 3. In T_k we remove all edges of P. The resulting graph is a union of vertex disjoint subtrees. Let \tilde{T}_i denote the subtree which contains the vertex v_i , $i = 1, \ldots, t-1$. Then for a fixed i, the set of labellings of vertices in \tilde{T}_i consists of consecutive integers. Moreover, the labelling on each \tilde{T}_i is optimal.

Property 4. Let \tilde{v} be the only vertex of T_k with valence 2. Then P passes through \tilde{v} .

Property 5. Let T'_k denote the tree which contains T_k as a subtree and T'_k has one more vertex than T_k , which is a leaf adjacent to \tilde{v} . Then $S(T'_k) = S_k + 2$.

From properties 1 to 5, the following recurrence relation holds:

$$S_k = 2^{k-1} + 4 + S_{k-1} + 2S_{k-2}$$
 for $k \ge 4$

and $S_2 = 2, S_3 = 8$.

It can easily be verified by induction that

$$S_k = 2^k (k/3 + 5/18) + (-1)^k (2/9) - 2$$

If we consider k-level complete p-nary trees T_k^p , some asymptotic estimates for S_k^p , the minimum sum of all edge weights of T_k^p over all labellings, have been obtained in [1] and [2]. We will briefly discuss the case p = 3.

Let $T_p(k, i)$ denote a k-level tree which has the root connected to i copies of a (k-1)level p-nary tree. For example the graph

is $T_3(3,2)$. Let $S_p(k,i)$ denote the minimum value of the sum of all edge weights of $T_p(k,i)$ over all labellings of $T_p(k,i)$.

It can be easily verified that

$$S_3(k,2) = 3S_3(k-1,2) + 2 \cdot 3^{k-2}$$
 for $k \ge 3$

and $S_3(2,2) = 2$. Therefore

$$S_3(k,2) = 2(k-1)3^{k-2}$$
 for $k \ge 2$

In general it can be shown that for p odd,

$$S_p(k,2) = k(p+1)p^{k-2}/2 + (-2p^k + 3p^{k-1} + p^{k-2} + p - 3)/(2(p-1))$$

and

$$S_p(k, p-1) = (k-1)(p^2 - 1)p^{k-2}/4$$

The recurrence relation for S_k^3 is as follows: Let f(k) be the integer l satisfying

$$(l-1)3^{l-2} + l + 1 \le k < l3^{l-1} + l \quad k \ge 3$$

Then we have

$$S_k^3 = 3^{k-2}(2k - 1/2) - 1/2 + k - f(k) + S_{k-1}^3$$
 for $k \ge 3$

and $S_2^3 = 4$.
This complicated recurrence relation reveals the possible difficulty in getting an explicit expression for S_k^p for general p.

REFERENCES

 M. A. Iordanskiĭ, Minimal numberings of the vertices of trees, Soviet Math. Dokl., 15(1974) 1311–1315

[2] M. A. Šeĭdvasser, The optimal numbering of the vertices of a tree, *Diskret. Analiz.*, 17(1970) 56–74; *MR* bf45 #105.

Also solved by W. F. SMYTH (Winnipeg, Manitoba) who sent a copy of a long paper, *A labelling algorithm for minimum edge weight sums of complete binary trees*, which had been submitted to Comm. ACM, whose interests were felt to be more diectly related to the subject matter.

[[RKG couldn't trace this paper. A MathSciNet search on "labelling algorithm" kielded 7 hits and on "labeling algorithm" 13. None had Smyth as an author. Maybe it was rejected?]]

An abstract of the paper is as follows:

Given a K-level complete binary tree $T_K = (V_K, E_K)$ on $2^K - 1$ vertices and a set $\mathcal{W}_K = \{N+1, N+2, \ldots, N+2^K-1\}$ of integers, it is desired to label the vertices V_K from the set \mathcal{W}_K without replacement, in such a way that the sum $S_K = \sum |n(u) - n(v)|$ taken over all edges $(u, v) \in E_K$ is a minimum, where n(u) denotes the label assigned to vertex u. The labelled tree is called a valuation tree and, corresponding to a minimum labelling, a minimum valuation tree. An algorithm for this purpose is specified, with execution time $O(2^K-1)$. An expression is derived for S_K^{\min} and it is shown that in fact the algorithm achieves this minimum. Connections to the minimum bandwidth and minimum profile problems are outlined. Some open problems are stated.

ANALYSIS

Differential equations: systems of equations

SIAM Rev., **19**(1977) 736.

Problem 77-17, A system of second order differential equations, by L. CARLITZ (Duke University)

Solve the following system of differential equations:

$$F''(x) = F(x)^{3} + F(x)G(x)^{2}$$
$$G''(x) = 2G(x)F(x)^{2}$$

where F(0) = G'(0) = 1, F'(0) = G(0) = 0

SIAM Rev., **20**(1978) 859.

Editorial note by MSK. Margolis, in her solution, first noted that it was easy to find a solution $F(x) = \sec x$, $G(x) = \tan x$ and then establish uniqueness. The proposer obtained the system of equations by considering generating functions associated with up-down and down-up permutations of $\{1, 2, \ldots, n\}$.

STATISTICS

Covariance

SIAM Rev., **19**(1977) 736–737.

Problem 77-18, *An infinite summation*, by A. M. LIEBETRAU (Johns Hopkins University).

Show that

$$\sum_{j=1}^{\infty} \alpha_j^{-6} \left[\frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j} \right]^2 = \frac{1}{80}$$

where the α_j are the positive solutions to the equation

$$(\cos\alpha)(\cosh\alpha) + 1 = 0$$

This identity follows from a problem in statistics, that of finding the distribution of a certain functional of a Gaussian process $\eta(t)$ with covariance kernel

$$E[\eta(t), \eta(u)] = K(t, u) = \begin{cases} tfrac23(3t^2u - t^3) & 0 \le t \le u \le 1\\ tfrac23(3u^2t - u^3) & 0 \le u \le t \le 1 \end{cases}$$

SIAM Rev., **20**(1978) 860–862.

Solution by the proposer.

In order to obtain the distribution of a certain random functional of a Poisson process [2], it became necessary to solve the following eigenvalue problem: Express the positive symmetric function

$$K(t,u) = \begin{cases} \frac{2}{3}(3t^2u - t^3) & 0 \le t \le u \le 1\\ \frac{2}{3}(3u^2t - u^3) & 0 \le u \le t \le 1 \end{cases}$$
(1)

in the form

$$\sum_{j=1}^{\infty} \lambda_j^{-1} f_j(t) f_j(u) \qquad (2)$$

where λ_j is an eigenvalue and $f_j(t)$ is the corresponding normalized eigenfunction of the system

$$f(t) = \lambda \int_0^1 K(t, u) f(u) \, du \qquad \int_0^1 f_j(t) f_k(t) = \delta_{jk} \qquad (3)$$

In (3) δ_{jk} is the Kronecker delta.

Substitution of (1) into the first equation of (3) yields

$$f(t) = \frac{2}{3}\lambda \left\{ \int_0^t (3u^2t - u^3)f(u) \, du + \int_t^1 (3t^2u - t^3)f(u) \, du \right\}$$
(4)

Successive differentiation of (4) with respect to t yields

$$f^{(4)}(t) = 4\lambda f(t) = \alpha^4 f(t)$$
 (5)

which is easily seen to have the solution

$$f(t) = c_1 e^{-\alpha t} + c_2 e^{\alpha t} + c_3 \cos(\alpha t) + c_4 \sin(\alpha t)$$
 (6)

for suitable constants c_1 , c_2 , c_3 , c_4 .

Boundary conditions for determining the c_j are obtained from considering f(0), f'(0), f''(0) and f'''(0). Substitution of the appropriate derivatives of (6) into (4) produces the following system of equations:

$$c_{1} + c_{2} + c_{3} = 0$$

$$c_{1} - c_{2} - c_{4} = 0$$

$$(\alpha + 1)e^{-\alpha}c_{1} + (1 - \alpha)e^{\alpha}c_{2} - (\alpha \sin \alpha + \cos \alpha)c_{3} + (\alpha \cos \alpha - \sin \alpha)c_{4} = 0$$

$$e^{-\alpha}c_{1} - e^{\alpha}c_{2} - (\sin \alpha)c_{3} + (\cos \alpha)c_{4} = 0$$

$$(7)$$

Elimination of c_1 and c_2 from (7) yields, after some manipulation:

 $(\cosh \alpha + \cos \alpha)c_3 + (\sinh \alpha + \sin \alpha)c_4 = 0 = (\sinh \alpha - \sin \alpha)c_3 + (\cosh \alpha + \cos \alpha)c_4 \quad (8)$

The equations (8) have nontrivial solution if and only if

$$(\cosh \alpha)(\cos \alpha) + 1 = 0 \tag{9}$$

Being a covariance function, (1) is positive definite, so it is only necessary to consider positive solutions to (9): let α_j denote the *j* th smallest, so that $\lambda_j = \frac{1}{4}\alpha_j^4$, $j = 1, 2, \dots$ From (7) and (9) it follows that

$$c_{3} = \frac{\cos \alpha_{j} + \cosh \alpha_{j}}{\sin \alpha_{j} - \sinh \alpha_{j}} c_{4} := \kappa_{j} c_{4}$$

$$c_{2} = -\frac{1}{2} (c_{3} + c_{4}) = -\frac{1}{2} (\kappa_{j} + 1) c_{4}$$

$$c_{1} = \frac{1}{2} (c_{4} - c_{3}) = -\frac{1}{2} (\kappa_{j} - 1) c_{4}$$

Finally, c_4 is chosen so that $\int_0^1 f_j^2(t) dt = 1$. A light but elementary calculation yields

$$c_4 = \kappa^{-1} = \frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cos \alpha_j} \tag{10}$$

hence

$$f_j(t) = -\frac{1}{2}(1 - \kappa_j^{-1})e^{-\alpha_j t} - \frac{1}{2}(1 + \kappa_j^{-1})e^{\alpha_j t} + \cos(\alpha_j t) + \kappa_j^{-1}\sin(\alpha_j t)$$

Now, by Mercer's theorem on positive definite kernels, it follows (see Churchill [1] or Riesz & Nagy [3], for example) that the series (2) converges absolutely and uniformly to K(t, u) in the unit square. Thus

$$\int_{0}^{1} \int_{0}^{1} K(t, u) dt du = \frac{4}{3} \int_{0}^{1} \int_{0}^{u} (3t^{2}u - t^{3}) dt du = \frac{1}{5}$$

$$= \int_{0}^{1} \int_{0}^{1} \sum_{j=1}^{\infty} \lambda_{j}^{-1} f_{j}(t) f_{j}(u) dt du = \sum_{j=1}^{\infty} \lambda_{j}^{-1} \int_{0}^{1} \int_{0}^{1} f_{j}(t) f_{j}(u) dt du$$

$$= \sum_{j=1}^{\infty} \lambda_{j}^{-1} \left[\int_{0}^{1} f_{j}(t) dt \right]^{2} = \sum_{j=1}^{\infty} \lambda_{j}^{-1} (2\lambda_{j}^{-1/2}\kappa_{j}^{-2}) \qquad (11)$$

$$= 2\sum_{j=1}^{\infty} \left(\frac{1}{4} \alpha_{j}^{4} \right)^{-3/2} \kappa_{j}^{-2} = 16 \sum_{j=1}^{\infty} \alpha_{j}^{-6} \kappa_{j}^{-2}$$

We conclude from (11) that

$$\sum_{j=1}^{\infty} \alpha_j^{-6} \kappa_j^{-2} = \sum_{j=1}^{\infty} \alpha_j^{-6} \left[\frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j} \right]^2 = \frac{1}{80}$$

where $\{\alpha_j\}_1^\infty$ are the positive solutions to (9) and κ_j^{-1} is given by (10). Moreover, since (9) is an even function of α ,

$$\sum_{j=-\infty}^{\infty} \alpha_j^{-6} \kappa_j^{-2} = \sum_{j=-\infty}^{\infty} \alpha_j^{-6} \left[\frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j} \right]^2 = \frac{1}{40}$$

where $\alpha_{-j} = -\alpha_j$.

REFERENCES

[1] R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill, New York, 1963.

[2] A. M. Liebetrau, Some tests of randomness based upon the second-order properties of the Poisson process, *Math. Sci. Tech. Report* **249**, The Johns Hopkins University, Baltimore, 1976. [[Compare *MR* **57** #18000, **58** #8106 and **81d**:62092.]]

[3] F. Riesz & B. Nagy, Functional Analysis, Ungar, New York, 1955.

Editorial note by MSK. The proof can be simplified by noting from (4) that f(0) = f'(0) = 0 = f''(1) = f'''(1). This leads to a simpler set of equations for (7) and (8). M. L. GLASSER (Clarkson College of Technology) sketches a proof by contour integrals for the related identities:

$$\sum_{j=1}^{\infty} \alpha_j^{-3} \left\{ \frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j} \right\} = 0$$
$$\sum_{j=1}^{\infty} (\alpha_j^4 - \lambda^4)^{-1} = \left\{ \frac{\sin \lambda \cosh \lambda - \sinh \lambda \cos \lambda}{1 + \cos \lambda \cosh \lambda} \right\} / 4\lambda^3 \qquad |\lambda| < |alpha_1|$$

He also indicates that the original equation can be derived in this way, but since the poles involved are second order, the calculations get messy.

TRIGONOMETRY

Inequalities: cos

SIAM Rev., **19**(1977) 737.

Problem 77-19^{*}, *Two inequalities*, by P. BARRUCAND (Université P. et M. Curie, Paris, France).

Let

$$F_1(\theta) = \sum_{n=1}^{\infty} \frac{\cos^n \theta \cos n\theta - \cos^{2n} \theta}{n(1 - 2\cos^n \theta \cos n\theta + \cos^{2n} \theta)}$$
$$F_2(\theta) = \sum_{\substack{n=1\\n\equiv 1(2)}}^{\infty} \frac{\cos^n \theta \cos n\theta - \cos^{2n} \theta}{n(1 - 2\cos^n \theta \cos n\theta + \cos^{2n} \theta)}$$

It is conjectured that $F_1(\theta)$ and $F_2(\theta)$ are negative for $0 < \theta < \pi/2$. The conjecture was found by a computer computation.

SIAM Rev., **21**(1979) 140.

 $Editorial\ note$ by MSK. W. Al Salam has shown that the conjectures are equivalent to showing that

$$\prod_{k=1}^{\infty} |1 - x^k e^{ik\theta}| < 1$$

and

$$\prod_{k=1}^{\infty} \left| \frac{1 + x^k e^{ik\theta}}{1 - x^k e^{ik\theta}} \right| < 1$$

where $x = \cos \theta$ and 0 < x < 1. SIAM Rev., **22**(1980) 509. solution by MSK.

ANALYSIS

Hermite interpolation

[[This item not necessarily connected with MSK]]

SIAM Rev., **20**(1978) 182.

Problem 78-2, *Two recurrence relations for Hermite basis polynomials*, by J. C. CAVENDISH and W. W. MEYER (General Motors Research Laboratories).

For p a positive integer, let $\Phi_k(x)$ denote a (2p + 1)-degree basis polynomial for (2p + 1)-Hermite interpolation on $0 \le x \le 1$. That is, for $n, k = 0, 1, \ldots, p$

$$\frac{d^{n}\Phi_{k}}{dx^{n}}\bigg\}_{x=0} = \begin{cases} 0 & \text{if } n \neq k\\ 1 & \text{if } n = k \end{cases}$$
$$\frac{d^{n}\Phi_{k}}{dx^{n}}\bigg\}_{x=1} = 0$$

Establish the following two recurrence relations for any $t \in [0, 1]$:

$$t\Phi_{k-1}(t) - k\Phi_k(t) = \frac{(2p-k+1)!}{p!(k-1)!(p-k+1)!}t^{p+1}(1-t)^{p+1} \qquad (0 < k \le p)$$

$$\Phi_{k-1}(t) - \Phi'_k(t) = \frac{(2p-k+1)!}{p!k!(p-k+1)!}t^p(1-t)^p(p+1-kt) \qquad (0 < k \le p)$$

SIAM Rev., **21**(1979) 144.

[[This item included by error. The comment was by Otto G. Ruehr, not MSK.]]

ALGEBRA

Determinants

[[This item not necessarily connected with Murray. The editorial comment is by Cecil C. Rousseau, not Murray.]]

SIAM Rev., **20**(1978).

Problem 78-3^{*}, *A conjecture on determinants*, by H. L. LANGHAAR and R. E. MILLER (University of Illinois).

A special case of a more general conjecture on determinants that has been corroborated numerically by operations with random determinants generated by a digital computer is expressed by the equation $\Omega = \Delta^{n+1}$ in which $\Delta = |a_1b_2 \cdots q_{n-1}r_n|$ is any *n* th order determinant and Ω is a determinant of order n(n+1)/2 constructed from the elements of Δ as follows: The first row of Ω consists of all terms that occur in the expansion of $(a_1 + a_2 + \cdots + a_n)^2$. A similar construction applies for rows 2, 3, ..., *n*. Row n + 1consists of expressions that occur in the expansion of

$$(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$$

A similar construction applies for the remaining rows. The letters in the columns in Ω are ordered in the same way as the subscripts in the rows. Prove or disprove the conjecture $\Omega = \Delta^{n+1}$

	$ a_1^2 $	a_{2}^{2}	• • •	a_n^2	$2a_1a_2$	$2a_1a_3$	• • •	$2a_{n-1}a_n$
	b_1^2	b_2^2	• • •	b_n^2	$2b_1b_2$	$2b_1b_3$	• • •	$2b_{n-1}b_n$
	• • •	•••	•••	• • •	• • •	• • •	•••	
		• • •	•••	•••	•••	•••	•••	•••
$\Omega =$	r_1^2	r_{2}^{2}	•••	r_n^2	$2r_{1}r_{2}$	$2r_{1}r_{3}$	• • •	$2r_{n-1}r_n$
	a_1b_1	a_2b_2	•••	$a_n b_n$	$a_1b_2 + a_2b_1$	$a_1b_3 + a_3b_1$	• • •	$a_{n-1}b_n + a_n b_{n-1}$
	a_1c_1	a_2c_2	•••	$a_n c_n$	$a_1c_2 + a_2c_1$	$a_1c_3 + a_3c_1$	•••	$a_{n-1}c_n + a_n c_{n-1}$
	• • •	• • •	•••	• • •	•••	•••	• • •	
		•••	•••	• • •	• • •	• • •	• • •	
	q_1r_1	$q_2 r_2$	• • •	$q_n r_n$	$q_1r_2 + q_2r_1$	$q_1r_3 + q_3r_1$	• • •	$q_{n-1}r_n + q_n r_{n-1}$

SIAM Rev., **21**(1979) 146.

Editorial note. The general conjecture referred to by the proposers is, in fact, the result which is proved by [David] Cantor. Jordan and Taussky point out that the theorem in question was proved by Schläfli in 1851 [2]. Other solvers noted that the theorem is a well-known result in multilinear algebra (see, e.g., [1, Chap.2])

REFERENCES

[1] M. Marcus, *Finite Dimensional Multilinear Algebra*, Part i. Marcel Dekker, New York, 1973.

[2] L. Schläfli, Über die Resultante eines Systems mehrerer algebraischen Gleichungen, Denkschr. d. k. Akad. d. Wiss. (Wien) Math.-Naturw. Cl., iv(2)(1851) 1–54.

NUMBER THEORY

Recurrences: arrays

SIAM Rev., **20**(1978) 394.

Problem 78-6, *A combinatorial identity*, by PETER SHOR (U.S.A. Mathematical Olympiad Team, 1977).

A function S(m, n) is defined over the nonnegative integers by

$$(A)S(0,0) = 1$$

(B)S(0,n) = S(m,0) = 0 for $m, n \ge 1$
(C)S(m+1,n) = mS(m,n) + (m+n)S(m,n-1)

Show that

$$\sum_{n=1}^m S(m,n) = m^m$$

SIAM Rev., **21**(1979) 259–260.

Comment by A. MEIR (University of Alberta).

By essentially the same inductive argument [as that of the proposer/solver] one can show that the more general recursion relation

$$S(m+1,n) = (m+z)S(m,n) + (m+n)S(m,n-1)$$

has a solution satisfying

$$\sum_{m=1}^{m} S(m, n) = (m+z)^{m}$$

Editorial note by MSK. Ruehr notes that his method can be applied to Meir's generalization. The differential recurrence equation becomes

$$F_{m+1} = [(m+z) + x(m+1)]F_m + x^2F'_m$$

whose solution is

$$F_m = \frac{1}{x^{m+1}} e^{(m+z)/x} \left[x^3 e^{-1/x} \frac{d}{dx} \right]^m x e^{-z/x}$$

The final step then requires the identity

$$\sum_{n=1}^{\infty} \frac{(n+z)^n \lambda^n e}{n!} = \frac{e^{\lambda z}}{1-\lambda}$$

which he also establishes.

It would be of interest to determine all pairs of polynomials P(m, n), Q(m, n) (in particular, linear ones) such that if

$$S(m + 1, n) = P(m.n)S(m, n) + Q(m, n)S(m, n - 1)$$

and subject to conditions (A) and (B), then also

$$\sum_{n=1}^m S(m,n) = m^m$$

PROBABILITY

Geometry: point spacing

SIAM Rev., **20**(1978) 394–395.

Problem 78-8, *Average distance in a unit cube*, by TIMO LEIP AL A (University of Turku, Turku, Finland).

Determine (a) the probability density, (b) the mean, and (c) the variance for the Euclidean distance between two points which are independently and uniformly distributed in a unit cube.

The numerical value of the mean is given in [1] and it is conjectured in [3] that an explicit closed form expression for it does not exist. The probability distribution for the distance in the interval [0,1] is given in [2].

REFERENCES

[1] R. S. Anderssen, R. P. Brent, D. J. Daley & P. A. P. Moran, Concerning $\int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2} dx_1 \cdots dx_k$ and a Taylor series method, *SIAM J. Appl. Math.*, **30**(1976) 22–30.

[2] É. Borel, *Principes et Formules Classiques du Calcul des Probabilités*, Gauthier-Villars, Paris, 1925, pp. 88–90.

[3] D. J. Daley, Solution to problem 75-12, this Review, **18**(1976) 498–499.

SIAM Rev., **21**(1979) 263.

Editorial note by MSK. The mean distance is also found for a rectangular box in E2629 by T. S.Bolis & D. P. Robbins, *Amer. Math. Monthly*, **85**(1978) 277–278.

COMBINATORICS

Coloring problems: tournaments

SIAM Rev., 20(1978) 593.

Problem 78-11, *Edge three-coloring of tournaments*, by N. MEGIDDO (University of Illinois).

We define an edge k-coloring of a tournament (i.e., a directed graph with a unique edge between every pair of vertices) to be that of coloring the edges in k colors such that every directed cycle of length n contains at least $\min(k, n)$ edges of distinct colors.

It can be easily seen that every tournament has a 2-coloring. Specifically, if the vertices are numbered 1, ..., $m \ (m \ge 2)$, then color every "ascending" edge black and every "descending" edge white.

We shall show that for each $k \ge 4$ there are tournaments which do not have k-colorings. Given $k \ge 4$, let m = k + 1 and consider a directed graph whose vertices are $1, \ldots, m, m+1, \ldots, 2m$ and whose edges are (i, m+i) $(i = 1, \ldots, m)$ and (m+j,i) $(i = 1, \ldots, m, j = 1, \ldots, m, i \ne j)$. Every pair of edges (i, m+i), (j, m+j) $(1 \le i < j \le m)$ lies on some cycle of length 4, namely (i, m+i), (m+i, j), (j, m+j), (m+j, i). Thus in every k-coloring the edges $(1, m+1), (2, m+2), \ldots, (m, 2m)$ must have distinct colors. This implies that there is no k-coloring for such graphs.

The remaining open question is: Does every tournament have a three-coloring? I conjecture that it does.

SIAM Rev., **21**(1979) 398.

—it Editorial note by MSK. G. K. KRISTIANSEN (Roskilde, Denmark) gives a counterexample, disproving the conjecture. It is the tournament T with vertices $1, 2, \ldots, 9$ that contains the directed edge (i, j) if and only f $j - i = 1, 2, 4, 6 \pmod{9}$. J. MOON (University of Alberta) simplifies Kristiansen's proof from one involving four cases to one of two cases. Although it is straightforward it still is a tedious exercise to show that T has no 3-coloring. L. L. KEENER (University of Waterloo) described a construction for a larger counterexample.

GEOMETRY

N-dimensional geometry: inequalities

SIAM Rev., **20**(1978) 856.

Problem 78-20, *A Volume Inequality for a Pair of Associated Simplexes*, by M. S. KLAMKIN (University of Alberta).

The lines loining the vertices $\{V_i\}$ i = 0, 1, ..., n of a simplex S to its centroid G meet the circumsphere of S again in points $\{V'_i\}$, i = 0, 1, ..., n. Prove that the volume of simplex S' with vertices V'_i is \geq the volume of S.

SIAM Rev., **21**(1979) 569–570.

Solution by the proposer.

Let V_i (V'_i) denote the volume of the simplex whose vertices are G and those of the face F_i (F'_i) of S (S') opposite A_i (A'_i) . It follows that

$$\frac{V_i'}{V_i} = \frac{GA_i}{GA_i'} \prod \frac{GA_j'}{GA_j}$$

where in the product here and subsequently (also sums), the index runs from 0 to n. By the power of a point theorem for spheres,

$$GA_i \cdot GA_i' = R^2 - OG^2 = k$$

where R, O are the circumradius and circumcenter, respectively, of S. Also, $V_i = V/(n+1)$ where V = volume of S. Then,

$$\frac{V_i'}{V} = \frac{k^n G A_i^2}{\prod G A_j^2}$$

and

$$V' = \sum V'_i = \frac{k^n V}{n+1} \frac{\sum GA_j^2}{\prod GA_j^2}$$

We now want to show that

$$\frac{k^n}{n+1}\sum GA_j^2 \ge \prod GA_j^2$$

By the arithmetic-geometric mean inequality, it suffices to establish the stronger inequality

$$\frac{k^n}{n+1}\sum GA_j^2 \ge \left\{\frac{\sum GA_j^2}{n+1}\right\}^{n+1}$$

Actually the latter is an equality suince it is known tha

$$k = \sum GA_i^2/(n+1)$$

and which follows from (where $\mathbf{A}_i = \overrightarrow{OA_i}$)

$$\sum_{i} \sum_{j} A_{i} A_{j}^{2} = \sum_{i} \sum_{j} (\mathbf{A}_{i} - \mathbf{A}_{j}) \cdot (\mathbf{A}_{i} - \mathbf{A}_{j})$$
$$= 2(n+1)^{2} R^{2} - 2 \sum_{i} \mathbf{A}_{i} \cdot \sum_{j} \mathbf{A}_{j}$$
$$= 2(n+1)^{2} (R^{2} - OG^{2})$$
$$= \sum_{i} \sum_{j} \{(\mathbf{A}_{i} - \mathbf{G}) - (\mathbf{A}_{j} - \mathbf{G})\}^{2}$$
$$= 2(n+1) \sum GA_{i}^{2}$$

V' = V if and only if GA_i = constant or equivalently O and G coincide. For n = 2 this implies the triangle is equilateral. This special case is knownand is ascribed to Janić [1]. For n = 3 the tetrahedron must be isosceles (opposite faces are congruent).

Also solved by L. GERBER (St. John's University) who additionally poses the problem of determining the limit of the volume (for n = 3) if the process is repeated indefinitely.

REFERENCE

[1] O. BOTTEMA, R. Z. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ and P. M. VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

PROBABILITY

Density functions

SIAM Rev., **21**(1979) 256.

Problem 79-6^{*}, A functional equation by L. B. KLEBANOV (Civil Engineering Institute, Leningrad, USSR)

Let f(x), g(x) be two probability densities on R^1 with g(x) > 0. Suppose that the condition

$$\int_{-\infty}^{\infty} (u-c) \prod_{j=1}^{n} f(x_j - u) g(u) \, du = 0$$

holds for all x_1, x_2, \ldots, x_n such that $\sum_{j=1}^n x_j = 0$ where $n \ge 3$ and c is some constant. Prove that

$$F9X) = \frac{1}{\sqrt{2\pi\sigma}} exp\{-(x-a)^2/2\sigma^2\}$$

Editorial note by MSK. This problem is related to a known theorem characterizing the normal distribution, i.e., if

$$\int_{-\infty}^{\infty} u \prod_{j=1}^{n} f(x_j - u)g(u) \, du + \int_{-\infty}^{\infty} \prod_{j=1}^{n} f(x_j - u)g(u) \, du = c_1 + c_2 \sum_{j=1}^{n} x_j$$

for all values of x_j , then the density f must be a normal density (A. M. Kagan, Y. V. Linnik & C. R. Rao, *Characterization Problems in Mathematical Sciences*, Wiley-Interscience, New York, 1973, p. 480, Thm. B.2.1.) The proposer notes that he can establish the result of his problem under further assumptions of regularity conditions. However, the assertion of the problem as is may not be correct. Nevertheless, even a proof of this would be of interest.

ANALYSIS

Functions: differentiable functions

SIAM Rev., **21**(1979) 257.

Problem 79-10^{*}, *Credibility functions*, by Y. P. SABHARWAL and J. KUMAR (Delhi University, India).

Determine the general form of a function F(x) satisfying the following conditions for $x \ge 0$:

(a) $0 \le F(x) \le 1$

(b)
$$\frac{d}{dx}F(x) > 0$$

(c)
$$\frac{d}{dx} \{F(x)/x\} < 0$$

These conditions arise in the construction of credibility formulas in casualty insurance work [1], [2].

REFERENCES

[1] F. S. Perryman, Experience rating plan credibilities, *Proc. Casualty Actuarial Soc.*, **24**(1937) 60 and **58**(1971) 143.

[2] H. L. Seal, Stochastic Theory of a Risk Business, John Wiley, New York, 1969, p. 83.

Editorial note by MSK. The proposers note that F(x) = x/(x+k) is a particular solution. However, it isn't difficult to extend this to F(x) = G(x)/(G(x)+k) where G(x) is a nonnegative increasing concave function of x, e.g., $G(x) = x^{\alpha}$ ($0 < \alpha \le 1$), $\ln(1+x)$, etc.

GEOMETRY

Triangle inequalities: medians and sides

SIAM Rev., **21**(1979) 559.

Problem 79-19, A Triangle Inequality, by M. S. KLAMKIN (University of Alberta).

If a_1 , a_2 , a_3 and m_1 , m_2 , m_3 denote the sides and corresponding medians of a triangle, respectively, prove that

$$(a_1^2 + a_2^2 + a_3^2)(a_1m_1 + a_2m_2 + a_3m_3) \ge 4m_1m_2m_3(a_1 + a_2 + a_3)$$
(1)

SIAM Rev., **22**(1980) 509–511.

Solution by the proposer.

To prove (1) as well as to give a dual inequality, we will use the known duality theorem that

$$F(a_1, a_2, a_3, m_1, m_2, m_3) \ge 0 \Leftrightarrow F\left(m_1, m_2, m_3, \frac{3a_1}{4}, \frac{3a_2}{4}, \frac{3a_3}{4}\right) \ge 0$$
(2)

This follows immediately from the fact that the three medians m_1 , m_2 , m_3 of any triangle are themselves sides of a triangle with respective medians $3a_1/4$, $3a_2/4$, $3a_3/4$. For a more general duality result, see [1].

We will now prove successively that

$$\sum a_1^2 \sum a_1 m_1 \ge \sum a_1 \sum a_1^2 m_1 \ge 4m_1 m_2 m_3 \sum a_1 \tag{3}$$

where the summations are to be understood as cyclic sums over the indices 1,2,3. The right-hand inequality of (3) follows immediately from the known inequality

$$a_1^2 m_1 + a_2^2 m_2 + a_3^2 m_3 \ge 4m_1 m_2 m_3 \tag{4}$$

which was obtained by Bager [2] by first establishing its dual, i.e.,

$$4\{a_1m_1^2 + a_2m_2^2 + a_3m_3^2\} \ge 9a_1a_2a_3 \tag{5}$$

However, if we make the substitutions

$$4m_1^2 = 2a_2^2 + 2a_3^2 - a_1^2$$
, etc.

we obtain

$$2\sum a_1 \sum a_1^2 \ge 3\{3a_1a_2a_3 + \sum a_1^3\}$$
(5')

which was obtained by Colins in 1870 [3, p.13].

To establish the left-hand inequality of (3), we expand out and collect terms to give

$$\sum a_1 a_2 (a_1 - a_2) (m_2 - m_1) \ge 0 \tag{6}$$

The latter inequality is valid, since if $a_1 \ge a_2 \ge a_3$ then $m_1 \le m_2 \le m_3$. There is equality in (1) if and only if the triangle is equilateral. However, if we allow degenerate triangles, there is another case of equality:

$$(a_1, a_2, a_3) = (2, 2, 0), \qquad (m_1, m_2, m_3) = (1, 1, 2)$$

The given inequality (1) has the following nice geometric interpretation. Let the medians of triangle $A_1A_2A_3$ be extended to meet the circumcircle again in points A'_1 , A'_2 , A'_3 . Then

$$\operatorname{Perimeter}(A_1', A_2', A_3') \ge \operatorname{Perimeter}(A_1 A_2 A_3) \tag{7}$$

For a related result concerning the repetition of the above operation, see [4]. Another related result due to Janić [3, p.90] is that

$$\operatorname{Area}(A_1'A_2'A_3') \ge \operatorname{Area}(A_1A_2A_3)$$

The latter result was extended to simplexes by the author (SIAM Rev., 21(1969) 569–590).

From (2), the dual of (1) is

$$4(m_1^2 + m_2^2 + m_3^2)(a_1m_1 + a_2m_2 + a_3m_3) \ge 9a_1a_2a_3(m_1 + m_2 + m_3)$$
(8)

There is equality if $a_1 = a_2 = a_3$ or $2a_1 = 2a_2 = a_3$.

The author has also shown [3, p.90] that if the angle bisectors of a triangle $A_1A_2A_3$ are extended to meet the circumcircle again in points A'_1 , A'_2 , A'_3 , then

$$\operatorname{Area}(A_1'A_2'A_3') \ge \operatorname{Area}(A_1A_2A_3)$$

We now also establish that

 $\operatorname{Perimeter}(A_1'A_2'A_3') \ge \operatorname{Perimeter}(A_1A_2A_3) \tag{9}$

In what follows O, I, R, r will denote the circumcenter, incenter, circumradius and inradius, respectively, of $A_1A_2A_3$. Let $R_i = A_iI$, $R'_i = A'_iI$; then from the power of a point property of circles,

$$R_i R'_i = R^2 - OI^2 = 2Rr \qquad (i = 1, 2, 3)$$

Then, by way of the law of cosines,

$$a_i' = \left\{\frac{2Rr}{R_1 R_2 R_3}\right\} a_i R_i$$

(for more extensive properties of this transformation and related ones, see [5]). Inequality (9) is now equivalent to

$$\frac{2Rr}{R_1R_2R_3}\{a_1R_1 + a_2R_2 + a_3R_3\} \ge a_1 + a_2 + a_3 \tag{9'}$$

From the known relations

$$r = R_i \sin A_i/2$$
 $a_i = 2R \sin A_i$ $r = 4R \prod \sin A_i/2$

we can transform (9') into the trigonometric form

$$\cos A_i/2 + \cos A_2/2 + \cos A_3/2 \ge \sin A_1 + \sin A_2 + \sin A_3$$

or, equivalently,

$$\sum \cos A_i/2 \ge \prod \cos A_i/2 \tag{9"}$$

Using the Arithmetic-Geometric Mean inequality, it suffices to prove that

$$(\sqrt{3}/2)^3 \ge \prod \cos A_i/2 \tag{10}$$

Although (1)) is a known inequality [3, p.26], another proof follows immediately from the concavity of $\log \cos x/2$.

Finally, it would be of interest to extend (7) and (9) to simplexes for which we consider total edge length as the perimeter.

REFERENCES

[1] M. S. KLAMKIN, Solution to Aufgabe 677, Elem. Math., 28(1973) 130.

 [2] A. BAGER, Some inequalities for the medians of a triangle, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 338–352(1971) 37–40; MR 44 #923.

[3] O. BOTTEMA, R. Z. DJORDJEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ and P. M. VASIĆ, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

[4] J. GARFUNKEL, A. BROUSSEAU and E. ITORS, Solution of Problem 913, Math. Mag., 48(1975) 246–247.

[5] M. S. KLAMKIN, Triangle inequalities via transformations, Notices Amer. Math. Soc., 19 No.1 (Jan. 1972) A-103–A-104.

[[as this fades into obscurity, I give the whole reference:

*691-26-3. MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan, 48121. Triangle inequalities via transformations.

In a previous paper, Oppenheim generates dual triangle inequalities for a triangle $A_1A_2A_3$ by reciprocation, inversion and isogonal conjugates with respect to an interior pont P. Letting $R_i = PA_i$, a_i = side opposite A_i , h_i = altitude from A_i , r_i = distance from P to a_i , $k = r_1r_2r_3$, $K = R_1R_2R_3$, w_i = angle bisector from P to a_i , $R'_i = PA'_i$,

etc., Oppenheim only considers the transformations of R_i , r_i into R'_i , r'_i . Here, we extend the scope of the duality by showing that if $F(a_i, h_i, R_i, r_i, w_i, R, r, \Delta) \geq 0$ is a valid triangle inequality, then so is $F(a'_i, h'_i, R'_i, r'_i, w'_i, R', r', \Delta') \geq 0$ where (1) under reciprocation $a'_i = a_i r_i R_i / 2R$, $R'_i = k/r_i$, $r'_i = k/R_i$, $\Delta' = k\Delta/2R$, R' = K/4R, $h'_i = kh_i/r_iR_i$, $r' = 2k\Delta/\sum a_i r_iR_i$; (2) under inversion $a'_i = a_iR_i$, $R'_i = K/R_i$, $r'_i = r_iR_i$, $w'_i = w_iR_i$, $R' = KR\Delta/\Delta'$, $h'_i = h_i\Delta'/\Delta R_i$, $\Delta' = k\Delta/2R$, $r' = 2k\Delta/\sum a_i r_iR_i$; (3) under isogonal conjugates $R'_i = 2\Delta r_iR_i/k\sum a_i/r_i$, $r'_i = 2\Delta/r_i\sum a_i/r_i$. Applying the latter transformations singly and in sequence to known inequalities, we generate numerous dual inequalities, many of them apparently are new. Additionally by specializing the point P to be the circumcenter, incenter, orthocenter, centroid, symmedian point or one of the Brocard points, we obtain easily other numerous inequalities. Again some of these are well known and others are apparently new. In particular, the conjecture of Stolarsky that $\sum a_i/R_i \geq a_1a_2a_3/R_1R_2R_3$ is pransformed into $\sum a_iR_i^2 \geq a_1a_2a_3$ by inversion and is then easily proved. (Received November 1, 1971.)]]

SIAM Rev., **22**(1980) 99.

Problem 80-5^{*}, A Dice Problem, by M. S. KLAMKIN and A. LIU (University of Alberta).

Given n identical polhedral dice whose faces are numbered identically with arbitrary integers:

(a) Prove or disprove that if the dice are tossed at random, the probability that the sum of the bottom n face numbers is divisible by n is at least $1/2^{n-1}$.

(b) Determine the maximum probability for the previous sum being equal to $k \pmod{n}$ for k = 1, 2, ..., n - 1.

In (a) the special case for n = 3 was set by the first author as a problem in the 1979 U.S.A. Mathematical Olympiad.

[[was a solution ever published ?]]

SIAM Rev., **22**(1980) 230.

Problem 80-10, Determinant of a Partitioned Matrix, by M. S. KLAMKIN, A. SHARMA and P. W. SMITH (University of Alberta).

Let D_i (i = 1, 2, ..., n) denote $k \times k$ matrices, a_{rs} denote the r, s th term of an $n \times n$ matrix A, and M denote the partitioned $kn \times kn$ matrix whose r, s th term (r, s = 1, 2, ..., n) is the matrix $a_{rs}D_s$. Prove that

$$|M| = |A|^n \prod |D_i|$$

where |M| denotes the determinant of M, etc.

SIAM Rev., **23**(1981) 259.

Solution by E. DEUTSCH (Polytechnic Institute of New York).

Let I_k denote the $k \times k$ identity matrix, let B be the partitioned $kn \times kn$ matrix whose r, s th term $a_{rs}I_k$ and let D be the partitioned $kn \times kn$ matrix whose diagonal terms are $D_1, \ldots D_n$ and whose offdiagonal terms are equal to the $k \times k$ zero matrix. Then it can be easily seen that M = BD and $B = A \otimes I_k$, where \otimes denotes the Kronecker product. Now

$$|M| = |B| |D| = |A|^k |I_k|^n |D| = |A|^k \prod |D_i|.$$

Also solved by [17 others] and the proposers.

SIAM Rev., **22**(1980) 364.

Problem 80-15, *An Identity for Complex Numbers, by* M. S. KLAMKIN *and* A. MEIR (University of Alberta).

Given that z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and

$$0 \le \arg z_1 \le \arg z_2 \le \arg z_3 \le \pi$$

prove that

$$\left(-z_{3}z_{1}+z_{1}z_{2}+z_{2}z_{3}\right)\left\{\left|z_{3}^{2}-z_{1}^{2}\right|+\left|z_{1}^{2}-z_{2}^{2}\right|+\left|z_{2}^{2}-z_{3}^{2}\right|\right\}=z_{2}^{2}\left|z_{3}^{2}-z_{1}^{2}\right|+z_{3}^{2}\left|z_{1}^{2}-z_{2}^{2}\right|+z_{1}^{2}\left|z_{2}^{2}-z_{3}^{2}\right|\right\}$$

SIAM Rev., **23**(1981) 395–396.

Solution by O. G. RUEHR (Michigan Technological University).

The identity can be rewritten in the form

$$I = (z_1 - z_2)(z_2 - z_3)|z_3^2 - z_1^2| + (z_2 - z_3)(z_1 + z_3)|z_1^2 - z_2^2| + (z_1 + z_3)(z_2 - z_1)|z_2^2 - z_3^2| = 0$$

LEMMA. If |z| = |w| = 1 and $0 \le \arg z \le \arg w \le \pi$, then

$$izw|z^2 - w^2| = w^2 - z^2$$

Then, using the lemma, we have

$$I = \frac{(z_3 + z_1)(z_1 - z_2)(z_2 - z_3)}{iz_1 z_2 z_3} \{ z_2(z_3 - z_1) - z_3(z_1 + z_2) + z_1(z_2 + z_3) \} = 0.$$

Also solved by [4 others] and the proposers using the same lemma as Ruehr.

SIAM Rev., **24**(1982) 340.

Problem 82-15, Flight in an Irrotational Wind Field, II, by M. S. KLAMKIN (University of Alberta).

It is a known result (see *Problem* 61-4, *SIAM Rev.*, 4(1962) 155) that if an aircraft traverses a closed curve at a constant air speed with respect to the wind, the time taken is always less when there is no wind, than when there is any bounded irrotational wind field.

(i) Show more generally that if the wind field is $k\mathbf{W}$ (**W** bounded and irrotational and k is a constant), then the time of traverse is a monotonic increasing function of $k \ (k \ge 0)$.

(ii) Let the aircraft be subject to the bounded irrotational wind field \mathbf{W}_i , i = 1, 2, and let T_i denote the time of flight over the same closed path. If $|\mathbf{W}_1| \leq |\mathbf{W}_2|$ at every point of the traverse, does it follow that $T_1 \leq T_2$?

SIAM Rev., **25**(1983) 407.

Solution by the proposer.

(i) Let the arc length s denote the position of the plane on its path and let w(s), $\theta(s)$ denote, respectively, the speed and the direction of the wind with respect to the tangent line to the path at position s. It is assumed that the wind field is continuous and that 1 > kw where the plane's speed is taken as 1. By resolving $k\mathbf{W}$ into components along and normal to the tangent line of the plane's path, the aircraft's ground speed is

$$\sqrt{1-k^2w^2\sin^2\theta}+kw\cos\theta$$

and then the time of flight id given by

$$T(k) = \oint \frac{ds}{\sqrt{1 - k^2 w^2 \sin^2 \theta} + kw \cos \theta}$$

From Problem 61-4, it is known that $T(k) \ge T(0)$ with equality if and only if $k\mathbf{W}=0$. We now show that T(k) is a strictly convex function of k which implies the desired result. Differentiating T(k) we get

$$\frac{dT}{dk} = -\oint \left\{ w\cos\theta - \frac{kw^2\sin^2\theta}{\sqrt{1 - k^2w^2\sin^2\theta}} \right\} \left\{ \sqrt{1 - k^2w^2\sin^2\theta} + kw\cos\theta \right\}^{-2} ds$$

Then $T'(0) = -\oint w \cos \theta \, ds = 0$ since **W** is irrotational. On differentiating again T'' > 0 since the integrand consists of positive terms. Thus T(k) is strictly convex (for $\mathbf{W} \neq 0$).

(ii) The answer here is negative. Just consider two constant wind fields, both habving the same wind speeds. Since the times of the traverses will in general be different, we cannot have both $T_1 \leq T_2$ and $T_2 \leq T_1$.

A problem related to part (i) is that the aircraft flies the same closed path twice with the second time in the reverse direction. All the other conditions of the problem are the same as before except that the wind field need not be irrotational. Then the total time of flight is an increasing function of k ($k\mathbf{W} \neq 0$). In this case if the aircraft only flew one loop, the time of flight could be less than the time of flight without wind (just consider a whirlwind). Here the total time of flight is

$$T(k) = \oint \frac{ds}{\sqrt{1 - k^2 w^2 \sin^2 \theta} + kw \cos \theta} + \oint \frac{ds}{\sqrt{1 - k^2 w^2 \sin^2 \theta} - kw \cos \theta}$$

By the A.M.-G.M. inequality, the sum of the integrands is $\geq 2(1-k^2w^2)^{-1/2} \geq 2$ which shows that $T(k) \geq T(0)$ with equality if and only if $k\mathbf{W} = 0$. Then as before T'' > 0.

SIAM Rev., **25**(1983) 98.

Problem 83-5^{*}, An Isoperimetric Inequality, by M. S. KLAMKIN (University of Alberta).

Given that A_1 is an interior point of the regular tetrahedron ABCD and that A_2 is an interior point of tetrahedron A_1BCD , it is conjectured that

$$I.Q.(A_1BCD) > I.Q.(A_2BCD)$$

where the isoperimetric quotient of a tetrahedron T is defined by

$$I.Q.(T) = \frac{Vol(t)}{[Area(T)]^{3/2}}$$

Also, one could replace $Area^{3/2}$ by total edge length cubed.

Prove or disprove the conjecture and also consider the analogous problems in E^n for a simplex for which there are many different isoperimetric quotients.

Editorial note. The 2-dimensional version of the problem was set by the proposer in the 1982 U.S.A. Mathematical Olympiad.

SIAM Rev., **26**(1984) 275–276.

Solution by the proposer.

Consider a tetrahedron $T(PA_1A_2A_3)$ with base $A_1A_2A_3$ an equilateral triangle of side 2 as in Fig.1.

	T	~
	ŀ	,
-	L	

 A_2

$$B_1$$
 B_0

 A_1

 A_3

Let $PB_0 = h$ be the altitude of T from P, $PB_1 = h_1$ the altitude of face PA_2A_3 from P, $x_1 = B_1B_0$ (which is then \perp to A_2A_3) and similarly for the other two faces having P as a vertex. The dihedral angle between the faces PA_2A_3 and $A_2A_3A_1$ is then $\angle PB_1B_0 = \alpha_1$, etc. Then

$$h = x_i \tan \alpha_i$$
 $(i = 1, 2, 3)$ $x_1 + x_2 + x_3 = \sqrt{3}$

Hence $h = \sqrt{3} / \sum \cot \alpha_i$ and

$$Vol = V(T) = 1 / \sum \cot \alpha_i$$

Area = $A(T) = \sqrt{3} + h_1 + h_2 + h_3$

or

$$A(T) = \sqrt{3} + h \sum \csc \alpha_i$$

Then, after some simplification.

I.Q.² =
$$\frac{V^2}{A^3} = \frac{1}{3\sqrt{3}} \frac{\sum \cot \alpha_i}{(\sum \cot \alpha_1/2)^3} = \frac{F}{\sqrt{3}}$$

The desired result will now follow by showing that F is an increasing function in each dihedral angle α_i for the range

$$\frac{1}{3} < \cos \alpha_i < 1 \quad \text{or} \quad \sqrt{\frac{2}{3}} < \cos \frac{\alpha_i}{2} < 1$$

(Note that the dihedral angles must be > 0 and less than those for a regular tetrahedron.)

It suffices to show $\partial F/\partial \alpha_i > 0$. For convenience, we replace the α_i by α , β , γ . Then $\partial F/\partial \alpha > 0$ is equivalent, after simplification, to

$$6\left(\cos^2\frac{\alpha}{2}\right)\left(\cot\alpha + \cot\beta + \cot\gamma\right) > \left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}\right)$$

Since $6\cos^2(\alpha/2) > 4$, it suffices to show that

$$4(\cot\alpha + \cot\beta + \cot\gamma) > \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2}$$

or that $4 \cot \alpha > \cot(\alpha/2)$. The latter is equivalent to

$$2\left(\tan\frac{\alpha}{2}\right)\left(1-2\tan^2\frac{\alpha}{2}\right) > 0$$

which follows since $\tan^2(\alpha/2) < \frac{1}{2}$.

SIAM Rev., **28**(1986) 88–89.

Solution by NOAM ELKIES (Harvard University).

We extend the previous solution [26(1984) 275–276] for tetrahedra to simplexes. Let S be a regular *n*-dimensional simplex of unit edge length and vertices $AB_1B_2...B_n$. Let S' be the simplex $A'B_1B_2...B_n$ where A' is an interior point of S. We will show that

$$[I.Q.(S')]^{n-1} = \frac{[Vol.(S')]^{n-1}}{[Area(S')]^n}$$

is an increasing function of each of the base angles of S'. By a base angle of S' we mean the angle between the (n-1)-dimensional face B_0 opposite A' and any other (n-1)-dimensional face. By the vol. and area of S' we mean its *n*-dimensional content and the sum of the (n-1)-dimensional contents of all its faces, respectively (see D. M. Y. Somerville, An Introduction to the Geometry of N Dimensions, Dover, New York, 1958).

Let h_0 be the altitude of a unit (n-1)-dimensional regular simplex and V_0 be the (n-2)-dimensional content of a unit (n-2)-dimensional regular simplex. Then the content of each (n-1)-dimensional face of $S = h_0 V_0/(n-1)$. Also let θ_0 be the angle between any two (n-1)-dimensional faces of S. Since $-\cos \theta_0$ is the inner product of any two of the unit vectors $\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_n$ normal to the faces of S, we have as well known that $\sum \mathbf{V}_i = 0$ and

$$\cos \theta_0 = -\mathbf{V}_0 \cdot \mathbf{V}_1 = \frac{-1}{n(n+1)} \sum_{i \neq j} \mathbf{V}_i \cdot \mathbf{V}_j = \frac{1}{n(n+1)} \left\{ \sum |\mathbf{V}_i|^2 - \left| \sum \mathbf{V}_i \right|^2 \right\} = \frac{1}{n}$$

Let B'_i be the face of S' opposite B_i , θ'_i be the angle between B'_i and B_0 , and H' be the altituse of S' from A'. Since A' is an interior point of S, $0 < \theta'_i < \theta_0$.

If P' is the projection of A' on B_0 , the distance from P' to the (n-2)-dimensional face of B_0 opposit B_i is $H' \cot \theta'_i$. Since B_0 is regular, $H' = h_o / \sum \cot \theta'_i$. Also, the (n-1)-dimensional content of B'_i is $H'V_0(\csc \theta'_i)/(n-1)$. Thus,

$$Area(S') = AreaB_0 + \sum AreaB_i = \left(\frac{V_0}{n-1}\right)(h+H'\sum \csc \theta'_i)$$
$$= \frac{V_0h_0}{n-1}\frac{\sum \cot(\theta'_i/2)}{\sum \cot \theta'_i}$$
$$Vol(S') = V_0h_0H'/(n(n-1))$$

Consequently,

$$[I.Q.(S')]^{n-1} = \frac{(n-1)h_0^{n-2}}{n^{n-1}V_0} \left\{ \sum \cot \theta'_i \right\} \left\{ \sum \cot(\theta'_i/2) \right\}^{-n}$$

We now show that $F(\theta_1, \theta_2, \ldots, \theta_n) \equiv {\cot \theta_i} {\sum \cot(\theta_i/2)}^{-n}$ is an increasing function in each of the variables θ_i for $0 < \theta_i < \cos^{-1}(1/n)$.

By symmetry, it suffices to show that $\partial F/\partial \theta_1 > 0$. Here,

$$\left\{\sum \cot(\theta_i/2)\right\}^{n+1} \frac{\partial F}{\partial \theta_1} = \frac{n}{2} \csc^2(\theta_1/2) \left\{\sum \cot\theta_i\right\} - \csc^2\theta_1 \left\{\sum \cot(\theta_i/2)\right\}$$

Clearly $\{\sum \cot(\theta_i/2)\}^{n+1} > 0$, and furthermore

$$\frac{n}{2}\csc^{2}(\theta_{1}/2) - (n+1)\csc^{2}\theta_{1} = \csc^{2}\theta_{1}\left\{2n\cos^{2}(\theta_{1}/2) - n - 1\right\}$$
$$= \frac{n\cos\theta_{1} - 1}{\sin^{2}\theta_{1}} > 0$$

Since also

$$(n+1)\sum \cot \theta_i - \sum \cot(\theta_i/2) = \sum \frac{n\cos \theta_i - 1}{\sin \theta_i} > 0$$

it follows that $\partial F/\partial \theta_1 > 0$.

Editorial note. There is a corresponding result for simplexes S'' which contain S. Let A'' be a point exterior to S such that the simplex $S'' : A''B_1B_2...B_n$ covers S. It then follows from the previous analysis that I.Q.(S'') is a decreasing function of each of the base angles θ'_i . Note that here $\theta'_i > \cos^{-1}(1/n)$. [M.S.K.]

SIAM Rev., **26**(1984) 273.

Problem 84-13, A Minimum Value, by M. S. KLAMKIN (University of Alberta). Determine the minimum value of

$$I = \frac{(y-z)^2 + (z-x)^2 + (x-y)^2}{(x+y+z)^2} \cdot \frac{(v-w)^2 + (w-u)^2 + (u-v)^2}{(u+v+w)^2}$$

subject to ux + vy + wz = 0 and all the variables are real.

SIAM Rev., 27(1985) 256.

Solution by MARK KANTROVITZ (Secondary school student, Maimonides School, Brookline, MA).

By Cauchy's inequality,

$$\sum (y-z)^2 \cdot \sum (v-w)^2 \ge \left\{ \sum (y-z)(v-w) \right\}^2 = \left\{ \sum x \cdot \sum u \right\}^2$$

where the sums arecyclic over x, y, z and u, v, w. Thus $I \ge 1$ with equality if and only if

$$(y - z, z - x, x - y) = k(v - w, w - u, u - v).$$

Solution by T. M. HAGSTROM (University of Wisconsin) and W. B. JORDAN (Scotia, NY).

Let $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{Q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, and $\mathbf{R} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$$I = \frac{(\mathbf{P} \times \mathbf{R})^2}{(\mathbf{P} \cdot \mathbf{R}^2)} \cdot \frac{(\mathbf{Q} \times \mathbf{R})^2}{(\mathbf{Q} \cdot \mathbf{R}^2)} \quad \text{and} \quad |bfP \cdot \mathbf{Q} = 0$$
(1)

If a is the angle between **P** and **R**, and b is the angle between **Q** and **R**, then $I = \tan^2 a \tan^2 b$. In terms of a spherical triangle whose sides are a, b, c with $c = \pi/2$, $I = \sec^2 C$. Clearly the minimum value is 1, occurring for $C = \pi$.

B. D. DORE (University of Reading, Reading, UK) shows that the minimum value 1 follows immediately from (1) by replacing $(\mathbf{P} \cdot \mathbf{R})^2 (\mathbf{Q} \cdot \mathbf{R})^2$ by its equivalent $\{(\mathbf{P} \times \mathbf{R}) \cdot (\mathbf{Q} \times \mathbf{R})\}^2$.

Editorial note. In the second solution, we can just as well use three vectors in \mathbb{R}^n , i.e., $\mathbf{P} = (x_1, x_2, \ldots, x_n)$, $\mathbf{Q} = (u_1, u_2, \ldots, u_n)$, $\mathbf{R} = (1, 1, \ldots, 1)$. Also, $(\mathbf{P} \times \mathbf{R})^2$ is to be replaced by its equivalent $[\mathbf{P}^2 \mathbf{R}^2 - (\mathbf{P} \cdot \mathbf{R})^2]$, etc. J. A. WILSON (Iowa State University) obtained this extension analytically. [M.S.K.]

Also solved by [18 others] and the proposer.

SIAM Rev., 27(1985) 250.

Problem 85-10, *An Identity*, *by* M. S. KLAMKIN (University of Alberta) and O. G. RUEHR (Michigan Technological University).

Let

$$S(x, y, z, m, n, r) \equiv \frac{x^{m+1}}{m!} \sum_{j=0}^{n} \sum_{k=0}^{r} \frac{y^j z^k (j+k+m)!}{j!k!}$$

Show that if x + y + z = 1, then

$$S(x, y, z, m, n, r) + S(y, z, x, n, r, m) + S(z, x, y, r, m, n) = 1.$$

SIAM Rev., **28**(1986) 243–244.

Solution by A. J. BOSCH and F. W. STEUTEL (Eindhoven University of Technology).

First let x > 0, y > 0, z > 0 (and x + y + z = 1) and consider the following probabilistic model.

Three urns labelled I, II and III contain m, n and r balls respectively. We perform independent drawings by choosing I, II and III with probabilities x, y and z and taking one ball from the urn chosen (without replacement). Let $P_{I}I$ be the probability defined by

 $P_{\rm I} = P({\rm I} \text{ is the first urn to be chosen when empty}).$

Then by elementary combinatorics we have

$$P_{I} = x \sum_{j=0}^{n} \sum_{k=0}^{r} \frac{(m+j+k)!}{m \; j!k!} x^{m} y^{j} z^{k}$$

The probabilities $P_{[II]}$ and P_{III} are defined similarly. Since eventually an empty urn will be chosen, we have $P_{I} + P_{II} + P_{III} = 1$. Finally, a polynomial in x and y that is identically 1 for x > 0, y > 0, x + y < 1 is also identically 1 without restriction.

Also solved by S. LJ. DAMJANOVI (TANJUG Telecommunication Center, Belgrade, Yugoslavia), [3 others] and the proposers.

Editorial note: Both Damjanovic and the proposers found the generalization

$$\sum_{i=1}^{n} x_i \sum_{m_1=0}^{p_1} \frac{x_1^{m_1}}{m_1!} \sum_{m_2=0}^{p_2} \frac{x_2^{m_2}}{m_2!} \cdots \sum_{m_n=0}^{p_n} \frac{x_n^{m_n}}{m_n!} \delta_{m_i, p_i} (m_1 + m_2 + \dots + m_n)! = 1 \quad (*)$$

using the generating function

$$\sum_{i=1}^{n} x_i \left\{ \frac{1}{\prod_{j=1, j \neq i}^{n} (1-u_j)} \right\} \left\{ \frac{1}{1 - \sum_{s=1}^{n} u_s x_s} \right\} = \frac{1}{\prod_{i=1}^{n} (1-u_i)} \quad (**)$$

The assertion is proved by repeated use of the following elementary identities:

$$\sum_{p=0}^{\infty} u^p \sum_{m=0}^{p} \frac{x^m}{m!} (m+k)! = \frac{1}{(1-u)} \frac{1}{(1-xu)^{k+1}}$$
$$\frac{1}{1-u} \sum_{q=0}^{\infty} v^q \sum_{s=0}^{q} \frac{y^s (s+k)!}{s!(1-xu)^{s+k+1}} = \frac{k!}{(1-u)(1-v)(1-xu-yv)^{k+1}}$$

Finally to show that (**) is an elementary algebraic identity, let

$$P_N = \prod_{i=1}^N (1-u), \quad L_N = \sum_{i=1}^N x_i, \quad R_N = \sum_{i=1}^N u_i x_i$$

Then, if $L_N = 1$, we have

$$\frac{1}{P_N} = \frac{L_N - 1 + 1 - R_n}{P_N (1 - R_N)} = \sum_{i=1}^N \frac{x_i (1 - u_i)}{P_N (1 - R_N)}$$
$$= \sum_{i=1}^N x_i \left\{ \frac{1}{\prod_{j=1, j \neq i}^N} (1 - u_j) \right\} \left\{ \frac{1}{1 - \sum_{s=1}^N u_s x_s} \right\}$$

which is (**).

A simpler derivation of (*) can be easily obtained by extending the probabilistic argument in the featured solution to n urns.

SIAM Rev., **27**(1985) 446.

Problem 85-15^{*}, Extension of Routh's Theorem to Spherical Triangles, by A. SHARMA and M. S. KLAMKIN (University of Alberta).

Routh [1] discovered that if the sides BC, CA, AB of a plane triangle ABC are divided at points L, M, N in the repective ratios $\lambda : 1$, $\mu : 1$, $\nu : 1$, the cevians AL, BM, CNintersect to form a triangle whose area is

$$\frac{(\lambda\mu\nu-1)^2}{(\lambda\mu+\lambda+1)(\mu\nu+\mu+1)(\nu\lambda+\nu+1)}$$

yimrs that of ABC. Determine the analogous area ratio formula for a spherical triangle (convex). Also, since it is unlikely that the area ratio here is independent of the sides of the triangle, determine the extreme values of this ratio. Even a solution of the special case BC = CA = AB, $\lambda = \mu = \nu = 2$ would be of interest. REFERENCE

[1] H. S. M. COXETER, Introduction to Geometry, John Wiley, New York, 1969, pp.211, 219-220.

Partial solution by W. B. JORDAN (Scotia, New York).

Since there is apparently no simple formula in the general case for the area of the inner triangle, we consider only the equilateral case.



Let the triangle have angles 2A and sides 2a, split by the cevians into a - w and a + w. Let the cevians intersect each other at an angle K and meet the sides at an angle C splitting 2A into B and 2A - B. Let m be the median. Then,

$$2\sin A\cos a = 1, \qquad 2\cos A\sin a = \sin m,$$

$$\cos K = -\cos B\cos C + \sin B\sin C\cos(a - w),$$

$$\cos 2A = -\cos B\cos C + \sin B\sin C\cos d$$

Also,

$$\cos K - \cos 2A = \sin B \sin C \left(\cos(a - w) - \cos d \right),$$
$$\cos d = \cos m \cos w,$$
$$\cos(a - w) - \cos d = \tan a \sin(a + w),$$

and

$$\frac{\sin B}{\sin(a-w)} = \frac{\sin C}{\sin 2a} = \frac{\sin 2A}{\sin d}$$

so that

$$\sin B \sin C = \sin(a - w) \sin 2a \sin^2 2A / \sin^2 d$$

$$\cos K - \cos 2A = \frac{\sin(a - w) \sin 2a \sin^2 2A \tan a \sin(a + w)}{1 - \cos^2 m \cos^2 w}$$

$$= \frac{\frac{1}{2} \sin^2 m (\sec^2 a \cos^2 w - 1)}{1 - \cos^2 m \cos^2 w}$$

The area of the original triangle is $6A - \pi$; of the inner triangle $3K - \pi$. Their ratio is expressible by means of an arccosine.

[[The following solution appeared much later.]]

SIAM Rev., **38**(1996) 319–320.

Solution by ZUO QUAN-RU (Yangzhou Teacher's Colege, Jiangsu, China).

THEOREM. Take the sphere to have unit radius and let the spherical triangle $A_1A_2A_3$ have sides α_1 , α_2 , α_3 (in radians). The arclengths A_2A_3 , A_3A_1 , A_1A_2 are divided by the points B_1 , B_2 , B_3 in the ratios $\lambda_1 : 1$, $\lambda_2 : 1$, $\lambda_3 : 1$ respectively. Suppose A_1B_1 intersects A_2B_2 at C_3 , A_2B_2 intersects A_3B_3 at C_1 and A_3B_3 intersects A_1B_1 at C_2 . If the areas of spherical triangles $A_1A_2A_3$ and $C_1C_2C_3$ are δ and δ' respectively, then

$$\frac{\sin(\delta'/2)}{\sin(\delta/2)} = \frac{(x_{12}x_{23}x_{31} - x_{13}x_{32}x_{21})^2\sqrt{(1 + \cos\alpha_1)(1 + \cos\alpha_2)(1 + \cos\alpha_3)}}{\sqrt{(|\mathbf{c}_2||\mathbf{c}_3| + \mathbf{c}_2 \cdot \mathbf{c}_3)(|\mathbf{c}_3||\mathbf{c}_1| + \mathbf{c}_3 \cdot \mathbf{c}_1)(|\mathbf{c}_1||\mathbf{c}_2| + \mathbf{c}_1 \cdot \mathbf{c}_2)}}$$
(1)

where

$$x_{i,i+1} = \frac{\sin\left(\frac{\alpha_i}{1+\lambda_i}\right)}{\sin\alpha_i}, \quad x_{i,i-1} = \frac{\sin\left(\frac{\lambda_i\alpha_i}{1+\lambda_i}\right)}{\sin\alpha_i}, \quad x_{i\pm3,j\pm3} = x_{i,j}$$

and

$$\mathbf{c}_{i} = x_{i+1,i} x_{i+2,i} \mathbf{a}_{i} + x_{i+1,i} x_{i+2,i} \mathbf{a}_{i+1} + x_{i+2,i} x_{i+1,i+2} \mathbf{a}_{i+2}$$

in which \mathbf{a}_i are unit vectors and $\mathbf{a}_i \cdot \mathbf{a}_j = \cos \alpha_k$ with *i*, *j*, *k* taken cyclically.

Proof. Let $\overrightarrow{OA_i} = \mathbf{a}_i$, $\overrightarrow{OB_i} = \mathbf{b}_i$, (i = 1, 2, 3), where O is the center of the sphere so $|\mathbf{a}_i| = |\mathbf{b}_i| = 1$. Since $A_1A_2A_3$ is a nontrivial spherical triangle, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly inependent set. Thus we can assume det $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) > 0$. Then

$$\mathbf{b}_i = x_{i1}\mathbf{a}_1 + x_{i2}\mathbf{a}_2 + x_{i3}\mathbf{a}_3$$

where $x_{11} = x_{22} + x_{33} = 0$

$$\begin{array}{rcl} x_{12} & = & \frac{\sin\left(\frac{\alpha_1}{1+\lambda_1}\right)}{\sin\alpha_1} & & x_{13} & = & \frac{\sin\left(\frac{\lambda_1\alpha_1}{1+\lambda_1}\right)}{\sin\alpha_1} \\ x_{23} & = & \frac{\sin\left(\frac{\alpha_2}{1+\lambda_2}\right)}{\sin\alpha_2} & & x_{21} & = & \frac{\sin\left(\frac{\lambda_2\alpha_2}{1+\lambda_2}\right)}{\sin\alpha_2} \\ x_{31} & = & \frac{\sin\left(\frac{\alpha_3}{1+\lambda_3}\right)}{\sin\alpha_3} & & x_{32} & = & \frac{\sin\left(\frac{\lambda_3\alpha_3}{1+\lambda_3}\right)}{\sin\alpha_3} \end{array}$$

This is because $\mathbf{a}_i \cdot \mathbf{b}_j = \cos \widehat{A_i B_j}$.

The plane OA_2B_2 and the plane OA_3B_3 intersect on the straight line OC_1 , whose direction is

$$\begin{aligned} (\mathbf{a}_2 \times \mathbf{b}_2) \times (\mathbf{a}_3 \times \mathbf{b}_3) &= (\mathbf{a}_2, \, \mathbf{b}_2, \, \mathbf{b}_3) \mathbf{a}_3 - (\mathbf{a}_2, \, \mathbf{b}_2, \, \mathbf{a}_3) \mathbf{b}_3 \\ &= (x_{23} x_3 \mathbf{a}_3 + x_{21} \mathbf{b}_3) (\mathbf{a}_1, \, \mathbf{a}_2, \, \mathbf{a}_3) \\ &= (x_{21} x_{31} \mathbf{a}_1 + x_{21} x_{32} \mathbf{a}_2 + x_{23} x_{31} \mathbf{a}_3) (\mathbf{a}_1, \, \mathbf{a}_2, \, \mathbf{a}_3) \end{aligned}$$

and we obtain

$$\mathbf{c}_{1} = x_{21}x_{31}\mathbf{a}_{1} + x_{21}x_{32}\mathbf{a}_{2} + x_{23}x_{31}\mathbf{a}_{3}$$

$$\mathbf{c}_{2} = x_{31}x_{12}\mathbf{a}_{1} + x_{32}x_{12}\mathbf{a}_{2} + x_{32}x_{13}\mathbf{a}_{3}$$

$$\mathbf{c}_{3} = x_{13}x_{21}\mathbf{a}_{1} + x_{12}x_{23}\mathbf{a}_{2} + x_{13}x_{23}\mathbf{a}_{3}$$

co-directional with $\overrightarrow{OC_1}$, $\overrightarrow{OC_2}$, $\overrightarrow{OC_3}$, respectively. Then we have $\cos \widehat{C_i C_j} = \mathbf{c_i} \cdot \mathbf{c_j}/|\mathbf{c_i}||\mathbf{c_j}|$. From the area formula for a spherical triangle, we have

$$\sin^2(\delta/2) = \frac{1}{2(1+\cos\alpha_1)(1+\cos\alpha_2)(1+\cos\alpha_3)} \begin{vmatrix} 1 & \cos\alpha_3 & \cos\alpha_2 \\ \cos\alpha_3 & 1 & \cos\alpha_1 \\ \cos\alpha_2 & \cos\alpha_1 & 1 \end{vmatrix}$$

and

$$\begin{aligned} \sin^{2}(\delta'/2) &= \frac{1}{2(1+\cos\widehat{C_{2}C_{3}})(1+\cos\widehat{C_{3}C_{1}})(1+\cos\widehat{C_{1}C_{2}})} \begin{vmatrix} 1 & \cos\widehat{C_{1}C_{2}} & \cos\widehat{C_{1}C_{3}} \\ \cos\widehat{C_{1}C_{3}} & \cos\widehat{C_{2}C_{3}} & 1 \\ \cos\widehat{C_{1}C_{3}} & \cos\widehat{C_{2}C_{3}} & 1 \end{vmatrix} \\ &= \frac{1}{2(|\mathbf{c}_{2}||\mathbf{c}_{3}|\mathbf{c}_{2}\cdot\mathbf{c}_{3})(|\mathbf{c}_{3}||\mathbf{c}_{1}|\mathbf{c}_{3}\cdot\mathbf{c}_{1})(|\mathbf{c}_{1}||\mathbf{c}_{2}|\mathbf{c}_{1}\cdot\mathbf{c}_{2})} \begin{vmatrix} \mathbf{c}_{1}\cdot\mathbf{c}_{1} & \mathbf{c}_{1}\cdot\mathbf{c}_{2} & \mathbf{c}_{1}\cdot\mathbf{c}_{3} \\ \cos\widehat{C_{1}C_{3}} & \cos\widehat{C_{2}C_{3}} & 1 \end{vmatrix} \\ &= \frac{(\mathbf{a}_{1},\mathbf{a}_{2},\mathbf{a}_{3})^{2}}{2(|\mathbf{c}_{2}||\mathbf{c}_{3}|\mathbf{c}_{2}\cdot\mathbf{c}_{3})(|\mathbf{c}_{3}||\mathbf{c}_{1}|\mathbf{c}_{3}\cdot\mathbf{c}_{1})(|\mathbf{c}_{1}||\mathbf{c}_{2}|\mathbf{c}_{1}\cdot\mathbf{c}_{2})} \begin{vmatrix} \mathbf{x}_{21}x_{31} & x_{21}x_{32} & x_{23}x_{31} \\ \mathbf{x}_{31}x_{12} & x_{32}x_{12} & x_{32}x_{13} \\ \mathbf{x}_{11}x_{21} & x_{12}x_{23} & x_{13}x_{23} \end{vmatrix} \end{vmatrix}^{2} \\ &= \frac{(x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32})^{4}}{2(|\mathbf{c}_{2}||\mathbf{c}_{3}|\mathbf{c}_{2}\cdot\mathbf{c}_{3})(|\mathbf{c}_{3}||\mathbf{c}_{1}|\mathbf{c}_{3}\cdot\mathbf{c}_{1})(|\mathbf{c}_{1}||\mathbf{c}_{2}|\mathbf{c}_{1}\cdot\mathbf{c}_{2})} \begin{vmatrix} 1 & \cos\alpha_{3} & \cos\alpha_{2} \\ \cos\alpha_{3} & 1 & \cos\alpha_{1} \\ \cos\alpha_{2} & \cos\alpha_{1} & 1 \end{vmatrix} \end{aligned}$$

So we have proved (1).

SIAM Rev., **27**(1985) 576.

Problem 85-26, Inequality for a Simplex, by M. S. KLAMKIN (University of Alberta).

If O, I, R denote the circumcenter, incenter, circumradius and inradius, respectively, of an n-dimensional simples, prove that

$$R^2 \ge n^2 r^2 + \overline{OI}^2$$

and with equality if and only if the simplex is regular.

SIAM Rev., 28(1986) 579–580.

Solution by the proposer.

First we derive a more general inequality. Let F_i denote the content of the (n-1)-dimensional face opposite vertex A_i of a given simplex, i = 0, 1, ..., n. If P is an arbitrary point, then by Cauchy's inequality,

$$\sum F_i \sum F_i \overline{PA_i}^2 \ge \left\{ \sum F_i \overline{PA_i} \right\}^2 \tag{1}$$

and with equality if $PA_0 = PA_1 = \cdots = PA_n$, i.e., if *P* coincides with *O*. Letting **P**. **A**_i, **I** denote the vectors from *O* to *P*, *A*_i, *I* respectively, we get

$$\sum F_i \overline{PA_i}^2 = \sum F_i (\mathbf{P} - \mathbf{A}_i) \cdot (\mathbf{P} - \mathbf{A}_i) = \sum F_i (R^2 + \mathbf{P}^2 - 2\mathbf{P} \cdot \mathbf{A}_i)$$
$$= F \left(R^2 + \mathbf{P}^2 - 2\mathbf{P} \cdot \sum F_i \mathbf{A}_i / \sum F_i \right)$$
$$= F \left(R^2 + \overline{OP}^2 - 2\mathbf{P} \cdot \mathbf{I} \right)$$

(here $F = \sum F_i$ and $\mathbf{I} = \sum F_i \mathbf{A}_i / F$). Then, since $2\mathbf{P} \cdot \mathbf{I} = \mathbf{P}^2 + \mathbf{I}^2 - (\mathbf{P} - \mathbf{I})^2$

$$\sum F_i \sum F_i \overline{PA_i}^2 = F^2 \left(R^2 + \overline{PI}^2 - \overline{OI}^2 \right)$$

Consequently,

$$\sum F_i \overline{PA_i} \le F \left\{ R^2 + \overline{PI}^2 - \overline{OI}^2 \right\}^{1/2}$$
(2)

and with equality if and only if P coincides with O. Now if h_i and r_i denote the distances from A_i and P, respectively, to the face F_i , $PA_i \ge h_i - r_i$. Thus,

$$\sum F_i \overline{PA_i} \ge \sum F_i (h_i - r_i) = \sum F_i h_i - \sum F_i r_i = (n+1)nV - nV = n^2 V \quad (3)$$

where V is the volume of the simplex. Coupling (2) and (3) and using nV = rF, we obtain

$$R^2 \ge n^2 r^2 + \overline{OI}^2 - \overline{PI}^2 \tag{4}$$
Finally, letting P coincide with I, we obtain the desired inequality

$$R^2 \ge n^2 r^2 + \overline{OI}^2 \tag{5}$$

For equality, $PA_i = h_i - r_i$ for all *i*, or equivalently, the simplex is orthocentric with the orthocentre coinciding with the circumcenter. This requires the simplex to be regular since

 $\mathbf{A}_i^2 = R^2$ and $\mathbf{A}_i \cdot (\mathbf{A}_j - \mathbf{A}_k) = 0 \Rightarrow (\mathbf{A}_i - \mathbf{A}_j)^2 = (\mathbf{A}_i - \mathbf{A}_k)^2$

for all i, j, k with $i \neq j, k$.

SIAM Rev., **29**(1987) 470-471.

Problem 87-15, An Optimum Multistage Rocket Design, by M. S. KLAMKIN (University of Alberta).

In a recent paper [1] on optimization in multistage rocket design, Peressini reduced his problem to minimizing the function

$$\ln\left\{\frac{M_0 + P}{P}\right\} = \sum_{i=1}^n \{\ln N_i + \ln(1 - S_i) - \ln(1 - S_i N_i)\}$$

subject to the constraint condition

$$\sum_{i=1}^{n} c_i \ln N_i = v_f \text{ (constant)}$$

Here the S_i and the c_i are given structural factors and engine exhaust speeds, respectively. He then obtains the necessary optimization equations using Lagrange multipliers. These equations are then solved explicitly for the special case when $S_i = S$, $c_i = c$ for all i.

For the latter special case, show how to determine the minimum in a simpler fashion without using calculus.

REFERENCE

[1] A. L.PERESSINI, Lagrange multipliers and the design of mutistage rockets, UMAP J., 7(1986) 249–262.

SIAM Rev., **30**(1988) 508–509.

Solution by the proposer.

More generally, the S_i need not be the same. Thus we wish to maximize $P \equiv \prod (1-x_i)$, where $x_i = S_i N_i$ and where the product and sums here and subsequently are all from i = 1 to n. Since $\sum \ln N_i = v_f/c = \text{constant}$, the S_i are specified, tacitly $0 < x_i < 1$, and our constraint condition is $\prod x_i = \text{constant}$.

Using the concavity of $\ln(1-x)$ for 0 < x < 1 and the A.M.-G.M. inequality, we have that

$$\sum \ln(1 - x_i) \le n \ln(1 - \sum x_i/n) \le n \ln(1 - \prod x_i^{1/n})$$

Thus the maximum is taken on for $x_i = S_i N - i = \text{constant}$ or $N_i = \lambda/S_i$, where $\lambda^n = e^{v_f/c} \prod S_i$. Finally,

$$Min(M_0 + P)/P = e^{v_f/c}(1 - \lambda)^{-n} \prod (1 - S_i).$$

SIAM Rev., **30**(1988) 125.

Problem 88-5, Characterizations of Parabolic Motion, by M. S. KLAMKIN (University of Alberta).

By a parabolic motion we mean the motion which ensues when a particle is projected in a uniform gravitational field and is not subject to any other frictional forces. Equivalently, the equations of motion are of the form x = at, $y = bt + ct^2$ and the trajectory is a parabola.

Let RP and RQ be two tangents to the trajectory and let RT be a vertical segment as in Fig. 1. y



FIG. 1

x

The following properties of the motion are known:

$$PR = V_P t$$
 and $RT = gt^2/2$ [1]

where V_P is the speed of the particle at P and t is the time for the particle to go from P to T.

$$PR/RQ = V_P/V_Q \qquad [2$$

and it takes the particle the same time to go from P to T as from T to Q.

Show that either properties [1] or [2] characterize parabolic motion, i.e., if either [1] or [2] holds for all points on a smooth trajectory, the motion must be parabolic.

SIAM Rev., **31**(1989) 130-131.

Solution by FRANK MATHIS (Baylor University).

Let x = x(t) and y = y(t) be a smooth trajectory, i.e., x and y are continuously differentiable functions of t. We may assume without loss of generality that there is

an interval I containing zero with x'(t) > 0 for t in I. Select P, R and T as in Fig. 1 with coordinates $(x(t_P), y(t_P))$, $(x(t_T), y_R)$ and $(x(t_T), y(t_T))$, respectively, so that t_P and t_T are in I. Then

$$V_P = \sqrt{x'(t_P)^2 + y'(t_P)^2}$$
(1)

and the tangent line PR may be parameterized by

$$x = x(t_P) + x'(t_P)s, \qquad y = y(t_P) + y'(t_P)s$$

for $0 \le s \le (x(t_T) - x(t_P))/x'(t_P)$. It follows that

$$y_R = y(t_P) + \frac{y'(t_P)}{x(t_P)} \left[x(t_T) - x(t_P) \right]$$
(2)
$$PR = \frac{x(t_T) - x(t_P)}{x'(t_P)} \sqrt{x'(t_P)^2 + y'(t_P)^2}$$
(3)

First we will assume that property [1] holds. By simple translations we may take $t_P = x(t_P) = y(t_P) = 0$. Set $t = t_T$ so that t is the time for the particle to go from P to T. Then since $PR = tV_P$, we have from (1) and (3) that

$$x(t) = x'(0)t \tag{4}$$

Thus x is linear for any t in I. But then x' is constant on I and so by the continuity of x', we may take I to be all of \mathbb{R} .

Now since $QR = \frac{1}{2}gt^2$, $y_R - y(t) = \frac{1}{2}gt^2$, and using (2) and (4), we find that

$$y(t) = y'(0)t - \frac{1}{2}gt^2$$

Therefore, property [1] implies parabolic motion.

Next we assume property [2]. Let Q be as in Fig. 1 with coordinates $(x(t_Q), y(t_Q))$ and t_Q in I. Then as above we have

$$V_Q = \sqrt{x'(t_Q)^2 + y'(t_Q)^2}$$

$$RQ = \frac{x(t_Q) - x(t_T)}{x'(t_Q)} \sqrt{x'(t_Q)^2 + y'(t_Q)^2}$$

$$y_R = y(t_Q) + \frac{y'(t_Q)}{x'(t_Q)} [x(t_Q) - x(t_T)]$$
(5)

Because the particle takes the same time to go from P to T as from T to Q, we may set $t_T = t$, $t_P = t - s$ and $t_Q = t + s$. Since $PR/RQ = V_P/V_Q$, we then have

$$x'(t+s)[x(t) - x(t-s)] - x'(t-s)[x(t+s) - x(t)] = 0$$
(6)

This condition holds for all t and s with $t \pm s$ in I. In particular, if we let t = s,

$$x'(2t) = x'(0) \left[\frac{x(2t) - x(t)}{x(t) - x(0)} \right]$$

and it follows that x is at least twice differentiable for t in $I, t \neq 0$.

Returning to (6), we may now differentiate both sides with respect to s and then divide by s to obtain

$$x''(t+s)\left[\frac{x(t) - x(t-s)}{s}\right] + x''(t-s)\left[\frac{x(t+s) - x(t)}{s}\right] = 0$$

Then letting $s \to 0$ and using the continuity of x'', we see that

$$2x''(t)x'(t) = 0$$

But x'(t) > 0 for t in I. So x'' must be identically zero for t in I, $t \neq 0$. Hence, using the continuity of x' as before, we conclude that x is linear for all t.

Finally, to show that y is quadratic in t, let us again set $t_P = x(t_P) = y(t_P) = 0$ and let $t = t_Q$ so that $t_T = t/2$, $x(t_T) = x'(0)t/2$ and $x(t_Q) = x'(0)t$. Then from (2) and (5) we have

$$y'(0)\frac{t}{2} = y_R = y(t) - y'(t)\frac{t}{2}$$

That is, y satisfies the differential equation

$$y' = \frac{2}{t}y - y'(0)$$

which has the solution

$$y = y'(0)t + ct^2$$

where c is an arbitrary constant. Thus we conclude that property [2] also implies a motion that is parabolic.

Also solved by SESHADRI SIVAKUMAR and the proposer.

SIAM Rev., **30**(1988) 504.

Problem 88-15, An Inverse Potential Problem, by M. S. KLAMKIN (University of Alberta).

The following is a known result and is given as a problem in [1]:

"Current enters an infinite plane conducting sheet at some point P and leaves at infinity. A circular hole, which does not include P, is cut anywhere in the sheet. Show that the potential difference between any two points on the edge of the hole is twice what it was between the same two points before the hole was cut."

(i) Show that the above result can be extended to a general two-dimensional flow of an inviscid incompressible fluid.

(ii)* Prove or disprove that only a circular hole has this property.

(iii) Show that in three dimensions, the analogous result for a sphere does not hold if the undisturbed flow is due to a source, but it does hold if the potential of the undisturbed flow is a spherical harmonic of degree n.

REFERENCE

[1] W. R. SMYTHE, Static and Dynamic Electricity, McGraw-Hill, New York, 1939, p.253, #51.

SIAM Rev., **31**(1989) 678–682.

Solution of (i) and (iii) by the proposer.

(i) Let f(z) denote the complex potential of the two-dimensional irrotational flow of an inviscid incompressible fluid in the (x, y) plane and let all the singularities of f(z)be a distance greater than a from the origin.

If a circular cylinder |z| = a is introduced into the flow, then by the Circle Theorem of Milne-Thompson [1], the complex potential of the disturbed flow becomes $f(z) + \bar{f}(a^2/\bar{z})$. Letting f(z) = f(x + iy) = u(x, y) + iv(x, y), we get

$$\bar{f}(a^2/\bar{z}) = \bar{f}(a^2(x+iy)/(x^2+y^2))$$

= $u(a^2x/(x^2+y^2), a^2y/(x^2+y^2)) - iv(a^2x/(x^2+y^2), a^2y/(x^2+y^2))$

Since the velocity potential is the real part of the complex potential and since for points (x, y) on the circle |z| = a, $a^2 x/(x^2 + y^2) = x$ and $a^2 y/(x^2 + y^2) = y$, we get the desired result.

(iii) Let $\phi_0(x, y, z)$ be the velocity potential of the irrotational flow of the incompressible inviscid fluid. The flow may contain singularities, but it is assumed that they all lie outside the sphere $S: r^2 = x^2 + y^2 + z^2 = a^2$. Then, by the Sphere Theorem of Weiss [2], if the sphere S is introduced into the flow, the flow is disturbed and its new velocity potential becomes

$$\phi_0(x, y, z) + \phi_1(x, y, z)$$

where

$$\phi_1(x, y, z) = (a/r)\phi_0(a^2 x/r^2, a^2 y/r^2, a^2 z/r^2) - (2/ar) \int_0^a \lambda \phi_0(\lambda^2 x/r^2, \lambda^2 y/r^2, \lambda^2 z/r^2) d\lambda$$

For a source m at the point $\mathbf{f} = (f, 0, 0), f > a, \phi_0 = m/|\mathbf{r} - \mathbf{f}|$. This does not give the analogous result that the potential difference across any two points of the sphere is increased by a constant factor for the disturbed flow.

If the potential of the undisturbed flow is a spherical harmonic of degree $n, \phi_0 = H_n(x, y, z)$. It then follows that

$$\phi_1 = \{n/(n+1)\}\{a^{2n+1/r^{2n+1}}\}H_n(x,y,z)$$

Thus, in the disturbed flow, the potential difference across any two points on the sphere is increased by the factor 1 + n/(n+1).

REFERENCES

[1] L. M. MILNE-THOMPSON, The circle theorem and the cylinder theorem, *Rev. Roumaine Math. Pures Appl.*, **14**(1969) 399–409.

[2] P. WEISS, On hydrodynamical images: Arbitrary irrotational flow disturbed by a sphere, *Proc. Cambridge Philos. Soc.*, **40**(1944) 259–261.

Solution of (ii)* by CARL C. GROSJEAN (State University of Ghent, Belgium).

This problem may be examined with the technique of conformal mapping. First, a typical example.

Let the infinite conducting plane, where a circular hole with radius R and center at O' has been cut out, be referred to an orthogonal frame O'x'y'. A second conducting plane contains an orthogonal frame Oxy. Let

$$z' = (z^2 + c^2)^{1/2}, \quad z' = x' + y'i, \quad z = x + yi$$
 (1)

with some positive real constant c as the conformal mapping relating (x, y) and (x', y') to one another. We wish the circular edge $x'^2 + y'^2 = R^2$ to be mapped on the oval of Cassini:

$$(x^{2} + y^{2})^{2} + 2c^{2}(x^{2} - y^{2}) + c^{4} = 4c^{4}$$

which constitutes the boundary between the convex and nonconvex curves comprised in the bundle

$$(x^2+y^2)^2+2c^2(x^2-y^2)+c^4=\lambda^4$$

where λ can take on any positive value greater than c. (The limit case $\lambda = c$ yields the lemniscate of Bernoulli.) This happens for $4c^4 = R^4$. Hence (1) becomes

$$z'^{=(z^2+\frac{1}{2}R^2)^{1/2}} \tag{2}$$

and x' and y' show corresponding results.

When the point (x', y') runs over a circle with radius $\rho \ge R$ and center at O' in the (x', y')-plane, the point (x, y) describes a convex oval of Cassini with its focal points on the y-axis at $\pm R/sqrt2$.

If, in the (x', y')-plane, a point source of current is located at $P'((a^2 + \frac{1}{2}R^2)^{1/2}, 0)$, with $a > R/\sqrt{2}$, the disturbed potential is

$$\Phi(x',y') = -\frac{C}{2} \ln \left[a^2 + \frac{1}{2}R^2 - 2\left(a^2 + \frac{1}{2}R^2\right)^{1/2} x' + {x'}^2 + {y'}^2 \right] -\frac{C}{2} \ln \left[1 - \frac{2R^2(a^2 + \frac{1}{2}R^2)^{1/2}x' - R^4}{(a^2 + \frac{1}{2}R^2)({x'}^2) + {y'}^2} \right], \qquad {x'}^2 + {y'}^2 \ge R^2$$

while if P' is mapped into P(x = a, y = 0), the ;potential becomes

$$\Psi(x,y) = -\ln\left\{a^2 + \frac{1}{2}R^2 - 2\left(a^2 + \frac{1}{2}R^2\right)^{1/2}u(x,y) + \left[(x^2 + y^2)^2 + R^2(x^2 - y^2) + \frac{1}{4}R^4\right]^{1/2}\right\}$$
$$-\frac{C}{2}\ln\left\{1 - \frac{2R^2(a^2 + \frac{1}{2}R^2)^{1/2}u(x,y) - R^4}{(a^2 + \frac{1}{2}R^2)[(x^2 + y^2)^2 + R^2(x^2 - y^2) + \frac{1}{4}R^4]^{1/2}}\right\}$$

where $(x^2 + y^2)^2 + R^2(x^2 - y^2) - \frac{3}{4}R^4 \ge 0$ and $u(x, y) = \operatorname{Re}(z^2 + \frac{1}{2}R^2)^{1/2}$. This potential is singular at x = a, y = 0. At all other points of the plane, exterior to the oval edge, Ψ is a solution of the (homogeneous) Laplace equation, and at any point on the edge, the current component along the normal is equal to zero. At two arbitrarily chosen points (x_1, y_1) and (x_2, y_2) on the oval edge, the ratio of the disturbed potential diffrence

$$\Psi(x_1, y_1) - \Psi(x_2, y_2) = C \left\{ \ln \left[1 - \frac{\sqrt{2} \operatorname{sgn} x}{(a^2 + \frac{1}{2}R^2)^{1/2}} (\frac{3}{2}R^2 + x^2 - y^2)^{1/2} + \frac{R^2}{a^2 + \frac{1}{2}R^2} \right] \right\} \Big|_{x_1}^{x_2}$$

and the undisturbed potential difference

$$V(x_1, y_1) - V(x_2, y_2) = \frac{C}{2} \left[\ln \left(1 - 2\frac{x}{a} + \frac{x^2 + y^2}{a^2} \right) \right] \Big|_{x_1}^{x_2}$$

could be equal to 2 if and only if the expression

$$\frac{1 - (\sqrt{2}\operatorname{sgn} x/(a^2 + \frac{1}{2}R^2)^{1/2})(\frac{3}{2}R^2 + x^2 - y^2)^{1/2} + (R^2/a^2 + \frac{1}{2}R^2)}{1 - 2(x/a) + (x^2 + y^2)/a^2}$$

were constant for all points (x, y) on the oval, which it is not. Thus the property that holds for a circular hole does not hold for a hole bounded by an oval of Cassini.

The generalization may be formulated as follows. Let

$$z' = f(z), \quad z' = x' + y'i, \quad z = x + yi$$
 (3)

be a conformal mapping with the following properties:

(a) The mapping is bijective at least between the part of the complex z'-plane characterized by $|z'| \ge R$ and the part of the complex z-plane outside a closed curve C_R whose map in the z'-plane is the circle |z'| = R.

(b) When z' runs over a circle with radius $\rho \geq R(>0)$ and center at O', the point z describes a closed trajectory C_{ρ} in the z-plane whereby C_{ρ_1} lies entirely inside C_{ρ_2} when $\rho_2 > \rho_1$.

For (b) to hold, f(z) should at least be continuous.

Let (3) then be rewritten as

$$x' = \operatorname{Re} f(x+yi) \equiv u(x,y), \quad y' = \operatorname{Im} f(x+yi) \equiv v(x,y)$$

In this notation, the equation of C_{ρ} reads

$$u^{2}(x,y) + v^{2}(x,y) = \rho^{2}, \qquad \rho \ge R$$

Assume the Cartesian (x, y)-plane to be an infinite conducting sheet and let electric current enter at the point P(a, b) outside C_R . The difference of the (unperturbed) potentials existing at any two points (x_1, y_1) and (x_2, y_2) located on C_R is

$$V(x_1, y_1) - V(x_2, y_2) = \frac{C}{2} \{ \ln[(a - x_2)^2 + (b - y_2)^2] - \ln[(a - x_1)^2 + (b - y_1)^2] \}$$

with C some positive proportionality factor when V(x, y) is assumed to decrease with increasing distance between (x, y) and (a, b).

Now, let a hole with edge C_R be cut in the (x, y)-plane. Since C_R is transformed into the circle |z'| = R by (3), the disturbed potential $\Psi(x, y)$ can be directly deduced from the potential in the conducting (x', y')-plane resulting from the current that enters the plane sheet at P'(a', b'),

$$a' = u(a, b), \qquad b' = v(a, b)$$

and is disturbed by the circular hole with edge |z'| = R. In this way, we find, on the basis of the same principle as in the example, that the property of doubling the potential difference at any two points of C_R by cutting a hole bounded by that curve holds if and only if

$$\frac{u^2(a,b) + v^2(a,b) - 2u(a,b)u(x,y) - 2v(a,b)v(x,y) + R^2}{a^2 + b^2 - 2ax - 2by + x^2 + y^2}$$
(4)

is a constant when the point (x, y) runs over the edge C_R . A type of conformal mapping for which this ratio is indeed a constant, so that the mentioned property exists, is the linear transformation

$$z' = (A+Bi)z + K + Li, \quad A+Bi \neq 0 \tag{5}$$

We easily verify that the ratio (4) is equal to $A^2 + B^2$ when (x, y) runs over C_R , whose map in the (x', y')-plane, deduced from (5), is |z'| = R. This is not astonishing because C_R is itself a circle.

For any (nondegenerate) circle

$$C_R: x^2 + y^2 + 2mx + 2ny + m^2 + n^2 = p^2, \quad p > 0$$
(6)

which we choose in the (x, y)-plane and which corresponds to |z'| = R in the (x', y')-plane, infinitely many conformal mappings of the form (5) may be used.

We can show that (4) is a constant solely for all possible circular holes in the (x, y)-plane for which P is lying outside, as follows. The numerator in (4) represents the square of the distance between P' and some point Q' on |z'| = R. Similarly, the denominator represents the square of the distence between P and the corresponding point Q on C_R . Taking into account that conformal mapping conserves the angle between any two directions, we see that (4) can only be a constant if the figure comprising C_R , Q, P and the straight line PQ in the (x, y)-plane differs only from that consisting of |z'| = R, Q', P' and the straight line P'Q' in the (x', y')-plane by a scale factor. Therefore C_R is a circle, and the scale factor between $\{C_R, P\}$ and $\{|z'| = R, P'\}$ is p/R, where p is the radius of C_R . The conclusion is: only the circular hole in an infinite conducting plane sheet gas the "doubling" property. [[In the following problem, and at *SIAM Rev.*, 32(1990) 308, and again at *SIAM Rev.*, 33(1991) 118, in the display after the quote, there is printed

$$\angle A = \angle BOC, \dots$$

whereas surely it should be 2A ??]] SIAM Rev., **31**(1989) 320.

Problem 89-10, Three Particles Moving on a Circle, by MURRAY S. KLAMKIN (University of Alberta).

The following is a result ascribed to Newton:

"Three equal particles A, B, C move on the arc of a given circle in such a way that their center of gravity remains fixed: prove that, in any position, their velocities are as $\sin 2A : \sin 2B : \sin 2C$."

We note that we can deduce that the angles are given by

 $\angle 2A = \angle BOC, \quad \angle 2B = \angle COA, \quad \angle 2C = \angle AOB$

where O is the center of the circle, and that the velocities are signed angular speeds. Show, conversely, that if A, B, C are moving on a circle such that their angular speeds are as $\sin \angle BOC$: $\sin \angle COA$: $\sin \angle AOB$, respectively, then the centroid of A, B, Cremains fixed.

Given the circle, the centroid, and the position of A at any time, show how to find the corresponding positions of B and C.

SIAM Rev., **32**(1990) 308.

Solution by W. WESTON MEYER (General Motors Research Laboratories).

The proposition holds true not only for circular orbits but for logarithmic spirals as well. That is to say:

In polar coordinates, let Γ_a be the logarithmic spiral $r = \exp(a\theta)$ and let O be the origin r = 0. If A, B, C are moving on Γ_a such that their linear speeds are as $\sin \angle BOC$: $\sin \angle COA$: $\sin \angle AOB$, respectively, then the centroid of A. B, C remains fixed.

We will geometrize in the complex plane, taking $z = r \exp(i\theta)$. Spiral Γ_a is the locus of

$$z = e^{(a+i)\theta} \qquad (-\infty < \theta < \infty)$$

and the speed of z, as a point moving along Γ_a , is

$$s(z) = \left| \frac{dz}{d\theta} \right| \dot{\theta} = |(a+i)e^{(a+i)\theta}| \dot{\theta} \Longrightarrow \dot{z} = \frac{(a+i)}{|a+i|}e^{i\theta}s(z)$$

where the dot accent signifies differentiation with respect to time.

Let z_1, z_2, z_3 be three points on Γ_a with centroid $z_c = \frac{1}{3}(z_1 + z_2 + z_3)$. The velocity of z_c is expressible as

$$\dot{z}_c = \frac{(a+i)}{3|a+i|} [e^{i\theta_1} s(z_1) + e^{i\theta_2} s(z_2) + e^{i\theta_3} s(z_3)]$$

Abbreviating $\omega_j = \exp(i\theta_j)$, assume that

$$s(z_j) = \sin(\theta_l - \theta_k) = \frac{1}{2i} \left(\frac{\omega_j}{\omega_k} - \frac{\omega_k}{\omega_l} \right) \qquad \{(j, k, l) = (1, 2, 3), (2, 3, 1), (3, 1, 2)\}$$

Then

$$\dot{z}_c = \frac{(a+i)}{6i|a+i|} \left[\omega_1 \left(\frac{\omega_3}{\omega_2} - \frac{\omega_2}{\omega_3} \right) + \omega_2 \left(\frac{\omega_1}{\omega_3} - \frac{\omega_3}{\omega_1} \right) + \omega_3 \left(\frac{\omega_2}{\omega_1} - \frac{\omega_1}{\omega_2} \right) \right] \equiv 0$$

We refer to the unit circle Γ_0 in explaining how to locate B and C for a given centroid G and a given A such that $\overline{AG} \leq \delta = (2 - 2 \cdot \overline{OG}^2)^{1/2}$:

Produce line AG half the distance \overline{AG} beyond G to point A_1 so that $\overline{AG} = 2 \cdot \overline{GA_1}$. Construct a chord of Γ_0 having A_1 as its midpoint. This is side BC of triangle ABC. If A_1 coincides with O, then BC can be any diameter of the circle. Otherwise, it is the unique chord through A_1 perpendicular to OA_1 .

If and only if $\overline{AG} = \delta/2$ will one of the other two points B, C coincide with A. But no point of Γ_0 lies at a distance less than $\delta/2$ or greater than δ from G, unless $\overline{OG} > \frac{1}{3}$. Then an arc of Γ_0 is inaccessible to A, B, C and there are two locations where two of the points, moving in opposite directions, can coincide.

SIAM Rev., **33**(1991) 118.

Editorial note: As a supplement to the previous solution we have the following related results by W. WESTON MEYER (General Motors Research Laboratories).

If A', B', C' move on a circle such that their angular speeds are as $\cot \frac{1}{2} \angle A'$: $\cot \frac{1}{2} \angle B'$: $\cot \frac{1}{2} \angle C'$, respectively, then the incircle of triangle A'B'C' remains fixed.

We shall not prove this, but the converse, linking it to the result "ascribed to Newton". Our figure shows a triangle A'B'C' with circumscribed circle Γ' and inscribed circle Γ , the latter contacting A'B'C' in points A, B, C. Line segments AA'', BB'', CC'' are the altitudes of ABC, produced so as to be chords of Γ . They concur in point H, the orthocenter of ABC. Since arcs B''A and AC'' on Γ subtend equal angles at

B and *C*, respectively (both angles are complementary to $\angle A$), it follows that *A* is the midpoint of arc B''C'', chord B''C'' is parallel to the tangent B'C' and A''A is the bisector of $\angle A''$. But each vertex of *ABC* has the same attributes, so triangles A''B''C'' and A'B'C' are homothetic—similar with corresponding sides parallel—and point *H* is the incenter of A''B''C''.



FIG. 1

Let us regard the two circles as fixed in position and size, Γ' centred in O', Γ centred in O. A classic theorem [1, p.86] says that vertex A' can fall anywhere on Γ' . Hence it is possible for A', B', C' to move on Γ' so that the incircle of A'B'C' remains Γ . Homothecy and the fixation of Γ as circumcircle require A''B''C'' to move about a fixed incircle as well. Point H is stationary. So is the centroid G of ABC which, according to a celebrated theorem of Euler [1, p.101], lies one third of the way from O to H on the line connecting this circumcenter of ABC to this orthocenter.

We observe (perhaps in company with Newton) that for A, B, C to remain on Γ , while G remains stationary, the velocities of A, B, C must parallel the vectors B'C', C'A', A'B', respectively, and match these vectors in summing to zero. Inevitably, then, the speeds of A, B, C are as the sides of A'B'C':

$$s(A) : s(B) : s(C) :: |B'C'| : |C'A'| : |A'B'|$$
 (1)

Let t(X) signify the length of a tangent to circle Γ from an arbitrary point X. Since the lengths of the sides of A'B'C' are

$$t(B') + t(C'), \quad t(C') + t(A'), \quad t(A') + t(B')$$

and the speeds of A, B, C are

$$\frac{s(B'') + s(C'')}{2}, \quad \frac{s(C'') + s(A'')}{2}, \quad \frac{s(A'') + s(B'')}{2}$$

by reason of the arc-bisection property (A the midpoint of B''C'' and so forth), we infer from (1) that

$$s(A'') : s(B'') : s(C'') :: t(A') : t(B') : t(C')$$
(2)

Here, of course, A'', B'', C'' can be replaced by A', B', C'. Moreover,

$$t(A') \cdot \tan \frac{1}{2} \angle A' = t(B') \cdot \tan \frac{1}{2} \angle B' = t(C') \cdot \tan \frac{1}{2} \angle C' = \text{ the radius of } \Gamma.$$

So, in conclusion,

$$s(A') : s(B') : s(C') :: \cot \frac{1}{2} \angle A' : \cot \frac{1}{2} \angle B' : \cot \frac{1}{2} \angle C'$$
 (3)

The arc-bisection property has another implication for ABC that may have escaped Newton's attention. Side CA, as bisector of $\angle C''CB''$, is the mid-perpendicular of the segment HB''. In other words, B'' is the mirror image of H in CA. The optical distance from O to H, under reflection in CA, is the distance |OB''|. But the envelope of all such reflecting lines CA is an ellipse Ψ with O and H as foci, |OB''| as major axis. Thus, moving with fixed centroid, triangle ABC circumscribes the fixed ellipse Ψ .

REFERENCE

[1] N. ALTSHILLER-COURT, College Geometry, Barnesand Noble, New York, 1952.

[[At SIAM Rev., 32(1990) 141 there is the following:]]

Editor's note. To liven up the problem section, we invite readers to submit "Quickies" (preferably of an applied nature). Mathematical Quickies were initiated by C. W. Trigg in 1950 when he was editor of the problem section in *Mathematics Magazine*. These are problems that can be solved laboriously, but with proper insight and knowledge can be disposed of quickly. The next problem is a Quickie. In subsequent issues, these problems will not be identified as such except in their solutions appearing at the end of the same problem section.

An Integral Inequality

Problem 90-5, by M. S. KLAMKIN (University of Alberta).

Prove that

$$\int_0^\infty \left(\frac{1}{a+\lambda} + \frac{1}{b+\lambda} + \frac{1}{c+\lambda}\right) \frac{d\lambda}{\sqrt{\lambda(a+\lambda)(b+\lambda)(c+\lambda)}} \geq \frac{4}{(abc)^{2/3}}$$

where a, b, c > 0.

[[then at p.154 of the same issue we find]]

Problem 90-5: Quickie.

Letting $a, b, c, \rightarrow a^2, b^2 c^2$, the inequality is equivalent to

$$S \ge 4\pi (abc)^{2/3}$$

where

$$S = \pi (abc)^2 \int_0^\infty \left(\frac{1}{a+\lambda} + \frac{1}{b+\lambda} + \frac{1}{c+\lambda} \right) \frac{d\lambda}{\sqrt{\lambda(a+\lambda)(b+\lambda)(c+\lambda)}}$$

It is known [1] that S is the area of an ellipsoid of semi-axes a, b, c. The rest follows from the known isoperimetric inequality that, for all three-dimensional bodies, the maximum of Volume²/Surface³ is achieved for a sphere, whence

$$\frac{(4\pi abc/3)^2}{S^3} \le \frac{(4\pi/3)^2}{(4\pi)^3}$$

REFERENCE

[1] J. EDWARDS, A Treatise on the Integral Calculus, Chelsea, New York, p.843.

[[At SIAM Rev., **32**(1990) 302 the Editor's note is repeated, beneath the following thinly disguised item:]]

Minimum Value of an Integral

Problem 90-10, by K. S. MURRAY (Brooklyn, NY).

Determine the minimum value of

$$I = \int_0^1 \sqrt[n]{F'(t)^n + t'''} \, dt$$

where $F'(t) \ge 0$, F(0) = a, F(1) = b, n is a constant greater than 1, and m is a constant greater than or equal to zero.

SIAM Rev., **32**(1990) 309.

Solution.

More generally, we will find the minimum of

$$J = \int_0^1 \{F'(t)^n + A'(t)^n\}^{1/n} dt$$

where additionally A(t) is given with A(0) = c, A(1) = d and $A'(t) \ge 0$.

By Hölder's inequality,

$$\{F'(t)^n + A'(t)^n\}^{1/n} \cdot \{(b-a)^n + (d-c)^n\}^{(n-1)/n} \ge \{F'(t)(b-a)^{n-1} + A'(t)(d-c)^{n-1}\}$$

Thus,

$$J \ge \int_0^1 \{F'(t)(b-a)^{n-1} + A'(t)(d-c)^{n-1}\} dt + \{(b-a)^n + (d-c)^n\}^{(n-1)/n}$$

or

$$J \ge \{(b - a)^n + (d - c)^n\}^{1/n}$$

There is equality if and only if F'(t)/(b - a) = A'(t)/(d - c).

For the special case n = 2, the problem is equivalent to finding the minimum length curve y = F(t), x = A(t), with endpoints (a, c) and (b, d).

[[At SIAM Rev., 32(1990) 673-674 there's a Quickie immediately followed by its solution. Presumably the solution was intended to be held until the end of the Solutions:]]

Cyclic Pursuit with Lead Angle

Problem 90-19, by M. S. KLAMKIN (University of Alberta).

Four bugs A, B, C, D, starting from the consecutive vertices of a unit square, pursue each other cyclically with the same unit speeds. Bug A always heads directly for Bwith a constant lead angle of R radians, and the same cyclically for the other bugs. If $\pi/2 \ge R \ge 0$, show that the four bugs will meet simultaneously and determine the length of time for this to occur.

[[solution follows immediately on p.674.]]

If we denote the positions of the four bugs by z_a , z_b , z_c , z_d , respectively, in the complex plane, the equations of motion are

$$\dot{z_a} = e^{iR}(z_b - z_a)/|z_b - z_a|$$

etc., where \cdot indicates differentiation with respect to time. We assume the initial positions to be given by

$$z_a = e^{i\pi/4} \quad z_b = iz_a \quad z_c = iz_b \quad z_d = iz_c$$

Since the equations are locally Lipschitz, the motion is unique and we find it by using the symmetry of the configuration, i.e.,

$$z_a = re^{i\theta} \quad z_b = iz_a \quad z_c = iz_b \quad z_d = iz_c$$

We then obtain

$$\dot{r} + ir\dot{\theta} = \frac{(i-1)(\cos R + i\sin R)}{\sqrt{2}}$$

or

$$\dot{r} = -\sin(R + \pi/4)$$
 $r\dot{\theta} = \cos(R + \pi/4)$

Thus

$$r = 1 - t\sin(R + \pi/4) \quad \theta = \pi/4 - \cot(R + \pi/4)\ln(1 - t\sin(R + \pi/4))$$

It now follows that the four bugs meet simultaneously at the center of the initial square in a time $t = \csc(R + \pi/4)$. It is to be noted that for R = 0 or $\pi/2$ we have the classic four bug cyclic pursuit problem. For $R = \pi/4$ the four bugs move on the diagonals of the initial square directly to the center.

The results generalize for n bugs starting out on the n vertices of a regular n-gon.

SIAM Rev., 33(1991) 474.

Point of Minimum Temperature

Problem 91-15^{*}, by M. S. KLAMKIN (University of Alberta).

A homogeneous convex centrosymmetric body with constant thermal properties is initially at temperature zero and its boundary is maintained at a temperature $T_b > 0$. Prove or disprove that at any time t > 0, the point of minimum temperature is the center. Also, prove or disprove that the isothermal surfaces are convex and centrosymmetric. Note that the convexity of the isothermal surfaces will imply that the center is the point of minimum temperature.

SIAM Rev., 34(1992) 120-121.

Extreme Gravitational Attraction

Problem 92-5^{*}, by M. S. KLAMKIN (University of Alberta).

It follows by symmetry that the inverse square law attraction of any uniform homogeneous regular polyhedron on a unit test particle at its centroid is zero.

(i) Determine the location of the test particle, within or on a regular polyhedron, in particular the cube, which maximizes the attraction.

(ii) Consider the same problem (and also the minimum attraction) for a uniform homogeneous torus.

(iii) Determine the dimensions of a uniform homogeneous rectangular parallelepiped of unit volume so that its attraction for a unit test particle located at the center of a face is a maximum (minimum).

SIAM Rev., **38**(1996) 515–520.

Solution by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium).

(i) When a test particle is located inside a regular polyhedron or even any convex body with constant mass density, there is an amount of cancellation between forces exercised on it by the masses comprised in some elementary volumes belonging to noninfinitesimal three-dimensional regions of the body. In the case that the test particle is located at the centroid of a uniform homogeneous regular polyhedron, the cancellation is perfect, as mentioned above; otherwise, it is only partial. For instance, consider in \mathbb{E}^3 referred to a rectangular cartesian coordinate system Oxyz, the uniform homogeneous cube in $-a \leq x, y, z \leq a$ (a > 0) and a test particle P at (b, 0, 0) whereby 0 < b < a, say. The the net attraction on P stems from the mass in the region $-a \leq x \leq 2b - a, -a \leq y, z \leq a$. In general, since the test particle is exactly or approximately at the centroid of the volume in which cancellation of forces takes place, this cancellation has two consequences: (1) the net force on the test particle is caused by only a fraction of the mass present;

(2) all elementary parts of that mass fraction are located at some noninfinitesimal distance, more or less far, from the test particle.

From this qualitative argument, it follows that the force exercised on an internal test particle is always less than the force on a test particle located at a point of the surface of the three-dimensional convex body, since in this case cancellation of forces stemming from nonvanishing amounts of mass cannot occur.

In the case of the above-mentioned cube, the attractive force on a test particle of unit mass at x = a, y = z = 0, is given by

$$f_{0,0} = \frac{M}{8a^3} \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} \frac{a - x}{[(a - x)^2 + y^2 + z^2]^{3/2}} dx dy dz$$

$$= \frac{M}{8a^2} \int_{-1}^{1} dz' \int_{-1}^{1} dy' \int_{-1}^{1} \frac{1 - x'}{[(1 - x')^2 + y'^2 + z'^2]^{3/2}} dx'$$

$$= \frac{M}{2a^2} \int_{0}^{1} dz' \int_{0}^{1} \left[\frac{1}{(y'^2 + z'^2)^{1/2}} - \frac{1}{(4 + y'^2 + z'^2)^{1/2}} \right] dy'$$

$$= \frac{M}{a^2} \left[\ln(\sqrt{2} + 1) + \frac{\pi}{2} - \int_{0}^{1/\sqrt{2}} \frac{(5 - 4t^2)^{1/2}}{1 - t^2} dt \right]$$

$$= \frac{M}{a^2} \left[\frac{\pi}{2} + \ln\left(\frac{(\sqrt{2} + 1)(\sqrt{6} - 1)}{\sqrt{5}}\right) - 2 \arcsin\sqrt{\frac{2}{5}} \right]$$

$$= 0.649224 \dots \frac{M}{a^2}$$

where M is the mass of the cube (with edge 2a). In this formula and those which follow, the gravitational constant which is merely a proportinality factor, unimportant within the present context, is left out.

When the unit test particle is located at the vertex x = y = z = a, we get similarly

$$\begin{split} f_{a,a} &= \frac{M\sqrt{3}}{8a^3} \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} \frac{a-x}{[(a-x)^2 + (a-y)^2 + (a-z)^2]^{3/2}} \, dx dy dz \\ &= \frac{M}{8a^2} \int_{-1}^{1} dz' \int_{-1}^{1} dy' \int_{-1}^{1} \frac{1-x'}{[(1-x')^2 + y'^2 + z'^2]^{3/2}} \, dx' \\ &= \frac{M\sqrt{3}}{8a^2} \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} \frac{u}{(u^2 + v^2 + w^2)^{3/2}} \, du dv dw \\ &= \frac{M\sqrt{3}}{8a^2} \int_{0}^{2} dw \int_{0}^{2} \left[\frac{1}{(v^2 + w^2)^{1/2}} - \frac{1}{(4 + v^2 + w^2)^{1/2}} \right] \, dv \\ &= \frac{M\sqrt{3}}{2a^2} \left[\ln(\sqrt{2} + 1) + \frac{\pi}{4} - \int_{0}^{1/\sqrt{2}} \frac{(2 - t^2)^{1/2}}{1 - t^2} \, dt \right] \\ &= \frac{M\sqrt{3}}{2a^2} \left[\ln(\sqrt{2} + 1) + \frac{\pi}{12} - \frac{1}{2} \ln(2 + \sqrt{3}) \right] \\ &= 0.419757 \dots \frac{M}{a^2} < F_{0,0} \end{split}$$

When the unit test particle is located at $(a, \eta a, \zeta a)$ $(0 < \eta, \zeta < 1)$, the attractive force is, in absolute value, equal to

$$f_{\eta a,\zeta a} = (f_1^2 + f_2^2 + f_3^2)^{1/2}$$

in which

$$f_{1} = \frac{M}{8a^{2}} \int_{\zeta-1}^{\zeta+1} \left[\frac{1}{(v^{2}+w^{2})^{1/2}} - \frac{1}{(4+v^{2}+w^{2})^{1/2}} \right] dv$$

$$f_{2} = \frac{M}{8a^{2}} \int_{\zeta-1}^{\zeta+1} \left[\frac{1}{(u^{2}+(1-\eta)^{2}+w^{2})^{1/2}} - \frac{1}{(u^{2}+(1+\eta)^{2}+w^{2})^{1/2}} \right] du$$

$$f_{3} = \frac{M}{8a^{2}} \int_{\eta-1}^{\eta+1} \left[\frac{1}{(u^{2}+v^{2}+(1-\zeta)^{2})^{1/2}} - \frac{1}{(u^{2}+v^{2}+(1+\zeta)^{2})^{1/2}} \right] du$$

Because $(v^2 + w^2)^{-1/2} - (4 + v^2 + w^2)^{-1/2}$ is infinite at u = v = w = 0 and decreases toward zero approximately like $2/(v^2 + w^2)^{3/2}$ in all (v, w)-directions with growing $(v^2 + w^2)$, f_1 decreases from $0.649224 M/a^2$ to $0.242347 M/a^2$. Similarly f_2 and f_3 increase from 0 to $0.242347 M/a^2$. As a result, starting from $\eta = \zeta = 0$, $f_{\eta a, \zeta a}$ at first decreases slowly with growing $\eta^2 + \zeta^2$, but suddenly decreases rather rapidly in a neighborhood of $\eta = \zeta = 1$. Qualitatively, his behavior can be ascribed to more matter being present in the vicinity of the position $(\eta a, \zeta a)$ when $\eta^2 + \zeta^2 \ll 2$, because then $(\eta a, \zeta a)$ lies in a plane of the face at x = a relatively far from the edges, and this is not the case when $(\eta a, \zeta a)$ is located close to a vertex. The same facts and reasoning also hold for the uniform homogeneous regular polyhedrons other than the cube. In the case of a tetrahedron with edge 2a, we have in absolute value – for the gravitational attraction at the center of a face

$$\left[\sqrt{6}\ln(2+\sqrt{3}) + \frac{2\pi}{3\sqrt{3}} - \frac{2\sqrt{6}}{3}\ln(\sqrt{3}+\sqrt{2})\right]\frac{M}{a^2} = 2.56331\dots\frac{M}{a^2}$$

– for the gravitational attraction at a vertex:

$$6\sqrt{3}\left(\frac{\pi}{3} - \arctan\sqrt{2}\right)\frac{M}{a^2} = 3\sqrt{3}\left(\frac{\pi}{6} - \arctan\frac{\sqrt{2}}{4}\right)\frac{M}{a^2} = 0.95485\dots\frac{M}{a^2}$$

Similarly, in the case of an octahedrn with edge 2a, there comes

- at the center of a face: $1.07630 \dots M/a^2$,
- at a vertex:

$$\left\{\sqrt{2}\ln[3(\sqrt{2}-1)] + \arctan\frac{M}{a^2} = 0.64705\dots\frac{M}{a^2}\right\}$$

To make the numerical coefficients for the tetrahedron, the cube and the octahedron comparable to that corresponding to a point on the sphere, Dr. H. De Meyer suggested expressing all rsults in terms of $\rho V^{1/3}$, where ρ is the constantmass density. In this way, we find for the attractive force, still in absolute value,

- at a point on the sphere: $(4\pi/3)^{2/3}\rho V^{1/3} = 2.5985\rho V^{1/3}$
- at the center of a face of the tetrahedron: $2.4646\rho V^{1/3}$
- at a vertex of the tetrahedron: $0.9181\rho V^{1/3}$
- at the center of a face of the cube: $2.5969\rho V^{1/3}$
- at a vertex of the cube: $1.6790\rho V^{1/3}$
- at the center of a face of the octahedron: $2.6077\rho V^{1/3}$
- at a vertex of the octahedron: $1.5677\rho V^{1/3}$

Conclusion. The attraction is maximized at the centre of each face of any uniform homogeneous regular polyhedron.

(ii) The same qualitative argument as expounded in (1), namely, that for masses lying in opposite locations with respect to a test particle there is cancellation between attractive forces to some extent, can be made use of in the case of a torus. This argument and some considerations of symmetry lead to the conclusion that for a torus with equation

$$(\rho - R)^2 + z^2 = a^2$$
 $0 < a < R$

in cylindrical coordinates

– the maximum attraction is exercised on any point of the outer circle $\rho=R+a,\,z=0$

– the minimum attraction is exercised on any point of the inner circle $\rho = R - a, z = 0$

In absolute value, the maximum attractive force is given by

$$F_{R+a} = \frac{2M}{\pi^2 a^2 R} \int_{R-a}^{R+a} \rho \, d\rho \int_0^{\pi} d\phi \int_0^{[a^2 - (\rho - R)^2]^{1/2}} \frac{(R+a-\rho\cos\phi) \, dz}{[(R+a-\rho\cos\phi)^2 + \rho^2\sin^2\phi + z^2]^{3/2}}$$

which can be reduced to a single integral depending on just one parameter $\alpha \ (= a/R)$. Its integrand consists of two parts that involve complete elliptic integrals of the first and third kinds. Also in absolute value, the minimum attractive force is represented by

$$F_{R-a} = \frac{2M}{\pi^2 a^2 R} \int_{R-a}^{R+a} \rho \, d\rho \int_0^{[a^2 - (\rho - R)^2]^{1/2}} \frac{(R - a - \rho \cos \phi) \, dz}{[(R - a - \rho \cos \phi)^2 + \rho^2 \sin^2 \phi + z^2]^{3/2}}$$

If we write $F_{R+a} = f(\alpha)M/R^2$, then the minimum attractive force is

$$F_{R-a} = -f(-\alpha)\frac{M}{R^2} = |f(-\alpha)|\frac{M}{R^2}$$

In the integral for F_{R+a} the two parts represent positive expressions that add up; in the case of F_{R-a} they form a subtraction by virtue of the coefficient in front of the second part being negative ($-\alpha$ instead of α). This reflects compensation of forces stemming from the parts of the torus on either side of the tangent plane at the point (R-a, 0, 0).

(iii) Let the space occupied by the parallelepiped with constant mass density ρ be described by $-a \leq x \leq a, -b \leq y \leq b, -c \leq z \leq c$ whereby 8abc = 1. In absolute value, the attractive force acting on a unit test particle at the point x = a, y = z = 0 is given by

$$F(a,b,c) = \rho \int_{-c}^{c} dz \int_{-b}^{b} dy \int_{-a}^{a} \frac{(a-x) dx}{[(a-x)^{2} + y^{2} + z^{2}]^{3/2}}$$

= $4\rho \int_{0}^{c} dz \int_{0}^{b} \left[\frac{1}{(y^{2} + z^{2})^{1/2}} - \frac{1}{(4a^{2} + y^{2} + z^{2})^{1/2}} \right] dy$

Extremizing F(a, b, c) under the constraint 8abc = 1 can be effectuated with the help of the Lagrange multiplier metho. This consists of constructing the function $\Phi(a, b, c, \lambda) = F(a, b, c) + (8abc - 1)\lambda$ and setting its four partial derivatives equal to zero:

$$\frac{\partial F}{\partial a} + 8bc\lambda = 0, \qquad \qquad \frac{\partial F}{\partial b} + 8ac\lambda = 0, \\ \frac{\partial F}{\partial c} + 8ab\lambda = 0, \qquad \qquad 8abc - 1 = 9.$$

Explicitly, this gives

$$16\rho a \int_0^c dz \int_0^b \frac{dy}{(4a^2 + y^2 + z^2)^{3/2}} + 8bc\lambda = 0$$
$$4\rho \int_0^c \left(\frac{1}{(b^2 + z^2)^{1/2}} - \frac{1}{(4a^2 + b^2 + z^2)^{1/2}}\right) dz + 8ac\lambda = 0$$
$$4\rho \int_0^b \left(\frac{1}{(y^2 + c^2)^{1/2}} - \frac{1}{(4a^2 + y^2 + c^2)^{1/2}}\right) dy + 8ab\lambda = 0$$

or, after taking the constraint into account,

$$4\rho a \left(\pi - 2 \arcsin \frac{2ac}{(4a^2 + b^2)^{1/2}(b^2 + c^2)^{1/2}} - 2 \arcsin \frac{2ab}{(4a^2 + c^2)^{1/2}(b^2 + c^2)^{1/2}}\right) = -\lambda$$

$$4\rho b \left\{ \ln[c + (b^2 + c^2)^{1/2}] - \ln b - \ln[c + (4a^2 + b^2 + c^2)^{1/2}] + \ln(4a^2 + b^2)^{1/2} \right\} = -\lambda$$

$$4\rho c \left\{ \ln[b + (b^2 + c^2)^{1/2}] - \ln c - \ln[b + (4a^2 + b^2 + c^2)^{1/2}] + \ln(4a^2 + c^2)^{1/2} \right\} = -\lambda$$

in which a should be replaced by 1/8bc. The elimination of λ by subtraction leads to two transcendental equations. Setting $c = \mu b$ so that $a = 1/8b^2\mu$, further simplification is attained by putting $1/4b^3\mu = x$ and so the final form of the transcendental system in μ and x reads

$$\mu \ln \left(\frac{(x^2 + \mu^2)^{1/2} [1 + (\mu^2 + 1)^{1/2}]}{\mu [1 + (x^2 + \mu^2 + 1)^{1/2}]} \right) = \ln \left(\frac{(x^2 + 1)^{1/2} [\mu + (\mu^2 + 1)^{1/2}]}{\mu + (x^2 + \mu^2 + 1)^{1/2}} \right)$$
$$= x \left[\frac{\pi}{2} - \arcsin \frac{\mu x}{(\mu^2 + 1)^{1/2} (x^2 + 1)^{1/2}} - \arcsin \frac{x}{(\mu^2 + 1)^{1/2} (x^2 + \mu^2)^{1/2}} \right] \quad (1)$$

In terms of μ and x, the edges of the parallelepiped are expressed by

$$2a = \left(\frac{x^2}{4\mu}\right)^{1/3} \qquad 2b = \left(\frac{2}{\mu x}\right)^{1/3} \qquad 2c = \left(\frac{2\mu^2}{x}\right)^{1/3} \qquad (2)$$

If (x, μ) is a positive solution of the transcendental system, such is also the case with $(x/\mu, 1/\mu)$ and the latter solution yields the same parallelepiped. Hence, in studying

system (1), μ can be restricted to the interval (0,1]. Straightforward analysis shows that, for any x > 0,

$$\mu \ln \left(\frac{(x^2 + \mu^2)^{1/2} [1 + (\mu^2 + 1)^{1/2}]}{\mu [1 + (x^2 + \mu^2 + 1)^{1/2}]} \right) > \ln \left(\frac{(x^2 + 1)^{1/2} [\mu + (\mu^2 + 1)^{1/2}]}{\mu + (x^2 + \mu^2 + 1)^{1/2}} \right)$$

holds when $0 < \mu < 1$, whereas both sides are equal to

$$\ln\left(\frac{(x^2+1)^{1/2}(1+\sqrt{2})}{1+(x^2+2)^{1/2}}\right)$$
(3)

when $\mu = 1$. Hence, the first equality in system (1) can only hold for $\mu = 1$, in which case it is an identity. Consequently (1) reduces to one transcendental equation in x:

$$\ln\left(\frac{(x^2+1)^{1/2}(1+\sqrt{2})}{1+(x^2+2)^{1/2}}\right) = x\left(\frac{\pi}{2} - 2\arcsin\frac{x}{\sqrt{2}(x^2+1)^{1/2}}\right)$$
$$= x\arcsin\frac{1}{x^2+1} \qquad (4)$$

As x increases from 0 onward,

– the left-hand side starts at zero and increases monotonically to $\ln(1+\sqrt{2})$ with a gradually decreasing slope,

- the right-hand side initially increases much faster than the left-hand side, attains an absolute maximum around x = 0.9, and after that decreases continuously towards 0.

Therefore (4) admits only one positive solution which must be calculated numerically. The result with five significant decimals is x = 1.84935. In turn, this gives

$$2a = 0.94913, \qquad 2b = 2c = 1.02645$$
 (5)

A verification can be carried out by calculating the attractive force on a unit test particle located at (a, 0, 0) stemming from a rectangular parallelepiped with constant mass density ρ characterized by $-a \leq x \leq a, -b \leq y, z \leq b$. The force has the value

$$F = \rho \int_{-b}^{b} dz \int_{-b}^{b} dy \int_{-a}^{a} \frac{a - x}{[(a - x)^{2} + y^{2} + z^{2}]^{3/2}} dx$$

= $4\rho \left[a\pi + 2b \ln \frac{(\sqrt{2} + 1)[((4a^{2}/b^{2}) + 2)^{1/2} - 1]}{((4a^{2}/b^{2}) + 1)^{1/2}} - 4a \arcsin \frac{a\sqrt{2}/b}{((4a^{2}/b^{2}) + 1)^{1/2}} \right]$ (6)

Note that this formula generalizes that for a cube worked out in (i). With (5) inserted into it, one finds $F = 2.59928\rho$, to be compared with $F = 2.59690\rho$ holding in the case

of the cube with edge 2a = 1. The fact that (5) yields the maximum attraction can be confirmed analytically by first replacing 2a by $(x/2)^{2/3}$ and 2b by $(2/x)^{1/3}$ in (6) according to (2) with $\mu = 1$, and after that by calculating the first and second derivatives with respect to x. By setting the first derivative equal to zero, (4) is retrieved. By inserting x = 1.84935 into the second derivative, a negative value isobtained confirming the maximum.

Remark. That $F = 2.59928\rho$ exceeds the value 2.59852ρ for the unit sphere is not an error or a contradiction. Additional calulations prove that, for instance, for an oblate ellipsoid with rotational symmetry and of unit volume, the maximum 2.65578ρ is attained for the ratio of its axes equal to 0.71952.

I thank Dr. H. De Meyer for considerable computational assistance.

SIAM Rev., **34**(1992) 308.

A Set of Maxima Problems

Problem 92-8, by K. S. MURRAY (Brooklyn, N.Y.).

Determine the maximum values of (a) x^2 , (b) y^2 , (c) $x^2 + y^2$, (d) $x^2 + z^2$, (e) $x^2 + y^2 + z^2$ and (f) $x^2 + y^2 + z^2 + w^2$ for all real x, y, z, w satisfying

$$x^{2} + y^{2} + z^{2} + w^{2} - xy - yz = k^{2}$$
(1)

SIAM Rev., **34**(1992) 323–324.

Problem 92-8 (Quickie).

(a) Since (1) can be written as

$$(2x - 3y)^2/12 + (2z - y)^2/4) + w^2 + 2x^2/3 = k^2,$$

max $x^2 = 3k^2/2$

(b) Since (1) can be rewritten as

$$(x - y/2)^2 + (z - y/2)^2 + w^2 + y^2/2 = k^2,$$

max $y^2 = 2k^2$

(c) We can write (1) in the form

$$\frac{(ax-by)^2}{2ab} + \left(z - \frac{y}{2}\right)^2 + w^2 + \left(1 - \frac{a}{2b}\right)x^2 + \left(\frac{3}{4} - \frac{b}{2a}\right)y^2 = k^2.$$

Then, setting $1 - a/2b = \frac{3}{4} - b/2a$, we find that $1 - a/2b = (7 - \sqrt{17})/8$, so that

$$\max (x^2 + y^2) = \frac{8k^2}{(7 - \sqrt{17})} = \frac{(7 + \sqrt{17})k^2}{4}$$

(d) Since (1) can be rewritten as

$$(2y - x - z)^2/4) + (x - z)^2/4 + w^2 + (x^2 + z^2)/2 = k^2,$$

max $(x^2 + z^2) = 2k^2$

(e) Since (1) can be rewritten as

$$\frac{(x-y/sqrt2)^2}{\sqrt{2}} + \frac{(z-y/sqrt2)^2}{\sqrt{2}} + w^2 + (x^2+y^2+z^2)\left(1-\frac{1}{\sqrt{2}}\right) = k^2$$

$$\max (x^2 + y^2 + z^2) = \frac{k^2 \sqrt{2}}{\sqrt{2} - 1} = k^2 (2 + \sqrt{2})$$

(f) Since in (e)

$$w^{2} + (x^{2} + y^{2} + z^{2})(1 - 1/\sqrt{2}) = \frac{w^{2}}{\sqrt{2}} + (x^{2} + y^{2} + z^{2} + w^{2})\left(1 - \frac{1}{\sqrt{2}}\right),$$
$$\max(x^{2} + y^{2} + z^{2} + w^{2}) = k^{2}(2 + \sqrt{2})$$

Comments. Similarly, since $\max z^2 = \max w^2$ and $\max w^2 = k^2$, it follows that the figure whose equation is (1) is bounded, and so it must be a four-dimensional ellipsoid.

SIAM Rev., 35(1993) 136.

Ellipsoid of Inertia of a Regular Simplex

Problem 93-3, by K. S. MURRAY (Brooklyn, N.Y.).

Prove that the ellipsoid of inertia about the centroid of a uniform regular simplex is a sphere.

SIAM Rev., **35**(1993) 141.

Problem 93-3 (Quickie).

Our proof is indirect. Assume that the ellipsoid of inertia which is unique is not a sphere. Then, by considering the group of motions which take the simplex into itself, we would generate a number of different ellipsoids of inertia. This gives the necessary contradiction. The same proof applies to any uniform regular polytope.

[[Let me know if you want to use any of

SIAM Rev., **35**(1993) 299–304.

This is Problem 92-6 by J.O.Fellman:

Maximum Value of a Shear Stress

a solution by Russell L. Mallett, and more than 2 pages of Editorial Note by M.S.K.]]

SIAM Rev., **35**(1993) 642.

[[Another of Murray's many disguises. Compare 92-5* at SIAM Rev., 34(1992) 120–121 above.]]

Maximum Gravitational Attraction

Problem 93-19^{*}, by K. M. SEYMOUR (Toronto, Ontario).

It it conjectured that if a uniform ellipsoid is divided into two subsets such that the gravitational attractive force between them is a maximum, then the two sets must be congruent hemiellipsoids formed by a plane containing two of the axes of the ellipsoid. Prove or disprove.

SIAM Rev., **36**(1994) 660–661.

Comments by C. C. GROSJEAN (University of Ghent, Ghent, Belgium).

The problem can be treated in a semi-quantitative manner as follows. In mechanics it occurs that one can characterize a rigid body by its mass and its center of mass, and that these data suffice to calculate certain mechanical qualities, for instance, the force exercised on the body by a homogeneous force field such as the earth's gravitational field taken locally in a comparatively small volume around a point of th earth's surface.

When a uniform ellipsoid is subdivided into two adjacent parts, one can acquire a qualitative approximation of the gravitational attractive force between them by calculating:

$$F = \gamma \frac{M_1 M_2}{\bar{R}^2} \tag{1}$$

where γ is the gravitational constant, M_1 and M_2 represent the masses of the two parts and \bar{R}^2 denotes the square of the distance between their centers of mass. The formula would be rigorous if each part were collapsed into its center of mass. In the case of the ellipsoid, $M_1 + M - 2 = M = (4\pi/3)abc\rho$ where a > b > c are the three half-axes and the constant ρ is the mass-density of the uniform ellipsoid.

The more symmetric[ally] the ellipsoid is divided, the smaller M_1M_2 and the larger \bar{R}^2 turn out to be. Because of $M_1 + M_2 = M$, the product M_1M_2 is at its maximum when $M_1 = M_2 = M/2$, which is for instance the case when the ellipsoid is cut in half by a plane passing through its center.

Now, to examine the behavior of \bar{R}^2 , consider the ellipsoid with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a > b > c$$

in a rectangular cartesian frame of refrence and the plane x = f with f a constant satisfying 0 < f < a. The mass and the center of mass of the part of the ellipsoid lying between x = f and x = a are given by, respectively,

$$M_1 = \frac{\rho \pi b c (a-f)^2 (2a+f)}{3a^2}$$
 and $y = z = 0, \ x = \frac{3}{4} \frac{(a+f)^2}{2a+f}$

Similarly the mass and the center of mass of the part of the ellipsoid lying between x = -a and x = f are described by, respectively,

$$M_2 = \frac{\rho \pi b c (a+f)^2 (2a-f)}{3a^2} \quad \text{and} \quad y = z = 0, \ x = -\frac{3}{4} \frac{(a-f)^2}{2a-f}$$

Hence, when the ellipsoid is divided by means of x = f, the gravitational attractive force between the two parts is approximately

$$F = \gamma \frac{\rho^2 \pi^2 b^2 c^2}{9a^4} \frac{(a-f)^2 (a+f)^2 (2a+f)(2a-f)}{\left[\frac{3}{4} \frac{(a+f)^2}{2a+f} + \frac{3}{4} \frac{(a-f)^2}{2a-f}\right]^2}$$
$$= \gamma \frac{M^2}{144a^2} \left(1 - \frac{f^2}{a^2}\right)^2 \left(4 - \frac{f^2}{a^2}\right)^3 \tag{2}$$

Note that F increases from 0 to a largest value when f decreases from a to 0. For f = 0, the local maximum is described within the present framework based upon (1) by

$$F = \gamma \frac{4M^2}{9a^2} \tag{3}$$

When the ellipsoid is cut by the plane y = g, the force is described by the same formula as (2) except that a is replaced by b and f by g. It is therefore also a function of g which increases from 0 to

$$F = \gamma \frac{4M^2}{9b^2} \tag{4}$$

when g decreases from b to 0. Since b < a, the ellipsoid cut in half by the Oxz-plane yields a stronger force between the two halves than by cutting by means of the Oyzplane. When we let the plane which cuts the ellipsoid move from Oyz to Oxz by rotation around O, the geometric center of the ellipsoid, F varies continuously fro (3) to (4).

Finally, the same can be repeated with a plane z = h, where h decreases from c to 0. A plane rotating around O from Oxz to Oxy gives rise to a force varying continuously from (4) to

$$F = \gamma \frac{4M^2}{9c^2} \tag{5}$$

since c < b. This is approximately the maximum force obtained via the formula (1). The reasons for that maximum are: symmetry entails $M_1 = M_2 = M/2$ and the oblate shapes of the two parts entail the smallest distance between the two centers of mass.

The approximation consisting of making use of the mass centers in (1) cannot be too rough because the gravitational fields emanating from the two hemiellipsoids are approximately homogeneous close to their flat side.

Conclusion. The maximal gravitational attractive force is produced by the two sets being hemiellipsoids separated by the plane Oxy which is the plane containing the largest symmetry axis and the symmetry axis of intermediate length of the ellipsoid (OA of length a and OB of length b). Although the above treatment is based on approximate formulas, the obtained solution is exact because no other symmetric subdivision of the uniform ellipsoid can be imagined for which the quantities of matter contained in the two subsets are on the average closer to one another, so as to give rise to a stronger attractive force.

Remark. A completely exact treatment requires the evaluation of a number of difficult multiple integrals of elliptic type.

SIAM Rev., **36**(1994) 107.

A Conjectured Heat Flow Problem

Problem 94-1^{*}, by M. S. KLAMKIN (University of Alberta).

Consider the unsteady heat flow problem

$$\partial T/\partial t = \alpha \left\{ \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2 \right\}$$

for a smooth, convex, homogeneous body that is initially at temperature 0 and whose boundary is maintained at temperature 1 for all t > 0. If the point of the body at minimum temperature for all t > 0 remains fixed, it is conjectured that the body must be centrosymmetric about this point. Prove or disprove.

[[Compare *Problem* 91-15^{*} at *SIAM Rev.*, **33**(1991) 474 above.]]

Solution by R. GULLIVER & N. B. WILLMS (University of Minnesota).

In this note, we disprove Klamkin's conjecture. We begin by examining the *analytic* solution for the two-dimensional equilateral triangle. We show that the "cold spot" for this body remains stationary, although the body is not centrosymmetric. The symmetry properties of the equilateral triangle motivate us to construct a large class of non-centrosymmetric, strictly convex bodies with analytic boundaries for which the point of minimum temperature remains fixed in space for all time t > 0. In fact we shall prove that if a body in \mathbb{R}^n , n > 1, has n or more independent reflection symmetries, and is quasi-convex in the directions orthogonal to the hyperplanes of reflective symmetry, then the point of minimum temperature will be the intersection of the reflection hyperplanes, and will remain fixed for all time. Of course, if these directions are mutually orthogonal, the the body will be centrosymmetric about their intersection point. Thus one way to weaken Klamkin's conjecture would be to propose that lack of movement over time of the "cold spot" implies that the body has at least n hyperplanes of reflective symmetry. We feel, however, that this is still too strong.

We begin by rephrasing the problem. Let the function $u : \Omega \times (0, \infty) \to \mathbb{R}$ be defined by $u = u(x, t) = 1 - T(x, t/\alpha)$. Then the problem

$$\begin{array}{rcl} \frac{\partial u}{\partial t} &=& \Delta u & (x,t) \in \Omega \times (0,\infty) \\ u(x,0) &=& 1 & x \in \Omega \end{array} \tag{1}$$

$$u(x,t) = 0 \qquad x \in \partial\Omega, \ t > 0 \tag{3}$$

i.e., the problem corresponding to the conjecture, has a solution $u \in C^2[\Omega \times (0,\infty)]$ given by

$$u(x,t) = \sum_{i=1}^{\infty} a_i e^{-\lambda_i t} u_i(x) \tag{4}$$

where (λ_i, u_i) are the eigenvalue-eigenvector pairs for the Dirichlet Laplacian in Ω . It is well known that the eigenvalues positive and haveno accumulation point. Without loss of generality, we shall take the eigenfunctions to be orthonormal, and shall abel the eigenvalues in order of increasing magnitude with respect to multiplicity, i.e.,

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \tag{5}$$

The coefficients in (4) must be chosen to satisfy the initial condition (2). Thus,

$$a_i = \int_{\Omega} u_i(x) \, dx \qquad i = 1, 2, \dots \tag{6}$$

As a consequance of the strong maximum principle for parabolic equations, the solution u is positive in Ω for all time $0 < t < \infty$. Clearly, as $t \to \infty$ all heat is removed from the body and the solution becomes constant: $u \equiv 0$. Moreover, for large time the solution becomes increasingly dominated by the scaled shape of the first eigenfunction, u_1 , corresponding to the simple eigenvalue λ_1 . Since the body was assumed to be convex, we can conclude that u_1 is log-concave, and therefore must have convex level sets and, in particular, a *unique* critical point in Ω , a positive maximum. Thus, if the "hot point" (the point of Ω for which u is maximized at a given time t > 0) moves spatially over time, it must come to rest, as $t \to \infty$, at the unique point where u_1 attains its maximum in Ω . We label this point P_H .

Klamkin's conjecture now becomes: if the problem (1)-(3) is overdetermined by the requirement that the body's "hot spot" remains stationary for all positive times, that is,

$$\{P_H\} = \{\xi \in \Omega \mid u(\xi, t) = \max_{x \in \Omega} u(x, t), \ t > 0\}$$
(7)

then the only domains for which there exist solutions to the overdetermined problem are centrosymmetric about P_H . For completeness, we include the definition.

A body is *centrosymmetric* about a point P if for every point A on the boundary. there exists another point A' on the boundary such that P is the midpoint of the line segment AA'.

The equilateral triangle does not possess centrosymmetry. Nevertheless, it has enough symmetry for us to intuitively expect that the hot spot should remain at the triangle's center for all time. We now verify this expectation.

Let $\Omega \subset \mathbb{R}^2$ be the equilateral triangle $\{(x, y) \mid 0 < y < \sqrt{3\min(x, 1-x)}\}$. Pinsky [1] has compiled acomplete list of the eigenvalues/vectors for this domain, from which we can construct the solution to (1)–(3) via(4). Let R be the rotation operator

$$R(x,y) \to \left(1 - \frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} - \frac{y}{2}\right)$$

An eigenvalue u_i is said to be symmetric if $u_i \circ R = u_i$. An eigenfunction is said to be complex if $u_i \circ R = e^{\pm 2\pi i/3}u_i$. Let $D = \{(x, y) \mid 0 < y < \min(x, 1-x)/\sqrt{3}\}$. If u_i is a complex eigenfunction, then $u_i \circ R = \sigma u_i$ where $1 + \sigma + \sigma^2 = 0$, so by (6),

$$a_i = (1 + \sigma + \sigma^2) \iint_D u_i(x, y) \, dx \, dy = 0$$

Hence complex eigenfunctions make no contribution to our solution. By [1, Corollaries 1 and 2, p.820], we find that the solution is

$$u(x, y, t) = \left(\frac{27}{2}\right)^{1/4} \sum_{k=1}^{\infty} \frac{e^{-16\pi^2 k^2 t/3}}{k} \{\sin(2\pi k d_1) + \sin(2\pi k d_2) + \sin(2\pi k d_3)\}$$

where d_1, d_2, d_3 are the normalized altitudes of the point $(x, y) \in \Omega$, i.e.,

$$d_1 = \frac{y}{\sqrt{3}}, \qquad d_2 = x - \frac{y}{\sqrt{3}}, \qquad d_3 = 1 - x - \frac{y}{\sqrt{3}}$$
 (9)

Clearly, the triangle's center, $d_1 = d_2 = d_3 = 1/3$, i.e., the point $x = 1/2, y = \sqrt{3}/6$, is a critical point for each of the symmetric eigenfunctions, and hence of u for all t > 0. We will be able to conclude that the "hot spot" remains stationary when we show that the triangle's center is the only critical point *interior* to $\overline{\Omega}$ for any positive time. To do this, let $a = 4\pi/\sqrt{3}$ and define

$$\psi(x,t) = \sum_{k=1}^{\infty} e^{-a^2k^2t} \cos(2\pi kx)$$
(10)

Then a necessary and sufficient condition for $(x, y) \in \Omega$ to be a critical point of u given by (8) is that

$$\psi(d_1, t) = \psi(d_2, t) = \psi(d_3, t) \tag{11}$$

We shall show, however, that for any fixed positive time, ψ decreases on [0,1/2]. Thus, since ψ is symmetric about x = 1/2, fulfilling the requirement (11) can only be accomplished for $d_1 = d_2 = d_3 = 1/3$, i.e., the triangle's center.

To prove that ψ decreases on [0,1/2], we use some facts concerning theta functions [3, pp.469–472]. The theta function

$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz)$$

has the product expansion

$$\vartheta_3(z,q) = G \prod_{n=1}^{\infty} (1 + 2q^{2n-1}\cos(2z) + q^{4n-2}), \qquad G = \prod_{n=1}^{\infty} (1 - q^{2n})$$

Thus,

$$\frac{1}{\vartheta_3}\frac{\partial\vartheta_3}{\partial z} = -4\sin(2z)\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+2q^{2n-1}\cos(2z)+q^{4n-2}}$$

from which it is clear that $\partial \vartheta_3/\partial z < 0$ for $0 < z < \pi/2$. Since $2\psi(x,t) + 1 = \vartheta_3(\pi x, e^{-a^2t})$, we see that for any fixed t > 0,

$$\frac{\partial \psi}{\partial x} < 0, \qquad 0 < x < 1/2$$

We have shown that the equilateral triangle in \mathbb{R}^2 has a unique hot spot (the center) for all positive time. We next consider general domains with the same property.

THEOREM. it Let $\omega \subset \mathbb{R}^n$ be a convex domain with C^2 boundary, possessing n independent (n-1)-dimensional hyperplanes, $\Pi_1, \Pi_2, \ldots, \Pi_n$, of reflective symmetry. The the solution $u : \Omega \times (0, \infty) \to \mathbb{R}$ of the problem (1) - (3) assumes its maximum for each fixed positive time only at the point of intersection of the hyperplanes, that is, the point $\Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_n$.

Our proof is based on the following result.

LEMMA. Suppose that the domain $\Omega \subset \mathbb{R}^n$ possessing C^2 boundary is symmetric about the hyperplane $\{x_n = 0\}$ and quasi-convex in the x_n -direction. Then for all t > 0 the solution $u(\cdot, t)$ of (1) - (3) satisfies

$$\frac{\partial u}{\partial x_n} < 0 \quad for x_n > 0 \qquad and \qquad \frac{\partial u}{\partial x_n} > 0 \quad for x_n < 0.$$

Proof. Because of the uniqueness of solutions to the well-posed problem (1)–(3), it follows that $u(x_1, \ldots, x_{n-1}, -x_n, t) = u(x_1, \ldots, x_{n-1}, x_n, t)$; as a consequence, $\frac{\partial u}{\partial x_n} = 0$ on $\Omega \cap \{x_n = 0\}$. By the paragolic boundary point lemma [2, Thm. 6, p.174], $u(\cdot, t)$ has a negative outward normal derivative, $\frac{\partial u}{\partial \nu}$, for each t > 0, and therefore $\frac{\partial u}{\partial x_n} \leq 0$ on $\partial \Omega \cup \{x_n = 0\}$ (strictly negative on those portions of the boundary for which the normal vector has a nonzero component in the x_n direction; zero on any portions orthogonal to the $\{x_n = 0\}$ hyperplane and, in particular, on the intersection set $\partial \Omega \cap \{x_n = 0\}$). Thus $\frac{\partial u}{\partial x_n}$ is a solution of the heat equation (1) with initial values $\frac{\partial u}{\partial x_n}(x,0) = 0$, and nonpositive boundary values on $\Omega_+ \times [0, \infty)$, where Ω_+ is the subdomain $\Omega \cap \{x_n > 0\}$. By the strong maximum principle [1, Thm. 5, p.173], $\frac{\partial u}{\partial x_n}(x,0) \leq 0$ on the whole

domain $\Omega_+ \times (0, \infty)$, and if $\frac{\partial u}{\partial x_n}(x_0, t_0) = 0$ for some x_0 in Ω_+ and $t_0 > 0$, then $\frac{\partial u}{\partial x_n} \equiv 0$ on $\Omega_+ \times [0, \infty)$. The boundary condition (3) would then imply $u \equiv 0$ on $\Omega \times (0, \infty)$, and the lemma follows from the assumed symmetry.

The proof of the theorem now follows by applying the lemma successively to each of the *n* hyperplanes of symmetry, noting in each case that critical points of the solution $u(\cdot, t)$ can only occur on the intersection of Ω with the hyperplane. Since the solution $u(\cdot, t)$ is positive and smooth on Ω and vanishes on the boundary, a maximum must therefore exist at the intersection point of the planes of symmetry for each positive time.

Remarks. 1. The convexity of Ω may be weakened without changing the conclusion. It is enough to assume that Ω is *quasi-convex* in directions orthogonal to each of the planes, $\Pi_1, \Pi_2, \ldots, \Pi_n$, that is, that Ω intersects any line orthogonal to one of the symmetry hyperplanes in an interval.

2. Centrosymmetry of Ω would follow from *n* orthogonal planes of symmetry which are not centrosymmetric. For example, any domain Ω whose symmetry group coincides with the symmetry group of the regular *n*-simplex (which is generated by the reflections in n(n+1)/2 independent, nonorthogonal hyperplanes in \mathbb{R}^n) has a stationary hot spot by the theorem, provided that Ω is quasi-convex in directions orthogonal to *n* of these planes.

3. The two-dimensional domain $\Omega = \{(r, \theta) \mid r \leq 11 + \cos(3\theta)\}$ is strictly convex, noncentrosymmetric, and has a real analytic boundary. By the theorem, this domain possesses a unique, stationary hot spot for all positive time, in contradiction to Klamkin's conjecture.

4. In three dimensions, a prismatic bar with cross section as above has four planes of symmetry, no three of which are mutually orthogonal. Rounding off the end in a symmetric and (strictly) convex manner wiuld therefore provide another type of counterexample to the conjecture.

5.. The proof of the theorem above follows immediately from the convexity of the level sets of the solution $u(\cdot, t)$ for each fixed time t > 0 (see B. Kawohl's solution below). However, the stronger conclusion contained in Remark 1 does not follow by this technique.

REFERENCES

[1] M. A. PINSKY, The eigenvalues of an equilateral triangle, SIAM J. Math. Anal., **11**(1980) 819–827; MR **82d**:35077.

[2] M. H. PROTTER & H. F. WEINBERGER, *Maximum Principles in Diffrential Equations*, Prentice-Hall, Englewood Cliffs NJ, 1967.

[3] E. T. WHITTAKER & G. N. WATSON, A Course in Modern Analysis, Cambridge Univ. Press, 1963.

Editorial note. In the following solution, proposed by B. KAWOHL, the problem has been reformulated as (1)–(3) exactly as done by GULLIVER & WILLMS.

The idea that Ω should be centrosymmetric is supported by the following physical reasoning. Heat flows away from the hot spot in all directions towrds $\partial\Omega$. If the rate of heat flow in one direction is less than the rate of heat flow in the opposite direction, then the hot spot is intuitively expected to drift away and move around as time goes on. So we might expect the flux $-\Delta u$, the spatial gradient of u, to be the same in opposite directions. Therefore, if the hot spot is stationary, it is not unreasonable to suspect that u is symmetrically decreasing on each line through the hot spot.

A counterexample to centrosymmetry is provided for n = 2 by an equilateral triangle, and for n = 3 by a regular tetrahedron. If there is objection to the fact that these examples have nonsmooth boundaries, one can mollify the corners and edges.

The fact that u has a stationary hot spot for bodies with at least n reflection symmetries follows easily from [3] as in [4]. Since the councerxamples are still symmetric, one might modify Klamkin's conjecture by removing the adjective "centro".

Is the fact that u develops only one spatial maximum for convex domains surprising? For nonconvex domains like barbells it is wrong, but for convex domains it has been long known that $v(x,t) = \log u(x,t)$ is concave in x [2, 6]. Thus the level sets $\{x \in D \mid u(x,t) \ge c\}$ of u are convex in space for every t > 0 and $c \ge 0$.

A problem related to the above conjecture was osed by L. Zalcman [7] and solved by G. Alessandrini [1]. Suppose that u solves (1)–(3) and that the level surfaces are invariant with respect to the time variable t, in other words, for any $z \in \Omega$ and $t_i > 0$ we have $\{x \in \Omega \mid u(x,t_1) = u(z,t_1)\} = \{x \in \Omega \mid u(x,t_2) = u(z,t_2)\}$. Suppose furthermore that $\partial\Omega$ is of class C^2 . Then Ω is a ball.

Another problem on hot spots which is still unsolved deals with Neumann rather than Dirichlet conditions. I learned it from J. Rauch in 1979. Consider the heat equation as in (1)-(3), but under the no-flux condition

$$\frac{\partial u}{\partial n}(x,t) = 0 \qquad \text{on} \qquad \partial \Omega \times \mathbb{R}^+$$

and with nonconstant initial data

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega$$

As $t \to \infty$ the solution *u* tends to its average, but (generically with respect to initial data)the hot spot moves to the boundary. More on this can be found in [5].
REFERENCES

[1] GIOVANNI ALESSANDRINI, Matzoh ball soup: a symmetry result for the heat equation, J. Analyse Math., **54**(1990) 229–236; MR **91d**:31001.

[2] HERM JAN BRASCAMP & ELLIOTT H. LIEB, On extensions of the Brunn-Minkowski and Prèkopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Functional Analysis, 22(1976) 366–389; MR 56 #8774.

[3] B. GIDAS, NI WEI MING & L. NIRENBERG, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, **68**(1979) 209–243; *MR* **80h**:35043.

[4] BERNHARD KAWOHL, A geometric property of level sets of solutions to semilinear elliptic Dirichlet problems, *Applicable Anal.*, **16**(1983) 229–233; *MR* **85f**:35083.

[5] BERNHARD KAWOHL, Rearrangements and convexity of level sets in PDE, *Lecture Notes in Mathematics*, **1150** Springer-Verlag, Berlin, 1985; *MR* **87a**:35001.

[6] NICHOLAS J. KOREVAAR, Convex solutions to nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.*, **32**(1983) 603–614; *MR* **85c**:35026.

[7] LAWRENCE ZALCMAN, Some inverse problems of potential theory, *Integral geometry (Brunswick ME*, 1984) 337–350; *Contemp. Math.*, **63**, Amer. Math. Soc., Providence RI, 1987; *MR* **88e**:31012.

Comment by the proposer. The paper by Alessandrini referred to above solves Problem $64-5^*$ in the affirmative and in the more general case for *n*-dimensions.

SIAM Rev., **36**(1994) 107.

A Possible Characterization of Uniformly Accelerated Motion

Problem 94-10^{*}, by M. S. KLAMKIN (University of Alberta).

It is a known result that if a particle is projected upwards in a uniform gravitational field with no resistance, then the product of the two times it takes to pass through any point of its path is independent of the initial velocity of the projection. Prove or disprove that this result cannot hold if additionally the particle was subject to a resistance as some function of the velocity.

SIAM Rev., **37**(1995) 249–250.

Solution by W. B. JORDAN (Scotia, NY).

We use subscripts 0, 1, m, 2 to denote the start, the specified point on the upward leg, the peak, and the specified point on the downward leg. Let f(v) be the acceleration due to resistance. On the upward leg (v positive upward) dv/dt = -g - f so

$$t = \int_{v}^{v_0} \frac{dv}{g+f}$$

and dy = v dt = v(dt/dv)dv = -v dv/(g+f) so

$$y = \int_v^{v_0} \frac{v \, dv}{g+f}$$

Thus

$$y_m = \int_0^{v_0} \frac{v \, dv}{g+f} = \text{max height} \text{ and } t_m = \int_0^{v_0} \frac{dv}{g+f}$$

Now $y = y_1$ when $v = v_1$, so

$$t_1 = \int_{v_1}^{v_0} \frac{dv}{g+f}$$
 and $y_1 = \int_{v_1}^{v_0} \frac{v \, dv}{g+f}$ (gives v_1)

On the downward leg (v positive downward) dv/dt = g - f

$$t = t_m + \int_0^v \frac{dv}{g - f}$$
 and $y = y_m - \int_{t_m}^t v \, dt = y_m - \int_0^v \frac{v \, dv}{g - f}$

Now $y = y_2$ when $v = v_2$, so

$$t_2 = \int_0^{v_0} \frac{dv}{g+f} + \int_0^{v_2} \frac{dv}{g-f} \quad \text{and} \quad y_2 = y_m - \int_0^{v_2} \frac{v \, dv}{g-f} \quad (\text{gives } v_2)$$

Let $P = t_1 t_2$, the product to be examined. We have

$$\frac{dP}{dv_0} = t_1 \frac{dt_2}{dv_0} + t_2 \frac{dt_1}{dv_0}$$

$$\frac{dt_1}{dv_0} = \frac{1}{g+f_0} - \frac{1}{g+f_1} \frac{dv_1}{dv_0}$$

$$0 = \frac{dy_1}{dv_0} = \frac{v_0}{g+f_0} - \frac{v_1}{g+f_1} \frac{dv_1}{dv_0} \quad (\text{gives } dv_1/dv_0)$$

$$\frac{dt_2}{dv_0} = \frac{1}{g+f_0} + \frac{1}{g-f_2} \frac{dv_2}{dv_0}$$

$$0 = \frac{dy_2}{dv_0} = \frac{v_0}{g+f_0} - \frac{v_2}{g-f_2} \frac{dv_2}{dv_0} \quad (\text{gives } dv_2/dv_0)$$

 \mathbf{SO}

$$(g+f_0)\frac{dP}{dv_0} = \left(1+\frac{v_0}{v_2}\right)\int_{v_1}^{v_0}\frac{dv}{g+f} + \left(1-\frac{v_0}{v_1}\right)\left[\int_0^{v_0}\frac{dv}{g+f} + \int_0^{v_2}\frac{dv}{g-f}\right]$$

If f(v) < g the integrands can be expanded in powers of f. The leading term in the expansion is

$$(g+f_0)\frac{dP}{dv_0} = \frac{(v_0+v_2)(v_0-v_1)}{g}\left(\frac{1}{v_2}-\frac{1}{v_1}\right)$$

If there is zero resistance, then f = 0, $v_1 = v_2$, $dP/dv_0 = 0$ and the proposer's "known result" is verified. But for nonzero resistance $v_0 > v_1 > v_2$, so $dP/dv_0 > 0$ and the product of the two times is no longer independent of v_0 .

[[SIAM Rev., 36(1994) Number 3 is the first issue of SIAM Review that doesn't have Murray's name at the head of the Problems Section — $35 \ 1/2$ years of continuous editing!! Even so, as you can see below, it was not the end of his contributions!]]

SIAM Rev., **37**(1995) 99.

Non-Symmetric Cyclic Pursuit on a Sphere

Problem 95-3^{*}, by M. S. KLAMKIN (University of Alberta).

Three bugs A, B, C, starting from the vertices of an arbitrary spherical triangle, pursue each other cyclically at the same constant speeds, i.e., A always heads directly towards B, while B heads towards C, and C heads towards A, along minor great circular arcs. Prove or disprove that there is simultaneous capture. For the plane case, it is known that there is simultaneous capture, and upper and lower bound are given for the time to capture [1].

REFERENCE

[1] M. S. KLAMKIN & D. J. NEWMAN, Cyclic pursuit or "the three bugs problem", Amer. Math. Monthly, **78**1971) 631–639.

[[Is it clear (true?) that the paths of the bugs are great circle arcs?? — Later: Is this question answered in the following solution? RKG]]

SIAM Rev., 38(1996) 153-155.

Solution by H. E. DE MEYER and C. C. GROSJEAN (University of Ghent, Belgium).

From the six differential equations describing in spherical coordinates the instantaneous motion of the three bugs on the sphere, one easily obtains by elementary trigonometric calculations another set of six first-order differential equations expressing the rate of change of the arcs a, b, c and the angles A, B, C of the spherical triangle with the bugs as vertices. On the unit sphere and with the assumption that the bugs move at unit velocity, whereby A heads towards B, B heads towards C, and C heads towards A, the latter set of equations is

$$\dot{a} = -1 - \cos C \qquad \dot{b} = -1 - \cos A \qquad \dot{c} = -1 - \cos B \qquad (1)$$

$$\dot{A} = \frac{\sin A \cos b}{\sin b} - \frac{\sin B}{\sin c} \qquad \dot{B} = \frac{\sin B \cos c}{\sin c} - \frac{\sin C}{\sin a} \qquad \dot{C} = \frac{\sin C \cos a}{\sin a} - \frac{\sin A}{\sin b} \qquad (2)$$

The initial configuration of the bugs on the sphere is an arbitrary nondegenerate spherical triangle, i.e., 0 < A, B, $C < \pi$ and 0 < a, b, $c < \pi$. It is remarkable that (1) are exactly the same as those for the cyclic pursuit in the plane (a, b, c being then the sides of the triangle), whereas (2) reduce to the corresponding equations in the plane by leaving out the cosine-factors and by replacing the sines of the arcs by the corresponding sides of the triangle. Due to this great similarity between the cyclic pursuit on the sphere and in the plane, the prof of the mutual capyure of the bugs on the sphere can be largely inspired by the proof for the mutual capture in the plane given by Klamkin and Newman (see [1] in the problem). We first notice that on account of (1) the three arcs are decreasing monotonically. Also, for all finite t, the three angles lie in $]0, \pi[$. Indeed, if any angle became 0 or π , then the bugs would be in three different positions on the same great circle of the sphere. By the uniqueness for the system of differential equations (1)–(2), these positions on a great circle could only be attained if the bugs were always moving on that great circle, which contradicts the initial conditions.

Next we show that at least one of the arcs becomes zero in finite time. Indeed, let us assume that none of the arcsbecome zero in finite time. Since the arcs are monotonically decreasing they coverge to a nonnegative value, i.e.,

$$\lim_{t \to \infty} a = a_{\infty} \ge 0 \quad \lim_{t \to \infty} b = b_{\infty} \ge 0 \quad \lim_{t \to \infty} c = c_{\infty} \ge 0$$

From (1) it follows that

$$\lim_{t \to \infty} A = \lim_{t \to \infty} B = \lim_{t \to \infty} C = \tau$$

which yields a contradiction since initially $A + B + C < 3\pi$ for any nondegenerate spherical triangle, and the sum of the angles is decreasing monotonically, as can be verified by taking the sum of (2).

Let t_0 be the finite time at which an arc, say c, becomes zero first, i.e.,

$$c(t_0) = 0 \tag{3}$$

We will prove that the assumption

$$a(t_0) > 0$$
 and $b(t_0) > 0$ (4)

leads to a contradiction, yielding mutual capture at t_0 as the only valid alternative.

From (3) and (4) and on account of the monotonicity of c, it follows that for t sufficiently close to t_0 , $\sin a / \sin b - \sin b / \sin c < 0$, and hence, by the application of the law of sines, that

$$\frac{\sin A}{\sin b} - \frac{\sin B}{\sin c} < 0 \tag{5}$$

Since $\sin A / \sin b > 0$ for all $t < t_0$, an even stronger inequality than (5) is obtained by multiplying the first term by $\cos b$. Hence, for t sufficiently close to t_0 , we have

$$\frac{\sin A \cos b}{\sin b} - \frac{\sin B}{\sin c} < 0 \tag{6}$$

It now follows from (2) that there exists a time $t' < t_0$ such that, from t' onwards, A is monotonically decreasing. Hence,

$$\lim_{t \to t_0} A \quad \text{exists and is} < \pi \tag{7}$$

We next prove that $\lim_{t\to t_0} B = 0$. If $a(t_0) < \pi/2$, then for t sufficiently close to t_0 we have $\sin A > \sin a(t_0)$ and from (2) it follows that

$$\dot{B} - \frac{\sin B \cos c}{\sin c} = -\frac{\sin C}{\sin a} > -\frac{1}{\sin a(t_0)} \tag{8}$$

Thus, $\dot{B} > -1/\sin a(t_0)$, so that $B + t/\sin a(t_0)$ is increasing. Since it is bounded by $\pi + t_o/\sin a(t_0)$, the limit $\lim_{t\to t_0} B$ exists and is finite. Integrating (8) from time zero to time t_0 we then obtain that the integral

$$\int_{0}^{t_0} \frac{\sin B \cos c}{\sin c} dt \tag{9}$$

converges. Similarly, if $\pi/s \leq a(t_0) < \pi$, then for all $t < t_0$ we have $\sin a > \sin a(0)$ and it can be shown again that the integral (9) converges.

If we regard the monotone decreasing c as the new independent variable, we may use (1) to change the integral (9) to

$$\int_{0}^{c(0)} \frac{\sin B \cos c}{\sin c} \frac{dc}{1 + \cos B} = \int_{0}^{c(0)} \tan \frac{B}{2} \frac{dc}{\tan c}$$

Since $\lim_{t\to t_0} B$ is finite, the integrability of $(\tan B/2)/(\tan c)$ around c = 0 requires that $\lim_{t\to t_0} B = 0$.

Finally, we recall from (3)–(4) that eventually c is the smallest arc. Hence C is eventually the smallest angle, and consequently $\lim_{t\to t_0} C = 0$. Therefore, under the assumptions (3)–(4) we have obtained

$$\lim_{t \to t_0} (A + B + C) < \pi$$

which gives a contradiction since for any nondegenerate spherical triangle $A + B + C > \pi$, and in the limit the sum of the angles cannot be smaller than π . Consequently there must be mutual capture of the bugs at time t_0 if their initial configuration is a nondegenerate spherical triangle. Since $-3 < \cos A + \cos B + \cos C < 3/2$ for any nondegenerate spherical triangle, we have from (1), $-9/2 < \dot{a} + \dot{b} + \dot{c} < 0$, and a lower bound of the time of capture t_0 follows:

$$t_0 > \frac{2}{9}a(0) + b(0) + c(0).$$

SIAM Rev., **37**(1995) 441–442.

[[Compare the very similar SIAM Rev., **36**(1994) 107:

A Possible Characterization of Uniformly Accelerated Motion

Problem 94-10^{*}, by M. S. KLAMKIN (University of Alberta). above. R.]]

A Characterization of Uniformly Accelerated Motion

Problem 95-15, by M. S. KLAMKIN (University of Alberta).

It is a known result that if a particle moves along a straight line with constant acceleration, the space-average of the velocity over the distance of any segment is

$$\frac{2(v_1^2 + v_1v_2 + v_2^2)}{3(v_1 + v_2)} \tag{1}$$

where v_1 and v_2 are the velocities at the beginning and at the end of the segment. Prove that this property characterizes uniformly accelerated motion, i.e., if a particle moves along a straight line such that the space average of its velocity over any segment is given by (1), then the motion is one of constant acceleration.

SIAM Rev., **38**(1996) 526.

Solution by W. B. JORDAN (Scotia, New York).

Let x = 0 at $v = v_1$, and let $v_2 = uv_1$. Then

$$\frac{2v_1(u^2+u+1)}{3(u+1)} = \bar{v} = \frac{1}{x} \int_0^x uv_1 \, dx$$

 \mathbf{SO}

$$\frac{3}{2}u = \frac{d}{dx}\left(\frac{x(u^2+u+1)}{u+1}\right)$$

which reduces to

$$\frac{dx}{x} = \frac{2u\,du}{u^2 - 1}$$

so $x = C(u^2 - 1)$ (C a constant). Then

$$a = a(x) = \frac{dv_2}{dt} = \frac{dv_2}{dx}\frac{dx}{dt} = v_2\frac{dv_2}{dx} = v_1^2u\frac{du}{dx} = \frac{v_1^2}{2C} = \text{ constant.}$$

Solution by MICHAEL RENARDY (Virginia Tech).

The condition can be written in the form

$$3(v(t_1) + v(t_2)) \int_{t_1}^{t_2} v^2 dt = 2(v(t_1)^2 + v(t_1)v(t_2) + v(t_2)^2) \int_{t_1}^{t_2} v dt$$

Now diffrentiate three times with respect to t_2 and set $t_2 = t_1$. The outcome of this calculation is

$$v^2 v'' = 0.$$

Hence v'' = 0, i.e., the motion has constant acceleration.

Also solved by [5 others] and the proposer.

[[For completeness I mention that there's a comment by Murray on pp.142–143 of $SIAM~Rev.,~{\bf 40}(1998)$ on a problem

Existence and Uniqueness for a Variational Problem

Problem 97-4* by Yongi Wang

A solution by ERIK VERRIEST runs over pages 132–142.]]

[[From 40(1998) the *SIAM Rev.* discontinued its Problems and Solutions section, and only Solutions (to past problems) appeared.]]

Murray Klamkin, Miscellaneous

Richard K. Guy

June 22, 2006

This file updated 2006-05-23.

I note the appearance of

Branko Grünbaum & Murray S. Klamkin, Euler's ratio-sum theorem and generalizations, *Math. Mag.*, **79**(2006) 122–130.

Inside the cover of the April, 2006 issue [of Math. Mag.] there appears:

Murray Seymour Klamkin was born in the United States. He received his undergraduate degree in chemistry, served four years in the U.S. Army during WW2, and later earned an M.S. in physics. In his career he worked both in industry and in universities, and for several years was Chair of the Mathematics Department at the University of Alberta in Edmonton. He is well-known to readers of *Mathematics Magazine* (and many other journals) through his activities in Olympiads and in the problems sections. More details about the life and achievements of Klamkin can be found in the eulogies published in *Focus* (November 2004, page 32) and in the *Notes of the Canadian Mathematical Society* (November 2004, pages 19–20).

Grünbaum and Klamkin first met in 1967, at a CUPM conference in Santa Barbara. They remained in contact over the years, but the present paper is their first joint publication. The *Focus* article is on the same page as an In Memoriam for Howard Eves. It has a photo of Murray and reads:

Murray Klamkin, 1921–2004

By Steven R. Dunbar

Murray Klamkin, prolific mathematical problem poser and solver, professor of mathematics, and member of the MAA since1948. passed away on August 6, 2004 at the age of 83. Murray Klamkin received a B.Ch.E. from the Cooper School of Engineering in 1942, then spent 4 years in the U.S. Army. After receiving an M.S. in Physics at the Polytechnic Institute of Brooklyn in 1947, he spent 1947–48 studying mathematics at Carnegie-Mellon. From there, he returned as an instructor to the Polytechnic Institute, then held positions successively at AVCO Research, SUNY-Buffalo, the University of Minnesota, Ford Motor Company, the University of Waterloo, and the University of Alberta, where he was chair of the Departentof Mathematics from 1976 to 1981. Murray Klamkin is best known for editing the Problems columns of many journals: *SIAM Review*, the *Pi Mu Epsilon Journal, School Science and Mathematics Journal, Crux Mathematicorum*, the American Mathematical Monthly, Mathematics Magazine, and most recently, Math Horizons. Klamkin is one of the three greatest contributors to the *SIAM Review* Problems and Solutions Section. Murray also served the MAA as a visiting lecturer, a comittee member, and on the Board of Governors.

Not surprisingly, he was also on the Putnam Competition Committee, and was instrumental in starting the USA Mathematical Olympiad. The standards he set as the Chair of the USAMO Committee from 1972–85 and the coach of the USA Team at the International Mathematical Olympiads from 975–1984 were significant to the continued success of the program. Under his leadership, the USA team delightfully surprised the mathematics community by doing well despite having to compete against countries that had been participating in the IMO since its beginning in 1959. In Steve Olson's recent book *Count Down*, Klamkin is quoted as saying "A lot of people were deadset against it, they thought a US team would be crushed" In the 2001 IMO in Washington DC, he returned as an Honorary Member of the Problem Selections Committee and a guest lecturerat the summer training program for the USA team.

Mathematicians and students of mathematics will long appreciate hiscreation of brilliantproblems and lucid solutions and the standardsthat he set. "The best prblems," he said, "are elegant in statement, elegant in result, and elegant in solution. Such problems are not easy to come by." Murray found them consistently and shared them generously throughout his long and fruitful career. Here's the article from the CMS Notes:

OBITUARY

Murray Seymour Klamkin (1921–2004)

by Andy Liu, University of Alberta

Let me first make it clear that this is not a eulogy. By my definition, a eulogy is an attempt to make the life of the departed sound much better than it was. In the present case, it is not only unnecessary, it is actually impossible. Murray Seymour Klamkin had a most productive and fulfilling life, divided between industry and academia.

Of the early part of his life I know little except that he was born in 1921 in Brooklyn, New York, where his father owned a bakery. This apparently induced in him his life long fondness for bread. I read in his curriculum viae that his undergraduate degree in Chemical Engineering was obtained in 1942 from the Cooper Union's School of Engineering. During the war he was attached to achemical warfareunit stationed in Maryland, as his younger sister Mrs. Judith Horn informed me.

In 1947, Murray obtained a Master of Science degree from the Polytechnic Institute of New York, and taught there until 1957 when he joined AVCO's Research and Advanced Development Division.

In 1962, Murray returned briefly to academia as a professor at SUNY, Buffalo, and then became a visiting professor at the University of Minnesota. In 1965, he felt again the lure of industry and joined Ford Motor Company as the Principal Research Scientist, staying there until 1976

During all this time, Murray had been extremely active in the field of mathematics problem solving. His main contribution was serving as editor of the problem section of SIAM Review. He had a close working relation with the Mathematical Association of America, partly arising from his involvement with the William Lowell Putnam Mathematics Competition.

In 1972, the MAA started the USA Mathematical Olympiad, paving the way for the country's entry into the International Mathematical Olympiad in 1974, hosted by what was still East Germany.

Murray was unable to obtain from Ford release time to coach the team. Disappointed, he began to look elsewhere for an alternative career. This was what brought him to Canada, at first as a Professor of Applied Mathematics at the University of Waterloo. However, it was not until the offer came from the University of Alberta that he made up his mind to leave Ford. I did not know if Murray had been to Banff before, but he must have visited this tourist spot during the negotiation period, fell in love with the place and closed the deal.

As Chair, Murray brought with him a management style from the private sector. Apparently not everyone was happy with that, but he did light some fires under several pairs of pants, and rekindled the research programs of the wearers.

Murray had always been interested in Euclidean Geometry. He often told me about his high school years when he and a friend would challenge each other to perform various Euclidean constructions. Although the Chair had no teaching duties at the time, Murray took on a geometry class himself.

At the same time, Murray began editing the Olympiad Corner in Crux Mathematicorum, a magazine then published privately by Professor Leo Sauvé of Ottawa. It is now an official journal of the Canadian Mathematical Society. Murray also introduced the Freshmen and Ungergraduate Mathematics Competitions in the Department.

Geometry, mathematics competitions and Crux Mathematicorum were what brought me to Murray's attention. At the time, I was a post-doctoral fellow seeking employment, having just graduated from his Department. Thus I was ready to do anything, and it happened that my interests coincided with those of Murray. I was holding office hours for his geometry class, helping to run the Department's competitions and assisting him in his editorial duty.

I remember being called into his office one day. He had just received a problem proposal for Crux Mathematicorum. 'Here is a nice problem," he said, "but the proposer's solution is crappy. Come up with a nice solution, and I need it by Friday afternoon!"

As much as I liked problem-solving, I was not sure that I could produce results by an industrial schedule. Nevertheless, I found that I did respond to challenges, and although I was not able to satisfy him every time, I managed to do much better than if I was left on my own, especially after I had got over the initial culture shock.

The late seventies were hard times for academics, with few openings in post-secondary institutions. I was short-listed for every position offered by the Department, but always came just short. Eventually, I went elsewhere for a year as sabbatical replacement. Murray came over to interview me for a new position, pushed my appointment through the Hiring Committee and brought me back in 1980.

Murray had been the Deputy Leader for the USA National Team in the IMO since 1975. In 1981, USA became the host of the event, held outside Europe for the first time. Sam Greitzer, the usual leader, became the chief organizer. Murray took over as the leader, and secured my appointment as his Deputy Leader.

I stayed in that position for four years, and in 1982 made my first trip to Europe because the IMO was in Budapest. This was followed by IMO 1983 in Paris, and IMO 1984 Prague. I was overawed by the international assembly, but found that they in turn were overawed by Murray's presence. He was arguably the most well-known mathematics problem-solver in thw whole world.

We both retired from the IMO after 1984, even though I would later return to it. His term as Chair also expired in 1981. Thus our relationship became collegial and personal. He and his wife Irene had no children, but they were very fond of company. I found myself a guest at their place at regular intervals, and they visited my humble abode a few times.

It wa during this period that I saw a different side of Murray. Before, I found him very businesslike, his immense talent shining through his incisive insight and clinical efficiency.

Now I found him a warm person with many diverse interests, including classical music, ballroom dancing, adventure novels, kung-fu movies and sports, in particular basket-ball.

Although Murray had been highly successful in everything he attempted, he will probably be remembered the most for his involvement in mathematics problem-solving and competitions. He had authored or edited four problem books, and left his mark in every major journal which had a problem section. He had received an Honorary Doctorate from the University of Waterloo and was a Fellow of the Royal Society of Belgium. He had won numerous prizes, and had some named after him.

Murray had enjoyed remarkably good health during his long life. It began to deteriorate in September 2000 when he underwent a bypass operation. After his release from the hospital, he continued to exert himself, walking up to his office on the sixth floor, and skating in the West Edmonton Mall.

His heart valve gave in November, fortunately while he was already in hospital for physiotherapy. He was in coma for some time. One day, when I visited him, he was bleeding profusely from his aorta. The doctor indicated to me that he did not expect Murray to last through the day.

Somehow, the inner strength of Murray came through, and on my next visit, he was fully conscious. He told me to make arrangements for his eightieth birthday party, stating simply that he would be out of the hospital by that time. It was a good thing that I took his words seriously, for he was out of the hospital by that time, ready to celebrate.

One of the last mathematical commitments that he made was to edit the problem section in the MAA's new journal *Math Horizons*. During this difficult time, he asked me to serve with him as joint-editor. Later, he passed the column onto me, but his finger-prints were still all over the pages.

Now I have to try to fill his shoes without the benefit of his wisdom. His passing marks the end of an era in mathematics competition and problem-solving. He will be deeply missed. I now (2006-Feb-01) have Don Albers's copy of Rabinowitz (& Bowron) 1975–1979, from which I'll add relevant items.

This is the (lost count!) of a number of files listing problems, solutions and other writings of Murray Klamkin.

The easiest way to edit is to cross things out, so I make no apology for the proliferation below. Just lift out what you want.

Problems which are in other files (e.g., *Monthly*, *SIAM Rev.*, ...) are just referenced, not typed out.

First I'll go through Stanley Rabinowitz's Index to Mathematical Problems 1980-1984

[[Stanley almost certainly has other files, because he was going to cover other periods as well. Would it be a good idea to approach him about what he might be willing to make available?]]

I'll note all the Murray entries, but will copy them out only if they are NOT one of

AMM Amer. Math. Monthly

(TY)CMJ (Two-Year) Coll. Math. J.

CRUX *Crux Mathematicorum* [[we should get rid of 'Mayhem' from the title, which devalues the magazine.]]

MM Mathematics Magazine

SIAM SIAM Review

[[This is a useful step forward towards classification, if that is the way we are to go. If I live long enough, and the enthusiasm doesn't dwindle, and I get no advice to the contrary, I may go over the earlier files (AMM, MM, SIAM, CMJ) and endeavor to classify according to SR's scheme. — R.]]

A first pass having been made, I'll endeavor to run through again putting in the vol & page numbers of the periodicals, which are:

MI The Mathematical Intelligencer MSJ The Mathematics Student Journal SSM School Science and Mathematics

CMB Canadian Mathematical Bulletin

AMATYC The AMATYC Review PME The Pi Mu Epsilon Journal

[[The Foreword to the (Stanley Rabinowitz) book is by Murray — here it is.]]

FOREWORD

Throughout the centuries, from the time of the Sumerian and Babylonian civilizations (roughly 2500 B.C.), one can find no end of mathematical problems and questions — since problems and questions beget more problems and questions in an unending cycle. These problems and questions are the lifeblood of mathematics. Smaller problems lead to larger problems, which in turn lead to substantial mathematical research. For example, consider the mathematics produced during attempts to prove Fermat's last theorem — which in itself is not an important result, even if true. The following metaphor, attributed to Allen Shields, is particularly apropos: "A mathematical problem is a 'jackpot' which gains in value as more of us throw our quarters into it."

Mathematical problems challenge and interest even those who are outside the profession. Just consider the large number of problems sections found in various journals, magazines and newspapers throughout the world. Although there is much mathematical information sequestered in the immense problem literature, unfortunately there is no easy way to access this material. It is rarely reviewed and existing indexes are not particularly useful.

In this regard, I am highly critical of the cavalier treatment with which some journals treat their problem sections. For example, consider journals such as *The American Mathematical Monthly, Mathematics Magazine* and *The College Mathematics Journal*. All of these publications contain valuable and interesting problem sections. The yearly problem indexes published by the first two journals only contain a listing of problem numbers and their corresponding proposers and solvers and their page numbers. *The College Mathematics Journal* does not include any yearly index at all. This is a sorry state of affairs for any journal!

I would like to see (and so, I am sure, would others) at least a return to the indexing system used in the first 19 volumes of *The American Mathematical Monthly*. Here the indexes also included problem titles and were arranged by various fields, such as Diophantine analysis, algebra, geometry, calculus, mechanics, averages and probability, and miscellaneous. With modern computers and word processing programs, this should be relatively easy to do.

In order to remedy this very sad state of problem indexing, the author, who is an ardent problemist, has taken on the very arduous task of producing a rather complete index system forproblems published in a large number of different journals from 1980 through 1984. He also plans to publish similar works for the years 1975 through 19789, 1985 through 1989, etc.

Since it is not easy to classify a problem, the author has sorted each problem by topic (e.g. Geometry/triangles; Analysis/series) and in almost all cases includes the complete statement of each problem. This explicit representation of the sorted problems together with the many other listings included, enables one to locate problems and their solutions (if available) in a relatively easy fashion.

This is a must book for problemists as well as problem editors. I only wish it had been available a long time ago.

MURRAY S. KLAMKIN PROFESSOR EMERITUS UNIVERSITY OF ALBERTA

ALGEBRA

Complex numbers

MM 1093 by M. S. Klamkin Math. Mag., 53(1980) 112. **CRUX 830** by M. S. Klamkin MI 83-10 5(1983) No.3, p.45 by K. S. Murray If |kw+z| + |w-z| = |kw-z| + |w+z| wher w, z are complex numbers and k is real, prove that either |kw + z| = |kw - z| or |kw + z| = |w + z|MM Q666 by M. S. Klamkin Math. Mag., **53**(1980) 301, 305. SIAM 80-15 by M. S. Klamkin and A. Meir SIAM Rev., **22**(1980) 364. SIAM Rev., **23**(1981) 395–396.

Factorization

MSJ 504 **27**(1979) No.1 p.5 Factor: $x^5 + y^5 + (z - x - y)^5 - z^5$

Functional equations: polynomials

PME 411 *Pi Mu Epsilon J.*, **6**(1977) 421 by R. S. Luthar

by Murray S. Klamkin

Find all polynomials P(x) such that

$$P(x^{2}+1) - [P(x)]^{2} - 2xP(x) = 0$$

and P(0) = 1.

Pi Mu Epsilon J., 6(1978) 558. Solution by MSK.

Identities

MI 83-7 5(1983) No.2 p.27 by M. S. Klamkin If a + b + c = 0 and x + y + z = 0, prove that $4(ax + by + cz)^3 - 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$

$$-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y) - 54abcxyz = 0$$

Inequalities: degree 2

CRUX 323 4(1978) 65 by Jack Garfunkel and M. S. Klamkin If xyz = (1-x)(1-y)(1-z) where $0 \le x, y, z \le 1$, show that

$$x(1-z) + y(1-x) + z(1-y) \ge 3/4$$

[Crux, 4(1978) 255 has an MSK solution & comment]

Inequalities: exponentials

Amer. Math. Monthly, **81**(1974) 660.

E 2483. Proposed by M. S. Klamkin, Ford Motor Company See MONTHLY file

MM Q658 CMB P261 by M. S. Klamkin by R. Schramm

[same problem submitted by 2 different people to two different places??] Math. Mag., **52**(1979) 114, 118. **Q 658**. Submitted by M. S. Klamkin, University of Alberta If a, b > 0, prove that $a^b + b^a > 1$.

by M. L. Glasser & M. S. Klamkin

Pi Mu Epsilon J., 6(1976) 306

Show that

PME 378

 $\left\{\frac{x^x}{(1+x)^{1+x}}\right\}^x > (1-x) + \left\{\frac{x}{1+x}\right\}^{1+x} > \frac{1}{(1+x)^{1+x}}$

for 1 > x > 0.

AMM S6 Amer. Math. Monthly, 86(1979) 222.

Inequalities: finite products

AMM 6294 Amer. Math. Monthly, **87**(1980) 309.

CMB P270

Prove that

$$2^{n} P\left\{\frac{x_{1}^{n} + x_{2}^{n} + \dots + x_{n}^{n}}{n}\right\}^{n-1} \ge \prod_{i=1}^{n} \{x_{i}^{n} + P\}$$

where $P = x_1 x_2 \cdots x_n$, $x_i \ge 0$, and there is equality if and only if x_i is constant.

Inequalities: finite sums

CMB P248 Canad. Math. Bull., 19(1976) 121
 by M. S. Klamkin

 Let
$$S = x_1 + x_2 + \dots + x_n$$
 where $x_i > 0$, $T_0 = 1/S$ and
 $T_r = \sum_{sym} S - x_1 - x_2 - \dots - x_r^{-1}$
 $1 \le r \le n - 1$

 Prove that $(n - r)^2 T_r / \binom{n-1}{r}$ is monotonically increasing in r from 0 to $n - 1$.
 MM Q664 Math. Mag., 52(1979) 317, 323.
 by M. S. Klamkin

 Inequalities: fractions
 AMM E2603 Amer. Math. Monthly, 83(1976) 483.
 by M. S. Klamkin

 MM Q608 Math. Mag., 47(1974) 52, 58.
 by M. S. Klamkin

 MM Q618 Math. Mag., 47(1974) 117, 122.
 by M. S. Klamkin

by Murray S. Klamkin

by M. S. Klamkin & A. Meir

by M. S. Klamkin

CRUX 413 5(1979) 47

If $a, b, c \ge 0$, prove that

 $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}$

[MSK solution at 5(1979) 302]

Inequalities: radicals

CRUX 805

MM Q688 Math. Mag., **57**(1984) 110, 115.

Maxima and minima

AMM E3573 See Amer. Math. Monthly, 83(1976) 54 in MONTHLY file.

CRUX 487 5(1979) 266 by Dan Sokolowski If a, b, c and d are positive real numbers such that $c^2 + d^2 = (a^2 + b^2)^3$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \ge 1$$

with equality if and only if ad = bc. [MSK solution at *Crux* **6**(1980) 259]

 SIAM 84-13
 by M. S. Klamkin

 SIAM Rev., 26(1984) 273.
 siAM Rev., 27(1985) 256.

 SSM 4009 84(1984) p.534
 by M. S. Klamkin

Determine the maximum value of

 $|4(z_1^8+z_2^8+z_3^8+z_4^8)-(z_1^4+z_2^4+z_3^4+z_4^4)^2|\\$

where z_1, z_2, z_3 and z_4 are comply numbers such that $z_1 + z_2 + z_3 + z_4 = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$.

by G. C. Giri

by M. S. Klamkin by M. S. Klamkin AMM E2573 Amer. Math. Monthly, 83(1976) 54.

E 2573. Proposed by Murray S. Klamkin, University of Waterloo

Means

MM 1000 Math. Mag., 49(1976) 253.

1000. Proposed by Murray S. Klamkin, University of Waterloo

Polynomials: integer coefficients

CRUX 254 Crux Math., 3(1977) 155

by M. S. Klamkin

(a) If P(x) denotes a polynomial with integer coefficients such that

 $P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 4000$

prove that the zeros of P(x) cannot be integers.

(b) Prove that there is no such polynomial if

 $P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 1000$

Radicals: approximations

CRUX 207 by Ross Honsberger Prove that $\frac{2r+5}{r+2}$ is always a better approximation to $\sqrt{5}$ than r. Comment & solution by Murray at *Crux Math.*, **3**(1977) 144 Rate problems: cars

OSSMB 75-3

by Murray Klamkin & Rodney Cooper

Ontario Secondary School Math. Bull., **11**(1975/1) 16.

Al leaves at noon and drives at constant speed back and forth from town A to town B. Bob also leaves at noon, driving at 40 mph back and forth from town B to town A on the same highway as Al. Al arrives at town B twenty minutes after first passing Bob, whereas Bob arrives at town A 45minutes after first passing Al. At what time do Al and Bob pass each other for the n th time?

Ontario Secondary School Math. Bull., **11**(1975/2) 21. Solution by Murray Klamkin Also at Ontario Secondary School Math. Bull., **12**(1976/1) 16 ??

Solutions of equations: degree 2

CRUX 489 5(1979) 266

Find all real numbers x, y and z such that

$$(1-x)^{2} + (x-y)^{2} + (y-z)^{2} + z^{2} = \frac{1}{4}$$

[MSK solution at Crux 6(1980) 263]

Solution of equations: determinants

CRUX 398 **4**(1978) 284

by Murray S. Klamkin

Find the roots of the $n \times n$ determinantal equation

$$\left|\frac{1}{x\delta_{rs}+k_r}\right| = 0$$

where δ_{rs} is the Kronecker delta.

by V. N. Murty

Solution of equations: radicals

CRUX 287 Crux Math., 3(1977) 251

by M. S. Klamkin

Determine a real value of x satisfying

 $\sqrt{2ab + 2ax + 2bx - a^2 - b^2 - x^2} = \sqrt{ax - a^2} + \sqrt{bx - b^2}$

if x > a and b > 0.

[RKG thinks that this should be x > a, b > 0]

Sum of powers

CMB 332 **26**(1983) p.250

by M. S. Klamkin

For each positive integer n, let $S_n = x^n + y^n + z^n$. Prove that

 $(x+y+z)S_{p+1} \ge 2(yz+zx+xy)S_p - 3xyzS_{p-1}$

where p is a positive integer and x, y, z are real numbers.

Systems of equations: sums of powers

 AMM 6312
 by M. S. Klamkin

 Amer. Math. Monthly, 87(1980) 675.

Theory of equations

 MM 1172
 by M. S. Klamkin

 Math. Mag., 56(1983) 177.
 by M. S. Klamkin

 TYCMJ 208
 by M. S. Klamkin

 Two-Year Coll. Math. J., 13 No.1 (Jan., 1982) 65.
 by M. S. Klamkin

 Two-Year Coll. Math. J., 14 No.3 (June, 1983) 261–262.
 by M. S. Klamkin

ANALYSIS

Differential equations

CMB 331 **26**(1983) p.126

Solve the differential equation

 $x^4y'' - (x^3 + 2axy)y' + 4ay^2 = 0$

CMB 340 **26**(1983) p.251

Determine the general solution of the differential equation

$$\{D^n x^{n+1/2} D^{n+1} - 2^{-2n-1}\}y = 0$$

[[The x-exponent looks ambiguous to me. - R.]]

Functions: continuous functions

CMB P281 Canad. Math. Bull., **22**(1979) 519

It is well know that if $a, c \ge 0, b^2 \le 4ac$ then

$$ax^2 + bxy + c^2 \ge 0 \tag{1}$$

and

$$ax^4 + bx^2y^2 + c^4 \ge 0 \tag{2}$$

for all real x and y. Assume (that) a, b and c are continous functions of x and y.

(a) Given that b > 0, that $a, c \ge 0$ and that (2) is valid for all real x and y, is it necessary that $b^2 - 4ac \ge 0$?

(b) Given that $a, c \ge 0$ and that (1) is valid for all real x and y is it necessary that $b^2 - 4ac \ge 0$?

by M. S. Klamkin

by M. S. Klamkin

by M. S. Klamkin

Functions: dependent functions

CRUX 299 Crux Math., $\mathbf{3}(1977)$ 298 by M. S. Klamkin

If

$$F_{1} = (-r^{2} + s^{2} - 2t^{2})(x^{2} - y^{2} - 2xy) - 2rs(x^{2} - y^{2} + 2xy) + 4rt(x^{2} + y^{2})$$

$$F_{2} = 2rs(x^{2} - y^{2} + 2xy) + (r^{2} + s^{2} - 2t^{2})(x^{2} - y^{2} - 2xy) + 4st(x^{2} + y^{2})$$

$$F_{3} = -2rt(x^{2} - y^{2} - 2xy) - 2st(x^{2} - y^{2} + 2xy) + (r^{2} + s^{2} + 2t^{2})(x^{2} + y^{2})$$

show that F_1 , F_2 and F_3 are functionally dependent and find their functional relationship. Also, rreduce the five-parameter representation of F_1 , F_2 and F_3 to one of two parameters.

Functions: differentiable functions

MI 84-12 6(1984) No.3 pp.28,37

by K. S. Murray

It has been stated in several texts that if the first approximation x_0 is sufficiently close to a root of F(x) then the successive approximants in the Newton-Raphson iteration scheme $x_{n+1} = x_n - F(x_n)/F'(x_n)$ will converge to that root. Prove or disprove that result if F(x) is a function with a continuously turning tangent.

Integral inequalities

MM Q622 – See Math Mag file.

Integrals: evaluation

CRUX 88 1(1975) 85

Evaluate the indefinite integral

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}}$$

[MSK comment at 5(1979) 48]

by F. G. B. Maskell

Integrals: improper integrals

AMM 6440by M. S. Klamkin, J. McGregor & A. MeirAmer. Math. Monthly, 90(1983) 569.

CRUX 273 Crux Math., 3(1977) 226

Prove that

$$\lim_{n \to \infty} \int_c^\infty \frac{(x+a)^{n-1}}{(x+b)^{n+1}} \, dx = \int_c^\infty \frac{(x+a)^{-1}}{x+b} \, dx \qquad (a,b,c>0)$$

without interchanging the limit with the integral.

Maxima and minima: radicals

CRUX 358 4(1978) 161

Determine the maximum of x^2y , subject to the constraints

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = k \text{ (constant)} \quad x, y \ge 0$$

[MSK solution at 5(1979) 84]

CRUX 347 4(1978) 134

Determine the maximum value of

$$\sqrt[3]{4 - 3x + \sqrt{16 - 24x + 9x^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + 9x^2 - x^3}}$$

in the interval $-1 \le x \le 1$.

Maxima and minima: unit circle

MM Q662 — See Math Mag file.

Power series

MM 1100 Math. Mag., **53**(1980) 180. by M. S. Klamkin & M. V. Subbarao

by Murray S. Klamkin

by M. S. Klamkin

by M. S. Klamkin

APPLIED MATHEMATICS

Airplanes

SIAM 82-15
SIAM Rev., 24(1982) 340.
SIAM Rev., 25(1983) 407.

Physics: projectiles

AMM E2535 Amer. Math. Monthly, **82**(1975) 520–521. by M. S. Klamkin

by M. S.Klamkin

COMBINATORICS

Card shuffling

MI 83-2 5(1983) No.1 p.59

by M. S. Klamkin & G. Moran

A deck of cards numbered 1 to N is shuffled. If the top card is numbered k, then remove the k-th card (counted from the top) and place it on the top of the pile. This process is repeated with the new top card and so on. Does the card numbered 1 always come to the top? When it does, determine the maximum number of moves this can take.

Counting problems: jukeboxes

 $\mathrm{CRUX}~280$

by L. F. Meyers

A jukebox has N buttons.

(a) If the set of N buttons is subdivided into disjoint subsets, and a customer is required to press exactly one button from each subset in order to make a selection, what is the distribution of buttons which gives the maximum possible number of different selections?

(b) What choice of n will allow the greatest number of selections if a customer, in making a selection, may press any n distinct buttons out of the N? How many selections are possible then?

[MSK comment at Crux 4(1978) 112]

GAME THEORY

Selection games: arrays

OSSMB 75-2

A penny is placed at each vertex of a regular n-gon. The pennies are removed alternately by two players, each move consisting of the withdrawal of a single penny or of two pennies that occupy adjacent vertices. The player to take the last penny wins the game. Determine a winning strategy for the second player.

Ontario Secondary School Math. Bull., 12(1976/1) 16. Solution by Murray Klamkin

GEOMETRY

Billiards

NAvW 475

Let E be an ellipse and n be an integer greater than or equal to 3. We think of E as the rim of a billiard table, the objective being to determine all closed billiard ball paths Π_n that are closed convex *n*-gons. This requires that, at each vertex of Π_n , the angle of incidence with E be equal to the angle of reflection. Prove the following:

(a) there is a 1-parameter family F_n of *n*-gons Π_n inscribed in *E* with the reflection property, the initial vertex of Π_n being chosen arbitrarily on *E*.

(b) All these Π_n are circumscribed to a fixed ellipse E_n confocal to E.

(c)All *n*-gons of the family F_n have the same (maximal) perimeter.

[MSK comment at Nieuw Archief voor Wiskunde 26(1978) 248

NAvW 476

by I. J. Schoenberg

by M. S. Klamkin

Let E be an ellipse that we think of as the rim of a billiard table, the objective being to determine all convex quadrilaterals $Q = A_1 A_2 A_3 A_4$ that are closed billiard ball paths. Equivalently, Q should have equal incidence and reflection angles at each

[MSK comment at Nieuw Archief voor Wiskunde 26(1978) 248

Conics

 $\mathrm{CRUX}~520$

Constructions: lines

CRUX 488* 5(1979) 266

by Kesiraju Satyanarayana

Given a point P within a given angle, construct a line through P such that the segment intercepted by the sides of the angle has minimum length.

[MSK solution at Crux 6(1980) 260]

by I. J. Schoenberg

Equilateral triangles: interior point

OSSMB 75-7 Ontario Secondary School Math. Bull., **11**(1975/?) ??. by Maurice Poirier

Given a point P interior to equilateral triangle ABC such that PA, PB, PC have lengths 3, 4, 5 respectively, find the area of $\triangle ABC$.

Ontario Secondary School Math. Bull., 12(1976/1) 16. Check if there's a solution by Murray Klamkin

Inequalities: polygons

CRUX 506	by M. S. Klamkin
MM Q686	by M. S. Klamkin
Math. Mag., 56(1983) 240, 244.	

Inequalities: quadrilaterals

CRUX 106 Crux Math., 2(1976) 6

Prove that, for any quadrilateral with sides a, b, c, d

 $a^2 + b^2 + c^2 \ge \frac{1}{3}d^2$

[MSK solution at 2(1976) 78]

Lattice points: collinear points

CRUX 408* 5(1979) 16

by Michael W. Ecker

A zigzag is an infinite connected path in a Cartesian plane fromed by starting at the origin and moving successively one unit right or up. Prove or disprove that for every zigzag and for every positive integer k, there exist (at least) k collinear lattice points on the zigzag.

[MSK solution at 5(1979) 295. No previous solution?]

by Viktors Linis

Maxima and minima: rectangles

CRUX 427 5(1979) 77

by G. P. Henderson

A corridor of width a intersects a corridor of width b to form an "L". A rectangular plate is to be taken along one corridor, around the corner and along the other corridor with the plate being kept in a horizontal plane. Among all the plates for which this is possible, find those of maximum area.

[MSK comment at 6(1980) 49]

Maxima and minima: semicircles

OSSMB 76-4

A semicircle is drawn outwardly on chord AB of the unit circle with centre O. Prove that the point P on this semicircle that sticks out of the given circle the farthest is on the radius OD that is perpendicular to AB.

The farther AB is moved from the centre O, the smaller it gets, accordingly yielding a smaller semicircle. Determine the chord AB that makes OC a maximum.

Ontario Secondary School Math. Bull., 12(1976/2) 22. Solution by Murray Klamkin

N-dimensional geometry

MI 84-9 6(1984) No.2 p.39

by Murray S. Klamkin

Determine the maximum volume of an *n*-dimensional simplex if at most r edges and greater in length than 1 (for r = 1, 2, ..., n-1).

N-dimensional geometry: inequalities

CMB P244 **18**(1975) 616

by P. Erdős & Murray Klamkin

Let P be an point within or on a given *n*-dimensional simplex $A_1, A_2, \ldots, A_{n+1}$. The point P is "reflected" across each face of the simplex along rays parallel to the respective medians to each face producing an associated simplex $A'_1, A'_2, \ldots, A'_{n+1}$ (PA'_i is parallel to the median from A_i and is bisected by the face opposite A_i). Show that

 $n^n \operatorname{Volume}(A'_1, A'_2, \dots, A'_{n+1}) \le 2^n \operatorname{Volume}(A_1, A_2, \dots, A_{n+1})$

with equality if and only if P is the centroid of the given simplex.

SIAM 78-20 SIAM Rev., **20**(1978) 329.

N-dimensional geometry: simplexes

AMM E2548 Amer. Math. Monthly, 82(1975) 756.

CRUX 224 Crux Math., 3(1977) 65

by M. S.Klamkin

by M.S. Klamkin

by Murray S. Klamkin

Let P be an interior point of an n-dimensional simplex with vertices $A_1, A_2, \ldots, A_{n+1}$. Let P_i $(1 = 1, 2, \ldots, n + 1)$ denote points on A_iP such that $A_iP_i/A_iP = 1/n_i$. Finally, let V_i be the volume of the simplex cut off from the given simplex by a hyperplane through P_i parallel to the face of the given simplex opposite A_i . Determine the minimum value of $\sum V_i$ and the location of the corresponding point P.

Murray solution at Crux Math., 3(1977) 203

Pentagons

CRUX 232

submitted by Viktors Linis

Given are five points A, B, C, D and E in the plane, together with the segments joining all pairs of distinct points. The areas of the five triangles BCD, EAB, ABC, CDE and DEA being known, find the area of the pentagon ABCDE.

Murray comment at Crux Math., 3(1977) 240

Polygons

AMATYC D-4 **4**(1983) No.2 p.63 SSM 4018 **84**(1984) p.626 by Murray S. Klamkin by M. S. Klamkin

What is the largest number of reflex angles that can occur in a simple *n*-gon?

[[Evidently same problem published in two places. — R.]]

Polygons: convex polygons

AMM E2514

by G. A.Tsintsifas

Amer. Math. Monthly, 83(1976) 200.

Area of a Convex Polygon

E 2514. Proposed by G. A. Tsintsifas, Thessaloniki, Greece

Quadrilaterals: circumscribed quadrilateral

 $\mathrm{CRUX}\ 189$

If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

Evidently a Murray comment at Crux Math., 3(1977) 75

CRUX 199

by H. G. Dworschak

by Kenneth S. Williams

If a quadrilateral is circumscribed about a circle, prove that its diagonals and the two chords joining the points of contact of opposite sides are all concurrent.

Evidently a Murray solution at Crux Math., 3(1977) 112

Squares: 2 squares

CRUX 464 5(1979) 200

by J. Chris Fisher and E. L. Koh

(a) If the two squares ABCD and AB'C'D' have the vertex A in common and are taken with the same orientation, then the centres of the squares together with the midpoints of BD' and B'D are the vertices of a square.

(b) What is the analogous theorem for regular n-gons?

[MSK solution at Crux 6(1980) 186]

Triangle inequalities: angle bisectors and medians

PME 421 Pi Mu Epsilon J., 6(1978) 483

by Murray S. Klamkin

If F(x, y, z) is a symmetric, increasing function of x, y, z, prove that for any triangle in which w_a, w_b, w_c are the internam angle bisectors and m_a, m_b, m_c the medians, we have

$$F(w_a, w_b, w_c) \le F(m_a, m_b, m_c)$$

with equality if and only if the triangle is equilateral.

Pi Mu Epsilon J., 679 631, Solution by MSK

Triangle inequalities: angles

AMM E2958 Amer. Math. Monthly, **89**(1982) 498. Amer. Math. Monthly, **92**(1985) 289.

CMJ 265
Two-Year Coll. Math. J., 15 No.1 (Jan., 1984) 68.
Coll. Math. J., 16 No.4 (Sept., 1985) 310.

CRUX 908

CRUX 958

PME 394 Pi Mu Epsilon J., **6**(1977) 366 by Erwin Just and Bertram Kabak

Prove that, if A_1 , A_2 , A_3 are the angles of a triangle, then

 $s\sum_{i=1}^{3}\sin^2 A_i - 2\sum_{i=1}^{3}\cos^3 A_i \le 6$

Pi Mu Epsilon J., 6(1978) 493. Solution by MSK.

Triangle inequalities: interior point

PME 410 Pi Mu Epsilon J., 6(1977) 420 by Murray S. Klamkin

If x, y, z are the distances of an interior point of a triangle ABC to the sides BC, CA, AB, show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{2}{r}$$

where r is the inradius of the triangle.

Pi Mu Epsilon J., 6(1978) 557. [does vol # 6 spread over 1977 and 1978?] Solution by MSK

by M. S. Klamkin

by M. S. Klamkin

by M. S. Klamkin

by M. S. Klamkin
Triangle inequalities: medians and sides	
MM Q638	by Murray S. Klamkin
See Math Mag file	
Triangle inequalities: radii	
MM 1043	by M. S. Klamkin
See Math Mag file	
Triangle inequalities: sides	
SIAM 77-10	by M. S. Klamkin
See SIAM Rev file.	
Triangles: angle bisectors	
MSJ 540 28 (1980) No.1 p.2	by Murray S. Klamkin & A. Meir
Triangles: cevians	

CRUX 485 5(1979) 265

Given three concurrent cevians of a triangle ABC intersecting at a point P, we construct three new points A', B', C' such that AA' = kAP, BB' = kBP, CC' = kCP, where $k > 0, k \neq 1$ and the segments are directed. Show that A, B, C, A', B', C' lie on a conic if and only if k = 2.

CRUX 548

Triangles: exradii

PME 540 7(1983) p.543

If the radii r_1 , r_2 , r_3 of the three escribed circles of a given triangle $A_1A_2A_3$ satisfy the equation

$$\left(\frac{r_1}{r_2} - 1\right) \left(\frac{r_1}{r_3} - 1\right) = 2$$

determine which of the angles A_1 , A_2 , A_3 is the largest.

by M. S. Klamkin

by M. S. Klamkin

Triangles: medians

AMM E2505

by Jack Garfunkel

Amer. Math. Monthly, 83(1976) 59–60.

Extended Medians of a Triangle

E 2505 [1974, 1111]. Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York

Triangles: inscribed triangles

CRUX 210 Crux Math., 3(1977) 10

by Murray S. Klamkin

Let P, Q and R denote points on the sides BC, CA and AB respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad (k \neq 0, \frac{1}{2}, 1)$$

then PQR (in some order) is similar to ABC.

Solution by Murray at Crux Math., 3(1977) 163

Triangles: medians

CRUX 383 4(1978) 250

Let m_a , m_b and m_c be respectively the medians AD, BE and CF of a triangle ABC with centroid G. Prove that

(a) if $m_a : m_b : m_c = a : b : c$, then $\triangle ABC$ is equilateral;

(b) if $m_b/m_c = c/b$, then either (i) b = c or (ii) quadrilateral AEGY is cyclic;

(c) if both (i) and (ii) hold in (b), then $\triangle ABC$ is equilateral.

[MSK solution at 5(1979) 174]

by Daniel Sokolowsky

LINEAR ALGEBRA

Determinants

SIAM 80-10
SIAM Rev., 22(1980) 230.
SIAM Rev., 23(1981) 259.

by M. S. Klamkin, A. Sharma & P. W. Smith

Matrices

CMJ 278
Coll. Math. J., 15 No.3 (June, 1984) 267.
Coll. Math. J., 17 No.2 (Mar., 1986) 185–186.

NUMBER THEORY

Approximations

 $\mathrm{CRUX}\ 202$

by Daniel Rokhsar

by M. S. Klamkin

Prove that any real number can be approximated within any $\epsilon > 0$ as the difference of the square roots of two natural numbers.

Evidently a Murray solution at Crux Math., 3(1977) 137

Arithmetic progressions

TYCMJ 248 *Two-Year Coll. Math. J.*, **14** No.2 (Mar., 1983) 173. *Two-Year Coll. Math. J.*, **16** No.2 (Mar., 1985) 155–157.

Digit problems: arithmetic progressions

CRUX 378 4(1978) 226

by Allan Wm. Johnson Jr.

(a) Find four positive decimal integers in arithmetic progression, each having the property that if any digit is changed to any other digit, the resulting number is always composite.

 $(b)^*$ Can the four integers be consecutive?

[MSK solution at 5(1979) 147]

Digit problems: base systems

OSSMB 75-6 Ontario Secondary School Math. Bull., **11**(1975/?) ??. by Michael Webster

Prove that

$$\frac{x_1 x_2 x_3 \dots x_n - \sum_{k=1}^n x_k}{\sum_{k=0}^{n-2} (x_1 + x_2 + x_3 + \dots + x_{n-k-1}) R^k} = R - 1$$

where $x_1 x_2 x_3 \dots x_n$ is an *n*-digit numeral, base $R, n \ge 2$.

Ontario Secondary School Math. Bull., 12(1976/1) 16. Check if there's a solution by Murray Klamkin

Digit problems: squares

MI 84-1 6(1984) No.1 p.32 by K. S. Murray For which k are there arbitrar[il]y large squares containing exactly k even digits (base 10)?

Digit problems: sum of digits

MM Q679 by M. S. Klamkin & M. R.Spiegel *Math. Mag.*, **55**(1982) 300, 307.

Diophantine equations: radicals

CRUX 969

Divisibility: factorials

MM 1089 Math. Mag., **53**(1980) 49.

Fractional parts

 $\operatorname{CRUX}\,269$

by Kenneth M. Wilke

by M. S. Klamkin

by M. S. Klamkin

Let $(\sqrt{10})$ denote the fractional part of $\sqrt{10}$. Prove that for any positive integer n there exists a positive integer I_n such that

$$(\sqrt{10})^n = \sqrt{I_n + 1} - \sqrt{I_n}$$

[There's reputedly an MSK comment at Crux 4(1978) 81]

Normal numbers

CMB R6 **25**(1982) p.126

by Murray Klamkin

Is the number, whose decimal expansion

 $0.248163264128256\ldots$

is obtained by juxtaposing the powers of 2, normal?

Powers

MSJ 565 **28**(1981) No.6 p.2

by Murray Klamkin

Can one find triplets (a, b, c) of real numbers such that none of the three numbers is the cube of an integer but

 $a^{n/3} + b^{n/3} + c^{n/3}$

is integral for every positive integer n?

Primes: generators

OSSMB 75-4 Ontario Secondary School Math. Bull., 11(1975/?) ??

Prove that, for all integers x, $x^2 + x + 41$ is never divisible by any natural number between 1 and 41.

Ontario Secondary School Math. Bull., $\mathbf{12}(1976/1)$ 16. Check if there's a solution by Murray Klamkin

Pythagorean triples: odd and even

CRUX 460 5(1979) 167

by Clayton W. Dodge

Can two consecutive integers ever be the sides of a Pythagorean triangle? Show how to find all such Pythagorean triangles.

[MSK solution at Crux 8(1980) 160]

Series: inequalities

CRUX 459 5(1979) 167

If n is a positive integer, prove that

$$\sum_{i=1}^{\infty} \frac{1}{k^{2n}} \leq \frac{\pi^2}{8} \cdot \frac{1}{1 - 2^{-2n}}$$

[MSK solution at Crux 6(1980) 158]

by V. N. Murty

Square roots

PME 427 *Pi Mu Epsilon J.*, **6**(1978) 539 by Jackie E. Fritts If *a*, *b*, *c* and *d* are integers, with $u = \sqrt{a^2 + b^2}$, $v = \sqrt{\sqrt{(a-c)^2 + (b-d)^2}}$ and $w = \sqrt{c^2 + d^2}$, then prove that

$$\sqrt{(u+v+w)(v+w-u)(w+u-v)(u+v-w)}$$

is an even integer.

PME 427 Pi Mu Epsilon J., 7(1979) 63. Solution by MSK

PROBABILITY

Dice problems

SIAM 80-5 SIAM Rev., **22**(1980) 99. SIAM Rev., **22**(1980) 230.

Inequalities

MSJ 545 **28**(1980) No.2 p.2

by M. S. Klamkin & A. Liu

by Murray S. Klamkin

Given that each random triplet of integers (x, y, z) with $1 \le x, y, z \le n$ is equally likely, determine the probability that

$$\frac{x-y}{x+y} + \frac{y-z}{y+z} + \frac{z-x}{z+x} > 0$$

CRUX 484 5(1979) 265

by Gali Salvatore

Let A and B be two independent events in a sample space, and let X_A , X_B be their characteristic functions. If $F = X_A + X_B$, show that at least one of the three numbers

 $a = P(F = 2), \quad b = P(F = 1), \quad c = P(F = 0)$

is not less than 4/9.

[MSK solution at $Crux \ 6(1980) \ 253$] [MSK comment at $Crux \ 6(1980) \ 285$]

Selection problems: sums

MATYC 122 The MATYC Journal 12(1978) 253

by Gene Zirkel

A sequence of real numbers $x_1, x_2, x_3, \ldots x_n$ are picked at random from the interval [0,1]. This random selection is continued until their sum exceeds one and is then stopped. It is known that the expectated number of reals chosen is given by E(n) = e.

What is the expected value of n if we instead continue until the sum exceeds two?

[MSK solution at The MATYC Journal 13(1979) 217]

Selection problems: unit interval

PME 429 Pi Mu Epsilon J., 6(1978) 540

by Richard S. Field

Let P denote the product of n random numbers selected from the interval (0,1). Is the expected value of P greater or less the expected value of the nth power of a single number randomly selected from the interval (0,1)?

Pi Mu Epsilon J., 7(1979) 65. Solution by MSK

RECREATIONAL MATHEMATICS

Logic puzzles: incomplete information

CRUX 400 4(1978) 284

by Andrejs Dunkels

In the false bottom of a chest which had belonged to the notorious pirate Captain Kidd was found a piece of parchment with instructions for finding atreasure buried on a certain island. The essence of the directions was as follows.

"Start from the gallows and walk to the white rock, counting your paces. At the rock turn left through a right angle and walk the same number of paces. Mark the spot with your knife. Return to the gallows. Count your paces to the black rock, turn right through a right angle and walk the same distance. The treasure is midway between you and the knife."

However, when the searchers got to the island they found the rocks but no trace of the gallows remained. After some thinking they managed to find the treasure anyway. How?

[RKG posed this problem, set in Bermuda, using a stone marked for Ferdinand & Isabella and some other landmark, and a tree that no longer existed, more than 50 years ago. MSK solution at 5(1979) 243]

Polyominoes: maxima and minima

CRUX 429 5(1979) 77

by M. S. Klamkin & A. Liu

On a 2n by 2n board we place $n \times 1$ polyominoes (each covering exactly n unit squares of the board) until no more $n \times 1$ polyominoes can be accommodated. What is the maximum number of squares that can be left vacant?

[This problem is a generalization of the next one. Solution by MSK at Crux 6(1980)51]

$\mathrm{CRUX}\ 282$

by Erwin Just & Sidney Penner

On a 6×6 board we place 3×1 trominoes until no more trominoes can be accommodated. What is the maximum number of squares that can be left vacant?

 $[MSK \text{ comment at } Crux \ \mathbf{4}(1978) \ 115]$

SOLID GEOMETRY

Analytic geometry

Amer. Math. Monthly, 84(1977) 218–219. Volume and Surface Area of a Solid

E 2563 [1975, 937]. Proposed by J. Th. Korowine, Athens, Greece

Let f_1 and f_2 be non-negative periodic functions of period 2π and let h > 0. Let $P_1(\theta)$ and $P_2(\theta)$ be the points whose cylindrical coordinates are $(f_1(\theta), \theta, 0)$ and $(f_2(\theta), \theta, 0)$ respectively. Find integrals for the volume and surface area of the solid bounded by the planes z = 0, z = h and the lines $P_1(\theta)P_2(\theta)$.

Solution by M. S. Klamkin, University of Waterloo, Ontario, Canada.

Paper folding

CRUX 375 4(1978) 225

by M. S. Klamkin

A convex *n*-gon *P* of cardboard is such that if lines are drawn parallel to all the sides at distances *x* from them so as to form within *P* another polygon *P'*, then *P'* is similar to *P*. Now let the corresponding consecutive vertices of *P* and *P'* be A_1, A_2, \ldots, A_n and A'_1, A'_2, \ldots, A'_n respectively. From A'_2 , perpendiculars A'_2B_1, A'_2B_2 are drawn to A_1A_2, A_2A_3 respectively, and the quadrilateral $A'_2B_1A_2B_2$ is cut away, Then quadrilaterals formed in a similar way are cut away from all the other corners. The remainder is folded along $A'_1A'_2, A'_2A'_3, \ldots, A'_nA'_1$ so as to form an open polygonal box of base $A'_1A'_2\cdots A'_n$ and of height *x*. Determine the maximum volume of the box and the corresponding value of *x*,

[Crux, 5(1979) 142 has MSK's solution]

Polyhedra: combinatorial geometry

OSSMB 75-8 Ontario Secondary School Math. Bull., 11(1975/2) 18 by Murray Klamkin

Show that in every simple polyhedron there always exist two pairs of faces that have the same number of edges.

Ontario Secondary School Math. Bull., 12(1976/1) 16. Check if there's a solution by Murray Klamkin

Regular tetrahedra

SIAM 83-5 SIAM Rev., **25**(1983) 98. SIAM Rev., 26(1984) 275–276.

Spherical geometry

MM Q685 Math. Mag., 56(1983) 178, 182.

AMM E2981 Amer. Math. Monthly, **90**(1983) 54. Amer. Math. Monthly, **93**(1986) 300–302.

Tetrahedra

AMM E2962 Amer. Math. Monthly, **92**(1985) 433–434. Amer. Math. Monthly, **90**(1983) 569.

Tetrahedra: faces

CRUX 478 5(1979) 229

by Murray S. Klamkin

Prove that if the circumcircles of the four faces of a tetrahedron are mutually congruent, then the circumcenter O of the tetrahedron and its incenter I coincide.

[MSK comments at 11(1985) 189 and at 13(1987) 151]

by M. S. Klamkin

by M. S. Klamkin

by Murray S. Klamkin

CRUX 330 4(1978) 67

by M. S. Klamkin

It is known that if any one of the following three conditions holds for a given tetrahedron then the four faces of the tetrahedron are mutually congruent (i.e., the tetrahedron is isosceles):

- 1. The perimeters of the four faces are mutually equal.
- 2. The areas of the four faces are mutually equal.
- 3. The circumcircles of the four faces are mutually congruent.

Does the condition that the incircles of the four faces be mutually congruent also imply that the tetrahedron be isosceles?

TRIGONOMETRY

Approximations

TYCMJ 261 *Two-Year Coll. Math. J.*, **14** No.5 (Nov., 1983) 439. *Coll. Math. J.*, **16** No.4 (Sept., 1985) 306–307.

Identities

CMJ 282 by M. S. Klamkin & Gregg Patruno Coll. Math. J., **15** No.4 (Sept., 1984) 346–347. Coll. Math. J., **17** No.3 (June, 1986) 250–251.

Inequalities

MM 1137 Math. Mag., **55**(1982) 44.

Infinite series

CRUX 235

Prove Gauss's Theorema elegantissimum: If

$$f(x) = 1 + \frac{1}{2} \cdot \frac{1}{2}x^2 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}x^4 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{5}{6}x^6 + \cdots$$

show that

$$\sin\phi f(\sin\phi)f'(\cos\phi) + \cos\phi f(\cos\phi)f'(\sin\phi) = \frac{2}{\pi\sin\phi\cos\phi}$$

42

Murray solution at Crux Math., 3(1977) 258

Maxima and minima

MM 1107 Math. Mag., **53**(1980) 300.

by Viktors Linis

by M. S. Klamkin

by M. S. Klamkin

Solution of equations

SSM 4017 **84**(1984) p.626 AMATYC D-3 **4**(1983) No.2 p.63 by M. S. Klamkin by Murray S. Klamkin

Determine all values of $x \in [0, 2\pi)$ such that

$$81\sin^{10}x + \cos^{10}x = 81/256$$

[[another duplicate? - R.]]

Triangles

CRUX 493 5(1979) 291

(a) Let A, B and C be the angles of a triangle. Prove that there are positive x, y and z, each less than 1/2, simultaneously satisfying

$$y^{2} \cot \frac{B}{2} + 2yz + z^{2} \cot \frac{C}{2} = \sin A$$
$$z^{2} \cot \frac{C}{2} + 2zx + x^{2} \cot \frac{A}{2} = \sin B$$
$$x^{2} \cot \frac{A}{2} + 2xy + y^{2} \cot \frac{B}{2} = \sin C$$

(b) In fact, 1/2 may be replaced by a smaller k > 0.4. What is the least value of k? [MSK comment at *Crux* 7(1981) 51]

by R. C. Lyness

There follow a few items which need completion:

NAvW 377 Nieuw Archief voor Wiskunde, 23(1975) 90. Solution by Murray.

OSSMB 74-12 Ontario Secondary School Math. Bull., **11**(1975/1) 24. Comment by Murray Klamkin

PME 313 *Pi Mu Epsilon J.*, **6**(1975) 109. Solution by MSK.

Suggestions concerning Murray Klamkin volume

Richard K. Guy

June 22, 2006

This file is being [has been, 2006-05-31] continuously amended. To save having several editions, I'll just try to remember to keep the date here current:

Wed 2006-05-31

In fact, I'll freeze the following. Skip through to next —

I earlier (2005-02-04) wrote the following, which needs editing into the main part of this document:

the book should commemorate the whole of Murray's contribution to the problem scene.

As an indication, Rabinowitz's 1980–84 Index pp.352–353 lists about 200 problems from AMATYC, AMM, CMB, CMJ, Crux, MI, MM, MSJ, PME, SIAM, TYCMJ.

There may be more than a thousand problems in all. These should be sifted, with just the best of each type appearing, but we could also do quite a bit of 'See also' and 'Compare', so that the keen student can access almost all of Murray's work.

I think that they could be fairly easily classified into sections, since Murray concentrated on a comparatively small number of themes – though I shall no doubt be pleasantly surprised by the diversity when it comes to the actual event. Quickies would be in a separate section, but might be numerous enough to require subdivision. Perhaps we should also have a section for problems by other setters where Murray's solution was selected for publication by the editors. ... and on 2005-03-03:

I'm quite vague about the probable format of the volume – perhaps everyone else is too?

Classify chronologically? by source? by topic? by problem type? by none of the above

how complete are the solutions to be? All at the end, or with the problems?

What sort of indexes? references?

.

This file [[SIAM Review]] has only been very lightly proof-read. It contains some classic stuff, stimulated by Murray.

Iso Schoenberg on splines.

Shanks & Atkin on 3x+1 relatives.

Knuth on Conway's 'Topswaps'

...

I can produce similar (though not so long, I hope!) files for the Monthly, Math Mag, Coll Math J, etc.

[[These have been produced. The Monthly file is longer, and others are comparable in length.]]

Perhaps Andy is better qualified and prepared than I to do Math Horizons ? E.g., he knows most of Murray's many aliases, and he may have most of it in LATEXalready.

and, in a message to Jon Borwein, 2005-03-28:

Of course, the LaTeX files are best to work from from the editing and production point of view but are not so legible. Will Beverly Ruedi have any part in the process? She is excellent at her job and I've already worked with her on three books.

I will also send you a list of suggestions, queries, comments, etc. that I'm compiling. Better to discuss these with just one person at first.

I must enquire of the Strens collection to see if there's any correspondence twist Charles Trigg & Murray.

[[I set Polly Steele on investigating this on 2005-06-01]]

The original file starts here. Entries have been added to it on an ongoing basis.

- 1. Collect all references at end in a separate bibliography, to save space and gain consistency. Note: I have occasionally, but not at all consistently, changed the format of references to that of MR, and sometimes added the MR number. This will save space, and avoid 'et al' and other abbreviations.
- 2. Classify all problems by subject and/or keywords and exhibit in an index. This would enable us to publish all of Murray's problems, even if some were merely a reference to a journal. [2005-06-28: this can perhaps be done using Stanley Rabinowitz's classification.]
- 3. Agree on an orthography and typography which will be consistent throughout the volume. Trouble is one will have to alter some of the original statements. This could be explained in the Preface.
- 4. Perhaps include a glossary of Cauchy-Schwarz inequality, Brun-Minkowski inequality, Bernoulli's inequality, Hölder's inequality, Schwarz-Buniakowski inequality, Minkowski's inequality, Jensen's inequality, power mean inequality, rearrangement inequality, Ptolemy's inequality, Muirhead's inequality, the majorization inequality, Heron's inequality, Lagrange's identity, Chebyshev's inequality, formula of Faà di Bruno, Pedoe's inequality, etc., etc.

Choices (my preference first – though in one or two I'm fairly ambivalent— consistency is more important than any one person's particular fad). Note that these choices ought to be made, since I've almost always copied verbatim. We can say in the Preface that there has been occasional minor editing from the original in order to attain consistency.

[[items in 2nd & 3rd cols (and the 1st) are taken from the actual text in journals.]]

well-known	well known	
nonnegative	non-negative	non negative
nonintegers	non-integers	non integers
nonzero	non-zero	non zero
$\operatorname{nonacute}(?)$	non-acute	
nonobtuse(?)	non-obtuse	
midpoint	mid-point	
endpoint	end-point	
insoluble	unsolvable	insolvable
centre	center	
centred	centered	
analog	analogue	
integer(adj.)	integral	
moduli(pl.)	modules (pl. of 'modulus'!)	
proved(pp.)	proven(=tested)	
formulas	formulae	
the a_i	the a_i s	$a_i s$ [sic]
?? (reword?)	gotten	
zeroes	zeros	
A, B and C	A, B, and C	

Pappus's	Pappus'	Pappu's [sic]
Leibniz's	Leibniz	Leibniz'
Descartes's rule	Descartes' rule	
Chebyshev	Chebychev, etc.	
L'Hôpital's rule	l'Hospital's Rule	
right side	right-hand side	r.h.s., etc.
minor semi-axis	semi-minor axis	
vertical semi-angle	semi-vertical angle	
arccos	\cos^{-1}	
$\ln x$	$\log x$	
$(\ln x)^k$	$\ln^k x$	
$\ln \ln \ln x$	$\ln_3 x$	$\ln^3 x$
one-parameter	one parameter	
angle-bisector	angle bisector	
cross-section	cross section	
vector triple product	triple vector product	
cannot	can not	can't
North, etc.	north, etc.	
Hölder's inequality	Hölder's Inequality	
Cauchy-Schwarz inequality	Cauchy-Schwarz Inequality	
Fermat's principle	Fermat's Principle	
arithmetic-geometric	Arithmetic-Geometric	AM-GM
counterexamples	counter-examples	
integer-valued	integer valued	
$(\log x)^2$	$\log^2 x$	
p.73	p. 73	Page 73
if and only if	iff	
partitioned	divided	
semiperimeter?	semi-perimeter	
$294001 \ (> 3 \text{ digits})$	294,001	294001
Smith & Jones	S. and J. (joint work)	
bc + ca + ab	ab + bc + ca	ab + ac + bc
edges	sides (of n -gon)	
faces	sides (of polyhedron)	
cancelled	canceled	
bracketted	bracketed	
simplexes	simplices	
$m \perp n$	(m,n) = 1	
onto	on to	
Editors' note.	Editor's Note: (etc.)	

Typographical points;

- 1. I apologize for the inept placing of formula labels. There are two problems: (a) I prefer them on the right, whereas the places I'm copying from are variable. (b) I don't know how to reset the counter to (1) each time I start a new item.
- 2. We should standardize or omit names and addresses of individuals, although it's interesting to see the changes down the years.
- 3. Sometimes there's a *Remark* and sometimes there's a REMARK.
- 4. Should book titles be capitalized? E.g., *Theory and Application of Infinite Series*, rather than *Theory and application of infinite series*. And, indeed, shd they be italicized?
- 5. I haven't punctuated displayed formulas. The idea of punctuation is to advise the reader where to pause, and displaying does just that. She shouldn't have to puzzle over the difference between

$$\sum_{i'} x_{i'}$$
 and $\sum_i x_i$,

should she?