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Strange Series and High Precision Fraud

J. M. Borwein and P. B. Borwein

INTRODUCTION. Five of the following twelve series approximations are exact. The remaining seven are not identities but are approximations that are correct to at least 30 digits. One in fact is correct to over 18,000 digits and another to in excess of a billion digits. The reader is invited to separate the true from the bogus. (For answers see the end of the introduction.) Most of these series are easily amenable to high precision calculation in one's favorite high precision environment, such as Maple or MACSYMA, and provide examples of "caveat computat." Things are not always as they appear.

Sum 1

$$\sum_{n=1}^{\infty} \frac{a(2^n)}{2^n} \doteq \frac{1}{99}$$

where $a(n)$ counts the number of odd digits in odd places in the decimal expansion of n. ($a(901) = 2$, $a(210) = 0$, $a(811) = 1$, here the 1st digit is the 1st to the left of the decimal point.)

Sum 2

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} \doteq \frac{10}{99}$$

where $a(n)$ is as above.

Sum 3

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \doteq \frac{25\pi^2}{297}$$

where $b(n)$ counts the number of odd digits in n ($b(901) = 2$, $b(811) = 2$, ($b(406) = 0$).

Sum 4

$$\sum_{n=1}^{\infty} \frac{c(n)}{2^n} \doteq \frac{511}{8184}$$

where $c(n) := 32c_1(n) - c_2(n)/32$, and $c_1(n)$ counts the number of nines in n, while $c_2(n)$ counts the number of eights in n ($c(8199) = 32 \cdot 2 - 1/32$).

Sum 5

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^{\delta(n)}}{16^{\delta(n)}(D(n))^4} \doteq 2\frac{(e^{\pi/2} - e^{-\pi/2})}{\pi^2}$$

where $\delta(n)$ is the number of ones in the binary expansion of n and $D(n)$ is the product $\prod_i \max\{i\delta_i(n), 1\}$ where $\delta_i(n)$ is the ith binary digit of n ($\delta(1011_2) = 3$, $D(1011_2) = 4 \cdot 2 \cdot 1 = 8$).

Sum 6

$$\sum_{n=1}^{\infty} \frac{e(n)}{n(n+1)} \doteq \frac{10}{99} \log 10$$

where $e(n)$ "reflects" n through the decimal point ($e(123) = .321$, $e(90140) = .04109$).

Sum 7

$$\sum_{n=1}^{\infty} \frac{b(2^n)}{2^n} \doteq \frac{1}{9}$$

where $b(n)$ counts the number of odd digits in n (as in Sum 3).

Sum 8

$$\sum_{n=1}^{\infty} \frac{e(2^n)}{2^n} \doteq \frac{3166}{3069}$$

where $e(n)$ counts the number of even digits in n.

Sum 9

$$\sum_{n=1}^{\infty} \frac{\lfloor n \tanh \pi \rfloor}{10^n} \doteq \frac{1}{81}$$

where $\lfloor\ \rfloor$ is the greatest integer function ($\lfloor 3.7 \rfloor = 3$).

Sum 10

$$\sum_{n=1}^{\infty} \frac{\lfloor ne^{\pi\sqrt{163/9}} \rfloor}{2^n} \doteq 1280640$$

Sum 11

$$\sum_{-\infty}^{\infty} \frac{1}{10^{(n/100)^2}} \doteq 100\sqrt{\frac{\pi}{\log 10}}$$

Sum 12

$$\left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-(n^2/10^{10})}\right)^2 \doteq \pi$$

These sums break into four types. Sums 2, 3, 4, 5, and 6 are all specializations of generating functions for digit sums, more-or-less of the type:

$$\prod_{n=0}^{\infty}(1+q^{2^n}) = \sum_{n=0}^{\infty} x^{\delta(n)}q^n \tag{1.1}$$

where $\delta(n)$ counts the number of ones in the binary expansion of n. These are treated in section 2. See also [**14**].

Sums 1 and 7 are related to a problem independently due to E. Levine (*College Math Journal*, Vol. 19, number 5, 1989) and to D. Bowman and T. White (*Amer. Math. Monthly*, Vol. 96 1989, p. 743), which asks if

$$\sum_{n=0}^{\infty} \frac{g(2^n)}{2^n} = \frac{2}{9}$$

where $g(n)$ counts the number of digits ≥ 5 in n. The key to the solution we provide is due to our colleague A. C. Thompson. See section 3.

The sums 8, 9 and 10 revolve around the fact that

$$\sum_{n=0}^{\infty} w^{\lfloor n\alpha \rfloor}q^n$$

has a particularly attractive and rapidly convergent generating function that is related to the continued fraction expansion of α. This is essentially an observation of Mahler's [**11**], though the development we offer in section 4 is quite distinct. See also [**10**], [**3**]. This is closely related to problem $\#E3353$ in the *MAA Monthly* due to H. Diamond [**6**].

The last section deals with series like Sums 11 and 12. There are consequences of the fact that $f(t) := \sum_{n=-\infty}^{\infty} e^{-n^2 t\pi}$ is a modular form and satisfies a simple functional equation linking $f(t)$ and $f(1/t)$.

The fradulent series are: Sum 2 (correct to 99 digits), Sum 4 (correct to 240 digits), Sum 8 (correct to 30 digits), Sum 9 (correct to 267 digits), Sum 10 (correct to at least half a billion digits), Sum 11 (correct to at least 18,000 digits), and Sum 12 (correct to at least 42 billion digits).

GENERATING FUNCTIONS—PART ONE. Many digit sums are generated by the following type of argument.

Example 2.1. Let $b(n)$ count the number of odd digits in n base 10 (as in Sums 3 and 7). Then for $|q| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} x^{b(n)}q^n &= \prod_{n=0}^{\infty}(1 + xq^{10^n} + q^{2\cdot 10^n} + xq^{3\cdot 10^n} + q^{4\cdot 10^n} + xq^{5\cdot 10^n} + q^{6\cdot 10^n} \\ &\qquad + xq^{7\cdot 10^n} + q^{8\cdot 10^n} + xq^{9\cdot 10^n}) \\ &=: \prod_{n=0}^{\infty} r(x, q^{10^n}).\end{aligned} \tag{2.1}$$

To see this, observe that in the expansion of the product each power of q^m arises in exactly one way. This is just the unique expansion of m base 10. The coefficient of q^m is just a product of x's, one for each odd digit in m. If we differentiate (2.1) with respect to x as is legitimate since $b(n) = O(n)$ and the derivatives converge

uniformly, we get

$$\frac{\sum_{n=0}^{\infty} b(n) x^{b(n)-1} q^n}{\sum_{n=0}^{\infty} x^{b(n)} q^n} = \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + q^{5\cdot 10^n} + q^{7\cdot 10^n} + q^{9\cdot 10^n}}{1 + xq^{10^n} + q^{2\cdot 10^n} + \cdots + q^{8\cdot 10^n} + xq^{9\cdot 10^n}} \tag{2.2}$$

and at $x := 1$

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} b(n) q^n}{(1-q)^{-1}} &= \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + \cdots + q^{9\cdot 10^n}}{1 + q^{10^n} + q^{2\cdot 10^n} + \cdots + q^{9\cdot 10^n}} \\ &= \sum_{n=0}^{\infty} \frac{q^{10^n}}{1 + q^{10^n}} \\ &=: \sum_{n=0}^{\infty} R(q^{10^n}) \end{aligned} \tag{2.3}$$

where the second last equality follows on factoring each term. It is apparent from this representation for example that

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{1}{1-q}\left(\frac{q^1}{1+q^1} + \frac{q^{10}}{1+q^{10}}\right) + O(q^{100}). \tag{2.4}$$

We need the following observation which we encapsulate as Lemma 2.1.

Lemma 2.1. *Suppose $R(q)$ is a non-negative, measurable function on* $[0, 1]$. *If $b > 1$ and*

$$f(q) := \sum_{n=0}^{\infty} R(q^{b^n}) \qquad |q| < 1$$

then

$$\int_0^1 \frac{f(q)}{q}\, dq = \frac{b}{b-1} \int_0^1 \frac{R(q)}{q}\, dq.$$

Proof:

$$\begin{aligned} \int_0^1 \frac{f(q)}{q}\, dq &= \int_0^1 \sum_{n=0}^{\infty} \frac{R(q^{b^n})}{q}\, dq \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{R(q^{b^n})}{q}\, dq \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{S(q^{b^n}) q^{b^n}}{q}\, dq \end{aligned}$$

where $S(q) := R(q)/q$.

Now set $u = q^{b^n}$ and observe that

$$\int_0^1 \frac{f(q)}{q}\, dq = \sum_{n=0}^{\infty} \int_0^1 \frac{S(u)}{b^n}\, du$$

and the lemma is proved. (The interchange of sum and integral is just the monotone convergence theorem.) ■

From (2.3) we have

$$\sum_{n=0}^{\infty} b(n)q^{n-1}(1-q) = \sum_{n=0}^{\infty} \frac{R(q^{10^n})}{q} \tag{2.5}$$

and with Lemma 2.1,

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{10}{9}\int_0^1 \frac{1}{1+q}\,dq$$

or

$$\sum_{n=1}^{\infty} \frac{b(n)}{n(n+1)} = \frac{10}{9}\log 2. \tag{2.6}$$

Indeed this process iterates, in the sense that we can keep dividing by q and integrating in (2.5). This yields with some effort the following

Sum 13. For k a positive integer

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \frac{10^k}{10^k - 1}\alpha(k)$$

where, α is the alternating zeta function,

$$\alpha(s) := (1 - 2^{1-s})\zeta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Note that Sum 3 is just the $k := 2$ case of the above, while $k := 1$ gives (2.6).

A direct derivation of Sum 13 valid for non-integer k can be based on the fact that:

$$\alpha(s)\sum_{n=1}^{\infty} \alpha_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$$

if and only if

$$\sum_{n=1}^{\infty} \alpha_n \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} b_n x^n.$$

This identity is now coupled with (2.3). See [**18**].

Example 2.2. The generating function for q, the number of odd digits in odd places (as in Sums 1 and 2), is given by

$$\sum_{n=0}^{\infty} x^{a(n)}q^n = \prod_{n=0}^{\infty} r\left(x, q^{10^{2n}}\right)$$

where

$$\begin{aligned} r(x,q) := {} & \left(1 + xq + q^2 + xq^3 + q^4 + \cdots + xq^9\right) \\ & \cdot\left(1 + q^{10} + q^{2\cdot 10} + q^{3\cdot 10} + \cdots + q^{9\cdot 10}\right) \end{aligned}$$

and leads, as in (2.3), to the series

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{q^{100^n}}{1+q^{100^n}}. \tag{2.7}$$

Sum 2 now appears on taking $q := \frac{1}{10}$ and using the first term of the above expansion. It is apparent that the remainder is positive of size very close to $\frac{1}{9} \cdot 10^{-99}$. This gives the nature of the estimate in Sum 2.

In similar fashion

$$\sum_{n=0}^{\infty} A_k(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{(10^k)^n}}{1 + q^{(10^k)^n}} \tag{2.8}$$

is the generating function for the number of odd digits in the 1st, $(k+1)$th, $(2k+1)$th places of k. So with $k = 10$, for example

$$\sum_{n=0}^{\infty} \frac{A_{10}(n)}{10^n} = \frac{10}{99} + \varepsilon_n \tag{2.9}$$

where $0 < |\varepsilon_n| < \frac{10}{9} \cdot 10^{-10^{10}}$, and the above approximation is correct to over a billion digits. ■

Example 2.3. The number of times the digit $i > 0$ occurs in n has generating function

$$\sum_{n=0}^{\infty} g(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{i \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

So the generating function for $c(n)$ in Sum 4 is just

$$\sum_{n=0}^{\infty} c(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{32 q^{9 \cdot 10^n} - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

At $q := \frac{1}{2}$, the second term vanishes to give

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{c(n)}{2^n} &= \frac{1}{1-q} \left(\frac{32 q^9 - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + \cdots + q^9} \right) + O(q^{800}) \\ &= \frac{511}{8184} + \varepsilon \end{aligned}$$

where $\varepsilon < 10^{-241}$.

Example 2.4. The generating function which reverses digits, as in Sum 6, is

$$\sum_{n=0}^{\infty} x^{e(n)} q^n = \prod_{n=0}^{\infty} \left(1 + x^{1/10^{n+1}} q^{10^n} + \cdots + x^{9/10^{n+1}} q^{9 \cdot 10^n}\right). \tag{2.10}$$

So

$$\sum_{n=0}^{\infty} e(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{\dfrac{1}{10^{n+1}} q^{10^n} + \cdots + \dfrac{9}{10^{n+1}} q^{9 \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}} \tag{2.11}$$

and as in Lemma 2.1

$$\sum_{n=1}^{\infty} \frac{e(n)}{n(n+1)} = \frac{10}{99} \log 10.$$

There are very many analogues of these results. All have variations in different bases. The binary digit counting functions δ has generating function

$$\sum_{n=0}^{\infty} x^{\delta(n)} q^n = \prod_{n=0}^{\infty} (1 + xq^{2^n}) \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} \delta(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{2^n}}{1+q^{2^n}} \tag{2.13}$$

whence

$$\sum_{n=1}^{\infty} \frac{\delta(n)}{n(n+1)} = 2 \log 2. \tag{2.14}$$

(See the Putnam examinations of 1981, 1984 and 1987.) As in Example 2.1 we have Sum 14.

Sum 14. Let $\delta(n)$ denote the sum of the binary digits of n. Then

$$\sum_{n=1}^{\infty} \delta(n) \left(\frac{1}{n^k} - \frac{1}{(n+1)^k} \right) = \left(\frac{2^k}{2^k - 1} \right) \alpha(k)$$

where $\alpha(k)$ is the alternating zeta function.

The sum of the decimal digits of n denoted $s(n)$ has generating function

$$\sum_{n=0}^{\infty} x^{s(n)} q^n = \prod_{n=0}^{\infty} (1 + xq^{10^n} + x^2 q^{2 \cdot 10^n} + \cdots + x^9 q^{9 \cdot 10^n}) \tag{2.15}$$

from which we deduce that

$$\sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)} = \frac{10}{9} \log 10. \tag{2.16}$$

Loxton and van der Poorten [**10**] and Mahler [**11**] treat transcendence questions for functions, with power series expansions at zero which satisfy functional equations. From these results, one knows that if f, holomorphic at zero and not an algebraic function, satisfies a function equation of the form

$$f(q^m) = f(q) + R(q) \tag{2.17}$$

where m is an integer and R is a rational function, then $f(\alpha)$ is transcendental for algebraic α. From this we deduce that the exact answers in Sum 2, Sum 4 and Sum 8, are transcendental. This can also be deduced easily from Roth's Theorem [**8**].

GENERATING FUNCTIONS—PART TWO. A second type of digit function arises as follows.

Example 3.1. Let $\delta(n)$ as before, denote the sum of the binary digits of n, and let $\rho(n) := \prod\{S_i\colon i\text{th binary digit of } n \neq 0\}$ and $\rho(0) := 1$, where S_i is a given sequence and the product is taken over those binary digits of n which equal one. Then formally

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)} q^n}{\rho(n)} = \prod_{n=0}^{\infty} \left(1 + \frac{x}{S_{n+1}} q^{2^n} \right) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)}}{\rho(n)} = \prod_{n=0}^{\infty} \left(1 + \frac{x}{S_{n+1}}\right).$$

Example 3.2. Let $\delta(n)$ denote the sum of the binary digits of n, and let

$$D(n) = \prod i$$

where the product is taken over those i where the ith binary digit of n is non-zero (as in Sum 5). So, if $0 < n_1 < n_2 < \cdots < n_k$,

$$D(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}) = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

Then as in Example 3.1, starting with

$$F_q(x) := x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} q^{2^{n-1}}\right) = x \prod_{n=0}^{\infty} \left(1 - \frac{x^2}{(n+1)^2} q^{2^n}\right) \tag{3.2}$$

we have, for $|x| < 1$,

$$F_1(x) = \frac{\sin \pi x}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{2\delta(n)+1}}{[D(n)]^2} \tag{3.3}$$

and at $x := \frac{1}{2}$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{4^{\delta(n)}[D(n)]^2}. \tag{3.4}$$

Similarly, starting with

$$\frac{(\sin \pi x)(\sinh \pi x)}{\pi^2} = x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^4}{n^4}\right) \tag{3.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{4\delta(n)+2}}{[D(n)]^4},$$

we have, at $x := \frac{1}{2}$,

$$2\left(\frac{e^{\pi/2} - e^{-\pi/2}}{\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{16^{\delta(n)}[D(n)]^4}, \tag{3.6}$$

which is Sum 5.

Example 3.3. Let $t(n) := \sum i$, where the sum is taken over the non-zero digits on n base 2. So $t(1011_2) = 4 + 0 + 2 + 1 = 7$. Then

$$\prod_{n=0}^{\infty} (1 - x^{n+1} q^{2^n}) = \sum_{n=0}^{\infty} (-1)^{\delta(n)} x^{t(n)} q^n. \tag{3.7}$$

So

$$\sum_{n=0}^{\infty} (-1)^{\delta(n)} x^{t(n)} = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{-\infty}^{\infty} (-1)^n x^{(3n+1)n/2} \tag{3.8}$$

on using Euler's pentagonal number theorem [2] and on integrating, from zero to one,

$$\sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{t(n)+1} = \sum_{-\infty}^{\infty} \frac{2(-1)^n}{3n^2+n+2}. \tag{3.9}$$

4. CONTINUED FRACTION EXPANSIONS. The identities of this section are based on the two functions

$$G_\alpha(z,w) := \sum_{n=1}^{\infty} z^n w^{\lfloor n\alpha \rfloor} \tag{4.1}$$

and

$$F_\alpha(z,w) := \sum_{n=1}^{\infty} z^n \sum_{m=1}^{\lfloor n\alpha \rfloor} w^m \tag{4.2}$$

where α is a non-negative real number and $\lfloor n\alpha \rfloor$ is the integer part of $n\alpha$, while z and w are complex with modulus so as to ensure convergence. The function F_α was studied by Mahler [**11**] and is obviously related to G_α by

$$F_\alpha(z,w) + \frac{w}{1-w} G_\alpha(z,w) = \frac{zw}{(1-z)(1-w)} \tag{4.3}$$

for $|z|, |w| < 1$. Van der Poorten [**10**] comments that Mahler's paper has been largely overlooked. In [**3**] we explore these matters further. Note that for positive z and w, F_α is strictly increasing as a function of α.

For irrational α we will use the infinite continued fraction approximations generated by

$$\begin{aligned} &\text{(a)} \qquad p_{n+1} := p_n a_{n+1} + p_{n-1} \qquad p_0 := a_0 = \lfloor \alpha \rfloor, \quad p_{-1} := 1 \\ &\text{(b)} \qquad q_{n+1} := q_n a_{n+1} + q_{n-1} \qquad q_0 := 1, \qquad\qquad q_{-1} := 0 \end{aligned} \tag{4.4}$$

for $n \geq 0$ where

$$\begin{aligned} \alpha &= [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots] \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{a_3 + \cdots}}} \end{aligned}$$

so that each a_i is integral, $a_0 \geq 0$ and $a_n \geq 1$ for $n \geq 1$. Then for $n \geq 0$ p_{2n}/q_{2n} increases to α while p_{2n+1}/q_{2n+1} decreases to α and

$$\frac{1}{q_n(q_n+q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{4.5}$$

All of this is standard and may be found in [**8**], [**9**], or [**16**]. We will avoid using finite continued fractions which arise only for rational α. Let us write $q_n\alpha - p_n$ as ε_n. By (4.5) and (4.4)

$$|\varepsilon_{n+1}| < \frac{1}{q_n + q_{n+1}} < |\varepsilon_n| < \frac{1}{q_{n+1}} \leq 1.$$

A key lemma is:

Lemma 4.1. *For irrational $\alpha > 0$ and n, N in* $\mathbf{N}$

(a) $$\lfloor n\alpha + \varepsilon_N \rfloor = \lfloor n\alpha \rfloor \qquad \text{for } n < q_{N+1}$$

(b) $$\lfloor n\alpha + \varepsilon_N n \rfloor = \lfloor n\alpha \rfloor + (-1)^N \quad \text{for } n = q_{N+1}.$$

Proof: Suppose N is even (the odd case is entirely parallel). Then $\varepsilon_N > 0$ and (a) fails when

$$n\alpha + \varepsilon_N > m > n\alpha \quad \text{for some } m \text{ in } \mathbf{N}. \tag{4.6}$$

As $\alpha > p_N/q_N$, we have an integer k with

$$(n + q_N)\varepsilon_N > mq_N - np_N = k > 0.$$

If $k \geq 2$ then $n + q_n > 2/\varepsilon_N > 2q_{N+1}$ and $n > q_{N+1}$.

If $k = 1$ we have

$$p_N q_N - q_N p_N = 0, \qquad p_{N+1}q_N - q_{N+1}p_N = 1,$$

so that the linear Diophantine equation $mq_N - np_N = 1$ has general solution $m = p_{N+1} + sp_N$, $n = q_{N+1} + sq_N$ for s integer. However, $n + q_N > 1/\varepsilon_N > q_{N+1}$ so that s is non-negative. This establishes (a). For $n = q_{N+1}$ we have

$$q_{N+1}\alpha < p_{N+1} < q_{N+1}\alpha + \varepsilon_N < p_{N+1} + 1$$

since $p_{N+1} > q_{N+1}\alpha$ and $0 < \varepsilon_{N+1} + \varepsilon_N < 1$. This yields (b). ■

Theorem 4.1.

(a) *For rational $\alpha = p/q$ (reducible or irreducible)*

$$(1 - z^q w^p)G_\alpha(z,w) = \sum_{j=1}^{q} z^j w^{\lfloor jp/q \rfloor}.$$

(b) *For irrational α and $N > 0$*

$$(1 - z^{q_N} w^{p_N})G_\alpha(z,w)$$
$$= \sum_{n=1}^{q_n} z^n w^{\lfloor n\alpha \rfloor} + (-1)^N \left(\frac{w-1}{w}\right) z^{q_N} w^{p_N} z^{q_{N+1}} w^{p_{N+1}} + R_N(z,w)$$

with

$$|R_N(z,w)| \leq |1 - w| \frac{|z|^{q_{N+1}+q_N+1}}{1 - |z|}.$$

Proof:

(a) $$G_\alpha(z,w) = \sum_{k=0}^{\infty} \sum_{j=1}^{q} z^{qk+j} w^{kp+\lfloor j(p/q) \rfloor}$$
$$= \sum_{k=0}^{\infty} (z^q w^p)^k \sum_{j=1}^{q} z^j w^{\lfloor j(p/q) \rfloor},$$

(b) $$(1 - z^{q_N} w^{p_N})G_\alpha(z,w) - \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N} \{ w^{\lfloor (n+q_N)\alpha \rfloor} - w^{p_N + \lfloor n\alpha \rfloor} \}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N} w^{\lfloor n\alpha \rfloor + p_N} \{ w^{\lfloor n\alpha + \varepsilon_N \rfloor - \lfloor n\alpha \rfloor} - 1 \}.$$

By the proof of Lemma 4.1, the first non-zero term in this last expression is $(-1)^N(w-1)/w)z^{q_N+q_{n+1}}w^{p_N+p_{N+1}}$ while the other terms are dominated by $|z|^n|1-w|$ with $n > q_N + q_{N+1}$. ■

For fixed $\alpha > 0$ we write

$$P_N := \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}, \qquad Q_N := 1 - z^{q_N}w^{p_N}$$

and observe that Theorem 4.1 shows that

$$G_\alpha - \frac{P_N}{Q_N} = (-1)^N\left(\frac{w-1}{w}\right)\frac{z^{q_N}w^{p_N}z^{q_{N+1}}w^{p_{N+1}}}{Q_N} + O(z^{q_N+q_{N+1}}+1) \quad (4.7)$$

for α irrational (while $G_\alpha = P_N/Q_N$ for rational α). Thus as a function of z P_N/Q_N is the main diagonal Padé approximation to G_α of order q_N.

Corollary 4.1. *For irrational* $\alpha > 0$

$$G_\alpha(z,w) = \frac{zw^{p_0}}{1-zw^{p_0}} - \frac{1-w}{w}\sum_{n=0}^{\infty}(-1)^n\frac{z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.8)$$

Proof: Let $A_N := P_{N+1}Q_N - Q_{N+1}P_N$. Then A_N is a polynomial of degree at most $q_{N+1} + q_N$ in z. From (4.7) we see that

$$\frac{P_{N+1}}{Q_{N+1}} - \frac{P_N}{Q_N} = \frac{A_N}{Q_NQ_{N+1}} = (-1)^N\left(\frac{w-1}{w}\right)\left\{\frac{z^{q_N}w^{p_N}z^{q_{N+1}}w^{p_{N+1}}}{Q_NQ_{N+1}}\right\}.$$

On summing from zero to infinity we produce (4.8). ■

This is derived by Mahler for $\alpha \in (0,1)$ in [**11**].

Corollary 4.2. *For irrational* $\alpha > 0$ *and for* $w \neq 1$

$$F_\alpha(z,w) = \frac{zw}{(1-z)(1-w)}\frac{1-w^{p_0}}{1-zw^{p_0}} + \sum_{n=0}^{\infty}\frac{(-1)^n z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.9)$$

In particular, for $w = 1$, *the spectrum of* α [**7**] *is generated by*

$$\sum_{n=1}^{\infty}\lfloor n\alpha\rfloor z^n = \frac{p_0 z}{(1-z)^2} + \sum_{n=0}^{\infty}(-1)^n\frac{z^{q_n}z^{q_{n+1}}}{(1-z^{q_n})(1-z^{q_{n+1}})}. \quad (4.10)$$

Proof: Equation (4.9) follows from (4.8) and (4.3). Equation (4.10) is now obtained by letting w tend to 1. ■

If F_N denotes the truncation of the right-hand side of (4.9)

$$\frac{zw}{(1-z)(1-zw^{p_0})}\left(\frac{1-w^{p_0}}{1-w}\right) + \sum_{n=0}^{N-1}(-1)^n\frac{z^{q_n}w^{p_n}z^{q_{n+1}}w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}$$

we observe that (4.7) and (4.3) show that

$$F_N = \frac{\left(\frac{w}{1-w}\right)[zQ_N - (1-z)P_N]}{(1-z)(1-z^{q_N}w^{p_N})} \quad (4.11)$$

and some manipulation shows that, for $q_N > 1$, the numerator may be rewritten as

$$B_N := wz \sum_{n=1}^{q_N} z^n \left(\frac{w^{\lfloor (n+1)\alpha \rfloor} - w^{\lfloor n\alpha \rfloor}}{1-w} \right) + wz \left(\frac{1-w^{p_0}}{1-w} \right) (1 - z^{q_N} w^{p_N}) \quad (4.12)$$

so that B_N is a very simple integer polynomial in w and z (of degree $q_N + 1$ in z), while

$$F_\alpha - F_N = O(z^{q_N + q_{N+1}}).$$

Note that F_N is especially simple for $w := 1$ and $0 < \alpha < 1$.

Example 4.1. (a) Let $\alpha := \pi/2$ in (4.11) or (4.10). As

$$\frac{\pi}{2} = [1, 1, 1, 31, \ldots]$$

we have $p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $p_3 = 11$, $p_4 = 344$ and $q_0 = 1$, $q_1 = 1$, $q_2 = 2$, $q_3 = 7$, $q_4 = 219$. Thus

$$\begin{aligned} F_{\pi/2}(z,1) &= \sum_{n=1}^{\infty} \left\lfloor \frac{\pi}{2} n \right\rfloor z^n \\ &= \frac{z}{(1-z)^2} + \frac{z^2}{(1-z)^2} - \frac{z^3}{(1-z)(1-z^2)} \\ &\quad + \frac{z^9}{(1-z^2)(1-z^7)} - \frac{z^{226}}{(1-z^7)(1-z^{219})} + \cdots \end{aligned}$$

and the approximation F_4 is also expressible as

$$\frac{z(z^7 + z^6 + 2z^5 + z^4 + 2z^3 + z^2 + 2z + 1)}{(1-z^7)(1-z)}$$

and has an error like z^{226}. In particular

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{\pi}{2} n \right\rfloor}{2^n} \doteq \frac{339}{127}$$

with error less than 10^{-68}.

(b) Sum 9 follows from using (4.10) for $\tanh(\pi) = [0, 1, 267, \ldots]$. This produces

$$\sum_{n=1}^{\infty} \lfloor n \tanh \pi \rfloor z^n = \frac{z^2}{(1-z)^2} - \frac{z^{269}}{(1-z)(1-z^{268})} + \cdots .$$

(c) Sum 10 follows similarly from (4.10) with one of our favorite transcendental numbers $\alpha := e^{\pi\sqrt{163/9}} = [640320, 1653264929, \ldots]$.

(d) Let $\alpha := \log_{10}(2) = [0, 3, 3, 9, \ldots]$. Then (4.11) with $N := 3$, $z := \frac{1}{2}$ and $w := 1$ gives

$$\sum_{n=1}^{\infty} \frac{\lfloor n \log_{10}(2) \rfloor}{2^n} \doteq \frac{146}{1023}$$

to 30 places since $q_0 = 1$, $q_1 = 3$, $q_2 = 10$, $q_3 = 93$. Thus, as the number of even digits in 2^n is $\lfloor n \log_{10}(2) \rfloor + 1$ less the number of odd digits in 2^n, the "false" Sum 8 follows from Sum 7 and this "false" identity. In fact, see below, Sum 8 is transcendental while Sum 7 is rational. ■

Other lovely approximations follow from

$$\log_{10}(6) = [0,1,3,1,1,32,\dots]$$
$$\tanh(1) = [1,3,7,9,11,\dots]$$
$$\frac{e-1}{2} = [0,1,6,10,14,\dots]$$

and other simple transcendental numbers. Thus

$$\sum_{n=1}^{\infty} \frac{\lfloor n\zeta(3)\rfloor}{2^n} \doteq \frac{64}{31}$$

to 30 places.

Example 4.2. Many other related sums can be derived from (4.8) and (4.9). We indicate some classes.

(a) For irrational $\alpha > 0$

$$G_\alpha(1,w) = \sum_{n=1}^{\infty} w^{\lfloor n\alpha\rfloor} = \left(\frac{1-w}{w}\right) F_{1/\alpha}(w,1),$$

and more generally

$$G_\alpha(z,w) = \left(\frac{1-w}{w}\right) F_{1/\alpha}(w,z).$$

This follows either from the elementary identity in [**11**]

$$F_\alpha(z,w) + F_{\alpha^{-1}}(w,z) = \frac{zw}{(1-z)(1-w)} \tag{4.13}$$

or from Theorem 2 in [**13**], when $z = 1$.

(b) Letting $w := -1$ in (4.9) produces a Lambert-like series for $\sum_{\lfloor n\alpha\rfloor \text{ odd}} z^n$. As an example,

$$\sum \left\{ \frac{1}{2^n} | \text{length}(2^n) \text{ even} \right\} \doteq \frac{114}{1025}$$

to 30 places.

(c) Observe that

$$\sum_{k=0}^{M} \frac{(-1)^k \binom{M}{R} G_\alpha(z,w^k)}{(1-w)^M} = \sum_{n=1}^{\infty} \left(\frac{1-w^{\lfloor n\alpha\rfloor^M}}{1-w}\right) z^n$$

so that on letting w tend to unity we obtain the approximation

$$\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^M z^n = \frac{\Delta_N^M(z)}{(1-z)(1-z^{q_N})^M} + O(z^{q_N+q_{N+1}})$$

where Δ_N^M is an integer polynomial in z of degree $MqN+1$. In particular

$$\begin{aligned}\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^2 z^n = \sum_{n=0}^{\infty} &\frac{z^{q_n+q_{n+1}}}{(1-z^{q_n})^2(1-z^{q_{n+1}})^2} \\ &\times\{(2p_n + 2p_{n+1} - 1) - z^{q_n} z^{q_{n+1}} \\ &\quad -(2p_n - 1)z^{q_{n+1}} - (2p_{n+1} - 1)z^{q_n}\}\end{aligned}$$

for $0 < \alpha < 1$, α irrational. Thus

$$\sum_{n=1}^{\infty} \frac{(\text{length}(6^n))^2}{6^n} \doteq \frac{196669}{37303}$$

to 88 places.

(d) Similarly, if w is a primitive Nth root of unity

$$\frac{1}{N} \sum_{k=1}^{N} G_\alpha(z, w^k) \overline{w}^{Mk} = \sum_{\lfloor n\alpha \rfloor \equiv M \pmod N} z^n$$

[compare (b)]. Thus

$$\sum_{3 \mid \lfloor n \log_{10} 2 \rfloor} \frac{1}{3^n} \doteq \frac{3554}{7381}$$

to 50 places.

(e) Let $w := e^{i\theta}$(θ real) in (4.9). We obtain

$$\sum_{n=1}^{\infty} \cos(\lfloor n\alpha \rfloor) z^n = \frac{\sum_{n=1}^{q_N} \cos(\lfloor n\alpha \rfloor \theta) z^n - \sum_{n=1}^{q_N} \cos(p_N - \lfloor n\alpha \rfloor \theta) z^{n+q_N}}{1 - 2z^{q_N} \cos(p_N \theta) + z^{2q_N}} + O(z^{q_N + q_{N+1}}),$$

with a similar expression for sin replacing cos. ■

The rational counterpart to (4.13) is

$$F_{p/q}(z, w) + F_{q/p}(w, z) = \frac{zw}{(1-z)(1-w)} + \frac{z^q w^p}{1 - z^q w^p}, \tag{4.14}$$

for p and q relatively prime.

We consider $F(\alpha) := F_\alpha(z, w)$ as a function of α, and observe that $F(\alpha)$ is continuous at each irrational. Moreover, $\lim_{\alpha \downarrow p/q} F(\alpha) = F(p/q)$. Thus, on using (4.13) and (4.14) $\lim_{\alpha \uparrow p/q} F(\alpha) = F(p/q) - z^q w^p/(1 - z^q w^p)$. In consequence, F is discontinuous at every rational and $F(1) - F(0) = \sum_{0 < p/q < 1} \{F(p/q) - F(\frac{p}{q} -)\}$ so that dF is a "pure jump measure" on the rationals in $[0, 1]$. [This observation was made by H. Diamond.] Explicitly the jumps are expressed as

$$\begin{aligned} J &:= \sum_{s=1}^{\infty} \sum_{\substack{1 \le r \le s \\ (r,s)=1}} \frac{z^s w^r}{1 - z^s w^r} \\ &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} z^{sk} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{rk}. \end{aligned} \tag{4.15}$$

Now, on setting $n = sk$, this yields

$$\sum_{n=1}^{\infty} z^k \left\{ \sum_{s/n} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{(r/s)n} \right\}.$$

Equation (16.2.3) in **[8]** applies with $F(w) := w^n$ and shows that the bracketed term is just $\sum_{m=1}^{n} w^m$. Hence $J = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} w^m = F_1(z, w)$ as claimed. This is valid for $|z| < 1$, $|w| \le 1$. ■

We have also shown, using Theorem 4.1(a) and $\lfloor n\alpha \rfloor = \lfloor n(p_N/q_N) \rfloor$ for $n < q_N$, that for $0 < \alpha < 1$

$$F_N = \begin{cases} F_{p_N/q_N} & N \text{ even} \\ F_{p_N/q_N} - \dfrac{z^{q_N} w^{p_N}}{1 - z^{q_N} w^{p_N}} & N \text{ odd.} \end{cases} \tag{4.16}$$

Clearly $F\colon Q \to Q$. In **[10]**, **[11]** (4.9) is used to obtain transcendence estimates by functional equation methods. For $w := \pm 1$ and $z := 1/b$, $b = 2, 3, 4, \ldots$ we can get very accessible estimates for F_α or G_α from Roth's theorem **[2]**, **[9]**, **[15]**.

First, observe that Corollary 4.2 shows F_α is irrational when α is irrational and w, z are rational. It is convenient to introduce

$$s := s(\alpha) = \limsup_{n \to \infty} a_n.$$

Thus s is infinite when α has unbounded continued fraction coefficients. For b and w as above, we have from (4.12)

$$0 < \left| F(\alpha) - \frac{P_N}{Q_N} \right| \le O\left(\frac{1}{b^{q_N + q_{N+1}}} \right) \le O\left(\frac{1}{Q_N^{(1 + q_{N+1}/q_N)}} \right) \tag{4.17}$$

for integers P_N and $Q_N := (b-1)(b^{q_n} - w^{p_N})$. Hence, Roth's theorem shows $F(\alpha)$ is transcendental when

$$\limsup_{n \to \infty} \frac{q_{N+1}}{q_N} > 1,$$

and clearly α is Liouville when $s(\alpha) = \infty$. Since almost all numbers have unbounded coefficients, $F(\alpha)$ is Liouville in almost all cases and F maps Liouville numbers to Liouville numbers as they have $s =$ infinity. When $s(\alpha)$ is finite, we have $q_{N+1} \le s q_N + q_{N-1} \le (s+1) q_N$ eventually and so infinitely often

$$q_{N+1} \ge s q_N + q_{N-1} \ge \frac{s^2 + s + 1}{s + 1} q_N$$

and (4.17) shows $F(\alpha)$ is approximable to order at least $(s+1) + (1/(s+1)) \ge 5/2$. If $s = 1$ then α is equivalent to $(\sqrt{5} + 1)/2$. In every other case $F(\alpha)$ is approximable to order $10/3$. In summary $F(\alpha)$ is never algebraic, indeed never has the expected rate of rational approximation and is usually Liouville (**[2]**, **[8]**, **[15]**). In fact almost all irrationals have only finitely many solutions to

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2 (\log q)^{1+}}.$$

Example 4.3. (a) Arguing similarly from Example 4.2 we see that for almost all α,

$$\sum_{n=1}^{\infty} \frac{p(\lfloor n\alpha \rfloor)}{b^n}$$

is a Liouville number, for any integer polynomial p.

It is hard to find explicit numbers with unbounded continued fraction coefficients but e and $\tanh(1)$ are two examples:

$$\sum_{n=1}^{\infty} \frac{p(\lfloor ne \rfloor)}{b^n}$$

is Liouville for all p and b.

(b) Correspondingly, $\sum_{n=1}^{\infty} p(\lfloor n\alpha \rfloor)/b^n$ is approximable to order at least

$$\frac{1+s(\alpha)}{\deg(p)}.$$

■

For irrational $0 < \alpha < 1$, $F_\alpha(z,w)$ may be computed entirely from the continued fraction expansion via

$$F_\alpha(z,w) = \sum_{n=0}^{\infty} (-1)^n \frac{z_n z_{n+1}}{(1-z_n)(1-z_{n+1})}$$

where $z_{n+1} := z_n^{a_{n+1}} z_{n-1}$, $z_0 := z$, $z_{-1} := w$. This follows from (4.9) and an easy induction.

We conclude with some remarks about iterates of $F(\alpha) := \sum_{n=1}^{\infty} \lfloor n\alpha \rfloor 2^{-n}$. For $\alpha = p/q$ $(0 < \alpha < 1)$ we have

$$F_\alpha(z,w) = zw \frac{\sum_{n=1}^{q} \left(\left\lfloor (n+1)\frac{p}{q} \right\rfloor - \left\lfloor n\frac{p}{q} \right\rfloor \right) w^{\lfloor n(p/q) \rfloor} z^n}{(1-z)(1-z^q w^p)} \tag{4.18}$$

either by direct computation or from (4.11) and (4.16). We now set $z := \frac{1}{2}$, $w := 1$ and observe that

$$F\left(\frac{p}{q}\right) + F\left(1 - \frac{p}{q}\right) = 1 + \frac{1}{2^q - 1}.$$

In particular $F(\frac{1}{2}) = \frac{2}{3}$. Moreover, (4.18) shows that

$$F\left(1 - \frac{1}{q}\right) = 1 - \frac{1}{2^q - 1}.$$

Let $q_0 := 2$ and $q_{n+1} := 1/(2^{q_n} - 1)$ to deduce that

$$F^{(n)}\left(\frac{1}{2}\right) = 1 - \frac{1}{q_{n+1}}$$

and so converges to 1. Similar analysis shows that

$$F\left(\frac{1}{q}\right) = \frac{2}{2^q - 1} < \frac{1}{2^{q-2}},$$

and so that

$$F^{(n)}\left(\frac{1}{3}\right) \to 0, \quad \text{because } F^{(2)}\left(\frac{1}{3}\right) = \frac{18}{127} < \frac{1}{7}.$$

Note that $\alpha \geq \frac{1}{2}$ implies $F^{(n)}(\alpha) \geq F^{(n)}(\frac{1}{2})$ and $\alpha < \frac{1}{2}$ implies $F^{(n+1)}(\alpha) \to 0$ for $0 \leq \alpha < \frac{1}{2}$. For rational α, the entire sequence is rational, otherwise it is entirely transcendental, usually Liouville.

5. RATIONAL DIGIT SUMS. This section is based on the following Lemma whose proof we owe to A. C. Thompson.

Lemma 5.1. *For* $0 < q < 1$ *and integer* $m > 1$

$$q = \sum_{n=1}^{\infty} \frac{\lfloor m^n q \rfloor \pmod{m}}{m^n}. \tag{5.1}$$

Proof: Consider the base m expansion of q

$$q = \sum_{k=1}^{\infty} \frac{a_k}{m^k} \qquad 0 \le a_k < m$$

where when ambiguous we take the terminating expansion. Then

$$m^n q = \sum_{k=1}^{n-1} m^{n-k} a_k + a_n + \theta_n$$

for some θ_n in $[0, 1[$. Thus a_n is the remainder of $\lfloor m^n q \rfloor$ modulo m, and (5.1) follows. ■

Let $F(q) := \sum_{n=1}^{\infty} c_n q^n$ be any formal power series.

Theorem 5.1. *For* $0 < q < 1/\limsup_{n\to\infty} |c_n|^{1/n}$,

$$F(q) = \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

where

$$f(n) = \sum_{k\ge 1} c_k(\lfloor m^n q^k \rfloor \bmod m).$$

Proof: From Lemma 5.1

$$F(q) = \sum_{k=1}^{\infty} c_k q^k = \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \frac{[m^n q^k] \bmod m}{m^n}$$
$$= \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

on exchanging order of summation, as is valid within the radius of convergence of F. ■

Theorem 5.1 can be extended so as to replace m^n by $\prod_{k=1}^{n} r_k$ where r_k are integers ≥ 2, and where the remainder is computed modulo r_n.

If we specialize Theorem 5.1 to the case where $q := 1/b$ and b is an integer divisible by m we may observe that $\lfloor m^n/b^k \rfloor \bmod m$ coincides with the coefficient (mod m) of b^k in the base b expansion of m^n (the $(k+1)^{th}$ digit).

Specializing further so that $m := 2$ and b is even we have

$$F\left(\frac{1}{b}\right) = \sum_{n=1}^{\infty} \frac{f_b(n)}{2^n} \tag{5.2}$$

where

$$f_b(n) := \sum \left\{c_k \mid 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd base } b\right\}.$$

Example 5.1. (a) Let $F(q) := q/(1-q)$. Then $f_b(n)$ counts the number of odd digits in 2^n base b. Sum 7 is established on setting $b := 10$.

(b) Sum 1 corresponds to taking $F(q) = q^2/(1-q^2)$ and $q = 1/10$.

(c) Let $F(q) = q/(1-q-q^2)$. Now F is the generating function of the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Again with $q := 1/10$, we

obtain for

$$f(n) := \sum \{F_k | 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd}\},$$

as in Bowman and White [4], that

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{10}{89}.$$

The generating function for F_k^2 is $\dfrac{q - q^2}{1 - 2q - 2q^2 + q^3}$ and so for

$$f(n) := \sum \{F_k^2 | 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd}\}$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{90}{781}.$$

(d) Let

$$F(q) = \sum_{n=1}^{\infty} q^{n^2} = \frac{\theta_3(q) - 1}{2}.$$

Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{\theta_3\left(\frac{1}{10}\right) - 1}{2}$$

where $f(n)$ counts the number of odd digits of 2^n in square positions (the second, fifth, tenth digits etc.).

(e) If we apply Theorem 5.1 to $F(q) := q/(1-q)$ with $b := 10$ and $m := 5$ we deduce that again

$$\sum_{n=1}^{\infty} \frac{f(n)}{5^n} = \frac{1}{9}$$

where $f(n)$ sums the digits (mod 5) of 5^n base 10 (e.g. $f(3125) = 6$). ■

6. THETA FUNCTION EXAMPLES. The underlying identity for this section is really just a modular transformation of $\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$. (See [2].)

Lemma 6.1. *For $\alpha, \beta > 0$ with $\alpha\beta = 2\pi$*

$$\sqrt{\alpha}\left[\sum_{n=-\infty}^{\infty} e^{-\alpha^2 n^2/2}\right] = \sqrt{\beta}\left[\sum_{n=-\infty}^{\infty} e^{-\beta^2 n^2/2}\right].$$

Example 6.1. From the Lemma, with $s = 2/\beta^2$ so $\alpha^2 = 2\pi^2 s$

$$\sqrt{\pi s} - \sum_{n=-\infty}^{\infty} e^{-n^2/s} = 2\sqrt{\pi s}\, e^{-\pi^2 s} + O(e^{-\pi^2 4s}) \tag{6.1}$$

$$\sim 2\sqrt{\pi s}\, 10^{-(4.2863\ldots)s}.$$

Now with $s := 10^{10}$ we get

$$\left|\sqrt{\pi} - \left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-n^2/10^{10}}\right)\right| \le 10^{-4.2\cdot 10^{10}}, \tag{6.2}$$

which is Sum 12.

If we set

$$s = \frac{1}{\log 10^{1/N}} = \frac{N}{\log 10}$$

we get

$$\sqrt{\frac{N\pi}{\log 10}} - \sum_{-\infty}^{\infty} \frac{1}{10^{n^2/N}} \sim 2 \cdot \sqrt{\frac{N\pi}{\log 10}}\, 10^{-(1.861\cdots)N} \tag{6.3}$$

and with $N := 10^4$ we get Sum 11.

Similarly we have

$$\sqrt{\frac{q\pi}{\log q}} - \sum_{-\infty}^{\infty} \frac{1}{q^{n^2/q}} \sim 2\sqrt{\frac{q\pi}{\log q}}\, e^{-\pi^2 q/\log q}. \tag{6.4}$$

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