ADDENDUM
for
A TABLE OF SERIES AND PRODUCTS
by Eldon Hansen

The following list of 446 series (and one product) supplement those in the book A Table of Series and Products by Eldon Hansen.

The equations in the book have three-part numbers from (5.1.1) through (93.1.7). In the following list, some equations have three-part numbers and some have four-part numbers. Those with three-part number are to be inserted in the book's list in a natural way. An equation with four-part number is to be inserted between series having the same (first) three-part number. For example, an equation numbered (5.9.6.1) is to be inserted between equations in the book numbered (5.9.6) and (5.9.7).

A few of the new series are generalizations of series in the book. When the new series generalizes a single series in the book, it is given the same number as the one it replaces. In some cases, more than one series in the book is a special case of a single new series. In such a case, the new series is given a new four-part number and is inserted in a relevant position.

Five of the new series do not fit into existing sections of the book. They have been numbered to be placed where appropriate new sections would occur. They are numbered (45.5.0), (49.0.1), (54.6.0), (69.1.5), and (76.1.4).

- $\frac{1}{2}x + \frac{1}{96}x^2 + \ldots$

(5.9.6.1) $\sum_{k=1}^{\infty} \frac{(k!)^2}{k^4(2k)!} = \frac{17\pi^4}{3240}$ JN20p.98(2.25)

- $\frac{1}{2}x + \frac{1}{48}x^2 + \frac{1}{540}x^3 + \ldots$

(5.9.8.1) $\sum_{k=1}^{\infty} \frac{(k!)^2}{k^3(2k)!}(-1)^k = -\frac{2}{5}\zeta(3)$ JN20p.92(1.1)

(5.9.8.2) $\sum_{k=1}^{\infty} \frac{(k!)^2}{k^3(2k)!}2^k = \frac{\pi^2}{8}\log(2) + \pi\lambda - \frac{35}{16}\zeta(3)$ JN20p.96(2.10)

(5.9.8.3) $\sum_{k=1}^{\infty} \frac{(k!)^2}{k^3(2k)!}4^k = \pi^2\log(2) - \frac{7}{2}\zeta(3)$ JN20p.96(2.12)

- $2x + 3x^2 + \ldots$

(5.15.4.1) $\sum_{k=1}^{\infty} \frac{(2k)!}{k(k!)^2}x^k = \sum_{k=1}^{\infty} \frac{(1/2)_k}{kk!}(4x)^k = -\log(x) - 2\,\mathrm{sech}^{-1}(2x^{1/2})$

- $\frac{1}{6}+\frac{1}{420}x^2+...$

(5.16.18.1) $\sum\limits_{k=0}^{\infty}\frac{(2k)!(2k+1)!}{(4k+3)!}x^{2k}=\frac{1}{x}(\frac{4}{x}+1)^{1/2}\sinh^{-1}(\frac{1}{2}x^{1/2})-$

$-\frac{1}{x}(\frac{4}{x}-1)^{1/2}\sin^{-1}(\frac{1}{2})$ GQ

- $\frac{1}{2}+\frac{1}{144}x+\frac{1}{5400}x^2+...$

(5.18.2.1) $\sum\limits_{k=0}^{\infty}\frac{(k!)^4}{(2k+1)!(2k+2)!}(16)^k=\pi\lambda-\frac{7}{4}\zeta(3)$ GQ

- $\frac{1}{2}+\frac{1}{96}+...$

(5.18.2.2) $\sum\limits_{k=0}^{\infty}\frac{(k!)^2}{(k+1)^2(2k+2)!}=\frac{17\pi^2}{3240}$ AE92p.453

- $\frac{1}{2}+\frac{1}{48}x+...$

(5.18.2.3) $\sum\limits_{k=0}^{\infty}\frac{(k!)^2}{(k+1)(2k+2)!}x^k=\frac{4}{x}\int\limits_0^{\frac{1}{2}x^{1/2}}\frac{1}{t}(\sin^{-1}t)^2dt$ AE92p.453

(5.18.2.4) $\sum\limits_{k=0}^{\infty}\frac{(k!)^2}{(k+1)(2k+2)!}=\frac{\pi}{72}3^{1/2}[\psi(2/3)-\psi(1/3)]-\frac{1}{3}\zeta(3)$ AE92p.453

(5.18.2.5) $\sum\limits_{k=0}^{\infty}\frac{(k!)^2}{(k+1)(2k+2)!}(-1)^k=\frac{2}{5}\zeta(3)$ AE92p.453

- $1-x^2-x^4+...$

(5.20.10.1) $\sum\limits_{k=-\infty}^{\infty}(-1)^k\exp[-2\pi(3k^2+k)]=2^{-7/8}\pi^{-5/4}e^{\pi/6}\Gamma(1/4)$

AS18p.18

- $1+\frac{1}{27}x+...$

(5.21.7.1) $\sum\limits_{k=0}^{\infty}\frac{(2k)!}{(k!)^2(2k+1)^3}(1/16)^k=\frac{7\pi^3}{216}$ JN20p.98(2.23)

- $1+\frac{2}{9}x+...$

(5.21.25.1) $\sum\limits_{k=0}^{\infty}\frac{(2k)!}{[k!(2k+1)]^2}(1/8)^k=\frac{\pi}{8}2^{1/2}\log(2)+2^{-1/2}\lambda$ JN20p.96(2.14)

(5.21.28) $\sum\limits_{k=0}^{\infty}\frac{(2k)!}{[k!(2k+1)]^2}(-1/32)^k=2^{1/2}\{\frac{\pi^2}{12}-\frac{1}{4}[\log(2)]^2\}$ JN20p.100(3.8)

- $1+\frac{2}{5}x+...$

(5.22.4.1) $\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^2(2k+1)}(1/4)^k=\frac{1}{4}(2\pi)^{-1/2}[\Gamma(1/4)]^2$ AS18p.14

- $1+x+x^4+...$

(5.23.7.1) $\sum_{k=0}^{\infty}(-1)^k e^{-\pi k^2/4}=\frac{7}{8}+2^{-15/4}\pi^{-3/4}(2^{1/4}-1)\Gamma(1/4)$

- $1+x+x^2+\frac{5}{6}x^3+...$

(5.24.10.1) $\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^2(k+1)!}x^k=e^{2x}[I_0(2x)-I_1(2x)]$

This series is a special case of (10.8.7).

- $1+8x+216x^2+...$

(5.25.23.1) $\sum_{k=0}^{\infty}\frac{[(2k)!]^3}{(k!)^6}(-1)^k 2^{-6k}=\frac{2^{1/2}}{8\pi^3}[\Gamma(1/4)]^4$

- $3-5x^2+...$

(5.27.8.1) $\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{n(n+1)}=1-x^{-1/4}\pi^{-3/2}[2kk'K(k)]^{1/2}$

SR15p.406

where $x=\exp[-\pi K'(k)/K(k)]$ and $k'=(1-k^2)^{1/2}$.

(6.3.2.1) $\sum_{k=0}^{n}k^m(n-k)^r$

This sum is studied in SQ19p.90.

(6.6.31.1) $\sum_{k=0}^{n}\frac{(m-2n-1)_k}{k!(2n+1-2k)}=\frac{(-1)^n\pi(2n+1-m)!}{4\Gamma(n+3/2)\Gamma(n-m+3/2)}+$

$$+\begin{cases} S_1 \text{ for } m=1,...,n+1 \\ 0 \text{ for } m=0 \\ S_2 \text{ for } m=-1,-2,-3,... \end{cases}$$

where $S_1=(-1)^n\frac{(2n)!(-n-1/2)_m}{(n!)^2(-2n-1)_m}\sum_{k=0}^{m-1}\frac{(-n)_k}{(1/2-n)_k(2n+1-k)}$,

$$S_2=(-1)^{n+1}\frac{\Gamma(n+3/2)(2n+1-m)!}{2n!(n+1)!\Gamma(n-m+3/2)}\sum_{k=0}^{-m-1}\frac{(n+3/2)_k}{(n+2)_k(2n+2+k)}$$

For $m=0$, see (6.6.32).

$$(6.6.34.1)\quad \sum_{k=0}^{\infty} \frac{(-m-a)_k}{(a)_k(2k+1)} = \frac{\Gamma(a+m+1)}{\Gamma(a+m+3/2)}\times$$

$$\times\left[\frac{\pi\Gamma(a)}{4\Gamma(a-1/2)} + \frac{\Gamma(a-1/2)}{2\Gamma(a-1)}\sum_{k=0}^{m+1}\frac{(a-1/2)_k}{(a)_k(2a+k-2)}\right]$$

$m=-1,0,1,2,...$; $\mathrm{Re}(a) > -m/2$.

$$(6.6.34.2)\quad \sum_{k=0}^{\infty} \frac{(m-a)_k}{(a)_k(2k+1)} = \frac{\Gamma(a-m+1)}{\Gamma(a-m+3/2)}\times$$

$$\left[\frac{\pi\Gamma(a)}{4\Gamma a-1/2)} - \frac{\Gamma(a-1/2)}{2a\Gamma(a-1)}\sum_{k=2}^{m-1}\frac{(-a)_k}{(1/2-a)_k(2a-1-k)}\right]$$

$\mathrm{Re}(a) > m/2$; $m=3,4,5,....$ For $m=2$, see (6.6.35).

$$(6.6.35.1)\quad \sum_{k=0}^{\infty} \frac{(b-m)_k}{(a)_k(a+b+k)} = \frac{b(1-b)_m}{a(1+a)_m}[\psi(a+b)-\psi(a-b-2)-$$

$$-\psi(\tfrac{a+b+1}{2})+\psi(\tfrac{a-b-1}{2})+\sum_{k=0}^{m+1}\frac{(a-1)_k}{(-b)_k(b-a-k+2)}]$$

$\mathrm{Re}(a-b) > -m-4$; $m=-1,0,1,2,....$

$$(6.6.35.2)\quad \sum_{k=0}^{\infty} \frac{(b+m)_k}{(a)_k(a+b+k)} = \frac{b(-a)_m}{a(b)_m}[\psi(a+b)-\psi(a-b-2)-$$

$$-\psi(\tfrac{a+b+1}{2})+\psi(\tfrac{a-b-1}{2})\ -\sum_{k=1}^{m-2}\frac{(b+1)_k}{(2-a)_k(b-a+k+2)}]$$

$\mathrm{Re}(a-b) > m-4$; $m=3,4,5,....$ For $m=2$, see (6.6.36).

$$(6.6.38.1)\quad \sum_{k=0}^{n-1} \frac{(m)_k}{(a)_k(k+1)} = \frac{1-a}{m-1}[\psi(m-a+1)-\psi(2-a)]+$$

$$+\frac{n!\Gamma(a)}{(m-1)\Gamma(a+n-1)}\sum_{k=0}^{m-2}\frac{(n+1)_k}{k!(k-a+2)}$$

$m=2,3,4,....$ For $k=2$, see (7.1.1).

$$(6.6.39.1)\quad \sum_{k=0}^{n-1} \frac{k!}{(-n)_k(k+1)} = \frac{(-1)^n-1}{n+1} + \frac{n+1}{4}\left\{[1+2(-1)^n]\psi'(\tfrac{n}{2}+1)+\right.$$

$$\left.+[1-2(-1)^n]\psi'(\tfrac{n+1}{2})\right\}$$

For n even, see (6.6.40).

$$(6.6.43)\quad \sum_{k=0}^{\infty}(-1)^k\frac{(a)_k}{k!(kx+y)} = \frac{1}{x}B_{1/2}(y/x, a-y/x)$$

(6.6.44.1) $\sum_{k=0}^{\infty}(-1)^k\frac{(2a-m)_k}{k!(k+a)} = \frac{\Gamma(a)\Gamma(a-m)}{2\Gamma(2a-m)}[1-$

$-\pi^{-3/2}m\frac{\Gamma(a-1/2)}{\Gamma(a)}\sum_{k=0}^{[\frac{m-1}{2}]}\frac{(1-m)_{2k}}{k!(3/2-a)_k(2k+1)}2^{-2k}]$

$m = 1, 2, 3,....$ See (7.5.4.2)

(6.6.44.2) $\sum_{k=0}^{\infty}(-1)^k\frac{(2a+m)_k}{k!(k+a)} = \frac{\Gamma(a)\Gamma(a+m)}{2\Gamma(2a+m)}\times$

$\times[1+\pi^{-1/2}m\frac{\Gamma(a+m-1/2)}{\Gamma(a+m)}\sum_{k=0}^{[\frac{m-1}{2}]}\frac{(1-m)_{2k}}{k!(3/2-a-m)_k(2k+1)}2^{-2k}]$

$m = 1, 2, 3,....$ See (7.5.4.1).

(6.6.47.1) $\sum_{k=1}^{\infty}\frac{k!}{(a)_k k^2} = \psi'(a)$ MN46p.216

(6.6.47.2) $\sum_{k=1}^{\infty}\frac{(1-a)_k}{(a)_k(k^2-c^2)} = \frac{1}{2c^2} - \frac{[\Gamma(a)]^2}{2c\Gamma(a+c)\Gamma(a-c)}\pi\csc(\pi c)$

$\mathrm{Re}(a) > 0.$

(6.6.47.3) $\sum_{k=1}^{\infty}\frac{(1-a)_k}{(a)_k k^2} = \frac{1}{2}[\psi'(a)-\psi'(1)]$ SQ23p.258(80.9)*

(6.6.64.1) $\sum_{k=0}^{\infty}(-1)^k\frac{(a+b)_k(a+b+2k)}{k!(a+k)(b+k)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ AE82p.414

For $a+b =$ negative integer, see (6.6.65).

(6.6.83.1) $\sum_{k=1}^{\infty}\frac{(a)_k(-a)_k}{(b)_k(1-b)_k k} = 2\psi(1-b)-\psi(1+a-b)-$

$-\psi(1-a-b)+\frac{\sin\pi a}{\sin\pi b}\left[\beta\left(\frac{1+a-b}{2}\right)+\beta\left(\frac{1-a-b}{2}\right)\right]$ MN46p.215(2)

(6.6.83.2) $\sum_{k=0}^{\infty}\frac{(k!)^2}{(1+a)_k(2-a)_k(k+1)} = a(1-a)[\psi''(1-a)-$

$-2\pi\beta'(1-a)\csc(\pi a)]$ MN46p.216

(6.6.87.1) $\sum_{k=0}^{\infty}\frac{(a+m)_k(a-c+1)_k}{k!(c)_k(2k+a)} = \frac{\Gamma(c)\Gamma(a/2+m)}{4\Gamma(a+m)}\times$

$\times\left[\frac{\Gamma(a/2)\Gamma(c-a)}{[\Gamma c-a/2)]^2}+2^{2c-a-2}\frac{\Gamma(a/2-1/2)\Gamma(c-a+1/2)}{\pi(c-a)\Gamma(2c-a-1)}S\right]$

where

$$S = \left\{ \begin{array}{c} S_1 \text{ for } m = 1, 2, 3, ..., \\ 0 \text{ for } m = 0, \\ S_2 \text{ for } m = -1, -2, -3, ... \end{array} \right. ,$$

$$S_1 = \sum_{k=1}^{m} \frac{(a-1)_k (a-c)_k}{(a/2)_k (2a-2c+1)_k}, \quad S_2 = -\sum_{k=0}^{-m-1} \frac{(1-a/2)_k (2c-2a)_k}{(2-a)_k (c-a+1)_k}.$$

For $m = 0$, see (6.6.87).

(6.6.87.2) $$\sum_{k=0}^{\infty} \frac{(a)_k (2b-2a)_k}{k!(b)_k (k+2a-1)} = \frac{\pi^{1/2} \Gamma(a-1/2)\Gamma(b)\Gamma(a-b+1)}{2\Gamma(a)\Gamma(b-a+1/2)\Gamma(2a-b+1/2)}$$

(6.6.87.3) $$\sum_{k=0}^{\infty} \frac{(1-a)_k (b)_k}{k!(2b-a)_k (k+a)} = 2^{1-2b} \pi \frac{\Gamma(a)\Gamma 2b-a)}{\Gamma(a+1/2)\Gamma(b)\Gamma(b-a+1/2)}$$

(6.6.91.1) $$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(a+m)_k (kx+y)} = (-1)^m \frac{\Gamma(1-a)\Gamma(1-b)}{x(a-y/x)_m} \times$$

$$\times \left[\frac{\Gamma(y/x)}{\Gamma(1-a-m)\Gamma(1-b+y/x)} + \frac{\Gamma(a+m)}{\Gamma(2-a-m)\Gamma(a-b+m)} \sum_{k=0}^{m-1} \frac{(1-b)_k (y/x-a-m+1)_k}{k!(2-a-m)_k} \right]$$

$m = 0, 1, 2, ...$ For $m = 0$, see (6.6.31).

(6.6.93) $$\sum_{k=0}^{n} \frac{(-n)_k (a+m)_k}{k!(a)_k (kx+y)} = \frac{n!(a-y/x)_m}{y(a)_m (kx+y)} + \left\{ \begin{array}{l} 0 \text{ if } m \le n, \\ S \text{ if } m > n \end{array} \right.$$

where $S = (-1)^n \frac{n!\Gamma(a)\Gamma(a+m-y/x)}{x\Gamma(a+m)\Gamma(a+n+a-y/x)} \sum_{k=0}^{m-n-1} \frac{(n+1)_k (a+n)_k}{k!(a+n+1-y/x)_k}$

$m = 0, 1, 2, ...$

(6.6.93.1) $$\sum_{\substack{k=0 \\ k \ne s}}^{n} \frac{(-n)_k (a+m)_k}{k!(a)_k (s-k)} = \frac{(a+s)_m}{(a)_m} \{ \frac{(-n)_s}{s!} [\psi(n-s+1) - \psi(s+1) +$$

$$+\psi(a+s+m) - \psi(a+s)] - (-1)^n \frac{n!\Gamma(a+s)}{\Gamma(a+s+n+1)} \sum_{k=0}^{m-n-1} \frac{(n+1)_k (a+n)_k}{k!(a+n+s+1)_k} \}$$

$s = 0, 1, ..., n$; $m = 0, 1, 2, ...$ For $m \le n$, omit the sum in the right member. For $m = 0$, see (6.6.37). For $s = 0$, compare (6.6.94). For $s = 0$ and $a =$ integer, see (6.6.84).

(6.6.94.1) $$\sum_{\substack{k=0 \\ k \ne s}}^{n} \frac{(-n)_k (a)_k}{k!(a+m)_k (s-k)} = \frac{(a)_m}{(a+s)_m} \{ \frac{(-n)_s}{s!} [\psi(n-s+1) -$$

$$-\psi(s+1) + \psi(1-a-s) - \psi(1-a-s-m)] +$$

$$+ \frac{n!\Gamma(a+m-1)}{\Gamma(a+m+n)} \sum_{k=0}^{m-1} \frac{(n+1)_k (1-a-m-s)_k}{k!(2-a-m)_k} \}$$

$s=0,1,...,n$; $m=1,2,....$ For $m=0$, see (6.6.37).

(6.6.94.2) $$\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{k!(a+m)_k k}=\psi(1)-\psi(1-b)-$$

$$-\frac{\Gamma(a+m)\Gamma(1-b)}{\Gamma(a+m-b)}\sum_{k=0}^{m-1}\frac{(1-b)_k}{k!(a+m-k-1)}$$

$m=1,2,3,....$

(6.6.96.1) $$\sum_{k=1}^{\infty}\frac{k!(a)_k}{(b)_k(1+2a-b)_k k^2}=\frac{1}{2}\{\psi'(1+2a-b)+\psi'(b)-$$

$$-[\beta(1+2a-b)+\beta(b)]^2\}$$ MN46p.215(3)

(6.6.110.1) $$\sum_{k=0}^{n}\frac{(-n)_k(b+m)_k(a)_k}{k!(a+m)_k(b-n)_k(b=k-1)}=\frac{(a-b)m!n!(a)_m(a-b+1)_n}{a(a+1)_n(a-b+1)_m(1-b)_n(b)_m}\times$$

$$\times\sum_{k=0}^{m-1}\frac{(n+1)_k(b)_k(a-b+1)_k}{k!(k+1)!(a+n+1)_k}$$

$m,\ n=1,2,3,....$

(6.7.36.2) $$\sum_{k=1}^{n}\frac{(-n)_k(a)_k}{k!(b)_k}k^m=\frac{(b-a)_n}{(b)_n}\sum_{k=1}^{p}\frac{(-n)_k(a)_k}{(a-b-n+a)_k}(-1)^k\ \mathcal{S}_m^{(k)}$$

where $p=\min\{m,n\}$; m = 1, 2, 3,....

For $a=-n$ and $b=1$, see (6.7.37).

(6.7.36.3) $$\sum_{k=1}^{n}\frac{(-n)_k(b+r)_k}{k!(b)_k}k^m=\begin{cases}0 & \text{if } r<n-m,\\ S_1 & \text{if } n-m\le r<n,\\ S_2 & \text{if } r\ge n\end{cases}$$

where $S_1=(-1)^n\frac{n!}{(b)_r}\sum_{k=0}^{q}\frac{(-r)_k(b+n)_k}{k!}(-1)^k\mathcal{S}_m^{(k+n-r)}$,

$$S_2=\frac{(-r)_n}{(b)_n}\sum_{k=1}^{p}\frac{(-n)_k(b+r)_k}{(r-n+1)_k}(-1)^k\mathcal{S}_m^{(k)},$$

$p=\min\{m,\ n\}$; $q=\min\{r,\ m+r-n\}$; $m=1,2,3,...,\ r=0,1,2,....$

(6.7.40.1) $$\sum_{k=1}^{n}\frac{(-n)_k(b+r)_{ak}}{k!(b)_{ak}}k^m=\begin{cases}0 & \text{if } m<n-r,\\ S_1 & \text{if } n-r\le m\le n,\\ S_2 & \text{if } m\ge n\end{cases}$$

where $S_1=(-1)^n a^{n-m}\frac{n!}{(b)_r}\sum_{k=0}^{r+m-n}\frac{(n-m+1)_k}{k!}S_r^{(k+n-m)}\times$

$$\times\sum_{p=0}^{k}\frac{(-k)_p}{(n-m+1)_p}(-a)^p(b+r-1)^{k-p}\mathcal{S}_{p+n}^{(n)},$$

$S_2=(-1)^n\frac{n!}{(b)_r}\sum_{k=0}^{r}\underline{S}_r^{(k)}\sum_{p=0}^{k}\frac{(-k)_p}{p!}(-a)^p(b+r-1)^{k-p}\mathcal{S}_{p+n}^{(n)}$

$m=1,2,3,...;r=0,1,2,....$

(6.7.46.1) $\sum_{k=1}^{n}\frac{(-n)_k}{k!(k+a)}k^m=(-1)^m a^{m-1}n![\frac{1}{(a+1)_n}-a^{-n}\sum_{k=0}^{m-n-1}(-a)^{-k}\mathcal{S}_{n+k}^{(n)}]$

where $m>n$. See (52.1.1.1).

(6.8.17.1) $\sum_{n=1}^{\infty}\frac{n}{x^{2n}-1}=\frac{1}{24}-\frac{1}{6\pi^2}K(k)[3E(k)+(k^2-2)K(k)]$

Here and in the series (6.8.17.2) $x=\exp[-\pi K'(k)/K(k)]$ or, eqivalently, $k=[\theta_2(0,\,x)/\theta_3(0,x)]^2$, $k'=(1-k^2)^{1/2}$,and $K'(k)=K(k)$.

(6.8.17.2) $\sum_{n=1}^{\infty}\frac{n^3}{x^{2n}-1}=\frac{1}{240}-\frac{1}{15\pi^4}(1-k^2+k^4)[K(k)]^4$

where x and k are related as in 6.8.17.1).

(6.8.30) $\sum_{n=1}^{\infty}\frac{x^{2n}}{n(x^{2n}+1)}=\frac{1}{2}\log\left[\frac{k}{2\pi x^{1/2}}K(k)\right]$ SR15p.408(T1.5)

where x and k are related as in (6.8.17.1).

(6.8.31) $\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n(x^{2n}+1)}=\frac{1}{2}\log\frac{1-k'}{2kx^{1/2}}$ SR15p.408(T1.8)

where x and k are related as in (6.8.17.1).

(6.8.32) $\sum_{n=1}^{\infty}\frac{x^{2n}}{n(x^{2n}-1)}=-\frac{1}{6}\log\left[\frac{2kk'}{\pi^3x^{1/2}}K(k)\right]$ SR15p.408(T1.1)

where x and k are related as in (6.8.17.1).

(6.8.33) $\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n(x^{2n}-1)}=-\frac{1}{12}\log\frac{k^2}{16k'x}$ SR15p.408(T1.2)

where x and k are related as in (6.8.17.1).

(6.9.12.1) $\sum_{k=1}^{\infty}(\pm1)^k\frac{k^m}{x^{2k}\pm1}\quad(m=0,1,2,....)$

Series of these forms are studied in SR9p.34 and SR10p.192.

(6.9.12.2) $\sum_{k=1}^{\infty}\frac{k^{2m-1}}{e^{\pi k}-(-1)^k}=\frac{1}{4m}[(2^{2m}-1)B_{2m}-\varepsilon_m]$ AKp.177

where $\varepsilon_1=1$ and $\varepsilon_m=0$ for $m=2,3,....$

(6.9.12.3) $\sum_{k=1}^{\infty} (\pm)^k \frac{k^m x^k}{x^{2k} \pm 1} \quad (m = 0, 1, 2, ...)$

Series of these forms are studied in SR9p.34 and SR10p.192.

(6.9.15.1) $\sum_{k=0}^{\infty} (\pm 1)^k \frac{(2k+1)^m}{x^{2k+1} \pm 1} \quad (m = 0, 1, 2, ...)$

Series of these forms are studied in SR9p.34 and SR10p.192.

(6.9.15.2) $\sum_{k=0}^{\infty} (\pm)^k \frac{(2k+1)^m x^k}{x^{2k+1} \pm 1} \quad (m = 0, 1, 2, ...)$

Series of these forms are studied in SR9p.34 and SR10p.192.

(6.11.11.1) $\sum_{k=0}^{\infty} \frac{(a)_k}{k!(k+m)} x^k = \frac{(m-1)!}{(1-a)_m} x^{-m} \times \left[1 - (1-x)^{1-a} \sum_{k=0}^{m-1} \frac{(1-a)_k}{k!} x^k\right]$

$(m = 1, 2, 3, ...)$

(6.11.39) $\sum_{k=0}^{\infty} \frac{(c)_{ak}}{k!(c)_{ak-k}(ak+b)} x^k = \frac{1}{b}(1+y)^{c-1} \times$

$\times \, {}_1F_1 \left(b - c + 1; b/a + 1; \frac{y}{1+y}\right)$ AP46p.75

where $x = y(1+y)^{-a}$

(6.13.7.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} (m+k)^r x^k$

This sum is studied in SN20p.400.

(7.4.1.1) $\sum_{k=1}^{n} \frac{(b)_k (c+m)_k}{(a)_k (c)_k} = \frac{\Gamma(a)\Gamma(c)}{\Gamma(b)\Gamma(c+m)} \times$

$\times [\frac{\Gamma(b+m+n+1)}{\Gamma(a+n)} \sum_{k=0}^{m} \frac{(-m)_k (c-b)_k}{k!(-b-m-n)_k (b-a+m-k+1)} -$

$- \frac{\Gamma(b+m+1)}{\Gamma(a)} \sum_{k=0}^{m} \frac{(-m)_k (c-b)_k}{k!(-b-m)_k (b-a+m-k+1)}]$

(7.4.6.1) $\sum_{k=0}^{n} \frac{(a)_k (n+1)_k}{k!(2a+n+1)_k} = \frac{(2a+1)_n}{(a+1)_n}$

For a = negative integer, see SQ37p.98(1).

(7.4.10.1) $\sum_{k=0}^{\infty} \frac{k!(a+b-1/2)_k}{(2a)_k (2b)_k} = \frac{2(2a-1)(2b-1)}{2a+2b-3} [\beta(2a-1) + \beta(2b-1)]$

MN46p.215(1)

(7.4.10.2) $\sum_{k=0}^{n} \frac{(2k)!}{(-2n)_{2k}} = (2n+1) 2^{-2n-2} \sum_{k=1}^{2n+2} \frac{2^k}{k}$ SQ25p.273(82-1)

(7.4.14.1) $$\sum_{k=0}^{n} \frac{(a)_k (m)_k}{k!(b)_k} = \frac{(-1)^{m-1}\Gamma(a+n+1)}{(m-1)!\Gamma(a-m+1)} \sum_{k=0}^{m-1} \frac{(1-m)_k (a+n+1)_k}{k!(a-m+1)_k (k+a-b+1)} + \frac{(1-b)_m}{(a-b+1)_m}$$

(7.4.22.1) $$\sum_{k=0}^{n} \frac{(b)_k (a-b-1)_k}{k!(a)_k} = \frac{(a-b)_n (b+1)_n}{n!(a)_n}$$

(7.5.4.1) $$\sum_{k=0}^{\infty} (-1)^k \frac{(b)_k (b-a-m)_k}{k!(a)_k} = \frac{\Gamma(a)}{2\Gamma(b)} \sum_{k=0}^{m+1} (-1)^k \frac{(-m-1)_k \Gamma(b/2+k/2)}{k!\Gamma(a-b/2+k/2)}$$

$m = 0, 1, 2, \ldots$; $\mathrm{Re}(b-a) \leq m/2$. For $b = 2a + m - 1$, see (6.6.44.2).

For $m = 0$, see (7.5.5).

(7.5.4.2) $$\sum_{k=0}^{n} (-1)^k \frac{(b)_k (b-a+m)_k}{k!(a)_k} = \frac{\Gamma(a)\Gamma(b-a+1)}{2\Gamma(b)\Gamma(b-a+m)} \times \sum_{k=0}^{m-1} \frac{(1-m)_k \Gamma[(b-m-k-1)/2]}{k!\Gamma[a+(1-b-m-k)/2]}$$

$m = 1, 2, 3, \ldots$.For $b = 2a - m - 1$, see (6.6.44.1). For $m = 1$, see (7.5.6).

(7.8.5.1) $$\sum_{k=0}^{n} \frac{(-n)_k (a+n-1/2)_k (b+n-1/2)_k (a+b+n-3/2)_k}{k!(a)_k (b)_k (a+b+2n-1/2)_k} =$$

$$= \frac{(1/2)_n (b-a+1/2)_n (a-b+1/2)_n (a+b-1/2)_{2n}}{(a)_n (b)_n (a+b-1/2)_{3n}}$$ SQ35p.646

(7.8.10.1) $$\sum_{k=0}^{[n/2]} \frac{[(a)_k]^2 (-n)_{2k}}{k!(-n)_k (2a+1)_{2k}} = \frac{a(a+1)_n}{(2a+1)_n} [\psi(a+n+1) - \psi(a)]$$ AJp.6

(7.11.3.1) $$\sum_{k=0}^{[n/m]} \frac{(r-n)_{km}}{(-n)_{km}}$$

This sum is studied in MN32p.1271.

(7.11.5.1) $$\sum_{k=0}^{n} (-1)^{km} \frac{(km)!}{(-mn)_{km}} = \frac{mn+1}{m} 2^m \sum_{k=0}^{mn+m-1} \frac{2^{-k}}{mn+m-k} \Big\{ \frac{1}{2} +$$

$$+ \sum_{s=1}^{\left[\frac{m-1}{2}\right]} (-1)^s \cos(\pi sk/m)\, [\sec(\pi s/m)]^{k-m+2} \Big\}$$

(7.11.11.1) $$\sum_{k=0}^{n} \frac{(-n)_k (a+m)_{kx}}{k!(a)_{kx}} = \frac{n!}{(a)_m} (-x)^n \sum_{k=0}^{m-n} \frac{(n+1)_k}{k!} \left(\frac{1-a-m}{x}\right)_k \times$$

$$\times (-x)^k \sum_{r=0}^{m-n-k} x^r \underline{S}_m^{(r+k+n)} \overline{S}_{r+k+n}^{(k+n)}$$

$m \geq n$. See (52.4.9).

This is a change of notation from the book. $\underline{S}_m^{(r+k+n)}$ and $\overline{S}_{r+k+n}^{(k+n)}$ are Stirling numbers of the first and second kind, resp.

(10.8.7.1) $\sum_{k=0}^{\infty} \frac{(1/2)_k}{k!(m)_k} x^k = e^{x/2}\left[2\sum_{k=0}^{m-1} \frac{(1-m)_k}{(m)_k} I_k(\tfrac{x}{2}) - I_0(\tfrac{x}{2})\right]$

$m = 1, 2, 3, ...$

(10.8.10.1) $\sum_{k=0}^{\infty} \frac{(m-\frac{1}{2})_k}{k!(m)_k} x^k = e^{x/2}\left[2\sum_{k=0}^{m-1} (-1)^k \frac{(1-m)_k}{(m)_k} I_k(\tfrac{x}{2}) - I_0(\tfrac{x}{2})\right]$

$m = 1, 2, 3, ...$

(10.11.4) $\sum_{k=0}^{\infty} \frac{(a+n)_k (1-a-n)_k}{k!(a)_k} x^k = \frac{n!}{(2a-1)_n} (1-x)^{a-1} C_n^{(a-1/2)}(1-2x)$

$= \frac{n!}{(a)_n} (-1)^n (-x/2)^{n/2} (1-x)^{a-1+n/2} C_n^{(1-a-n)} \left[\frac{1-2x}{2}(x^2-x)^{-1/2}\right]$

(10.12.2) $\sum_{k=0}^{\infty} \frac{(-a)_k (a+2)_k}{k!(a-n+1)_k} x^k = \frac{(n+1)!\Gamma(2a-2n+1)}{2(a+1)(2a-2n+1)\Gamma(2a-n+1)} (1-x)^{a-n-1} \times$

$\times \left[(2a-n) C_{n+1}^{(a-n-1/2)}(1-2x) + 2(2a-2n-1) C_n^{(a-n+1/2)}(1-2x)\right]$

$= \frac{\Gamma(a-n+1)}{\Gamma(a+2)} (n+1)! (-x)^{n/2} (1-x)^{a-n/2} \{ C_n^{(-a)} [\frac{1-2x}{2}(x^2-x)^{-1/2}] +$

$+ \left(\frac{x}{x-1}\right)^{1/2} C_{n+1}^{(-a)} [\frac{1-2x}{2}(x^2-x)^{-1/2}] \}$

(10.13.2) $\sum_{k=0}^{\infty} \frac{(-a)_k (a+3)_k}{k!(a-n+2)_k} x^k = \frac{(a-n+1)(n+1)!\Gamma(2a-2n+1)}{2x(a+1)(a+2)\Gamma(2a-n+2)} (1-x)^{a-n-1} \times$

$\times \{ (2a-n+1) C_n^{(a-n+1/2)}(1-2x) +$

$+ [2x(a+1)-n-1] C_{n+1}^{(a-n+1/2)}(1-2x) \}$

$= \frac{\Gamma(a-n+2)}{2\Gamma(a+3)} (-1)^n (n+1)! (-x)^{n/2-1} (1-x)^{a-1-n/2} \times$

$\times \{ \left(2x - \frac{n+1}{a+1}\right) [x(x-1)]^{1/2} C_{n+1}^{(-a-1)} [\frac{1-2x}{2}(x^2-x)^{-1/2}] -$

$- C_n^{(-a)} [\frac{1-2x}{2}(x^2-x)^{-1/2}] \}$

(10.15.2) $\sum_{k=0}^{\infty} \frac{(-n/2)_k (c+n/2-3/2)_k}{k!(c)_k} x^k = \frac{n!\Gamma(2c-1)}{(n-1)\Gamma(2c+n-3)} \times$

$\times \{ \frac{1}{2c+n-2} C_n^{(c-1/2)} [(1-x)^{1/2}] - \frac{1}{2c-3} C_n^{(c-3/2)} [(1-x)^{1/2}] \}$

$= (-1)^n (-x/4)^{n/2} \frac{n!}{(n-1)(c)_n} \{ (2c+n-3) C_n^{(1-c-n)} [(1-1/x)^{1/2}] -$

$-2(c+n-1) C_n^{(2-c-n)} [(1-1/x)^{1/2}] \}$

(10.16.4) $\sum_{k=0}^{\infty} \frac{(-n/2)_k (c+n/2-1/2)}{k!(c)_k} x^k = \frac{n!}{(2c-1)_n} C_n^{(c-1/2)} [(1-x)^{1/2}]$

$= (-1)^n(-x/4)^{n/2}\frac{n!}{(c)_n}C_n^{(1-c-n)}[(1-1/x)^{1/2}]$

For n even, see ES10.9(21), SZ(4.7.30). For $c = 1/2 - n$,see (10.16.7).

(10.16.5) $$\sum_{k=0}^{\infty}\frac{(-n/2)_k(n/2+1/2)_k}{(k!)^2} = P_n[(1-x)^{1/2}]$$

(10.18.2) $$\sum_{k=0}^{\infty}\frac{(1/2-n/2)_k(c+n/2)_k}{k!(c)_k}x^k = \frac{n!}{(2c-1)_n}(1-x)^{-1/2}\times C_n^{(c-1/2)}[(1-x)^{1/2}]$$

For n odd, see ES10.9(22), SZ(4.7.30).

(10.18.2.1) $$\sum_{k=0}^{\infty}\frac{(1/2-n/2)_k(1+n/2)_k}{(k!)^2}x^k = (1-x)^{-1/2}P_n[(1-x)^{1/2}]$$

(10.18.4) $$\sum_{k=0}^{\infty}\frac{[(1/2-n/2)_k]^2}{k!(1/2-n)_k}x^k = \frac{(n!)^2}{(2n)!}(-4x)^{n/2}\times(1-x)^{-1/2}P_n[(1-1/x)^{1/2}]$$

(10.19.2) $$\sum_{k=0}^{n}\frac{(-n)_k(2c+n-2)_k}{k!(c)_k}x^k = \frac{(c-1)n!}{2x(1-c-n)(2c-3)_n}\times$$

$$\times\left[\frac{n+1}{2c+n-3}C_{n+1}^{(c-3/2)}(1-2x) - C_n^{(c-3/2)}(1-2x)\right]$$

$$= \frac{n!}{2x(c)_n}(-x)^n(x^2-x)^{n/2}\left\{C_n^{(2-c-n)}[(1/2-x)(x^2-x)^{-1/2}]+\right.$$

$$\left.+\frac{n+1}{c+n-1}(x^2-x)^{1/2}C_{n+1}^{(1-c-n)}[(1/2-x)(x^2-x)^{-1/2}]\right\}$$

See (10.29.9.1).

(10.21.2) $$\sum_{k=0}^{n}\frac{(-n)_k(2a+n-1)_k}{k!(a)_k}x^k = \frac{n!}{(2a-1)_n}C_n^{(a-1/2)}(1-2x)$$

$$= (-1)^n\frac{n!}{(a)_n}(x^2-x)^{n/2}C_n^{(1-a-n)}[(1/2-x)(x^2-x)^{-1/2}]$$

AB15.4.5, ER3.15(2)*, ES10.9(20), RAp.279(15), SZ(4.7.6)

When summed in reverse order, this sum is equaivalent to (10.30.4).

(10.22.2) $$\sum_{k=0}^{n}\frac{(-n)_k(2c+n)_k}{k!(c)_k} = \frac{n!}{2(1-x)(2c-1)_n}\left[\frac{n+1}{2c+n-1}C_{n+1}^{(c-1/2)}(1-2x)+\right.$$

$$\left.+\,C_n^{(c-1/2)}(1-2x)\right]$$

$$= \frac{(-1)^n xn!}{2(c)_n}(x^2-x)^{n/2-1}\left\{\frac{n+1}{n+c}(x^2-x)^{1/2}C_n^{(-c-n)}\left[\frac{1-2x}{2}(x^2-x)^{-1/2}\right]-\right.$$

$$\left.-\,C_n^{(1-c-n)}\left[\frac{1-2x}{2}(x^2-x)^{-1/2}\right]\right\}$$

When summed in reverse order, this sum is equivalent to (10.31.2).

$$(10.23.2)\qquad \sum_{k=0}^{n}\frac{(-n)_k(2-c-n)_k}{k!(c)_k}x^k=\frac{n!\Gamma(2c-1)(1-x)^n}{(c+n-1)\Gamma(2c+n-2)(1+x)}\times$$

$$\times\left[C_n^{(c-1/2)}\left(\tfrac{1+x}{1-x}\right)-\frac{(2xc+n-3)(1-x)}{2(2c-3)}C_n^{(c-3/2)}\left(\tfrac{1+x}{1-x}\right)\right]$$

$$=(-1)^n\frac{n!x^{n/2}}{(c)_n(x+1)}\left[(x-1)C_n^{(2-c-n)}\left(\tfrac{x+1}{2x^{1/2}}\right)+\frac{2c+n-2}{c+n-1}C_n^{(1-c-n)}\left(\tfrac{x+1}{2x^{1/2}}\right)\right]$$

See (10.25.1).

$$(10.23.3)\qquad \sum_{k=0}^{\infty}\frac{(a+n)_k(2a+n-2)_k}{(k!(a)_k}x^k=\frac{(a-1)n!}{(2a-3)(a+n-1)(2a-2)_n}(1-x)^{1-2a-n}\times$$

$$\times\left[2(2a-3)C_n^{(a-1/2)}\left(\tfrac{1+x}{1-x}\right)-(n+1)(1-x)C_{n+1}^{(a-3/2)}\left(\tfrac{1+x}{1-x}\right)\right]$$

$$=\frac{n!}{(a+n-1)(a)_n}(-1)^nx^{n/2}(1-x)^{1-2a-2n}\times$$

$$\times\left[(2a+n-2)C_n^{(1-a-n)}\left(\tfrac{1+x}{2x^{1/2}}\right)+(n+1)C_{n+1}^{(1-a-n)}\left(\tfrac{1+x}{2x^{1/2}}\right)\right]$$

$$(10.24.6)\qquad \sum_{k=0}^{\infty}\frac{(a+n)_k(2a+n-1)_k}{k!(a)_k}x^{2k}=\frac{n!}{(2a-1)_n}(1-x^2)^{1-2a-n}\times C_n^{(a-1/2)}\left(\frac{1+x^2}{1-x^2}\right)$$

$$=\frac{n!}{(a)_n}(-x)^n(1-x^2)^{1-2a-2n}C_n^{(1-a-n)}\left(\frac{1+x^2}{2x}\right)$$

$$(10.25.1)\qquad \sum_{k=0}^{n}\frac{(-n)_k(-a-n)_k}{k!(a)_k}x^k=\frac{n!(x-1)^n}{(2a)_n(x+1)}\left[2xC_n^{(a+1/2)}\left(\tfrac{x+1}{x-1}\right)-\right.$$

$$\left.-\frac{2a+n-1}{2a-1}(x-1)C_n^{(a-1/2)}\left(\tfrac{x+1}{x-1}\right)\right]$$

$$=(-1)^nx^{n/2}\frac{n!}{(a)_n(x+1)}\left[(1-x)C_n^{(1-a-n)}\left(\tfrac{1+x}{2x^{1/2}}\right)+\right.$$

$$\left.+\frac{2a+n}{a+n}xC_n^{(-a-n)}\left(\tfrac{1+x}{2x^{1/2}}\right)\right]$$

$$(10.27.3.1)\qquad \sum_{k=0}^{\infty}\frac{(2c+n-1)_{2k}}{k!(c)_k}x^k=\frac{n!}{(2c-1)_n}(1-4x)^{1/2-c-n/2}\times$$

$$\times C_n^{(c-1/2)}[(1-4x)^{-1/2}]$$

$$=(-1)^n\frac{n!}{(c)_n}x^{n/2}(1-4x)^{1/2-c-n}C_n^{(1-c-n)}(\tfrac{1}{2}x^{-1/2})$$

$$(10.29.9.1)\qquad \sum_{k=0}^{n}\frac{(-n)_k(a)_k}{k!(2a+1)_k}x^k=\frac{n!\Gamma(a+1/2)}{\Gamma(a+n+3/2)}\left(\tfrac{x}{4}\right)^{n+1}\times$$

$$\times\left[(n+1)C_{n+1}^{(-a-n-1/2)}\left(1-\tfrac{2}{x}\right)+(2a+n+1)C_n^{(-a-n-1/2)}\left(1-\tfrac{2}{x}\right)\right]=$$

$$=\frac{n!}{2(2a+1)_n}(-1)^n(1-x)^{n/2}\left\{xC_n^{(a+1)}\left[\tfrac{x-2}{2}(1-x)^{-1/2}\right]-\right.$$

$- \frac{n+1}{a}(1-x)^{1/2}C_{n+1}^{(a)}\left[\frac{x-2}{2}(1-x)^{-1/2}\right]\Big\}$

When summed in reverse order, this series is equivalent to (10.19.2).

(10.30.7) $\sum_{k=0}^{\infty} \frac{(a)_k(2a+n)_k}{k!(2a)_k} x^k = \frac{n!}{(2a)_n}(1-x)^{-a-n/2} C_n^{(a)}\left[\frac{1-x/2}{(1-x)^{1/2}}\right] =$

$= (-1)^n \frac{n!}{(a+1/2)_n}\left(\frac{x}{4}\right)^n (1-x)^{-a-n} C_n^{(1/2-a-n)}\left(\frac{2-x}{x}\right)$

(10.31.2) $\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(2a-1)_k} x^k = \frac{n!\Gamma(a-1/2)}{\Gamma(a+n+1/2)(1-x)} \times$

$\times\left[(n+1)C_{n+1}^{(1/2-a-n)}\left(1-\frac{2}{x}\right) - (2a+n-1)C_n^{(1/2-a-n)}\left(1-\frac{2}{x}\right)\right]$

$= \frac{1}{2}(-1)^{n+1}\frac{n!}{(2a-1)_n}(1-x)^{n/2-1}\Big\{xC_n^{(a)}\left[\left(\frac{x}{2}-1\right)(1-x)^{-1/2}\right] +$

$+ \frac{n+1}{a-1}(1-x)^{1/2}C_{n+1}^{(a-1)}\left[\left(\frac{x}{2}-1\right)(1-x)^{-1/2}\right]$

See (10.22.2).

(10.33.1.1) $\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(2k)!} x^{2k} = \begin{cases} S_1 \text{ for } 0 < x^2 < 4, \\ S_2 \text{ for } |\arg(x^2)| < \pi \end{cases}$

where

$S_1 = \frac{1}{2}\pi^{-1/2}(4-x^2)^{1/4-a/2-b/2}\Gamma(1-a)\Gamma(1-b)\times$

$\times\left[P_{b-a-1/2}^{(a+b-1/2)}\left(\frac{x}{2}\right) + P_{b-a-1/2}^{(a+b-1/2}\left(-\frac{x}{2}\right)\right],$

$S_2 = \frac{1}{2}\pi^{-1/2}(x^2-4)^{1/4-a/2-b/2}\Gamma(1-a)\Gamma(1-b)\times$

$\times\left[\widetilde{P}_{b-a-1/2}^{(a+b-1/2)}\left(\frac{x}{2}\right) + \widetilde{P}_{b-a-1/2}^{(a+b-1/2)}\left(-\frac{x}{2}\right)\right]$

where $\widetilde{P}$ is the associated Legendre function of the first kind and P is the associated Legendre function of the first kind on the cut.

(10.33.1.2) $\sum_{k=0}^{\infty} \frac{(a)_k(n+1/2)_k}{(2k)!} x^{2k} = \frac{n!}{(a+1/2)_n} 2^{2a+2n}(4-x^2)^{-a-n}\times$

$\times C_{2n}^{(1/2-a-n)}\left(\frac{x}{2}\right)$

$= \frac{n!}{(1-a)_n}2^{2a}(4-x^2)^{-a}C_{2n}^{(a-n)}[x(x^2-4)^{-1/2}]$

(10.34.1.1) $\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(2k+1)!} x^{2k+1} = \begin{cases} S_1 \text{ for } 0 < x < 2, \\ S_2 \text{ for } x \text{ not in the interval } (0,2) \end{cases}$

where

$S_1 = \frac{1}{2}\pi^{-1/2}\Gamma(1-a)\Gamma(1-b)(4-x^2)^{3/4-a/2-b/2}\times$

$\times \left[P_{b-a-1/2}^{(b+a-3/2)}\left(-\frac{x}{2}\right) - P_{b-a-1/2}^{(b+a-3/2)}\left(\frac{x}{2}\right)\right],$

$S_2 = \frac{1}{2}\pi^{-1/2}\Gamma(1-a)\Gamma(1-b)(x^2-4)^{3/4-a/2-b/2}\times$

$\times \left[\widetilde{P}_{b-a-1/2}^{(b+a-3/2)}\left(-\frac{x}{2}\right) - \widetilde{P}_{b-a-1/2}^{(b+a-3/2)}\left(\frac{x}{2}\right)\right]$

(10.34.1.2) $\sum_{k=0}^{\infty} \frac{(a)_k (n+3/2)_k}{(2k+1)!} x^{2k+1} = -\frac{n!\Gamma(a-1/2)}{\Gamma(a+n+1/2)} \left(1-\frac{x^2}{4}\right)^{-a-n} \times C_{2n+1}^{(1/2-a-n)}\left(\frac{x}{2}\right)$

$= (-1)^n \ i\ 2^{2a-1}(4-x^2)^{1/2-a} \frac{n!\Gamma(a-n-1)}{\Gamma(a)} C_{2n+1}^{(a-n-1)}[x(x^2-4)^{-1/2}]$

(10.37.6.1) $\sum_{k=0}^{\infty} \frac{(1/2)_k}{k!(a+1)_k(-a)_k}(-1)^k x^{2k} = -1 - \pi x \csc(\pi a) J_a(x) J_{-a-1}(x)$

(10.37.10.1) $\sum_{k=0}^{\infty} \frac{(a+1/2)_k}{k!(2a)_k(a+1)_k}(-1)^k x^{2k} = \left(\frac{x}{2}\right)^{1-2a} \Gamma(a)\Gamma(a+1) J_a(x) J_{a-1}(x)$

(10.49.21.1) $\sum_{k=0}^{\infty} \frac{\Gamma(ak+b)}{k!} x^k = \int_0^{\infty} t^{b-1} \exp(xt^a - t)\ dt \qquad (0 \le a < 1)$

(11.3.1.1) $\sum_{k=0}^{\infty} \frac{(-1)^k}{e^{(2k+1)a} \pm 1}$

Series of these forms are summed for $a = \pi$, $3^{1/2}\pi$, *and* $3^{-1/2}\pi$ in SR9p.43.

(12.1.4) $\sum_{k=0}^{p-1} \exp[\pi i k^2(k^2+n)/p] = p^{1/2} \exp\left[\frac{\pi i}{4p}(p-n^2)\right]$ AKp.157

$p =$ prime, $n =$ odd integer.

(12.1.5) $\sum_{k=0}^{p-1} (-1)^k \exp(\pi i k^4/p) = p^{1/2} \exp\left[\frac{\pi i}{4}(1-p)\right]$ AKp.157

$p =$ prime.

(13.1.13) $\sum_{n=0}^{m} \prod_{k=0}^{n} \frac{1}{x^{2^k}-1} = \frac{1}{2}(x^{2^{m+1}}+1) \prod_{k=0}^{m} \frac{1}{x^{2^k}-1} - \frac{x+1}{2}$ GQ

(13.1.14) $\sum_{n=1}^{m-1} \prod_{k=1}^{n} (x^{2^{-k}}-1) = \frac{x-1}{2} - \frac{1}{2}(x^{2^{-m}}+1) \prod_{k=1}^{m} (x^{2^{-k}}-1)$ GQ

$m = 2, 3, 4, \ldots.$

(14.3.6.1) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 n^2 - m^2} \sin(kx) = \frac{1}{mn} \sum_{k=1}^{n-1} \sin[(x+k\pi)m/n] \times$

$\times \frac{\sin(\pi k m/n)}{\sin(\pi m/n)} \log\left|\frac{\sin\{[x+(2k+1)\pi]/(2n)\}}{\sin\{[x+(2k-1)\pi]/(2n)\}}\right|$ SQ29p.305(86-10)

(14.3.6.2) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 a^2 - b^2} \sin(\pi k m/n) = -\frac{\pi \sin[\pi m b/(na)]}{2ab \sin(\pi b/a)} +$

$+\frac{1}{2nab}\sum_{k=1}^{n-1}\sin(\pi km/n)\left\{\psi\left[\frac{n+b/a-(-1)^m k}{2n}\right]-\psi\left(\frac{b/a+k}{n}\right)\right\}$ SQ29p.304(86-10)

(14.14.13.1) $\sum_{k=-\infty}^{\infty}\frac{1}{(b)_{ak}(c)_{-ak}}\sin(kx+y)=\frac{\Gamma(b)\Gamma(c)}{a\Gamma(b+c-1)}2^{b+c-2}\times$

$\times\sum_{k=m}^{n}\sin\left[\frac{b-c}{2a}(2\pi k+x)-y\right]\left[\cos\left(\frac{2\pi k+x}{2a}\right)\right]^{b+c-2}$

where $m=-\left[\frac{\pi a+x}{2\pi}\right]$ and $n=\left[\frac{\pi a-x}{2\pi}\right]$, $\mathrm{Re}(b+c)>1$.

If $x=\pm\pi a$, then $\mathrm{Re}(b+c)>2$. See OA(5.3).

(16.1.15) $\sum_{k=1}^{\infty}\frac{1}{k}\sin(k^{1/2}x)=\pi+\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}(-1)^k\zeta\left(\frac{1}{2}-k\right)$

SQ37p.443(94-12)

(17.9.13.1) $\sum_{n=1}^{\infty}\frac{q^n}{n[1+(-q)^n]}\cos(nx)=\frac{1}{2}\log[cn(u)]-$

$-\frac{1}{2}\log[2q^{1/4}(k'k)^{1/2}\cos(x/2)]$ OA(2.21)

where $\pi u=xK(k)$ and $q=\exp[-\pi K'(k)/K(k)]$.

(17.9.21) $\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(2n+1)(1-q^{4n+2})}\cos[(2n+1)x]=\frac{1}{4}\log[dn(u)]-$

$-\frac{1}{8}\log(k')$ OA(2.22)

where $\pi u=xK(k)$ and $q=\exp[-\pi K'(k)/K(k)]$.

(17.14.9.1) $\sum_{k=0}^{\infty}\frac{(1/4)_k}{[(3/4)_k]^2}\cos(kx)=\frac{1}{2}+\frac{[\Gamma(3/4)]^2}{\Gamma(1/4)}[K(\sin\frac{x}{4})+$

$+K(\cos\frac{x}{4})]$

$0<x<2\pi$. See OA(4.1), (4,2), (4.3)*, and (4.4).

(17.14.9.2) $\sum_{k=0}^{\infty}\frac{(3/4)_k}{[(5/4)_k]^2}\cos[(2k+1)x]=\frac{[\Gamma(5/4)]^2}{\Gamma(3/4)}[K(\cos\frac{x}{2})-$

$-K(\sin\frac{x}{2})]$

$0<x<\pi$. See OA(4.1), (4.2), (4.3)*, and (4.4).

(17.14.16.1) $\sum_{k=-\infty}^{\infty}\frac{\cos(kx+y)}{(b)_{ka}(c)_{-ka}}=\frac{\Gamma(b)\Gamma(c)}{a\Gamma(b+c-1)}2^{b+c-2}\times$

$\times\sum_{k=m}^{n}\cos\left[\frac{(b-c)(2\pi k+x)}{2a}+y\right]\left(\cos\frac{2\pi k+x}{2a}\right)^{b+c-2}$

where $m=-\left[\frac{\pi a+x}{2\pi}\right]$ and $n=\left[\frac{\pi a-x}{2\pi}\right]$. See OA(5.3).

$\mathrm{Re}(b+c)>1$. If $x=\pm\pi a$, then $\mathrm{Re}(b+c)>2$.

(19.1.10) $\sum_{k=1}^{\infty}\frac{1}{k}\cos(k^{1/2}x)=-C-2\log(x)+\sum_{k=1}^{\infty}\frac{x^{2k}}{(2k)!}\zeta(1-k)$

SQ37p.443(94-12)

(23.1.4) $\sum_{k=1}^{\infty}\frac{(-1)^{k(n+1)}}{k^3}\csc(k\pi mp^{1/2})=\frac{\pi^3}{72(x^2\delta-1)}\left[\frac{7}{10}\left(x^3+\frac{1}{x}\right)+x\right]$ AE82p.680

where $x=n+mp^{1/2}$; m, n, and p are integers with $p>1$ and square free, and $n^2-pm^2=\delta$ with $\delta=\pm1$.

(23.1.5) $\sum_{k=1}^{\infty}\frac{1}{k^{2m+1}}\csc(2^{1/2}\pi k)=\frac{(-1)^m\pi^{2m+1}}{2a(1+a^{2m})}\times$

$\times\sum_{k=0}^{m}\frac{(2^{2k}-2)(2^{2m-2k+2}-2)}{(2k)!(2m-2k+2)!}a^{2k}B_{2k}B_{2m-2k+2}$ AE82p.681

where $a=2^{1/2}-1$ and $m=1,2,3,....$

(24.1.10) $\sum_{k=1}^{n-1}k\ \left[csc\left(\frac{k\pi}{n}\right)\right]^2=\frac{n}{6}(n^2-1)$ SQ29p.132(86.5)

(24.1.11) $\sum_{k=1}^{\infty}(\pm1)^k[\csc(kx)]^m$ $m=1,2,3,....$

These series are studied in SR10p.192.

(24.1.12) $\sum_{n=0}^{\infty}\{\csc[(2k+1)x]\}^{2m}$ $m=1,2,3,....$

This series is studied in SR10p.198.

(29.1.6) $\sum_{k=1}^{\infty}\frac{1}{k^3}\cot\left(k\pi\frac{1+5^{1/2}}{2}\right)=-\frac{1}{45}\pi^3 5^{-1/2}$ AE82p.681

(34.1.12) $\sum_{k=-\infty}^{\infty}\frac{1}{\Gamma(c+kx+y)\Gamma(c-kx-y)}\cos(2\pi kt)\cos[\pi(kx+y)]=$

$=\frac{2^{2c-3}}{x\Gamma(2c-1)}\sum_{k=-[x+t]}^{[x-t]}\cos[2\pi(k+t)y/x]|\sin[\pi(k+t)/x]|^{2c-2}$

If $t=\pm x$ or if $t=0$, then $\mathrm{Re}(c)>1$. If $x\pm t$ is an integer or zero, then the term in the sum in the member for $k=x\pm t$ must be halved. See OA(5.2).

(41.2.16.1) $\sum_{k=1}^{n}\frac{\cos(\pi km/n)}{\cos x-\cos(\pi k/n)}=\frac{1}{2}\left[\frac{1}{1-\cos x}+\frac{(-1)^m}{1+\cos x}\right]-$

$-n\csc x\ \csc(nx)\cos\left\{n-m+2nx\left[\frac{m}{2n}\right]\right\}$

(41.2.16.2) $\sum_{k=1}^{n} \frac{\cos(\pi km/n)}{\cosh x - \cos(\pi k/n)} = \frac{1}{2}\left[\frac{1}{1- \cosh x} + \frac{(-1)^m}{1+ \cosh x}\right] +$

$+n \operatorname{csch} x \operatorname{csch}(nx) \cosh\left\{n - m + 2nx\left[\frac{m}{2n}\right]\right\}$

(43.4.5) $\sum_{n=1}^{\infty} \frac{1}{a + \cosh nx} = \frac{y}{x} - \frac{1}{2(a+1)} - \frac{2}{x}K(k)Z(u,k)$

where $a = \cosh y$, $xu = 2yK(k)$, and $x = \pi K'(k)/K)k)$.

See SN15p.433.

(43.4.6) $\sum_{n=0}^{\infty} \frac{1}{a - \cosh(2n+1)x} = \frac{1}{\pi} \csc y \; K(k)Z(u,k)$

where $a = \cosh y$, $xu = 2yK(k)$, and $x = \pi K'(k)/K)k)$.

See SN15p.433.

(43.4.7) $\sum_{n=1}^{\infty} \frac{1}{\cosh 2nx + \cosh(2m+1)x} = \left(m + \frac{1}{2}\right)\operatorname{csch}(2m+1)x -$

$-\frac{1}{4}\left[\operatorname{sech}\left(n + \frac{1}{2}\right)x\right]^2$ MN28p.266

(43.5.5) $\sum_{n=1}^{\infty} \frac{1}{n}(1 - \tanh nx) = \frac{x}{2} + \log\left[\frac{kK(k)}{2\pi}\right]$ SR15p.406(T1.5)

where $x = \pi K'(k)/K(k)$.

(43.5.6) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(1- \tanh kx) = \frac{x}{2} + \log\left(\frac{1-k'}{2k}\right)$ SR15p.406(T1.8)

where $x = \pi K'(k)/K(k)$.

(43.7.3.1) $\sum_{n=0}^{\infty} (-1)^n \operatorname{csch}(2n+1)x = \frac{k}{\pi}K(k)$ SQ22p.231

where $x = \frac{\pi}{2}K'(k)/K(k)$.

(43.7.3.2) $\sum_{k=1}^{\infty} (\pm 1)^k k^m \operatorname{csch} kx$

These series are studied in SR10p.198.

(43.7.5.1) $\sum_{k=0}^{\infty} (\pm)^k (2k+1)^m \operatorname{csch}(2k+1)x$

These series are studied in SR9p.34 and SR10p.198.

(43.7.5.2) $\sum_{n=1}^{\infty} \frac{1}{n}\operatorname{csch} nx = \frac{x}{12} - \frac{1}{6}\log\frac{4(k')^2}{k}$ SR15p.406(T1.4)

where $x = \pi K'(k)/K(k)$.

(43.7.5.3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{csch} n = \frac{\pi}{12} + \frac{1}{6} \log \frac{kk'}{4}$ SR15p.406(T1.3)

where $x = \pi K'(k)/K(k)$.

(43.7.8.1) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{11}} \operatorname{csch} kx = \frac{4009\pi^{11}}{13621608000}$ BI13p.108

(43.7.12.1) $\sum_{n=1}^{\infty} (\operatorname{csch} n\pi)^4 = \frac{8}{15} \left\{ \frac{1}{(2\pi)^6} \left[\Gamma\left(\frac{1}{4}\right)\right]^8 - \frac{11}{48} + \frac{5\pi}{8} \right\}$ AS18p.18

(43.7.12.2) $\sum_{k=0}^{\infty} [\operatorname{csch}(2k+1)\pi\}^4 = \frac{1}{6} \left\{ 1 - \frac{1}{8\pi^5} \left[\Gamma\left(\frac{1}{4}\right)\right]^4 \right\}^2 + \frac{1}{6} \left(\frac{1}{\pi} - 1\right)$

AS18p.18

(43.7.12.3) $\sum_{k=1}^{\infty} (\pm 1)^k (\operatorname{csch} kx)^m \qquad (m = 1, 2, 3, ...)$

These series are studied in SR10p.197.

(43.7.12.4) $\sum_{k=0}^{\infty} (-1)^k [\operatorname{csch}(2k+1)x]^m \qquad (m = 1, 2, 3, ...)$

This series is studied in SR10p.201.

(43.7.15) $\sum_{k=1}^{\infty} (\pm 1)^k (\operatorname{csch} kx)^m \qquad (m = 1, 2, 3, ...)$

These series are studied in SR10p.197.

(43.7.16) $\sum_{k=1}^{\infty} (-1)^k [\operatorname{csch}\ (2k+1)x]^m \qquad (m = 1, 2, 3, ...)$

This series is studied in SR10p.201.

(43.8.1.1) $\sum_{n=0}^{\infty} \operatorname{sech}(2n+1)x = \frac{k}{\pi} K(k)$ SQ22p.231

where $x = \frac{\pi}{2} K'(k)/K(k)$.

(43.8.1.2) $\sum_{k=1}^{\infty} (\pm)^k\ k^m \operatorname{sech} kx \qquad (m = 1, 2, 3, ...)$

These series are studied in SR9p.34 and SR10p.192.

(43.8.1.3) $\sum_{k=0}^{\infty} (\pm 1)^k (2k+1)^m \operatorname{sech}\ (2k+1)x \qquad (m = 1, 2, 3, ...)$

These series are studied in SR9p.34, SR10p.192, and BI13p.110.

(43.8.2.1) $\sum_{m=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\ (2k+1)x = \frac{1}{2} \sin^{-1} k$ SR15p.406(T1.10) where $x = \frac{\pi}{2}K'(k)/K(k)$.

(43.8.6.1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{13}} \text{sech}\ \left(n+\frac{1}{2}\right)\pi = \frac{33661\pi^{13}}{245248819200}$ BI13p.105

(43.8.7.1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{6m+1}} \text{sech}[(2n+1)3^{1/2}\pi/2] = \frac{(-1)^{m-1}}{2}\pi^{6m+1}\times$

$\times \sum_{k=0}^{3m} \frac{E_{2k+1}B_{6m-2k}}{(2k+1)!(6m-2k)!} \cos\left[(2k+1)\frac{\pi}{3}\right]$ SR15p.413(23)

$m = 0, 1, 2, \ldots.$

(43.8.14.1) $\sum_{k=1}^{\infty} (\pm 1)^k (\text{sech}\ kx)^m \qquad (m = 1, 2, 3, \ldots.)$

These series are studied in SR10p.192.

(43.8.14.2) $\sum_{n=1}^{\infty} (\text{sech}\ 2^{1/2}\pi n)^6 = \frac{2^{3/2}}{15\pi} + \frac{2^{1/2}}{15}A + \frac{2^{1/2}-1}{12}A^2 +$

$+ \left(\frac{1}{60} - \frac{2^{1/2}}{40}\right) A^2 - \frac{1}{2}$ SQ18p.117

where $A = 2^{1/2}[\Gamma(1/8)]^4[4\pi\Gamma(1/4)]^{-1/2}$.

(43.8.14.3) $\sum_{k=0}^{\infty} (\pm 1)^k [\text{sech}(2k+1)x]^m \qquad (m = 1, 2, 3, \ldots.)$

These series are studied in SR10p.192.

(43.8.14.4) $\sum_{k=0}^{\infty} (\pm 1)^k (2n+1)^{2m}\ \text{sech}(2k+1)x \qquad (m = 1, 2, 3, \ldots.)$

These series are studied in SR10p.192.

(43.9.3.1) $\sum_{n=1}^{\infty} \frac{1}{n^{11}} \coth n\pi = \frac{1453\pi^{11}}{425675250}$ BI13p.104

(43.9.7) $\sum_{n=1}^{\infty} \frac{1}{n}(1-\coth nx) = \frac{x}{6} + \log[2\pi^{-3}kk'K(k)]$ SR15p.408(T1.1)

where $x = \pi K'(k)/K(k)$.

(43.9.8) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(1-\coth nx) = \frac{x}{6} + \frac{1}{6}\log\frac{k^2}{16k'}$ SR15p.408(T1.2)

where $x = \pi K'(k)/K(k)$.

(43.10.3) $\sum_{n=1}^{\infty} (-1)^n \text{csch}\ nx \coth nx = \frac{2k^2-1}{3\pi^2}K(k) - \frac{1}{12}$ SR15p.412(18)

where $x = \pi K'(k)/K(k)$.

(44.13.3) $\sum_{k=2}^{\infty} \frac{(-1)^k}{\log\ k} = 0.9242998972$ SQ32p.481(89-15*)

(45.1.0.1) $\sum_{k=0}^{n} P_k^{(a,0)}(x) = \frac{(a+2)_n}{n!}\ {}_3F_2\left(-n, a+n+2, \frac{a+1}{2}; a+1, \frac{a+3}{2}; \frac{1-x}{2}\right)$

AKp.413(8.10)

(45.1.10.1) $\sum_{k=0}^{\infty} \frac{(m+1)_k (a+b+m+1)_k}{k!(a+b+2m+2)_k} t^k P_{k+m}^{(a,b)}(x) = 2t^{-a-b-m-1} \frac{\Gamma(a+b+2m+2)}{\Gamma(a+m+1)\Gamma(b+m+1)} \times$

$\times P_m^{(a,b)}\left(\frac{1-r}{t}\right) Q_m^{(a,b)}\left(\frac{1-r}{t}\right)$ SQ28p.133

where $r = (1-2xt+t^2)^{1/2}$, $|t| < 1$.

(45.1.12.1) $\sum_{k=0}^{\infty} \frac{(c)_k (a+b+1)_k}{(a+1)_k (d)_k} (2k+a+b+1) P_k^{(a,b)}(x) =$

$= (a+b+1)(c+d-a-b-2) \frac{\Gamma(d)\Gamma(d-c-b-a-2)}{\Gamma(d-c)\Gamma(d-a-b-1)} \times$

$\times\ {}_4F_3\left[\begin{matrix} c,\ (a+b+3)/2,\ 1+(a+b)/2,\ a+b-d+2; \\ a+1,\ (a+b+c-d+3)/2,\ 2+(a+b+c-d)/2; \end{matrix}\ \frac{1-x}{2}\right]$

For $b = a$, $c = -n$, and $d = -2a-n-2$, see AKp.416(8.19).

(45.1.16.1) $\sum_{k=0}^{\infty} \frac{(m+1)_k (a-c)_k (a+b+2m+1)_k}{k!(a+m+1)_k (b+c+2m+2)_k} (a+b+2m+2k+1) P_{k+m}^{(a,b)}(x) =$

$= \frac{\Gamma(a+m+1)\Gamma(b+c+2m+2)}{\Gamma(a+b+2m+1)\Gamma(c+m+1)} \left(\frac{1-x}{2}\right)^{c-a} P_m^{(c,b)}(x)$

(45.3.5) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} P_m^{(a,b-k)}(x) = \left(\frac{x-1}{2}\right)^n P_{m-n}^{(a+m,b)}(x)$ $(m \geq n)$

(45.3.6) $\sum_{k=0}^{n} \frac{(-n)_k (2a+n+1)_k}{k!(2a+m+2)_k} P_m^{(a,-a-m-k-1)}(x) = \pi^{-1/2} 2^{2a} \times$

$\times \frac{n!\Gamma(a+m+1)\Gamma(a+1/2)\Gamma(2a+m+2)}{(m-n)!\Gamma(2a+m+n+2)\Gamma(2a+n+1)} C_n^{(a+1/2)}(x)$

(45.3.7) $\sum_{k=0}^{n} \frac{(-n)_k (a-b+n-m)_k}{k!(a-b+1)_k} P_m^{(a,b-k)}(x) = 0$ $(m < n)$

(45.3.8) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(a-b+k)} P_n^{(a,b-k)}(x) = \frac{n!(a+1)_n\ \Gamma a-b)}{(2a+1)_n\ \Gamma(a-b+n+1)} C_n^{(a+1/2)} x)$

(45.3.9) $\sum_{k=0}^{\infty} \frac{(c)_k}{(d)_k} \left(\frac{2}{x+1}\right)^k P_k^{(a,b-k)}(x) = \frac{\Gamma(d)\Gamma(d-c-b-a-1)}{\Gamma(d-b-a-1)\Gamma(d-c)} \times$

$\times\ {}_2F_1\left(-b,\ c;\ d-b-a-1; \frac{2}{x+1}\right)$

For $c =$ negative integer, see (45.2.6).

(45.3.10) $$\sum_{k=0}^{\infty} \frac{(c)_k}{(a+1)_k} \left(\frac{2}{x+1}\right)^k P_k^{(a,b-k)}(x) = \left(\frac{x+1}{x-1}\right)^c \frac{\Gamma(a+1)\Gamma(-b-c)}{\Gamma(a-c+1)\Gamma(-b)}$$

(45.3.11) $$\sum_{k=0}^{n} \frac{(-n)_k(2a-r)_k}{k!(a-r)_k} \left(\frac{2}{x+1}\right)^k P_m^{(a,r-m-2a-k)}(x) =$$

$$= \frac{(m+1)_n}{(a-r)_n} \left(\frac{-2}{x+1}\right)^n P_{m+n}^{(a-n,r-m-n-2a)}(x)$$

(45.3.12) $$\sum_{k=0}^{n} \frac{(-n)_k(-a-m)_k}{k!(-2a-m-r)_k} \left(\frac{2}{x+1}\right)^k P_m^{(a+r,a-k)}(x) =$$

$$= \frac{(m+n)!\Gamma(-2a-m-r)}{m!\Gamma(-2a-m-r+n)} \left(\frac{-2}{x+1}\right)^n P_{m+n}^{(a+r-n,a-n)}(x)$$

(45.3.13) $$\sum_{k=0}^{\infty} \frac{(c)_k(-b-n)_k}{k!(-b-m-n)_k} t^k P_n^{(a,b-k)}(x) = (-1)^m(1-t)^{-c-m} \left(\frac{x+1}{2}\right)^n \times$$

$$\times \frac{m!(a+1)_n}{n!(b+n+1)_m} \sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(a+1)_k} \left[\frac{x-1}{(x+1)(1-t)}\right]^k P_{m+k}^{(-b-m-n-1,-c-m-k)}(1-2t)$$

(45.4.6) $$\sum_{k=0}^{\infty} \frac{(c)_k}{(d)_k} P_k^{(a,b-k)}(x) = \frac{\Gamma(d)\Gamma(b-c+d)}{\Gamma(b+d)\Gamma(d-c)}\ {}_2F_1\left(a+b+1, c; b+d; \frac{x+1}{2}\right)$$

(45.4.7) $$\sum_{k=0}^{n} \frac{(-n)_k}{(d)_k} P_k^{(a,b-k)}(x) = \frac{n!}{(d)_n}(-1)^n P_n^{(a-d-n+1,b+d-1)}(x)$$

(45.4.8) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!} P_{m+k}^{(a,b-k)}(x) = (-1)^n P_{m+n}^{(a-n,b)}(x)$$

(45.4.9) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(c)_k} P_{m+k}^{(a,b-k)}(x) = \frac{n!\Gamma(c)}{m!\Gamma(c-m+n)} \times$$

$$\times \sum_{k=0}^{m} \frac{(-m)_k(n+1)_k}{k!(c-m+n)_k} P_{n+k}^{(b+c-1,a-c+m-n-k+1)}(-x)$$

(45.4.10) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(a+m+1)_k} P_{m+k}^{(a,b-k)}(x) = \frac{(a+b+m+1)_n}{(a+m+1)_n} \left(\frac{1-x}{2}\right)^n P_m^{(a+n,b)}(x)$$

(45.4.11) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(\frac{2}{x+1}\right)^k P_{m+k}^{(a,b-k)}(x) = \left(\frac{-2}{x+1}\right)^n P_{m+n}^{(a-n,b-n)}(x)$$

(45.4.12) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(c)_k} \left(\frac{2}{1+x}\right)^k P_{m+k}^{(a,b-k)}(x) = \frac{n!\Gamma(c)}{m!\Gamma(c-m+n)} \left(\frac{1+x}{2}\right)^m \times$$

$$\times \sum_{k=0}^{m} \frac{(-m)_k(n+1)_k}{k!(c-m+n)_k} P_{n+k}^{(c-a-b-2m-2,a-c+m-n-k+1)} \left(\frac{x-3}{x+1}\right)$$

(45.4.13) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(a+m+1)_k} \left(\frac{2}{1+x}\right)^k P_{m+k}^{(a,b-k)}(x) =$$

$= \frac{(-b-m)_n}{(a+m+1)_n} \left(\frac{x-1}{x+1}\right)^n P_m^{(a+n,b-n)}(x)$

(45.5.0) $\sum_{k=0}^{n} \frac{1}{c+n-k} P_k^{(a,b)}(x) P_{n-k}^{(c,-1)}(x) = \frac{1}{c} P_n^{(a+c,b)}(x)$

(45.5.8) $\sum_{k=0}^{n} \frac{[(a+b+1)_k]^2}{(a+1)_k(b+1)_k}(2k+a+b+1) P_k^{(a,b)}(x) P_k^{(b,a)}(x) =$

$= \frac{(a+b+1)[(a+b+2)_n]^2}{(2n+a+b+2)(a+1)_n(b+1)_n} \Big[(a+b+n+2) P_n^{(a+1,b+1)}(x) P_n^{(b,a)}(x)$

$-(n+1) P_{n-1}^{(a+1,b+1)}(x) P_{n+1}^{(b,a)}(x) \Big]$ SR14p.312(10.28)

$-1 < a < 0, -1 < b < 0$

(45.5.9) $\sum_{k=0}^{\infty} \frac{(a+b+1)_k(a+b-m+1)_k}{(a+1)_k(b+1)_k}(2k+a+b+1)(-1)^k P_k^{(a,b)}(x) P_{m+k}^{(a-m,b-m)}(y) =$

$= (-1)^m 2^{a+b-m+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\pi\Gamma(a+b+1)} |x+y|^{m-a-b-1} \begin{cases} \sin \pi a \text{ if } x+y > 0, \\ \sin \pi b \text{ if } x+y < 0 \end{cases}$

SQ14p.309(10.4)

$a > -1,\ b > -1,\ a+b < 0,\ a+b < m = -1, 0, 1, 2, ...$

For $m = -1$, set $P_{-1}^{(a+1,b+1)}(y) = 0$.

(45.5.10) $\sum_{k=0}^{\infty} \frac{k!(a+b+1)_k}{(a+1)_k(b+1)_k} t^k P_k^{(a,b)}(x) P_k^{(a,b)}(y) =$

$= (1+t)^{-a-b-1} F_4 \left[\frac{a+b+1}{2}, \frac{a+b+2}{2}, a+1, b+a, \frac{t(1-x)(1-y)}{(1+t)^2} \frac{t(1+x)(1+y)}{(1+t)^2}\right]$

HVp.116(48)

(45.5.11) $\sum_{k=0}^{n} \frac{k!(-n)_k(a+b+1)_k}{(a+1)_k(b+1)_k(a+b+n+2)_k}(-1)^k(2k+a+b+1) P_k^{(a,b)}(x) P_k^{(a,b)}(y) =$

$= (a+b+1) \frac{n!(a+b+2)_n}{(a+1)_n(b+1)_n} \left(\frac{x+y}{2}\right)^n P_n^{(a,b)} \left(\frac{1+xy}{x+y}\right)$ AY

(46.3.2) $\sum_{k=1}^{n} t^k P_{k+m}(x) = \frac{2}{t^2-1}\{(m+1)t[P_{m+1}(x) - tP_m(x)]+$

$+(m+n+1)t^{n+1}[tP_{m+n}(x) - P_{m+n+1}(x)]\}$

where $t^2 - 2tx + 1 = 0$.

(46.6.3.1) $\sum_{k=1}^{n-1} \frac{(m/2+1/2)_k}{(m/2+1)_k}(2m+4k+1)(-1)^k P_{m+2k}(x) = -\frac{m+1}{x} P_{m+1}(x) -$

$-\frac{m(m/2+1/2)_n}{x(m/2)_n}(-1)^n P_{m+2n-1}(x)$

(46.6.6.1) $$\sum_{k=0}^{\infty} \frac{(2m+1)_k(1-a)_k}{(m+1)_k(a+m+1)_k}(2k+2m+1)P_{k+m}(x) =$$

$$= \frac{(m!)^2\Gamma(a+m+1)}{(2m)!\Gamma(a-m)}(-1)^m \left(\frac{1-x}{2}\right)^{a-m-1} P_m(x)$$

(46.6.6.2) $$\sum_{k=0}^{\infty} \frac{(-a)_k(2m+1)_k}{k!(a+2m+2)_k}(2m+2k+1)P_{m+k}(x) =$$

$$= \frac{m!\Gamma(a+2m+2)}{(2m)!\Gamma(a+m+1)} \left(\frac{1-x}{2}\right)^a P_m^{(a,0)}(x)$$

(46.6.11.1) $$\sum_{k=0}^{\infty} \frac{(2m+1/2)_k(3/2-a)_k}{k!(a+2m)_k}(4k+4m+1)P_{2k+2m}(x) =$$

$$= 2^{2a-2}\pi^{-1/2}(1-x^2)^{a-3/2}\frac{(2m)!\Gamma(a-1)\Gamma(a+2m)}{\Gamma(2m+1/2)\Gamma(2a+2m-2)}C_{2m}^{(a-1)}(x)$$

$m = 0, 1, 2, ...$ For $m = 0$, see (46.6.12).

(46.6.12.1) $$\sum_{k=0}^{\infty} \frac{(2m+3/2)_k(5/2-a)_k}{k!(a+2m)_k}(4k+4m+3)P_{2k+2m+1}(x) =$$

$$= 2^{2a-4}\pi^{-1/2}(1-x^2)^{a-5/2}\frac{(2m+1)!\Gamma(a-2)\Gamma(a+2m)}{\Gamma(2m+3/2)\Gamma(2a+2m-3)}C_{2m+1}^{(a-2)}(x)$$

$m = 0, 1, 2, ...$ For $m = 0$, see (46.6.13).

(16.7.8.1) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!}(2x)^k P_{m+k}(x) = (2x)^{-m}\sum_{k=0}^{m} \frac{(-m)_k}{k!}P_{n+k}(1-2x^2)$$

(46.7.12) $$\sum_{k=1}^{n} \frac{m-1/2)_k}{(m)_k}(m+k)(2x)^k P_{m+k}(x) = x(2m-1)\times$$

$$\times \left[\frac{(m+1/2)_n}{(m)_n}(2x)^n P_{m+n-1}(x) - P_{m-1}(x)\right] \qquad m = 1, 2, 3, ...$$

(46.7.12) $$\sum_{k=1}^{n} \frac{m-1/2)_k}{(m)_k}(m+k)(2x)^k P_{m+k}(x) = x(2m-1)\times$$

$$\times \left[\frac{(m+1/2)_n}{(m)_n}(2x)^n P_{m+n-1}(x) - P_{m-1}(x)\right] \qquad m = 1, 2, 3, ...$$

(46.7.13) $$\sum_{k=1}^{n-1} \frac{(m+1)_k}{(m+3/2)_k}(m+k)(2x)^{-k} P_{k+m-1}(x) = (m+1)P_{m+1}(x) -$$

$$-x(2m+1)\frac{(m+n)!}{m!(m+1/2)_n}(2x)^{-n}P_{m+n}(x) \qquad m = 0, 1, 2, ...$$

(47.4.4.1) $$\sum_{k=1}^{n} \frac{(1/2)_k}{(q+1/2)_k}(q+2k)(-1)^k C_{2k}^{(q)}(x) = \frac{(3/2)_n}{2x(q+1/2)_n}(-1)^n C_{2n+1}^{(q)}(x) - q$$

(47.4.4.2) $$\sum_{k=0}^{n-1} \frac{k!}{(q+1)_k}(q+2k+1)(-1)^k C_{2k+1}^{(q)}(x) = \frac{q}{x}\left[1 - \frac{n!}{(q)_n}(-1)^n C_{2n}^{(q)}(x)\right]$$

(47.4.4.3) $\sum_{k=0}^{\infty} \frac{(2q+2)_k}{k!}(q+k+1)C_{k+1}^{(q)}(x) = -\frac{\pi^{1/2}}{\Gamma(q)\Gamma(1/2-q)}(1-x)^{-2q-1}$

(47.4.7.1) $\sum_{k=0}^{n} \frac{(-n)_k(2q+2)_k}{k!(2q+n+3)_k}(q+k+1)(-1)^k C_{k+1}^{(q)}(x) =$

$= 2^{-2q}\pi^{1/2}\frac{\Gamma(2q+n+3)}{(2q+1)\Gamma(q)\Gamma(q+n+3/2)}P_{n+1}^{(q-1/2,q-n-1/2)}(x)$

(47.4.7.2) $\sum_{k=0}^{\infty} \frac{(-a)_k(q+m+1/2)_k}{(a+2q+1)_k(q-m+1/2)_k}(q+k)C_k^{(q)}(x) = (-1)^m 2^{m-a-2q}(1-x)^{a-m}\times$

$\times\ \pi^{1/2}\frac{m!\Gamma(a+2q+1)\Gamma(q-m+1/2)}{\Gamma(q)\Gamma(a-m+q+1/2)\Gamma(q+m+1/2)}P_m^{(a-m,q-1/2)}(x)$

(47.4.7.3) $\sum_{k=0}^{\infty} \frac{(q-a)_k(2q+2m)_k}{(q+a+m+1)_k(2q+m)_k}(k+q+m)C_{k+m}^{(q)}(x) = 2^{-q-a-m}(1-x)^{a-q-m}\times$

$\times\ \pi^{1/2}\frac{m!\Gamma(q+a+m+1)(q-a)_m}{\Gamma(a+1/2)\Gamma(q+m)}C_m^{(q)}(x)$

(47.4.14) $\sum_{k=0}^{n-1} \frac{(k+1)!}{(q+2)_k}(2q+k)(2x)^{-k}C_k^{(q)}(x) = 2x(q+1)\Big[C_1^{(q)}(x)-$

$-\ \frac{(n+1)!}{(q+1)_n}(2x)^{-n}C_{n+1}^{(q)}(x)\Big]$

(74.4.15) $\sum_{k=1}^{n} \frac{(q-1)_k}{(2q-1)_k}k(2x)^k C_k^{(q)}(x) = \frac{(q)_n}{(2q-1)_n}(q-1)(2x)^{n+1}C_{n-1}^{(q)}(x)$

(47.4.16) $\sum_{k=0}^{n} \frac{(-n)_k}{k!}C_{m+k}^{(q)}(x) = \frac{(-1)^n\Gamma(2q+m)}{(m+n)!\Gamma(2q-n)}\ \times$

$\times\ {}_3F_2\left[-m-n,\ q,\ 2q+m;\ q-\frac{n}{2},\ q;\ \frac{1-x}{2}\right]$

(47.4.17) $\sum_{k=0}^{n} \frac{(-n)_k}{(a)_k}C_k^{(q)}(x) = \frac{(a-2q)_n}{(a)_n}\times$

$\times\ {}_3F_2\left(-n,\ q,\ 2q-a+1;\ q-\frac{a+n-1}{2},\ q+1-\frac{a+n}{2};\ \frac{1-x}{2}\right)$

(47.5.6.1) $\sum_{k=0}^{\infty} \frac{(b)_k}{(a)_k}(2x)^k C_k^{(q)}(x) = {}_3F_2(q,\ b,\ a-b;\ a/2,\ a/2+1/2;\ x^2)$

(47.5.13) $\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(2q+m)_k}t^k C_{m+k}^{(q)}(x) = \frac{n!\Gamma(2q+m)}{m!\Gamma(2q+n)}t^{-m}w^n\times$

$\times\sum_{k=0}^{m} \frac{(-m)_k(n+1)_k}{k!(2q+n)_k}w^k C_{n+k}^{(q)}\left(\frac{1-xt}{w}\right)$

where $w = (1-2xt+t^2)^{1/2}$

(47.5.14) $\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(2q+m)_k}t^k C_{k+m}^{(q)}\left(\frac{1+t^2}{2t}\right) =$

$$= \frac{(q)_n}{(2q+m)_n} t^{-m}(1-t^2)^n P_m^{(2q+n-1,-q-m)}(2t^2-1)$$

(47.5.15) $$\sum_{k=0}^{\infty} \frac{(b)_{ak}(2b)_{2ak}(b+1/2)_{k-ak}}{(2b)_{k+2ak}} t^k C_k^{(b-ak)}(x) = \frac{[(1+u)(1+v)]^{b+1/2}}{(2b-1)![1+a(u+v)+(2a-1)uv]}$$

where $u=(x+1)w$, $v=(X-1)w$, and $w = 2^{2a-1}t[(1+u)(1+v)]^{1-a}$.

(47.5.16) $$\sum_{k=0}^{n} \frac{(-m-n)_k(a-m)_k}{k!(a-n+1)_k} C_{2m}^{(k-m-n)}(x) = \frac{(-1)^n(m+n)!}{(m-n)!(-a)_n(1+a)_n} C_{2n}^{(-a-n)}(x)$$

(47.5.17) $$\sum_{k=0}^{n} \frac{(-m-n-1)_k(a-m-1)_k}{k!(a-n)_k} C_{2m+1}^{(k-m-n-1)}(x) = \frac{(m+n+1)!\Gamma(a-n)}{(m-n)!\Gamma(a+n+1)} C_{2n+1}^{(-a-n)}(x)$$

(47.5.18) $$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(a+2m-n+1)_k} C_{2m}^{(a+k)}(x) = \frac{(a)_n}{(-a-2m)_n} C_{2m-2n}^{(a+n)}(x) \qquad (m \geq n)$$

(47.5.19) $$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(a+2m+n+2)_k} C_{2m+2n+1}^{(a+k)}(x) = (-1)^n \frac{\Gamma(a+n)\Gamma(a+2m+n+2)}{\Gamma(a)\Gamma(a+2m+2n+2)} C_{2m+1}^{(a+n)}(x)$$

(47.5.20) $$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(a+n)_k(b+k)} C_{2n}^{(a+k)}(x) = \frac{n!(a)_n}{b(b+1)_n(a-b)_n} C_{2n}^{(a-b)}(x)$$

(47.5.21) $$\sum_{k=0}^{n} \frac{(-n)_k(a)_k}{k!(a+n+1)_k(b+k)} C_{2n+1}^{(a+k)}(x) =$$

$$= \frac{n!\Gamma(b)\Gamma(a-b)\Gamma(1-a)}{\Gamma(b+n+1)\Gamma(-a-n)\Gamma(a-b+n+1)} (-1)^{n-1} C_{2n+1}^{(a-b)}(x)$$

(47.5.22) $$\sum_{k=0}^{n} \frac{(-n)_k(b)_k}{k!(1-a)_k} C_{2m+2k}^{(a-k)}(x) = (-1)^m \frac{(a)_m(m-b+1)_n}{(m+n)!} \times$$

$$\times\ {}_3F_2(-m-n,\, a+m,\, b-m;\, 1/2,\, b-m-n;\, x^2)$$

(47.5.23) $$\sum_{k=0}^{n} \frac{(-n)_k(a+2m+n)_k}{k!(1-a)_k} C_{2m+2k}^{(a)}(x) = C_{2m+2n}^{(a)}(x)$$

(47.5.24) $$\sum_{k=0}^{n} \frac{(-n)_k(b)_k}{k!(1-a)_k} C_{2m+2k+1}^{(a-k)}(x) = 2ax(-1)^m \frac{(a+1)_m(m-b+1)_n}{)m+n)!} \times$$

$$\times\ {}_3F_2(-m-n,\, a+m+1,\, b-m;\, 3/2,\, b-m-n;\, x^2)$$

(47.5.25) $$\sum_{k=0}^{n} \frac{(-n)_k(a+2m+n+1)_k}{k!(1-a)_k} C_{2m+2k+1}^{(a-k)}(x) = C_{2m+2n+1}^{(a)}(x)$$

(47.6.11.1) $$\sum_{k=0}^{\infty} \frac{k!}{(2q)_k} \left(\frac{x}{y}\right)^k C_k^{(p)}(x) C_k^{(q)}(y) = (1-x^2)^{-p} \times$$

$$\times\ {}_2F_1\left[p,\ \tfrac{1}{2},\ q+\tfrac{1}{2};\ \tfrac{x^2(1-y^2)}{y^2(1-x^2)}\right]$$

$|x| < 1$

(47.6.16) $$\sum_{k=0}^{[n/2]} \frac{(1/2)_k(-q-n)_k(1-2q-n)_{2k}}{k!(1/2-q-n)_k(-n)_{2k}}(q+n-2k)\left[C_{n-2k}^{(q)}(x)\right]^2 =$$

$$= q\frac{(q+1)_n(2q)_n(2q+1)_n}{(n!)^2\Gamma(q)(q+1/2)_n}\ {}_3F_2(-n,\ n+2q+1,\ q;\ 2q,\ q+1;\ 1-x^2)$$

AKp.414p.(8.12)

(47.7.4) $$\sum_{k=0}^{n} \frac{(2k+2q-1)[(q)_k]^2}{(-n)_k(n+2q)_k}(-4\sin x\ \sin y)^k\times$$

$$\times C_{n-k}^{(q+k)}(\sin x)C_{n-k}^{(q+k)}(\sin y)C_k^{(q-1/2)}(\cos z) =$$

$$= \frac{(2q-1)_n}{n!}C_n^{(q)}(\cos x\ \cos y + \sin x\ \sin y\ \sin z)$$

SR10p.452(2)

(48.3.14.1) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(c-r+m+1)_k}L_{m+k}^{(c)}(x) = \frac{r!\Gamma(c-r+m+1)}{m!\Gamma(c-r+n+1)}(-1)^r x^{n-r}\times$$

$$\times\sum_{k=0}^{m} \frac{(-m)_k}{k!(c-r+n+1)_k}x^k L_r^{(n-r+k)}(x)$$

For $r=0$,see (48.3.15). For $n=0$, see (48.13.12).

(48.3.19) $$\sum_{k=0}^{\infty} \frac{(a)_k(b+c)_k}{(b)_k(c+1)_k}L_k^{(c)}(x) =$$

$$= \frac{\Gamma(b)\Gamma(-a-c)}{\Gamma(-c)\Gamma(b-a)}\ {}_2F_2(a,\ b+c;\ c+1,\ a+c+1;\ x)+$$

$$+x^{-a-c}\sin\pi a\frac{\Gamma(b)\Gamma(c+1)\Gamma(a+c)}{\pi\Gamma(b+c)}\ {}_2F_2(-c,\ b-a;\ 1-a,\ 1-a-c;\ x)$$

(48.5.3) $$\sum_{k=0}^{n} \frac{(m+2)_k}{(m+c/2-x/2+5/2)_k}(m+k+c+1)2^{-k}L_{m+k}^{(c)}(x) =$$

$$= (2m+c-x+3)L_{m+1}^{(c)}(x) - \frac{(m+3)_n}{(m+c/2-x/2=5/2)_n}(m+2)2^{-n}L_{m+n+2}^{(c)}(x)$$

(48.5.4) $$\sum_{k=1}^{n-1} \frac{(m/2+1/2)_k}{(m/2+c/2+1)_k}(2m+4k+c-x+1)(-1)^k L_{m+2k}^{(c)}(x) =$$

$$= \frac{(m/2+1/2)_n}{(m/2+c/2)_n}(m+c)(-1)^{n-1}L_{m+2n-1}^{(c)}(x) - (m+1)L_{m+1}^{(c)}(x)$$

(48.7.8.1) $$\sum_{k=1}^{n-1} \frac{(m+c/2-x/2-1/2)_k}{(m+c)_k}(m+k)2^k L_{m+k}^{(c)}(x) =$$

$$= \frac{(m+c/2-x/2-1/2)_n}{(m+c-1)_n}(m+c-1)2^n L_{m+n-2}^{(c)}(x) - (2m+c-x-1)L_{m-1}^{(c)}(x)$$

(48.7.8.2) $$\sum_{k=0}^{n} \frac{(m+2)_k}{(m+c/2-x/2+5/2)_k}(m+k+c+1)2^{-k}L_{m+k}^{(c)}(x) =$$

$$= (2m+c-x+5/2)L_{m+1}^{(c)}(x) - \frac{(m+3)_n}{(m+c/2-x/2+5/2)_n}(m+2)2^{-n}L_{m+n+2}^{(c)}(x)$$

$$(48.7.8.3)\qquad \sum_{k=0}^{n}\frac{(c/2-x/2+1/2)_k}{k!(c+1)_k}2^k L_{k+1}^{(c)}(x)=\frac{(c/2-x/2+3/2)_n}{n!(c+1)_n}(c-x+1)2^n L_n^{(c)}(x)$$

$$(48.8.2)\qquad \sum_{k=1}^{n-1}\frac{k!}{(c/2-x/2+3/2)_k}2^{-k}(c+k)L_{k-1}^{(c)}(x)=L_1^{(c)}(x)-$$

$$-\frac{n!}{(c/2-x/2+1/2)_n}2^{-n}(c-x+1)L_n^{(c)}(x)$$

$$(48.8.3)\qquad \sum_{k=1}^{n}\frac{1}{k!(c/2+1)_k}(-4)^{-k}(4k+c-x+1)L_{2k}^{(c)}(x)=$$

$$=\frac{(-4)^{-n}}{n!(c/2+1)_n}L_{2n+1}^{(c)}(x)-L_1^{(c)}(x)$$

$$(48.8.4)\qquad \sum_{k=0}^{n-1}\frac{1}{(3/2)_k(c/2+3/2)_k}(-4)^{-k}(4k+c-x+3)L_{2k+1}^{(c)}(x)=$$

$$=(c+1)\left[1-\frac{(-4)^{-n}}{(1/2)_n(c/2+1/2)_n}L_{2n}^{(c)}(x)\right]$$

$$(48.9.2)\qquad \sum_{k=0}^{n-1}L_m^{(c-k)}(x)=L_{m+1}^{(c)}(x)-L_{m+1}^{(c-n)}(x)$$

$$(48.10.3)\qquad \sum_{k=1}^{n-1}\frac{(x+c-1)_k}{(m+c)_k}L_m^{(c+k)}(x)=$$

$$=\frac{1}{x}\left[\frac{(x+c-1)_n}{(m+c-1)_n(m+c-1)}L_m^{(c+n-2)}(x)-(x+c-1)L_m^{(c-1)}(x)\right]$$

$$(48.10.4)\qquad \sum_{k=0}^{n}\frac{(-n)_k(c+n)_k}{k!(c+m+1)_k}L_m^{(c+k)}(x)=\frac{\Gamma(c+m+1)}{\Gamma(c+m+n+1)}(-1)^n L_{m-n}^{(c+2n)}(x)\qquad (m\geq n)$$

$$(48.10.5)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(m+1)_k}L_m^{(c+k)}(x)=\frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}\sum_{k=0}^{m}\frac{(-m)_k(1-a-b)_k}{(1-a)_k(1-b)_k}(-1)^k L_k^{(c-k)}(x)$$

$$(48.10.6)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(c)_k}{k1(c+m)_k}L_m^{(c+k+1/2)}(x)=$$

$$=-\frac{(c)_m\Gamma(c+m-1)\Gamma(1-a)}{8x(m-1)!\Gamma(c-a+m)}\sum_{k=0}^{m-1}\frac{(1-m)_k(1-a)_k}{k!(k+1)!(2-c-m)_k}(-4)^{-k}H_{2k+3}(x)$$

$(m=1,2,3,....)$

$$(48.10.7)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(c)_k}{k!(c+m)_k}L_m^{(c+k-1/2)}(x)=$$

$$=-\frac{(c)_m\Gamma(c+m-1)\Gamma(1-a)}{4(m-1)!\Gamma(c-a+m)}\sum_{k=0}^{m-1}\frac{(1-m)_k(1-a)_k}{k!(k+1)!(2-c-m)_m}(-4)^{-k}H_{2k+2}(x)$$

$(m=1,2,3,....)$

(48.10.8) $$\sum_{k=0}^{\infty} \frac{(a)_k (c)_k}{k!(a+m)_k} L_m^{(a+c+k-1)}(x) =$$

$$= \frac{\Gamma(1-c)\Gamma(a+m)}{(m-1)!\Gamma(a-c+1)} \sum_{k=0}^{m-1} \frac{(1-m)_k (1-c)_k}{k!(a-c+1)_k} (-1)^k L_{k-1}^{(c-k-1)}(x)$$

(48.10.9) $$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(m+1)_k} L_m^{(-m-k-1/2)}(x^2) =$$

$$= (-1)^m \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} \sum_{k=0}^{m} \frac{(-m)_k (1-a-b)_k}{k!(1-a)_k (1-b)_k} (-4)^{-k} H_{2k}(x)$$

$(m = 0, 1, 2, ...)$

(48.10.10) $$\sum_{k=0}^{\infty} \frac{(c)_k (a-m)_k}{k!(a)_k} L_m^{(c-a-k)}(x) =$$

$$= -\frac{\Gamma(a-1)\Gamma(1-c)}{(m-1)!\Gamma(a-c)} (1-a)_m \sum_{k=0}^{m-1} \frac{(1-m)_k (1-c)_k}{k!(2-a)_k} (-1)^k L_{k+1}^{(c-k-1)}(x)$$

(48.10.11) $$\sum_{k=0}^{\infty} \frac{(a)_k (c-m)_k}{k!(c)_k} L_m^{(1/2-c-k)}(x^2) =$$

$$= (-1)^m \frac{\Gamma(c)\Gamma(1-a)}{4(m-1)!\Gamma(c-a-m+1)} \sum_{k=0}^{m-1} \frac{(1-m)_k (1-a)_k}{k!(k+1)!(c-a-m+1)_k} (-4)^k H_{2k+1}(x)$$

$(m = 1, 2, 3, ...)$

(48.10.12) $$\sum_{k=0}^{\infty} \frac{(a)_k (c-m)_k}{k!(c)_k} L_m^{(3/2-c-k)}(x^2) = (-1)^m \frac{\Gamma(c)\Gamma(1-a)}{9x(m-1)!\Gamma(c-a-m+1)} \times$$

$$\times \sum_{k=0}^{m-1} \frac{(1-m)_k (1-a)_k}{k!(k+1)!(c-a-m+1)_k} (-4)^{-k} H_{2k+3}(x) \qquad (m = 1, 2, 3, ...)$$

(48.13.5.1) $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} t^k L_m^{(c+k)}(x) =$$

$$= (1-t)^{-a-m} \frac{(a)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k}{(1-a-m)_k} (1-t)^k L_k^{(c-a)}(x)$$

$$= (1-t)^{-a-m} \frac{(c+1)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k}{(c+1)_k} t^k L_k^{(c-a)} \left(x - \frac{x}{t}\right)$$

$$= (1-t)^{-a} \frac{(c+1)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k}{(c+1)_k} \left(\frac{t}{1-t}\right)^k L_k^{(-a-k)} \left(x - \frac{x}{t}\right)$$

For $c = -m$, see (48.13.7).

(48.13.5.2) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!} t^k L_m^{(r+k)}(x) = (-1)^m t^{-r} (1-t)^{n-m} \times$$

$$\times \sum_{k=0}^{m+r} \frac{(-m-r)_k}{k!} (1-t)^k L_m^{(n-m+k)} \left(x - \frac{x}{t}\right)$$

(48.13.9.1) $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!(b)_k} x^k L_m^{(c+k)}(x) =$$

$$= e^x \frac{(c+1)_m}{m!} \; {}_2F_2(b-a,\, c+m+1;\, b,\, c+1;\, -x)$$

For $a = -n$ and $b = c - n + 1$, see (48.13.10).

(48.13.9.2) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!(b)_k} x^k L_m^{(c+k)}(x) = \frac{n!(c+1)_m}{m!(b)_n} \sum_{k=0}^{m} \frac{(-m)_k}{k!(c+1)_k} x^k L_n^{(b-1+k)}(x)$$

For $c = -m$, see (48.13.12).

(48.13.9.3) $$\sum_{k=0}^{n} \frac{(-n)_k}{k!(c+m+1)_k} x^k L_m^{(c+k)}(x) = \frac{(m+n)!}{m!(c+m+1)_n} L_{m+n}^{(c)}(x)$$

For $c =$ negative integer, see (48.13.11).

(48.13.9.4) $$\sum_{k=0}^{\infty} \frac{(a)_k}{k!(a-n+1)_k} x^k L_m^{(k-n)}(x) = \frac{(n-1)!\Gamma(a-n+1)}{m!\Gamma(a-m)} (-1)^m e^x L_{n-m-1}^{(a-n)}(-x)$$

$n \geq m+1$

48.13.9.5) $$\sum_{k=0}^{\infty} \frac{(r)}{k!(m-n+1)_k} x^k L_m^{(k-n)}(x) = \frac{(m-n)!}{m!} (-x)^n L_{m-n-r}^{(n)}(x)$$

$r = 0,\ \pm 1,\ \pm 2, ...;\ m \geq n;\ m - r \geq n.$

(48.14.2) $$\sum_{k=1}^{n-1} \frac{m+n+k+1}{(a+x+2)_k} x^k L_m^{(a+k)}(x) = x L_m^{(a+2)}(x) - \frac{a+x+1}{(a+x+1)_n} x^n L_m^{(a+n+1)}(x)$$

(48.14.3) $$\sum_{k=1}^{n} \frac{a=x+2k}{(a/2+m/2+1)_k} \left(-\frac{x}{2}\right) L_m^{(a+2k)}(x) = -x L_m^{(a+1)}(x) -$$

$$- \frac{2(-x/2)^{n+1}}{(a/2+m/2+1)_n} \left(-\frac{x}{2}\right)^{n+1} \; L_m^{(a+2n+1)}(x)$$

(48.16.2.1) $$\sum_{k=0}^{\infty} \frac{(a)_k(m+1)_k}{k!(1-c-n)_k} (-1)^k L_{m+k}^{(c-k)}(x) = \frac{\Gamma(1-c-n)\Gamma(1-a)}{n!\Gamma 1-c-a-n)} (-x)^n \times$$

$$\times \sum_{k=0}^{\infty} \frac{(a+n)_k}{k!(n+1)_k} x^k L_m^{(c+n+k)}(x)$$

(48.16.2.2) $$\sum_{k=0}^{n} \frac{9-n)_k(m+1)_k}{k!(1-c-n)_k} (-1)^k L_{m+k}^{(c-k)}(x) = \frac{x^n}{(c)_n} L_m^{(c+n)}(x)$$

For $c =$ negative integer, see (48.16.3).

(48.16.4) $$\sum_{k=0}^{n} \frac{(-n)_k(a+c+m+n)_k}{k!(a)_k} (-1)^k L_{m+k}^{(c-k)}(x) =$$

$$= \frac{(a+c)_n\Gamma(c+m+n+1)}{(a)_n\Gamma(c+1)(m+n)!}\ {}_2F_2(-m-n,\, a+c+n;\, a+c,\, c+1;\, x)$$

For $a = c = 1 - n$ and $c =$ negative integer, see (48.46.3).

(48.16.5) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(a)_k}(-1)^k L_{m+k}^{(c-k)}(x) =$$

$$= (-1)^m \frac{n!\Gamma(a)}{m!\Gamma(a-m+n)} \sum_{k=0}^{m} \frac{(-m)_k(n+1)_k}{k!(a-m+n)_k} L_{n+k}^{(a+c-1)}(x)$$

For $m = 0$, see (48.16.1). For $n = 0$, see (48.3.8).

(48.16.6) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(1/2-a)_k}(-1)^k L_{m+k}^{(a-k)}(x^2) =$$

$$= (-1)^{m+n} 2^{-2n} \frac{\Gamma(1/2-a)}{m!\Gamma(1/2-a-m+n)} \sum_{k=0}^{m} \frac{(-m)_k}{k!(1/2-a-m+n)_k}(-1)^k 2^{-2k} H_{2n+2k}(x)$$

(48.16.7) $$\sum_{k=0}^{n} \frac{(-n)_k(m+1)_k}{k!(3/2-a)_k}(-1)^k L_{m+k}^{(a-k)}(x^2) =$$

$$= (-1)^{m+n} 2^{-2n-1} \frac{\Gamma(3/2-a)}{xm!\Gamma(3/2-a-m+n)} \sum_{k=0}^{m} \frac{(-m)_k}{k!(3/2-a-m+n)_k}(-1)^k 2^{-2k} H_{2n+2k+1}(x)$$

(48.19.19) $$\sum_{k=0}^{\infty} \frac{(c)_k}{(a+1)_{2k}}(4x)^k L_k^{(a+k)}(x) = {}_1F_1(2c;\, a+1;\, 2x)$$

(49.19.20) $$\sum_{k=0}^{n} \frac{(-n)_k}{(a+1)_{2k}}(4x)^k L_k^{(a+k)}(x) = \frac{(2n)!}{(a+1)_{2n}} L_{2n}^{(a)}(2x)$$

(49.0.1) $$\sum_{k=0}^{n} (2k-2x+3)H_k(x) = (2x-1)[H_{n+1}(x)-1)] - H_{n+2}(x) + 2x$$

(49.3.2) $$\sum_{k=0}^{n-1} (k+1)(2x)^{-k} H_k(x) = x[H_1(x) - (2x)^{-n} H_{n+1}(x)]$$

(49.4.2.1) $$\sum_{k=0}^{\infty} \frac{(2x)^k}{(a)_k} H_k(x) = {}_1F_2(a-1;\, a,\, a+1/2;\, x^2)$$

(49.4.2.2) $$\sum_{k=0}^{\infty} \frac{(2x)^k}{(k+1)!} H_k(x) = \frac{1}{x}\exp(x^2)\operatorname{erf}(x)$$

(49.4.2.3) $$\sum_{k=0}^{\infty} \frac{x^k}{(m+1)_k} H_k(x) = 1 + x^{-m}\exp(x^2)\gamma(m/2+1,\, x^2)$$

(49.4.2.4) $$\sum_{k=1}^{n} \frac{x^k}{(m)_k} H_{k+m}(x) = 2x\left[\frac{x^n}{(m)_n} H_{m+n-1}(x) - H_{m-1}(x)\right]$$

For $m = 1$, see AN88p.678.

(49.4.2.5) $\sum_{k=0}^{n} \frac{x^k}{k!} H_{2m+k+1}(x) = (-1)^m 2^{2m+1} x^{n+1} \frac{m!}{n!} \sum_{k=0}^{m} \frac{1}{k!} (-1)^k 2^{-2k} H_{n+2k}(x)$

(49.4.4.1) $\sum_{k=0}^{\infty} \frac{1}{(k+1)!} (-4)^{-k} H_{2k}(x) = x \exp(x^2) \Gamma(-1/2,\, x^2)$

(49.4.4.2) $\sum_{k=0}^{\infty} \frac{1}{(k+1)!} (-1)^k 2^{-2k} H_{2k+1}(x) = 2 \exp(x^2) \Gamma(1/2,\, x^2)$

(49.4.4.3) $\sum_{k=1}^{n} \frac{1}{(m+1/2)_k} (-1)^k 2^{-2k} H_{2m+2k}(x) = \frac{1}{4x^2} \{ -H_{2m+2}(x) -$

$-4(m+1) H_{2m}(x) + \frac{1}{(m+1/2)_n} (-1)^n 2^{-2n} [4(m+n+1) H_{2m+2n}(x) +$

$+ H_{2m+2n+2}(x)] \}$

(49.4.4.4) $\sum_{k=1}^{n} \frac{1}{(m+3/2)_k} (-1)^k 2^{-2k} H_{2m+2k+1}(x) =$

$= \frac{1}{4x^2} \{ -H_{2m+3}(x) - 4(m+1) H_{2m+1}(x) +$

$+ \frac{1}{(m+3/2)_n} (-1)^n 2^{-2n} [H_{2m+2n+3}(x) + 4(m+n+1) H_{2m+2n+1}(x)] \}$

(49.4.26.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(b)_k} (-1)^k 2^{-2k} H_{2m+2k}(x) =$

$= (-1)^{m+n} 2^{2m} \frac{n!\Gamma(b)}{\Gamma(b+n-m)} \sum_{k=0}^{m} \frac{(-m)_k (n+1)_k}{k!(b+n-m)_k} (-1)^k L_{n+k}^{(1/2-b-n+m-k)}(x^2)$

For $m = 0$, see (49.4.22).

(49.4.26.2) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(m+1/2)_k} (-1)^k 2^{-2k} H_{2m+2k}(x) =$

$= \frac{m!}{(m+1/2)_n} (-4)^m x^{2n} L_m^{(n-1/2)}(x^2)$

(49.4.26.3) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(m+1)_k} (-1)^k 2^{-2k} H_{2m+2k}(x) = (-1)^{m+n} 2^{2m} m! L_{m+n}^{(-n-1/2)}(x^2)$

(49.4.26.4) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(b)_k} (-1)^k 2^{-2k} H_{2m+2k+1}(x) =$

$= (-1)^{m+n} 2^{2m+1} x \frac{n!\Gamma(b)}{\Gamma(b-m+n)} \sum_{k=0}^{m} \frac{(-m)_k (n+1)_k}{k!(b-m+n)_k} (-1)^k L_{n+k}^{(3/2-b+m-n-k)}(x^2)$

For $m = 0$, see (49.4.25).

(49.4.26.5) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(m+3/2)_k} (-1)^k 2^{-2k} H_{2m+2k+1}(x) =$

$= \frac{2m!}{(m+3/2)_n} (-4)^m x^{2n+1} L_m^{(n+1/2)}(x^2)$

(49.4.26.6) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(m+1)_k}(-1)^k 2^{-2k} H_{2m+2k+1}(x) =$

$= (-1)^{m+n} 2^{2m+1} m!\ x\ L_{m+n}^{(1/2-n)}(x^2)$

(49.4.29) $\sum_{k=0}^{n} \frac{(-n)_k (a+m)_k}{k!(a)_k (b)_k}(-1)^k 2^{-2k} H_{2k}(x) =$

$= (-1)^m \frac{n!\Gamma(m-n-b+1)}{(a)_n \Gamma(1-b)} \sum_{k=0}^{m} \frac{(-m)_k (a+n)_k}{k!(b-m+n)_k}(-1)^k L_{n-m+k}^{(1/2-b+m-n-k)}(x^2)$

(49.4.30) $\sum_{k=0}^{n} \frac{(-n)_k (a+m)_k}{k!(a)_k (b)_k}(-1)^k 2^{-2k} H_{2k+1}(x) =$

$= 2x(-1)^n \frac{\Gamma(b)}{(a)_m \Gamma(b-m+n)} \sum_{k=0}^{m} \frac{(-m)_k (a+n)_k}{k!(b-m+n)_k}(-1)^k L_{n-m+k}^{(3/2-b+m-n-k)}(x^2)$

$(m \le n)$

(49.5.8) $\sum_{k=1}^{\infty} \frac{1}{k(1/2)_k}(-1)^k 2^{-2k}\ H_{2k}(x) = \psi(1/2) - 2 \log x$

(49.5.9) $\sum_{k=1}^{\infty} \frac{1}{k(3/2)_k}(-1)^k 2^{-2k} H_{2k+1}(x) = 2x[\psi(3/2) - 2 \log x]$

(49.6.15.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(m+1)_k} 2^{-k} H_{m+k}(x) H_{r+k}(x) =$

$= (-1)^n 2^{r-n} \frac{m!}{(m-r+n)!} \sum_{k=0}^{r} \frac{(-r)_k}{k!(m-r+n+1)_k}(-2)^{-k} H_{m-r+2n+2k}(x)$

$m \ge r$. For $r = 0$, see (49.6.16). For $n = 0$, see (49.4.17).

(49.6.17.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(3/2-n)_k} 2^{-k} H_k(x) H_{k+1}(x) = (1-2n)\frac{n!}{(2n)!} 2^{-1/2} H_{2n+1}(2^{1/2}x)$

(50.7.7.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} B_m(kx+y) = (-m)_n x^n \sum_{k=0}^{m-n} \frac{(m-n)_k}{(n+1)_k}(-x)^k \widehat{S}_{n+k}^{(n)} B_{m-n-k}(y)$

$m \ge n$. For $m = 0, 1, ..., n+1$, see (50.7.7). $\widehat{S}_{n+k}^{(n)}$ is a Stirling number of the second kind.

(50.7.13.1) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(ka+b)}[B_m(x-k) - B_m] = \frac{n!}{b(b/a+1)_n}[B_m(x+b/a) - B_m]$

$m \le n$

(50.7.13.2) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(r-k)}[B_m(r-k+1) - B_m] =$

$$= \begin{cases} 0 \text{ for } m \le n, \\ mn! \sum_{k=0}^{m-n-1} \frac{(n-r+1)_k (n+1)_k}{k!(n+k+1)} (-1)^k \widehat{S}_m^{(n+k+1)} \text{ for } m > n \end{cases}$$

$m = 2, 3, 4, ...; r > n$. $\widehat{S}_m^{(n+k+1)}$ is a Stirling number of the second kind.

(51.2.2) $$\sum_{k=1}^{\infty} \frac{1}{k}(-a)^{-k} E_k(x) \sim -2 \log \left\{ 2^{1/2} a^{-1/2} \frac{\Gamma[(a+x+1)/2]}{\Gamma[(a+x)/2]} \right\}$$

SQ26p.278(83.8)

This series is asymptotic as $a \to \infty$ for $|\arg(a)| < \pi$.

(52.1.1.1) $$\sum_{k=0}^{n-1} a^k \widehat{S}_{m+k}^{(m)} = \frac{(-a)^{-m}}{(1-1/a)_m} \left[1 - a^{m+n} \sum_{k=1}^{m} (-1)^k (-1/a)_k \widehat{S}_{m+n}^{(k)} \right]$$

$\widehat{S}$ is Stirling number of the second kind.

(52.1.13.1) $$\sum_{k=0}^{n} \frac{(-n)_k (m+1)_k}{k!(r+m+1)_k} (-1)^k \widehat{S}_{r+m+k}^{(r)} =$$

$$= (-1)^m \frac{(r+m)!n!}{(r+n)!m!} \sum_{k=0}^{m} \frac{(-m)_k (n+1)_k}{k!(r+n+1)_k} \widehat{S}_{r+n+k+1}^{(r+1)}$$

For $m = 0$, see (52.1.8). For $n = 0$, see (52.1.9).

$\widehat{S}$ is Stirling number of the second kind.

(52.1.14.1) $$\sum_{k=1}^{n} (-1)^k (x)_k \widehat{S}_m^{(k)} = (-x)^m \left[1 - (x+1)_n x^{-n} \sum_{k=0}^{m-n-1} (-x)^{-k} \widehat{S}_{n+k}^{(n)} \right]$$

$m > n$

(54.6.0) $$\sum_{k=0}^{\infty} \frac{x^k}{k!} \zeta(a-k) = \frac{1}{\Gamma(a)} \left[\pi(-x)^a \csc \pi a + \int_0^{\infty} \frac{t^{a-1}}{\exp(t-x)-1} dt \right]$$ DJp.73

(55.2.9) $$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} [\psi(k+1/2) - \psi(1/2)] = \frac{1}{2}(2^{2n+1}-1)\zeta(2n+1) -$$

$$- \sum_{k=1}^{n-1} (2^{2n-2k+1}-1)\zeta(2k)\zeta(2n-2k+1)$$ AM79p682(8)

$n = 1, 2, 3, ...$ For $n = 1$, omit the sum in the right member.

(55.2.10) $$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} [\psi(k+3/2) - \psi(1/2)] = (1-2^{-2n-1})\zeta(2n+1) +$$

$$+ 2(1-2^{-2n})\zeta(2n) \log 2 - 2^{-2n} \sum_{k=1}^{n-1} (2^{2k}-1)\zeta(2k)\zeta(2n-2k+1)$$

AM79p683(9)

$n = 1, 2, 3, \ldots$. For $n = 1$, omit the sum in the right member.

(55.4.2.1) $\sum_{k=1}^{\infty} \frac{(a)_k}{k!} [\psi(x+k) - \psi(x)] = -\frac{\Gamma(-a)\Gamma(x)}{\Gamma(x-a)}$

(55.4.2.2) $\sum_{k=2}^{\infty} \frac{(a)_k}{(b)_k} [\psi(k) - \psi(1)] = \frac{a}{a-b+1} [\psi(b-a+1) - \psi(b)]$

$\mathrm{Re}(b-a) > 1$

(55.4.2.3) $\sum_{k=1}^{n} \frac{(a)_k}{(b)_k} [\psi(b+k) - \psi(b)] = \frac{a}{(a-b+1)^2} \left[1 - \frac{(a+1)(a+2)_n}{b(b+1)_n}\right] +$

$+\frac{a(a+1)_n}{(a-b+1)(b)_n} [\psi(b+n+1) - \psi(b)]$

(55.4.2.4) $\sum_{k=1}^{n} \frac{(a)_k}{(b)_k} [\psi(a+k) - \psi(a)] = \frac{1}{(a-b+1)^2} \{b - 1 +$

$+a\frac{(a+1)_n}{(b)_n} [-1 + (a-b+1)(\psi(a+n+1) - \psi(a))]\}$

(55.4.5.1) $\sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{k!(a)_k} [\psi(a+k) - \psi(a)] =$

$= \frac{\Gamma(a)\Gamma(a-b-c)}{\Gamma(a-b)\Gamma(a-c)} [\psi(a-b) + \psi(a-c) - \psi(a) - \psi(a-b-c)]$ $\mathrm{Re}(c+b-a) < 1$

(55.4.5.2) $\sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{k!(a)_k} [\psi(c-k) - \psi(c)] =$

$= \frac{\Gamma(a)\Gamma(a-b-c)}{\Gamma(a-b)\Gamma(a-c)} [\psi(a-c) - \psi(a-b-c)]$

$\mathrm{Re}(c+b-a) < 1$

(55.5.2) $\sum_{k=1}^{\infty} \frac{(a)_k (a-b+1)_k}{k!(b)_k} (2k+a)[\psi(k+a/2) - \psi(a/2)] =$

$= \frac{\Gamma(b)\Gamma(b-a-1)}{2\Gamma(a)} \left[\frac{\Gamma(a/2)}{\Gamma(b-a/2)}\right]^2 - 2$ $\mathrm{Re}(a-b) \leq 3/4$

(56.1.9.1) $\sum_{k=1}^{n} t^k P_{p+k}^{(q)}(x) = \frac{2}{(2q+1)(t^2-1)} [(p-q+1)tP_{p+1}^{(q)}(x) -$

$(p+q+1)t^2 P_p^{(q)}(x) + (q-p-n-1)t^{n+1} P_{p+n+1}^{(q)}(x) +$

$(p+q+n+1)t^{n+2} P_{p+n}^{(q)}(x)]$ where $t^2 - 2xt + 1 = 0$.

(56.1.14.1) $\sum_{k=0}^{n-1} (-1)^k \frac{(p/2-q/2+1/2)_k}{(p/2+q/2+1)_k} (2p+4k+1) P_{p+2k}^{(q)}(x) =$

$= \frac{p+q}{x} \left[(-1)^{n-1} \frac{(p/2-q/2+1/2)_n}{(p/2+q/2)_n} P_{p+2n-1}^{(q)}(x) + P_{p-1}^{(q)}(x)\right]$

This equation is satisfied if $P_p^{(q)}(x)$ is replaced by any function satisfying the recursion relation $(p-q+1)P_{p+1}^{(q)}(x)-(2p+1)xP_p^{(q)}(x)+(p+q)P_{p-1}^{(q)}(x)=0$.

(56.1.14.2) $$\sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(a+k)} \left(\tanh \tfrac{x}{2}\right)^k \ \widehat{P}_p^{(q+k)}(\cosh x) =$$

$$= \Gamma(a) \left(\coth \tfrac{x}{2}\right)^a \ \widehat{P}_p^{(q-a)}(\cosh x)$$

($\widehat{P}$ denotes the associated Legendre function of the first kind.)

(56.1.14.3) $$\sum_{k=0}^{\infty} \tfrac{(-1)^k}{k!(a+k)} \left(\tan \tfrac{x}{2}\right)^k P_p^{(q+k)}(\cos x) = \Gamma(a) \left(\cot \tfrac{x}{2}\right)^a P_p^{(q-a)}(\cos x)$$

(56.1.16.1) $$\sum_{k=1}^{n-1} \tfrac{(p-1/2)_k}{(p+q)_k}(p-q+k)(2x)^k P_{p+k}^{(q)}(x) =$$

$$= \tfrac{(p-1/2)_n}{(p+q-1)_n}(p+q-1)(2x)^n P_{p+n-2}^{(q)}(x) \ -(2p-1)xP_{p-1}^{(q)}(x)$$

This equation holds if $P_p^{(q)}(x)$ is replaced by any function satisfying the recursion relation

$$(p-q+1)P_{p+1}^{(q)}(x)-(2p+1)xP_p^{(q)}(x)+(p+q)P_{p-1}^{(q)}(x)=0.$$

(56.1.16.2) $$\sum_{k=0}^{n} \tfrac{(p-q+2)_k}{(p+5/2)_k}(p+q+k+1)(2x)^{-k} P_{p+k}^{(q)}(x) =$$

$$= (2p+3)xP_{p+1}^{(q)}(x) - \tfrac{(p-q+3)_n}{(p+5/2)_n}(p-q+2)(2x)^{-n} P_{p+n+2}^{(q)}(x)$$

This equation holds if $P_p^{(q)}(x)$ is replaced by any function satisfying the recursion relation

$$(p-q+1)P_{p+1}^{(q)}(x)-(2p+1)xP_p^{(q)}(x)+(p+q)P_{p-1}^{(q)}(x)=0.$$

(56.3.17) $$\sum_{k=0}^{n} \tfrac{q+2k}{(q/2-p/2+1/2)_k(q/2+p/2+1)_k} 2^{-2k} P_p^{(q+2k)}(x) = \tfrac{1}{2x}(1-x^2)^{1/2} \times$$

$$\times \left[(q+p)(q-p-1)P_p^{(q-1)}(x) - \tfrac{2^{-2n}}{(q/2+p/2+1)_n(q/2-p/2+1/2)_n} P_p^{(q+2n+1)}(x)\right]$$

This equation holds if $P_p^{(q)}(x)$ is replaced by any function satisfying the recursion relation

$$P_p^{(q+1)}(x) + 2qx(1-x^2)^{-1/2} P_p^{(q)}(x) + (p-q+1)(p+q)P_p^{(q-1)}(x) = 0.$$

(56.3.18) $$\sum_{k=1}^{n} \tfrac{(q-1)_k}{(p+q)_k(q-p-1)_k} \left(\tfrac{4x^2}{1-x^2}\right)^{k/2} P_p^{(q+k)}(x) = (q-1)\left(\tfrac{4x}{1-x^2}\right)^{1/2} \times$$

$$\times \left[P_p^{(q-1)}(x) - \tfrac{(q)_n}{(q+p)_n(q-p-1)_n} \left(\tfrac{4x^2}{1-x^2}\right)^{n/2} P_p^{(q+n-1)}(x)\right]$$

This equation holds if $P_p^{(q)}(x)$ is replaced by any function satisfying the recursion relation

$$P_p^{(q+1)}(x)+q\left(\frac{4x^2}{1-x^2}\right)^{1/2}P_p^{(q)}(x)+(p-q+1)(p+q)P_p^{(q-1)}(x)=0.$$

(56.5.2) $$\sum_{k=0}^{n}\frac{(-n)_k}{k!(1/2-a)_k}(-1)^k[4(1-x^2)]^{-k/2}P_{a-n-k}^{(k-a)}(x)=\frac{[4(1-x^2)]^{-n/2}}{(1/2-a)_n}P_a^{(n-a)}(x)$$

(57.14.3.1) $$\sum_{k=0}^{n-1}\frac{1}{(a+2)_k}z^kZ_{a+k}(2z)=$$

$$=(a+1)\left[\frac{1}{z}Z_{a+1}(2z)-\frac{1}{(a+1)_n}Z_{a+n+1}(2z)\right]$$

(57.14.3.2) $$\sum_{k=1}^{n}\frac{1}{(2-a)_k}(-z)^kZ_{a-k}(2z)=Z_{a-2}(2z)-\frac{\Gamma(a-n)}{\Gamma(a-1)}z^{n-1}Z_{a-n-1}(2z)$$

(57.14.4.1) $$\sum_{k=1}^{n}\frac{(z/2)^k}{(a+3/2)_k}J_{a+k}(z)=\pi^{1/2}\Gamma(a+3/2)(2/z)^a[J_{a+n}(z)\mathbf{H}_{a+n+1}(z)-$$

$$-J_{a+n+1}(z)\mathbf{H}_{a+n}(z)+J_{a+1}(z)\mathbf{H}_a(z)-J_a(z)\mathbf{H}_{a+1}(z)]$$

(57.14.5.1) $$\sum_{k=1}^{n-1}(a-1)_kz^{-k}Z_{a+k}(2z)=$$

$$=(a-1)_nz^{-n}Z_{a+n-2}(2z)-\frac{a-1}{z}Z_{a-1}(2z)$$

(57.14.5.2) $$\sum_{k=0}^{n-1}(-a-1)_k(-z)^{-k}Z_{a-k}(2z)=$$

$$=(-a-1)_n(-z)^{-n}Z_{a-n+2}(2z)-Z_{a+2}(2z)$$

(57.14.7.1) $$\sum_{k=0}^{n}\frac{(-n)_k}{k!(b)_k}z^kJ_{a+k}(2z)=\frac{1}{\Gamma(a+1)}z^a\ {}_1F_2(b+n;a+1,b;-z^2)$$

(57.14.7.2) $$\sum_{k=0}^{\infty}\frac{(c)_k}{k!(b)_k}(-z)^kJ_{a-k}(2z)=$$

$$=\frac{\Gamma(b)\Gamma(a+b-c)}{\Gamma(a+1)\Gamma(b-c)\Gamma(a+b)}z^a_1F_2(a+b-c;a+1,a+b;-z^2)$$

(57.14.7.3) $$\sum_{k=0}^{\infty}\frac{(a-1/2)_k}{k!(a)_k}(-z)^kJ_{a-k}(2z)=z^{1-a}\Gamma(a)J_a(z)J_{a-1}(z)$$

(57.14.7.4) $$\sum_{k=0}^{\infty}\frac{(-a-1/2)_k}{k!(-2a)_k}(-z)^kJ_{a-k}(2z)=$$

$$=\frac{2^{-2a-1}}{\Gamma(a+1)}z^a[-1-\pi z\csc(\pi a)J_a(z)J_{-a-1}(z)]$$

(57.14.7.5) $$\sum_{k=0}^{\infty}\frac{(1/2-a)_k}{k!(1-2a)_k}(-z)^kJ_{a-k}(2z)=\frac{\pi}{\Gamma(a)}\left(\frac{z}{4}\right)^a\csc(\pi a)J_a(z)J_{-a}(z)$$

(57.14.7.6) $\sum_{k=0}^{\infty} \frac{(1/2)_k}{k!(1-a/2)_k}(-z)^k J_{a-k}(2z) = \frac{\Gamma(1-a/2)}{\Gamma(1/2-a/2)}\pi^{1/2}[J_{a/2}(z)]^2$

(57.14.7.7) $\sum_{k=0}^{\infty} \frac{(a+1/2)_k}{k!((a+1)_k}(-z)^k J_{a-k}(2z) = \Gamma(a+1)z^{-a}[J_a(z)]^2$

(57.14.7.8) $\sum_{k=0}^{\infty} \frac{(1/2)_k}{k!(3/2-a/2)_k}(-z)^k J_{a-k}(2z) =$

$= \frac{\Gamma(3/2-a/2)}{\Gamma(1-a/2)}\pi^{1/2} J_{a/2+1/2}(z) J_{a/2-1/2}(z)$

(57.14.7.9) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(1-a-n)_k}(-z)^k J_{a-k}(2z) = \frac{z^n}{(a)_n} J_{a+n}(2z)$

(57.14.8.1) $\sum_{k=0}^{\infty} \frac{1}{k!(b)_k} z^{2k} J_{a+2k}(4z) =$

$= \frac{1}{\Gamma(a+1)}(2z)^a \ {}_1F_2(b-1/2; 2b-1, a+1; -8z^2)$

For $b = 1/2$, see (57.11.5). For $b = 3/2$, see (57.11.6).

(57.14.8.2) $\sum_{k=0}^{\infty} \frac{1}{k!(a+3/2)_k} z^{2k} J_{a+2k}(4z) =$

$= (2\pi)^{-1/2} z^{-a-1}\Gamma(a+3/2) J_{2a+1}(2^{5/2}z)$

(57.14.8.3) $\sum_{k=0}^{\infty} \frac{1}{k!(b)_k} \left(\frac{z}{2}\right)^{2k} J_{a-2k}(2z) = \frac{\Gamma(b)\Gamma(a+b-1/2)}{\Gamma(a+1)\Gamma(a+2b-1)}\pi^{-1/2}2^{a+2b-2} z^a \times$

$\times \ {}_1F_2(a+b-1/2; a+1, a+2b-1; -2z^2)$

(57.14.8.4) $\sum_{k=0}^{\infty} \frac{1}{k!(1-a)_k} \left(\frac{z}{2}\right)^{2k} J_{a-2k}(2z) =$

$= \Gamma(1-a)\left(\frac{z}{2}\right)^a J_a(2^{1/2}z) J_{-a}(2^{1/2}z)$

(57.14.11.1) $\sum_{k=0}^{n} \frac{(-n)_k(b)_k}{k!} z^{-k} J_{a+k}(2z) = \frac{(a-b+1)_n}{\Gamma(a+n+1)} z^a \times$

$\times \ {}_1F_2(a-b+n+1; a-b+1, a+n+1; -z^2)$

(57.14.11.2) $\sum_{k=0}^{n} \frac{(-n)_k(a+m)_k}{k!} z^{-k} J_{a+k}(2z) = \frac{n!}{m!\Gamma(a+m+n+1)}(-1)^m z^{a+2m} \times$

$\times \ {}_1F_2(n+1; m+1, a+m+n+1; -z^2) \qquad (0 \le m \le n)$

(57.14.11.3) $\sum_{k=0}^{n} \frac{(-n)_k(a+n)_k}{k!} z^{-k} J_{a+k}(2z) = (-1)^n J_{a+2n}(2z)$

For $a = 0$, see (57.14.12).

(57.14.12.1) $$\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!} (-z)^{-k} J_{a-k}(2z) = \frac{(b)_n}{\Gamma(a-n+1)} z^{a-2n} \times$$

$$\times\ {}_1F_2\left(1-b; 1-b-n, a-n+1; -z^2\right)$$

(57.14.12.2) $$\sum_{k=0}^{n} \frac{(-n)_k (n-a)_k}{k!} (-z)^{-k} J_{a-k}(2z) = (-1)^n J_{a-2n}(2z)$$

(57.14.12.3) $$\sum_{k=0}^{\infty} \frac{(m+1)_k (b-a)_k}{k!(k+1)!(b+m)_k} z^k J_{a+k}(2z) = (-1)^m z^{a-b} \frac{\Gamma(b+m)\Gamma(b-a-1)}{m!\Gamma(b-a-m)\Gamma(a+m)} \times$$

$$\times \sum_{k=0}^{m} \frac{(-m)_k (b-1)_k (b-a-1)_k}{k!(b-a-m)_k} z^{-k} J_{b+k-2}(2z)$$

For m=1, see (57.14.10).

(57.15.5.1) $$\sum_{k=0}^{\infty} \frac{1}{k!(b+k)} (-z)^k J_{a-k}(2z) = \Gamma(b) z^{-b} J_{a+b}(2z)$$

For $a = -1/2$, see (57.15.9).

(57.25.11) $$\sum_{k=0}^{\infty} \frac{(1/2)_k (a)_k}{k!(a+1/2)_k} (2k+a)[J_{2k+a}(z)]^2 =$$

$$= 2^{-4a-1} \pi^{-1/2} \frac{\Gamma(a+1/2)}{\Gamma(a-1)} \int_0^{2z} J_{2a-1}(t)\,dt$$ AKp.414(8.15)

(57.29.1.1) $$\sum_{k=1}^{\infty} \frac{1}{a^2-k^2} (-1)^k J_{2m}(kx) = \frac{\pi}{2a} \csc(\pi a) J_{2m}(ax)$$

$0 < x \le \pi$; $m = 1, 2, 3, \ldots$. For $x = \pi$, see SQ22p.508.

(57.29.3) $$\sum_{k=1}^{\infty} \frac{k}{a^2-k^2} (-1)^k J_{2m+1}(kx) = \frac{\pi}{2} \csc(\pi a) J_{2m+1}(ax)$$

$0 < x \le \pi$; $m = 0, 1, 2, \ldots$. For $x = \pi$, see SQ22p.508.

(57.30.3) $$\sum_{k=1}^{\infty} k^{-a} J_a(kx) =$$

$$= -\frac{2^{-a-1} x^a}{\Gamma(a+1)} + \frac{2\pi^{1/2} (x/2)^a}{\Gamma(a+1/2)|x|} \left[\frac{1}{2} + \sum_{k=1}^{m} (1 - 4\pi^2 k^2/x^2)^{a-1/2}\right]$$

where m is an integer such that $2\pi m < |x| < 2\pi(m+1)$; $a > -1/2$. For $m = 0$, omit the sum in the right member. For $0 < x < 2\pi$, see OCp.23.

(57.30.4.1) $$\sum_{k=1}^{\infty} (-1)^k k^{1-a} J_a(kx) = \frac{1}{2\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(a)_k} (2^{2k}-1) \left(\frac{x}{2}\right)^{a+2k-2} B_{2k}$$

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$\mathrm{Re}(a) > 3/2$.

(57.30.6) $\sum_{k=1}^{\infty} k^{-2n-a} J_a(kx) = (-1)^n \left(\frac{x}{2}\right)^{a+2n} \Bigg[\frac{\pi}{|x|\Gamma(n+1/2)\Gamma(a+n+1/2)} +$

$$+ \frac{-1}{2n!\Gamma(a+n+1)} \sum_{k=0}^{n} \frac{(-n)_k(-a-n)_k}{(2k)!} \left(\frac{4\pi}{x}\right)^{2k} B_{2k} + S\Bigg]$$

$$\text{where } S = \begin{cases} 0 \text{ for } -2\pi \le x \le 2\pi, \\ \frac{(2\pi)^{1/2}}{|x|}(1 - 4\pi^2 x^{-2})^{a/2+n-1/4} P_{a-1/2}^{(1,2-a-2n)}\left(\frac{2\pi}{|x|}\right) \text{ for } 2\pi \le |x| \le 4\pi \end{cases}$$

$\mathrm{Re}(a) > 1/2 - 2n$; $n = 0, 1, 2, ...$ For $-2\pi \le x \le 2\pi$, see MH122p.51(10). For some special cases, see (57.28.1), (57.28.5), (57.30.1), and (57.30.3). Also, see (34.1.18.1).

(57.30.7) $\sum_{k=1}^{n} k^{2n-a} J_a(kx) = \frac{(x/2)^{a-2n}\Gamma(n+1/2)}{|x|\Gamma(a-n+1/2)} +$

$$+ \frac{(-1)^n(2n)!\Gamma(a-2n)(2x)^{a-2n}}{\Gamma(2a-2n)|x|} \sum_{k=1}^{m} (1 - 4\pi^2 k^2/x^2)^{a-2n-1/2} C_{2n}^{(a-2n)}\left(\frac{2\pi k}{|x|}\right)$$

where m is an integer such that $2\pi m < |x| < 2\pi(m+1)$; $\mathrm{Re}(a) > 2n + 1/2$; $n = 1, 2, 3, ...$ For $m = 0$, omit the second term in the right member. For $a = 2n$,see (57.27.1). For $a = 2n + 1$, see (57.28.2). For $n = 0$, see (57.30.3).

(57.30.8) $\sum_{k=1}^{\infty} (-1)^k k^{-2n-a} J_a(kx) =$

$$= \frac{(-1)^n (x/2)^{a+2n}}{2n!\Gamma(a+n+1)} \sum_{k=0}^{n} \frac{(-n)_k(-a-n)_k}{(2k)!} (2^{2k} - 2) \left(\frac{2\pi}{x}\right)^{2k} B_{2k} +$$

$$+ \begin{cases} 0 \text{ if } -\pi < x < \pi, \\ \frac{x^{a+2n}}{|x|}(-1)^n (2\pi)^{1/2} \left(1 - \frac{\pi^2}{x^2}\right)^{a/2+n-1/4} P_{a-1/2}^{(1/2-a-2n)}\left(\frac{\pi}{|x|}\right) \text{ if } \pi < |x| < 3\pi. \end{cases}$$

$\mathrm{Re}(a) \ge 1/2 - 2n$; $n = 0, 1, 2, ...$ For some special cases, see (57.27.2), (57.28.6), (57.30.2), and (57.30.4).

(57.30.9) $\sum_{k=1}^{\infty} (-1)^k k^{2n-a} J_a(kx) = (-1)^n \frac{(2x)^{a-2n}(2n)!\Gamma(a-2n)}{|x|\Gamma(2a-2n)} \times$

$$\times \sum_{k=1}^{m} [1 - (2k-1)^2\pi^2/x^2]^{a-2n-1/2} C_{2n}^{(a-2n)}[(2k-1)\pi/x]$$

$(2m-1)\pi < |x| < (2m+1)\pi$; $m = 0, 1, 2, ...$; $n = 1, 2, 3, ...$; $\mathrm{Re}(a) > 2n+1/2$.

For $m = 0$, the right member is zero. For $a = 2n + 1$, see (57.28.3). For $0 \le x \le \pi$, see (57.30.5).

(57.30.10) $\sum_{k=0}^{\infty} (2k+1)^{-2n-a} J_a](2k+1)x] = \frac{1}{2}(-1)^n \left(\frac{x}{2}\right)^{2n+a} \times$

$$\times\left[\frac{\pi}{|x|\Gamma(n+1/2)\Gamma(a+n+1/2)}-\frac{1}{2n!\Gamma(a+n+1)}\sum_{k=0}^{n}\frac{(-n)_k(-a-n)_k}{(2k)!}\left(\frac{2\pi}{x}\right)^{2k}(2^{2k+1}-2)B_{2k}\right]$$

-$\pi < x < \pi$; Re$(a) > 1/2 - 2n$; $n = 1, 2, 3, \ldots$. For $a = 0$ and $n = 1$, see (57.28.7).

(57.30.11) $$\sum_{k=0}^{\infty}(2k+1)^{2n-a}J_a[(2k+1)x]=\frac{(x/2)^{a-2n}\Gamma(n+1/2)}{2|x|\Gamma(a-n+1/2)}$$

$-\pi < x < \pi$; Re$(a) > 2n + 1/2$; $n = 1, 2, 3, \ldots$

(57.30.12) $$\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-a}J_a[(2k+1)x]=$$

$$=\frac{1}{2\Gamma(a+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(a+1)_k}\left(\frac{x}{2}\right)^{a+2k}E_{2k}$$ AKp.170

Re$(a) > 1/2$

(57.30.13) $$\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-a-2n-1}J_a[(2k+1)x]=$$

$$=\frac{(-1)^n\pi(x/2)^{a+2n}}{4n!\Gamma(a+n+1)}\sum_{k=0}^{n}\frac{(-n)_k(-a-n)_k}{(2k)!}\left(\frac{\pi}{x}\right)^{2k}E_{2k}$$

$-\pi/2 < x < \pi/2$; Re$(a) > -1/2$; $n = 0, 1, 2, \ldots$

(57.32.5) $$\sum_{k=1}^{\infty}\frac{k^{2n-a}}{k^2-c^2}J_a(kx)=\frac{(x/2)^a}{2c^2\Gamma(a+1)}\delta_{n,0}-\frac{\pi}{2}c^{2n-a-1}[\cot(\pi c)J_a(cx)+$$

$$+sgn(x)\mathbf{H}_a(cx)]-\frac{sgn(x)}{\Gamma(a+3/2)}\pi^{1/2}c^{2n}(x/2)^{a+1}\sum_{k=1}^{n}(-1)^k(-1/2)_k(-a-1/2)_k(cx/2)^{-2k}$$

$-2\pi < x < 2\pi$; Re$(a) > 2n - 3/2$; $n = 0, 1, 2, \ldots$

For $n = 0$, see (57.32.1). For $n = 1$, see (57.32.4).

(57.32.6) $$\sum_{k=1}^{\infty}\frac{k^{-2n-a}}{k^2-c^2}J_a(kx)=-\frac{\pi}{2}c^{-a-2n-1}[sgn(x)\mathbf{H}_a(cx)+\cot(\pi c)J_a(cx)]+$$

$$+\frac{(x/2)^a}{2c^{2n+2}}\sum_{k=0}^{n}(-1)^k\frac{(2\pi c)^{2k}}{(2k)!}B_{2k}\sum_{s=2}^{n-k}\frac{(-1)^s(cx/2)^{2s}}{s!\Gamma(a+s+1)}+\frac{\pi(x/2)^{a+2}}{|x|c^{2n}}\sum_{k=0}^{n-1}\frac{(-1)^k(cx/2)^{2k}}{\Gamma(k+3/2)\Gamma(a+k+3/2)}$$

$n = 1, 2, 3, \ldots$; Re$(a) > -2n - 3/2$, $-2\pi < x < 2\pi$

For $0 \le x < 2\pi$, see MH122p.51. For $n = 0$, see (57.32.1).

(57.32.7) $$\sum_{k=1}^{\infty}\frac{k^{-2n-a}}{k^2-c^2}(-1)^kJ_a(kx)=-\frac{\pi}{2}c^{-a-2n-1}\csc(\pi c)J_a(cx)+$$

$$+2^{-a-1}c^{-2n-2}x^a\sum_{r=0}^{n}(-1)^r\left(\frac{cx}{2}\right)^{2r}\sum_{k=0}^{r}\frac{(2-2^{2k})B_{2k}}{(2k)!(r-k)!\Gamma(a+r-k+1)}\left(\frac{2\pi}{x}\right)^{2k}$$

$-\pi \le x \le \pi$; $\mathrm{Re}(a) > -2n-3/2$; $n = 0, 1, 2, ...$

See ES7.15(57). For $a = n = 0$, see (57.29.2). For $n = 0$, see (57.32.2).

(57.38.8.1) $$\sum_{k=1}^{\infty} \frac{k^{2n-a-b}}{k^2-c^2} J_a(kx) J_b(kx) = -\frac{\pi}{2} c^{2n-a-b-1} \cot(\pi c) J_a(kc) J_b(kc) +$$

$$+\frac{1}{2} c^{2n} \left(\frac{x}{2}\right)^{a+b+1} \sum_{k=-n}^{\infty} \frac{\Gamma(-k-1/2)\Gamma(a+b+2k+2)}{\Gamma(a+k+3/2)\Gamma(b+k+3/2)\Gamma(a+b+k+3/2)} \left(\frac{cx}{2}\right)^{2k}$$

$0 < x < \pi$; $\mathrm{Re}(a+b) > 2n-2$; $n = 1, 2, 3, ...$

(57.34.8.2) $$\sum_{k=1}^{\infty} \frac{k^{-2n-a-b}}{k^2-c^2} J_a(kx) J_b(kx) = -\frac{\pi}{2} c^{-a-b-2n-1} \cot(\pi c) J_a(cx) J_b(cx) +$$

$$+\frac{c^{-2n-2}}{2\Gamma(a+1)\Gamma(b+1)} \left(\frac{x}{2}\right)^{a+b} \sum_{k=0}^{n} \frac{(a+b+1)_{2k}}{k!(a+1)_k (b+1)_k (a+b+1)_k} (-1)^k \left(\frac{cx}{2}\right)^{2k} \sum_{r=0}^{n-k} \frac{(-1)^r}{(2r!} (2\pi c)^{2r} B_{2r} +$$

$$+\frac{1}{2} c^{-2n} \left(\frac{x}{2}\right)^{a+b+1} \sum_{k=n}^{\infty} \frac{\Gamma(-k-1/2)\Gamma(a+b+2k+2)}{\Gamma(a+k+3/2)\Gamma(b+k+3/2)\Gamma(a+b+k+3/2)} \left(\frac{cx}{2}\right)^{2k}$$

$0 < x < \pi$; $\mathrm{Re}(a+b) > -2n-2$; $n = 0, 1, 2, ...$

(57.34.8.3) $$\sum_{k=1}^{\infty} \frac{k^{-2n+1/2}}{k^2-c^2} J_a(kx) J_{-a-1/2}(kx) =$$

$$= -\frac{\pi}{2} c^{-2n-1/2} \cot(\pi c) J_a(cx) J_{-a-1/2}(cx) - \frac{\pi}{2} c^{-2n-1/2} J_{a+1/2}(cx) J_{-a}(cx) +$$

$$+ \begin{cases} 0 \text{ for } n = -1, -2, -3, ..., \\ S \text{ for } n = 0, 1, 2, ... \end{cases}$$

where

$$S = \frac{c^{-2n-2}(2x)^{-1/2}}{\Gamma(a+1)\Gamma(-a+1/2)} \sum_{k=0}^{n} \frac{(-1)^k (1/2)_{2k} (cx)^{2k}}{(2k)!(a+1)_k(-a+1/2)_k} \sum_{r=0}^{n-k} \frac{(2\pi c)^{2r}}{(2r)!} (-1)^r B_{2r} +$$

$$+\frac{(\pi/2)^{3/2} c^{-2n}}{\Gamma(a+3/2)\Gamma(1-a)} \sum_{k=0}^{n-1} \frac{(-1)^k (3/2)_{2k} (cx)^{2k}}{(2k+1)!(a+3/2)_k (1-a)_k}$$

$0 < x < \pi$. For $n = 0$, see (57.34.11).

(57.34.11.1) $$\sum_{k=1}^{\infty} \frac{k^{-2n-a-b}}{k^2-c^2} (-1)^k J_a(kx) J_b(ky) =$$

$$= -\frac{\pi}{2} c^{-2n-a-b-1} \csc(\pi c) J_a(cx) J_b(cx) -$$

$$-\frac{1}{2} \left(\frac{x}{2}\right)^a \left(\frac{y}{2}\right)^b c^{-2n-2} \sum_{r=0}^{n} (-1)^r \left(\frac{cy}{2}\right)^{2r} \sum_{m=0}^{r} (2^{2m}-2) \frac{B_{2m}}{(2m)!} \left(\frac{2\pi}{y}\right)^{2m} \times$$

$$\times \sum_{k=0}^{r-m} \frac{(x/y)^{2k}}{k!(r-m-k)!\Gamma(a+k+1)\Gamma(b+r-m-k+1)}$$

$-\pi < x \pm y < \pi$; $\mathrm{Re}(a+b) > -2n-2; n = 0, 1, 2, \ldots$. For some special cases, see (57.34.7) and (57.34.12).

(57.34.11.2) $$\sum_{k=1}^{\infty} \frac{k^{2n-a-b}}{k^2-c^2}(-1)^k J_a(kx)J_b(ky) = -\frac{\pi}{2}c^{2n-a-b-1}\csc(\pi c)J_a(cx)J_b(cy)$$

$-\pi < x \pm y < \pi; \mathrm{Re}(a+b) > 2n-2; n = 1, 2, 3, \ldots$.

For some special cases, see (57.34.5) and (57.34.10).

(57.34.15.1) $$\sum_{k=1}^{\infty} k^{2n-a-b} J_a(kx)J_b(ky) =$$

$$= \frac{(x/2)^a(y/2)^{b-2n}\Gamma(n+1/2)}{|y|\Gamma(a+1)\Gamma(b-n+1/2)}\ {}_2F_1\left(n+\frac{1}{2}, n-b+\frac{1}{2}; a+1; \frac{x^2}{y^2}\right)$$

$-2\pi < x \pm y < 2\pi$; $|x| \le |y|$; $\mathrm{Re}(a+b) > 2n$; $n = 1, 2, 3, \ldots$.

(57.34.15.2) $$\sum_{k=1}^{\infty} k^{2n-a-b} J_a(kx)J_b(ky) = \frac{(x/2)^{a+b-2n}\Gamma(n+1/2)\Gamma(a+b-2n)}{|x|\Gamma(a-n+1/2)\Gamma(b-n+1/2)\Gamma(a+b-n+1/2)}$$

$-2\pi < x < 2\pi$; $\mathrm{Re}(a+b) > 2n$; $n = 1, 2, 3, \ldots$. For $0 < x < \pi$, see SQ22p.367(11). For some special cases, see (57.34.1), (57.34.3), and (57.34.4).

(57.34.15.3) $$\sum_{k=1}^{\infty} k^{-2n-a-b} J_a(kx)J_b(ky) = (-1)^n \frac{2^{-a-b-2n-1}x^a y^{b+2n}}{n!\Gamma(b+n+1)} \times$$

$$\times \left\{ 4\pi^{1/2} n(b+n)xy^{-2} \sum_{k=0}^{n-1} \frac{(1-n)_k(1-b-n)_k}{(3/2)_k(a+3/2)_k} \left(\frac{x}{y}\right)^{2k} \right.$$

$$\left. -\frac{1}{\Gamma(a+1)} \sum_{m=0}^{n} \frac{(-n)_m(-b-n)_m}{(2m)!} \left(\frac{4\pi}{y}\right)^{2m} B_{2m} \sum_{k=0}^{n-m} \frac{(m-n)_k(m-n-b)_k}{k!(a+1)_k} \left(\frac{x}{y}\right)^{2k} \right\}$$

(57.34.16) $$\sum_{k=1}^{\infty} (-1)^k k^{-2n-a-b} J_a(kx)J_b(ky) = \frac{1}{2}(-1)^n \left(\frac{x}{2}\right)^a \left(\frac{y}{2}\right)^{b+2n} \times$$

$$\times \sum_{k=0}^{n} \frac{1}{(n-k)!\Gamma(b+n-k+1)} \left(\frac{x}{y}\right)^{2k} \sum_{s=0}^{k} \frac{2^{2s}-2}{(2s)!(k-s)!\Gamma(a+k-s+1)} \left(\frac{2\pi}{x}\right)^{2s} B_{2s}$$

$-\pi \le x \pm y \le \pi$; $\mathrm{Re}(a+b) > -2n$; $n = 0, 1, 2, \ldots$.

(57.34.17) $$\sum_{k=1}^{\infty} (-1)^k k^{-2n-a-b} J_a(kx)J_b(kx) = \frac{1}{2}(-1)^n \left(\frac{x}{2}\right)^{a+b+2n} \times$$

$$\times \sum_{k=0}^{n} \frac{\Gamma(a+b+2n-2k+1)}{(2k)!(n-k)!\Gamma(a+n-k+1)\Gamma(b+n-k+1)\Gamma(a+b+n-k+1)} (2^{2k}-2)B_{2k} \left(\frac{2\pi}{x}\right)^{2k}$$

$-\pi/2 \le x \le \pi/2$; $\mathrm{Re}(a+b) > -2n$; $n = 0, 1, 2, \ldots$.

For $a+b=1/2$ and $n=0$, see OCp.23. For $n=1$, see OCp.40. For $b=-a$ and $n=1$, see (57.34.8)*. For $n=0$, see MN37p.501(13).

(57.34.18) $$\sum_{k=1}^{\infty}(-1)^k k^{2n-a-b} J_a(kx) J_b(ky) = 0$$

$-\pi \le x \pm y \le \pi$; $\mathrm{Re}(a+b) > 2n$; $n = 1, 2, 3, ...$

For $a=b=n+1/2$ and $y=x$, see (57.34.6).

(57.34.18.1) $$\sum_{k=1}^{\infty}(-1)^k (2k+1)^{2n-a-b-1} J_a[(2k+1)x] J_b[(2k+1)x] =$$

$$= \frac{\pi}{4\Gamma(a+1)\Gamma(b+1)} \left(\frac{x}{2}\right)^{a+b} \delta_{n,0}$$

$0 < x < \pi/2$; $\mathrm{Re}(a+b) > 2n-2$; $n = 0, 1, 2, ...$

For $a=b=n=$ odd integer, see MN37p.501(15).

(57.35.2) $$\sum_{k=1}^{\infty}(-1)^{k-1} \prod_{i=1}^{n} k^{-a_i} J_{a_i}(kx_i) = \frac{1}{2}\prod_{i=1}^{n} \frac{1}{\Gamma(a_i+1)} \left(\frac{x_i}{2}\right)^{a_i}$$

$0 < \sum_{i=1}^{n}|x_i| < \pi$; $\sum_{i=1}^{n}\mathrm{Re}(a_i) > 1 - n/2$; $n = 1, 2, 3, ...$

For $a_i = a$ $(i = 1, ..., n)$, see (57.35.1).

(58.6.2.1) $$\sum_{k=1}^{\infty} \frac{(1-a)_k}{(a)_k} I_k(z) = -\frac{1}{2} I_0(z) + \frac{1}{2} e^{-z}\ {}_1F_1(1/2; a; 2z)$$

(58.6.2.2) $$\sum_{k=1}^{\infty} (-1)^k \frac{(1-a)_k}{(a)_k} I_k(z) = -\frac{1}{2} I_0(z) + \frac{1}{2} e^{-z}\ {}_1F_1(a-1/2; a; 2z)$$

(59.2.1.1) $$\sum_{k=1}^{\infty} k^{-a-2n-1} \mathbf{H}_a(kx) = (-1)^n \left(\frac{x}{2}\right)^{a+2n} \left[\frac{\pi}{2n!\Gamma(a+n+1)}\right.$$

$$\left. - \frac{x}{4}\sum_{k=0}^{n} \frac{(4\pi/x)^{2k}}{(2k)!\Gamma(n-k+3/2)\Gamma(a+n-k+3/2)} B_{2k}\right]$$ MH122p.56(22)

$a > -3/2 : 0 < x < 2\pi$; $n = 0, 1, 2, ...$

(59.2.2.1) $$\sum_{k=1}^{\infty} (-1)^k k^{-a-2n-1} \mathbf{H}_a(kx) =$$

$$= (-1)^n \left(\frac{x}{2}\right)^{a+2n+1} \sum_{k=0}^{n} \frac{2^{2k-1}-1}{(2k)!\Gamma(n-k+3/2)\Gamma(a+n-k+3/2)} \left(\frac{2\pi}{x}\right)^{2k} B_{2k}$$

MH122p.54(18)

$a > -3/2$; $0 \le x < \pi$; $n = 0, 1, 2, ...$

(59.2.5.1) $\sum_{k=1}^{\infty} \frac{k^{-2n-a-1}}{k^2-b^2} \mathbf{H}_a(kx) = \frac{\pi}{2} b^{a-2n-2}[J_a(bx) - \cot(\pi b)\mathbf{H}_a(bx)] +$

$+\frac{1}{2} b^{-2n-2} \left(\frac{x}{2}\right)^a \sum_{s=0}^{n} (-1)^k \left(\frac{bx}{2}\right)^{2s} \left[\frac{-\pi}{s!\Gamma(a+s+1)} + \frac{x}{2} \sum_{k=0}^{s} \frac{(4\pi/x)^{2k} B_{2k}}{(2k)!\Gamma(s-k+3/2)\Gamma(a+s-k+3/2)}\right]$

MH122p.55

$a > -3/2;\ 0 < x < 2\pi;\ n = -1, 0, , 1, 2, ...$

(59.2.5.2) $\sum_{k=1}^{\infty} (-1)^k \frac{k^{-a-2n-1}}{k^2-b^2} \mathbf{H}_a(kx) = \frac{\pi}{2} b^{-a-2n-2} \csc(\pi b)\mathbf{H}_a(bx) +$

$+\sum_{s=0}^{n} b^{2s-2n-2} \left(\frac{x}{2}\right)^{a+2s+1} \sum_{k=o}^{s} \frac{2^{2k-1}-1}{(2k)!\Gamma(s-k+3/2)\Gamma(a+s-k+3/2)} (-1)^s \left(\frac{2\pi}{x}\right)^{2k} B_{2k}$

MH122p.55(19)

$a > -3/2;\ 0 \le x < \pi;\ n = -1, 0, 1, 2, ...$

(59.2.10) $\sum_{k=0}^{\infty} \frac{(-z)^k}{k!(b+k)} \mathbf{H}_{a-k}(2z) = \Gamma(b) z^{-b} \mathbf{H}_{a+b)}(2z)$

(59.2.11) $\sum_{k=0}^{\infty} \frac{(a)_k}{k!(a+n+1)_k} (-1)^k \mathbf{H}_{-n-k}(2x) = \begin{cases} 0 & \text{for } n = -1, -2, -3, ..., \\ S & \text{for } n = 0, 1, 2, ... \end{cases}$

where $S = \frac{\Gamma(a+n+1)}{n!} \left[(-1)^n x^{-a} \mathbf{H}_{a+n}(2x) + x^{1-n} \sum_{k=0}^{n-1} \frac{(-1)^k}{\Gamma(k-n+3/2)\Gamma(a+k+3/2)} x^{2k}\right]$

For $n = 0$, omit the sum in the definition of S.

(59.2.12) $\sum_{k=0}^{n} \frac{(-n)_k}{k!(1-a-n)_k} (-x)^k \mathbf{H}_{a-k}(2x) =$

$\frac{1}{(a)_n} \left[x^n \mathbf{H}_{a+n}(2x) + x^{a+1}(-1)^n \sum_{k=0}^{n-1} \frac{(-1)^k}{\Gamma(a+k+3/2)\Gamma(k-n+3/2)} x^{2k}\right]$

(59.2.13) $\sum_{k=0}^{\infty} \frac{(n)_k}{k!(1-a+n)_k} (-x)^k \mathbf{H}_{a-k}(2x) =$

$= (-1)^n (1-a)_n \left[x^{-n} \mathbf{H}_{a-n}(2x) - x^{a-2n+1} \sum_{k=0}^{n-1} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(a-n+k+3/2)} x^{2k}\right]$

(59.2.14) $\sum_{k=0}^{n} \frac{(-n)_k (a+n)_k}{k!} x^{-k} \mathbf{H}_{a+k}(2x) = (-1)^n \mathbf{H}_{a+2n}(2x) +$

$+\frac{2}{\Gamma(a+2n+3/2)} (-1)^n \pi^{-1/2} x^{a+2n+1} \sum_{k=1}^{n} (-1/2)_k (-a-2n-1/2)_k (-1)^k x^{-2k}$

(60.1.1.1) $\sum_{k=1}^{n} \frac{1}{(a/2+1)_k} (-2)^{-k} D_{a+2k}(x) =$

$= \frac{1}{x(a/2+1)_n}(-1)^{-n}D_{a+2n+1}(x) - \frac{1}{x}D_{a+1}(x)$

(60.1.6.1) $\sum_{k=}^{\infty}\frac{1}{k!(2k-a)}(-2)^{-k}D_{2k}(x) = 2^{-a/2-1}\Gamma(-a/2)D_a(x)$ IS26p.158

(60.1.6.2) $\sum_{k=0}^{\infty}\frac{1}{k!(2k+1-a)}(-2)^{-k}D_{2k+1}(x) = 2^{-a/2-1/2}\Gamma(1/2-a/2)D_a(x)$

IS26p.158

(61.1.6) $\sum_{k=0}^{n}\frac{(-n)_k}{k!}x^k\Gamma(k+1,\, y) = (-x)^n e^{-1/x}\Gamma(n+1,\, y-1/x)$

(61.1.7) $\sum_{k=0}^{n}\frac{(-n)_k}{k!(ka+b)}e^{kx}\Gamma(m,\, kx+y) = \frac{n!}{b(1+b/a)_n}e^{-xb/a}\Gamma(m,\, y-xb/a)$

$m = 1, 2, ..., n+1$

(61.2.3.1) $\sum_{k=0}^{n}\frac{(-n)_k}{k!}x^{-k}\gamma(m+k,\, x) = x^m\frac{(m-1)!n!}{(m+n)!}\ {}_1F_1(m;\, m+n+1;\, -x)$

(62.1.0) $\sum_{k=1}^{n}B_x(a+k,\, b) = B_x(a+1,\, b-1) - B(a+n+1,\, b-1)$

(62.1.0.1) $\sum_{k=1}^{n}B_x(a,\, b+k) = B_x(a-1,\, b+1) - B_x(a-1,\, b+n+1)$

(62.1.0.2) $\sum_{k=0}^{\infty}\frac{(c)_k}{k!}(-1)^k B_x(a+k,\, b-k) = B_x(a,\, b+c)$

(62.1.0.3) $\sum_{k=0}^{\infty}\frac{(c)_k}{k!}(-1)^k B_x(a-k,\, b+k) = B_x(a+c,\, b)$

(62.1.1.1) $\sum_{k=0}^{n}\frac{(-n)_k}{k!}x^k B_x(a-k,\, b+k) =$

$= (-1)^n x^a\frac{n!\Gamma(a-n)}{\Gamma(a+1)}\ {}_2F_1(1-b, a-n; a+1; x)$

(62.1.1.2) $\sum_{k=0}^{\infty}\frac{(c)_k}{k!}x^{-k}B_x(a+k,\, b) = x^a\frac{\Gamma(a)\Gamma(1-c)}{\Gamma(a-c+1)}\ {}_2F_1 1-b, a; a-c+1; x)$

(62.1.1.3) $\sum_{k=0}^{n}\frac{(-n)_k}{k!}(x-1)^{-k}B_x(a,\, b+k) =$

$= \frac{n!\Gamma(a)}{\Gamma(a+n+1)}x^{a+n}(x-1)^{-n}\ {}_2F_1(1-b, a; a+n+1; x)$

(62.1.2.1) $\sum_{k=0}^{n}\frac{(-n)_k}{k!}B_x(a,\, b-k) = (-1)^n B_x(a+n,\, b-n)$

(62.1.3.1) $\sum_{k=0}^{\infty}\frac{(c)_k}{k!}B_x(a,\, b+k) = B_x(a-c,\, b)$

(62.1.3.2) $\sum_{k=0}^{\infty} \frac{(c)_k}{k!} x^{-k} B_x(a+k,\ b) = \frac{\Gamma(a)\Gamma(1-c)}{\Gamma(a-c+1)} x^a \ {}_2F_1(1-b, a; 1+a-c; x)$

(62.1.3.3) $\sum_{k=1}^{n} \frac{(a+b)_k}{(a)_k} x^{-k} B_x(a+k, b) =$

$= \frac{a+b}{b(1-x)} \left[\frac{(a+b+1)_n}{(a)_n} x^{-n} B_x(a+n,\ b+1) - B_x(a,\ b+1) \right]$

(62.1.3.4) $\sum_{k=1}^{n} \frac{(a+b)_k}{(b)_k} (1-x)^{-k} B_x(a,\ b+k) =$

$= \frac{a+b}{ax} \left[\frac{(a+b+1)_n}{(b)_n} (1-x)^{-n} B_x(a+1,\ b+n) - B_x(a+1,\ b) \right]$

(62.1.6) $\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(a)_k} B_x(c+k, b-a-n+1) = x^c \sum_{k=0}^{\infty} \frac{(a-b)_k (a+n)_k}{k!(a)_k (c+k)}$

(62.1.7) $\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(a)_k} B_x(a+k,\ b-a-n+1) =$

$= \frac{x^a}{a} {}_2F_1(a-b, a+n; a+1; x)$

(62.1.8) $\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(a)_k} B_x(a-b+k-1,\ b-a-n+1) =$

$= \frac{x^{a-b-1}}{a-b-1} \ {}_2F_1(a-b-1, a+n; a; x)$

(62.1.9) $\sum_{k=0}^{n} \frac{(-n)_k (b)_k}{k!(a)_k} B_x(a+n+k-1,\ b-a-n+1) =$

$= \frac{x^{a+n-1}}{a+n-1} \ {}_2F_1(a-b, a+n-1; a; x)$

(62.1.10) $\sum_{k=0}^{n} \frac{(-n)_k (a+n-1/2)_k}{k!(a)_n} B_x(3/2-a-n-k,\ 1/2) =$

$= \frac{(2n-1)!\Gamma(a)}{(a+n-1)\Gamma(a+2n-1)} (-1)^n 2^{1-2n} x^{3/2-a-2n} C_{2n-1}^{(2-a-2n)} [(1-x)^{1/2}]$

$0 \le x \le 1;\ a < 3/2 - 2n;\ n = 1, 2, 3, \ldots.$

(62.1.11) $\sum_{k=0}^{\infty} \frac{(-c)_k}{k!} x^{-k} B_x(a+k,\ b) = \frac{\Gamma(a)\Gamma(c+1)}{\Gamma(a+c+1)} x^a \ {}_2F_1(a, 1-b; a+c+1; x)$

$\mathrm{Re}(a) > 0,\ \mathrm{Re}(c) > -1,\ 0 \le x < 1.$

(62.1.12) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} (1-x)^{-k} B_x(a,\ b+k) =$

$= \frac{n!}{a(a+1)_n} x^{a+n} (x-1)^{-n} \ {}_2F_1(a, 1-b; a+n+1; x)$

$\mathrm{Re}(a) > 0; 0 \le x < 1.$

(62.1.13) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} x^k B_x(a-k,\ b) = \frac{n!}{a(1-a)_n} x^a \ {}_2F_1(a-n, 1-b; a+1; x)$

(62.1.14) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} (1-x)^{-k/2} B_x(a,\ k/2+1/2) =$

$= (-1)^n 2^a n! \Gamma(a) x^{a/2+n/2} P_{a-1}^{(-a-n)}[(1-x)^{1/2}]$

(65.1.6) $\sum_{k=0}^{n-1} \frac{(c/2-a/2)_k}{(1-a/2)_k} (x-1)^{-k} [2a-c-4k-x(a-b-2k)] \times$

$\times \ {}_2F_1(a-2k, b; c; x) =$

$= a(1-x) \left[{}_2F_1(a+1, b; c; x) + \frac{(c/2-a/2)_n}{(-a/2)_n} (x-1)^{-n} \ {}_2F_1(a+1-2n, b; c; x) \right]$

(65.1.7) $\sum_{k=0}^{n} \frac{(-n)_k}{k!} \ {}_2F_1(a-k, b; c; x) = x^n \frac{(b)_n}{(c)_n} \ {}_2F_1(a, b+n; c+n; x)$

(65.1.8) $\sum_{k=0}^{n} \frac{(-n)_k (d)_k}{k!(1-a)_k} \ {}_2F_1(a-k, b; c; x) =$

$\frac{(1-a-d)_n}{(1-a)_n} \ {}_3F_2(b, a-d, a-n; c, a+d-n; x)$

(65.2.9) $\sum_{k=0}^{n-1} \frac{(c-a)_k (c-b)_k}{(2c)_{2k}} \left(\frac{4x}{1-x}\right)^k [2c+2k-1+(2a+2b-4c-4k+1)x] \times$

$\times \ {}_2F_1(2a, 2b; 2c+2k; x) = (2c-1)(1-x)[{}_2F_1(2a, 2b; 2c-1; x) -$

$- \frac{(c-a)_n (c-b)_n}{(2c-1)_{2n}} \left(\frac{4x}{1-x}\right)^n \ {}_2F_1(2a, 2b; 2c+2n-1; x)]$

(65.2.10) $\sum_{k=0}^{\infty} \frac{(d)_k}{k!} \ {}_2F_1(a, b; c+k; x) =$

$= abx \frac{\Gamma(c)\Gamma(1-d)}{\Gamma(c-d+1)} \ {}_3F_2(a+1, b+1, 1-d; 2, c-d+1; x)$

$\mathrm{Re}(d) < 1,\ d \neq 0$

(65.2.11) $\sum_{k=0}^{\infty} \frac{(d)_k (e)_k}{k!(c)_k} \ {}_2F_1(a, b; c+k; x) =$

$= \frac{\Gamma(c)\Gamma(c-d-e)}{\Gamma(c-d)\Gamma(c-e)} \ {}_3F_2(a, b, c-d-e; c-dc-e; x) \qquad \mathrm{Re}(d+e-c) < 1$

(65.4.5) $\sum_{k=0}^{\infty} \frac{(d)_k (1-c)_k}{k!(1-a)_k} \ {}_2F_1(a-k, b; c-k; x) = \frac{\Gamma(1-a)\Gamma(c-a-d)}{\Gamma(1-a-d)\Gamma(c-a)} \times$

$\times \ {}_2F_1(a+d, b; c; x)$

$\mathrm{Re}(a+d-c) < 1$

$$(65.4.6)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(d)_k}{k!(c)_k}\ {}_2F_1(a+k,b;c+k;x)=\frac{\Gamma(c)\Gamma(c-d-a)}{\Gamma(c-a)\Gamma(c-d)}\ {}_2F_1(a,b;c-d;x)$$

$$(65.5.7)\qquad \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}x^k\ {}_2F_1[a+k,c+k-1/2;2c+2k-1;4(x^{1/2}-x)]=$$

$$=x^{-a/2}\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1[a,a-c+1;c-b;(x^{-1/2}-1)^2]$$

$$(66.1.4)\qquad \sum_{k=0}^{n}\frac{(-n)_k(c)_k}{k!(1-a)_k}\ {}_1F_1(a-k;b;x)=\frac{(1-a-c)_n}{(1-a)_n}\ {}_2F_2(a+c,a-n;b,a+c-n;x)$$

$$(66.2.4.1)\qquad \sum_{k=0}^{\infty}\frac{(b)_k}{k!(c)_k}x^k\ {}_1F_1(a;c+k;x)={}_1F_1(a+b;c;x)$$

$$(66.2.10)\qquad \sum_{k=0}^{\infty}\frac{(1-b)_k(d)_k}{k!(c)_k}\ {}_1F_1(a;b-k;x)$$

$$=\frac{\Gamma(c)\Gamma(b+c-d-1)}{\Gamma(b+c-1)\Gamma(c-d)}\ {}_2F_2(a,b+c-d-1;b,b+c-1;x)$$

$$(67.1.21.1)\qquad \sum_{k=0}^{\infty}\frac{(c)}{k!}t^k\ {}_{p+1}F_{q+1}\left[\begin{matrix}-k,a_1,...,a_p;\\ 1-c-k,b_1,...,b_q;\end{matrix}\ x\right]=$$

$$=(1-t)\ {}_pF_q\left[\begin{matrix}a_1,...,a_p;\\ b_1,...,b_q;\end{matrix}\ xt\right]\qquad \text{SVp.209(10)}$$

$$(67.2.5)\qquad \sum_{k=0}^{\infty}\frac{(b_1)_k...(b_s)_k(e_1)_k...(e_r)_k(f)_{k+kd}}{k!(c_1)_k...(c_q)_k(f+1)_{kd}}(-x)^k\times$$

$$\times\ {}_{p+2}F_{r+2}\left[\begin{matrix}-k,1+f(d+1),a_1,...,a_p;\\ f+kd+1,f(d+1),e_1,...,e_r;\end{matrix}\ y\right]\times$$

$$\times\ {}_{r+s+1}F_q\left[\begin{matrix}f+k+kd,e_1+k,...,e_r+k,b_1+k,...,b_s+k;\\ c_1+k,...,c_q+k;\end{matrix}\ x\right]$$

$$=\ {}_{p+s}F_q\left[\begin{matrix}a_1,...a_p,b_1,...,b_s;\\ c_1,...,c_q;\end{matrix}\ xy\right]\qquad \text{AP46p.73(2.1)}$$

Either $p+s\leq q$ or else $p+s=q+1$ and $|xy|<1$. Also, either $r+s<q-1$ or else $r+s=q$ and $|x|<1$.

$$(68.1.17.1)\qquad \sum_{k=1}^{\infty}(-1)^k\frac{k}{a^4-k^4}\cos(kx)\cosh(kx)\csc h(kx)=$$

$$=\frac{1}{4\pi a^4}[\pi^2a^2\cos(ax)\csc(\pi a)\cosh(ax)\operatorname{csch}(\pi a)-1]$$

$$(68.1.18.1)\qquad \sum_{k=0}^{\infty}(-1)^k\frac{2k+1}{(2k+1)^4-a^4}\sin[(2k+1)x]\sinh[(2k+1)x]\operatorname{sech}[(2k+1)x]=$$

$$=\frac{\pi}{8a^2}\sin(ax)\sec(\pi a/2)\sinh(ax)\operatorname{sech}(\pi a/2)$$

(69.1.1.1) $\sum_{k=0}^{\infty} \sin(kx+y)P_k(\cos t) =$

$$= \begin{cases} (-1)^{m-n-1}\cos(y-x/2)[2(\cos t-\cos x)]^{-1/2} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi \\ (-1)^{m-n}\sin(y-x/2)[2(\cos x-\cos t)]^{-1/2} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \\ \qquad \text{or for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi \\ (-1)^{m-n}\cos(y-x/2)[2(\cos t-\cos x)]^{-1/2} \\ \qquad \text{for } 4\pi m \le x+t \le (4m+2)\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \end{cases}$$

$m = 0, \pm 1, \pm 2, ...; n = 0, \pm 1, \pm 2, ...$

For $y = 0$, see OA(3.58)*. For $x = 2y$, see OA(3.54)*

(69.1.1.2) $\sum_{k=0}^{\infty} \cos(kx+y)P_k(\cos t) =$

$$= \begin{cases} (-1)^{m-n}\sin(y-x/2)[2(\cos t-\cos x)]^{-1/2} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi \\ (-1)^{m-n}\cos(y-x/2)[2(\cos x-\cos t)]^{-1/2} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \\ \qquad \text{or for } 4m\pi \le x+t \le (4n+2)\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi \\ (-1)^{m-n-1}\sin(y-x/2)[2(\cos t-\cos x)]^{-1/2} \\ \qquad \text{for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \end{cases}$$

$m = 0, \pm 1, \pm 2, ...; n = 0, \pm 1, \pm 2, ...$

For $y = 0$, see OA(3.59)*. For $x = 2y$, see OA(3.55).

(69.1.5) $\sum_{k=0}^{\infty} \sin(kx+y)C_k^{(a)}(\cos t) =$

$$= \begin{cases} \sin[(2m+2n-1)\pi a - ax + y][2(\cos t-\cos x)]^{-a} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi, \\ \sin[2(m+n)\pi a - ax + y][2(\cos x-\cos t)]^{-a} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \\ \qquad \text{or for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi, \\ \sin[(2m+2n+1)\pi a - ax + y][2(\cos t-\cos x)]^{-a} \\ \qquad \text{for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi. \end{cases}$$

$m = 0. \pm 1, \pm 2, ...; n = 0, \pm 1, \pm 2, ; \mathrm{Re}(a) < 1.$

For $y = 0$, see OA(3.83)*. For $y = ax$, see SQ14p.313(10.34).

(69.1.6) $\sum_{k=0}^{\infty} \cos(kx+y)C_k^{(a)}(\cos t) =$

$$= \begin{cases} \cos[(2m+2n-1)\pi a - ax + y][2(\cos t - \cos x)]^{-a} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi, \\ \cos[2(m+n)\pi a - ax + y][2(\cos x - \cos t)]^{-a} \\ \qquad \text{for } (4m-2)\pi \le x+t \le 4m\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi \\ \qquad \text{or for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } (4n-2)\pi \le x-t \le 4n\pi, \\ \cos[(2m+2n+1)\pi a - ax + y][2(\cos t - \cos x)]^{-a} \\ \qquad \text{for } 4m\pi \le x+t \le (4m+2)\pi \text{ and } 4n\pi \le x-t \le (4n+2)\pi. \end{cases}$$

$m = 0. \pm 1, \pm 2, ...; n = 0, \pm 1, \pm 2, ...; \mathrm{Re}(a) < 1.$

For $y = 0$, see OA(3.84)*. For $y = ax$, see SQ14p.313(10.33).

(74.1.18) $$\sum_{k=1}^{\infty} k^{-a} \cos(ky) J_a(kx) = -\frac{(x/2)^a}{2\Gamma(a+1)} +$$

$$+ \begin{cases} 0 \text{ if } |x| \le |y|, \\ \frac{\pi^{1/2}(x/2)^a}{|x|\Gamma(a+1/2)} \left(1 - \frac{y^2}{x^2}\right)^{a-1/2} \text{ if } |x| \ge |y| \end{cases}$$

$-2\pi < y \pm x < 2\pi$; $\mathrm{Re}(a) > 1/2$. For $a = 0$ and $|y| \le x = \pi$, see (74.1.3). See OA(3.42).

(74.1.18.1) $$\sum_{k=1}^{\infty} k^{-2m-a} \cos(ky) J_a(kx) = \frac{(-1)^m 2^{-a-1} x^a y^{2m}}{(2m)!\Gamma(a+1)} \times$$

$$\times \left[\frac{2\pi m}{y} \sum_{k=0}^{m-1} \frac{(1-2m)_{2k}}{k!(a+1)_k} \left(\frac{x}{2y}\right)^{2k} - \sum_{k=0}^{m} \frac{(-2m)_{2k}}{(2k)!} \left(\frac{2\pi}{y}\right)^{2k} B_{2k} \sum_{n=0}^{m-k} \frac{(2k-2m)_{2n}}{n!(a+1)_n} \left(\frac{x}{2y}\right)^{2n} \right]$$

$0 \le y \pm x \le 2\pi$; $m = 0, 1, 2, ...$

(74.1.18.2) $$\sum_{k=1}^{\infty} k^{2n-a} \cos(ky) J_a(kx) = -\frac{(x/2)^a}{2\Gamma(a+1)} \delta_{n,0} + \begin{cases} 0 \text{ if } |x| \le |y|, \\ S \text{ if } |x| \ge |y| \end{cases}$$

where $S = (-1)^n \frac{(2x)^{a-2n}(2n)!\Gamma(a-2n)}{2|x|\Gamma(2a-2n)} \left(1 - \frac{y^2}{x^2}\right)^{a-2n-1/2} C_{2n}^{(a-2n)}\left(\frac{y}{x}\right)$

$\mathrm{Re}(a) > 2n - 1/2$; $-2\pi < y \pm x < 2\pi$; $n = 0, 1, 2, ...$

For $a = 2n + 1$, see (74.1.17). See OA(4.75)*.

(74.1.19.1) $$\sum_{k=1}^{\infty} (-1)^k k^{-2n-a} \cos(ky) J_a(kx) =$$

$$\frac{(-1)^{n-1}(x/2)^{a+2n}}{2n!\Gamma(a+n+1)} \sum_{k=0}^{n} \frac{(-n)_k(-a-n)_k}{(2k)!} \left(\frac{4\pi}{x}\right)^{2k} B_{2k}\left(\frac{y+\pi}{2\pi}\right)$$

$-\pi < x \pm y < \pi$; $\mathrm{Re}(a) > 1/2 - 2n$; $n = 0, 1, 2, ...$

For $n = 1$ and $y = x$,see (74.1.20).

(74.1.19.2) $\sum_{k=1}^{\infty}(-1)^k k^{2n-a}\cos(ky)J_a(kx) = -\frac{(x/2)^a}{2\Gamma(a+1)}\delta_{n,0}$

$-\pi < x \pm y < \pi; \mathrm{Re}(a) > 2n + 1/2; n = 0, 1, 2, ...$

(74.1.20.1) $\sum_{k=1}^{\infty} k^{-2n-1-a}\sin(ky)J_a(kx) =$

$$= \frac{(-1)^{n-1}\pi(x/2)^{a+2n}}{n!\Gamma(a+n+1)}\sum_{k=0}^{n}\frac{(-n)_k(-a-n)_k}{(2k+1)!}\left(\frac{4\pi}{x}\right)^{2k}B_{2k+1}\left(\frac{y}{2\pi}\right) +$$

$$+\begin{cases} 0 \text{ if } 0 \le y \pm x < 2\pi \text{ and } |x| \le |y|, \\ S_1 \text{ if } -2\pi < y \pm x \le 0 \text{ and } |x| \le |y|, \\ S_2 \text{ if } -2\pi < x \pm y < 2\pi \text{ and } |y| \le |x| \end{cases}$$

where $S_1 = (-1)^{n-1}\left(\frac{x}{2}\right)^{a+2n}\frac{\pi}{\Gamma(a+2n+1)}C_{2n}^{(-a-2n)}\left(\frac{y}{x}\right)$

and $S_2 = (-1)^{n-1}x^{a+2n}\left(\frac{\pi}{2}\right)^{1/2}\left(1-\frac{y^2}{x^2}\right)^{n+a/2+1/4}P_{a-1/2}^{(-a-2n-1/2)}\left(\frac{y}{|x|}\right)$

$\mathrm{Re}(a) > -2n - 1/2$; $n = 0, 1, 2, ...$ For $0 < x = y < \pi$ and $n = 0$, see (74.1.21).

(74.1.20.2) $\sum_{k=1}^{\infty} k^{2n-1-a}\sin(ky)J_a(kx) = \begin{cases} 0 \text{ if } |x| < |y|, \\ S \text{ if } |x| \ge |y| \end{cases}$

where $S = \frac{y\Gamma(n+1/2)}{|x|\Gamma(a-n+1/2)}\left(\frac{x}{2}\right)^{a-2n}{}_2F_1(n+1/2, n-a+1/2; 3/2; y^2/x^2)$

$-2\pi < x \pm y < 2\pi$; $\mathrm{Re}(a) > 2n - 1/2$; $n = 1, 2, 3, ...$

For $n = 1$, and $x = \pi$, see OA(3.34)*. The hypergeometric function in the definition of S is given by (10.34.1).

(74.1.20.3) $\sum_{k=1}^{\infty} k^{2n-1-a}\sin(kx)J_a(kx) = 0$

$-\pi < x < \pi$; $\mathrm{Re}(a) > 2n - 1/2$; $n = 1, 2, 3, ...$

(74.1.21) $\sum_{k=1}^{\infty} k^{-a-1}\sin(ky)J_a(kx) =$

$$= \begin{cases} \frac{(x/2)^a}{2\Gamma(a+1)}(\pi - y) \text{ if } 0 \le y \pm x < 2\pi \text{ and } |x| \le |y|, \\ -\frac{(x/2)^a}{2\Gamma(a+1)}(\pi + y) \text{ if } -2\pi < y \pm x \le 0 \text{ and } |x| \le |y|, \\ -\frac{(x/2)^a}{2\Gamma(a+1)}(\pi - y) + \frac{\pi^{1/2}(2x)^a}{\Gamma(a+1/2)}B_{\frac{|x|-y}{2|x|}}(a+1/2, a+1/2) \\ \text{if } -2\pi < x \pm y \le 0 \text{ and } |y| \le |x| \end{cases}$$

$\mathrm{Re}(a) > -1/2.$

(74.1.21.1) $\sum_{k=1}^{\infty}(-1)^k k^{-2n-1-a}\sin(ky)J_a(kx) =$

$= \frac{(-1)^{n-1}\pi(x/2)^{a+2n}}{n!\Gamma(a+n+1)}\sum_{k=0}^{n}\frac{(-n)_k(-a-n)_k}{(2k+1)!}\left(\frac{4\pi}{x}\right)^{2k}B_{2k+1}\left(\frac{y+\pi}{2\pi}\right)$

$-\pi < x \pm y < \pi$; $\mathrm{Re}(a) > -2n-1/2$; $n = 0, 1, 2, \ldots.$

For $n = 0$, and $y = x$, see (74.1.22).

(74.1.21.2) $\sum_{k=1}^{\infty}(-1)^k k^{2n+1-a}\sin(kx)J_a(kx) = 0$

$-\pi < x \pm y < \pi$; $\mathrm{Re}(a) > 2n-1/2$; $n = 0, 1, 2, \ldots.$

(74.1.24.1) $\sum_{k=1}^{\infty}\frac{k^{-2n-1-a}}{k^2-c^2}\sin(ky)J_a(kx) = \frac{\pi}{2}c^{-a-2n-2}\csc(\pi c)\sin[c(\pi-y)]J_a(cx)+$

$+2\pi c^{-2n-2}\left(\frac{x}{2}\right)^a\sum_{r=0}^{n}(-1)^r\left(\frac{cx}{2}\right)^{2r}\sum_{k=0}^{r}\frac{(4\pi/x)^{2k}}{(2k+1!(r-k)!\Gamma(a+r-k+1)}B_{2k+1}\left(\frac{y}{2\pi}\right)$

(74.1.24.2) $\sum_{k=1}^{\infty}\frac{k^{2n-1-a}}{k^2-c^2}\sin(ky)J_a(kx) = \frac{\pi}{2}c^{-a+2n-2}\csc(\pi c)\sin[c(\pi-y)]J_a(cx)$

$0 < y \pm x < 2\pi$; $\mathrm{Re}(a) > 2n-5/2$; $n = 1, 2, 3, \ldots.$

For $n = 1$, and $a = 0$, see (74.1.25). For $n = 1$ and $y = x$, see (74.1.29)

(74.1.25.1) $\sum_{k=1}^{\infty}\frac{k^{-2n-a}}{k^2-c^2}\cos(ky)J_a(kx) =$

$= -\frac{\pi}{2}c^{-a-2n-1}\csc(\pi c)\cos[c(\pi-|y|)]J_a(cx)+$

$+\frac{(x/2)^a}{2c^{2n+2}\Gamma(a+1)}\sum_{r=0}^{n}\frac{(-1)^r(cx/2)^{2r}}{r!(a+1)_r}\sum_{k=0}^{r}\frac{(-r)_k(-a-r)_k}{(2k)!}\left(\frac{4\pi}{x}\right)^{2k}B\left(\frac{|y|}{2\pi}\right)$

$-2\pi < y \pm x < 2\pi$; $|x| \le |y|$; $\mathrm{Re}(a) > -2n-3/2$; $n = 0, 1, 2, \ldots.$

For $n = 0$ and $a = 0$, see (74.1.24). For $n = 0$ and $y = x$, see (74.1.26).

(74.1.25.2) $\sum_{k=1}^{\infty}\frac{k^{2n-a}}{k^2-c^2}\cos(ky)J_a(kx) =$

$= -\frac{\pi}{2}c^{-a+2n-1}\csc(\pi c)\cos[c(\pi-|y|)]J_a(cx)$

$-2\pi < y \pm x < 2\pi$; $|x| \le |y|$; $\mathrm{Re}(a) > 2n-3/2$; $n = 1, 2, 3, \ldots.$

For $n = 1$ and $y = x$, see (74.1.28).

(74.1.26.1) $\sum_{k=1}^{\infty}(-1)^k\frac{k^{2n-a}}{k^2-c^2}\cos(ky)J_a(kx) =$

$= -\frac{\pi}{2}c^{-a+2n-1}\csc(\pi c)\cos(cy)J_a(cx)$

$-\pi < y \pm x < \pi$; $\mathrm{Re}(a) \geq -2n-3/2$; $n = 1, 2, 3, ...$

(74.1.26.2) $\sum_{k=1}^{\infty} (-1)^k \frac{k^{-2n-a}}{k^2-c^2} \cos(ky) J_a(kx) =$

$= -\frac{\pi}{2} c^{-a-2n-1} \csc(\pi c) \cos(cy) J_a(cx) +$

$+c^{-2n-2} \frac{(x/2)^a}{2\Gamma(a+1)} \sum_{r=0}^{n} \frac{(-1)^r}{r!(a+1)_r} \left(\frac{cx}{2}\right)^{2r} \sum_{k=0}^{r} \frac{(-r)_k(-a-r)_k}{(2k)!} \left(\frac{4\pi}{x}\right)^{2k} B_{2k}\left(\frac{y+\pi}{2\pi}\right)$

$-\pi \leq y \pm x \leq \pi$; $n = 0, 1, 2, ...$; $\mathrm{Re}(a) > -2n-3/2$

(74.1.27) $\sum_{k=1}^{\infty} (-1)^k \frac{k^{-2n-a}}{k^2-c^2} \cos(ky) J_a(kx) =$

$= -\frac{\pi}{2} c^{-a-2n-1} \csc(\pi c) \cos(cy) J_a(cx) +$

$+\frac{(x/2)^a}{2\Gamma(a+1)} c^{-2n-2} \sum_{r=0}^{n} \frac{(-1)^r}{r!(a+1)_r} \left(\frac{cx}{2}\right)^{2r} \sum_{k=0}^{r} \frac{(-r)_k(-a-r)_k}{(2k)!} \left(\frac{4\pi}{x}\right)^{2k} B_{2k}\left(\frac{y+\pi}{2\pi}\right)$

$-\pi \leq y \pm x \leq \pi$; $\mathrm{Re}(a) \geq -2n-3/2$; $n = 0, 1, 2, ...$

For $n = 0$ and $0 < x = y < \pi/2$, see OCp.13.

(74.1.29.1) $\sum_{k=1}^{\infty} (-1)^k \frac{k^{-2n-1-a}}{k^2-c^2} \sin(ky) J_a(kx) =$

$= -\frac{\pi}{2} c^{-a-2n-2} \csc(\pi c) \sin(cy) J_a(cx) +$

$+\pi c^{-2n-2} \left(\frac{x}{2}\right)^a \sum_{r=0}^{n} (-1)^r \left(\frac{cx}{2}\right)^{2r} \sum_{k=0}^{r} \frac{1}{(2k+1)!(r-k)!\Gamma(a+r-k+1)} \left(\frac{4\pi}{x}\right)^{2k} B_{2k+1}\left(\frac{y+\pi}{2\pi}\right)$

$-\pi < y \pm x < \pi$; $\mathrm{Re}(a) > -2n-5/2$; $n = 0, 1, 2, ...$

(74.1.29.2) $\sum_{k=1}^{\infty} (-1)^k \frac{k^{2n-1-a}}{k^2-c^2} \sin(ky) J_a(kx) = -\frac{\pi}{2} c^{2n-a-2} \csc(\pi c) \sin(cy) J_a(cx)$

$-\pi < y \pm x < \pi$; $\mathrm{Re}(a) > 2n-5/2$; $n = 1, 2, 3, ..$

For $n = 1$ and $y = x$, see (74.1.30).

(74.6.12.1) $\sum_{k=0}^{\infty} \cos(kt)[J_{kn}(x)]^2 = \frac{1}{2}[J_0(x)]^2 + \frac{1}{4n} \sum_{k=0}^{2n-1} J_0\left\{2x \sin\frac{t+2\pi k}{2n}\right\}$

(74.6.13.1) $\sum_{k=0}^{\infty} \cos(2kt) J_{k+a}(x) J_{k-a}(x) = \frac{1}{2} J_a(x) J_{-a}(x) +$

$+\frac{1}{2}\sin(\pi a) \mathbf{E}_{2a}(2x \sin t) + \frac{1}{2}\cos(\pi a) \mathbf{J}_{2a}(2x \sin t)$

For $a = 0$, see OA(4.29). For $a = 1, 2, 3, ...$, see OA(4.55). For $a = 1/4$, see OA(4.25).

(74.6.13.2) $\sum_{k=0}^{\infty}(-1)^k \cos(2kt) J_{k+a}(x) J_{k-a}(x) = \frac{1}{2} J_a(x) J_{-a}(x)+$

$+\frac{1}{2}\sin(\pi a)\mathbf{E}_{2a}(2x\cos t) + \frac{1}{2}\cos(\pi a)\mathbf{J}_{2a}(2x\cos t)$ OA(4.59)

For $a = 1/4$, see OA(4.26). For $a = 0, 1, 2, ...$, see OA(4.56).

(74.6.13.3) $\sum_{k=0}^{\infty}\sin[2k+1)t] J_{k+a+1}(x) J_{k-a}(x) = \frac{1}{2}\sin(\pi a)\mathbf{E}_{2a+1}(2x\sin t)+$

$+\frac{1}{2}\cos(\pi a)\mathbf{J}_{2a+1}(2x\sin t)$

For $a = 0$, see OA(4.33). For $a = -1/4$ or $a = -3/4$, see OA(4.28).

For $a = -1/2$, see OA(4.36). For $a = 0, 1, 2, ...$, see OA(4.58).

(74.6.13.4) $\sum_{k=0}^{\infty}(-1)^k \cos[2k+1)t] J_{k+a+1}(x) J_{k-a}(x) =$

$= \frac{1}{2}\sin(\pi a)\mathbf{E}_{2a+1}(2x\cos t) + \frac{1}{2}\cos(\pi a)\mathbf{J}_{2a+1}(2x\cos t)$ OA(4.60)

For $a = -1/4$ or $a = -3/4$, see OA(4.27). For $a = -1/2$, see OA(4.35).

For $a = -1$, see OA(4.34). For $a = 0, 1, 2, ...$, see OA(4.57)*.

(76.1.3) $\sum_{k=0}^{\infty}\frac{1}{k!}P_k(x)H_k(y) = \exp\left[\left(1+\frac{1}{x^2}\right)\frac{y^2}{2}\right] I_0\left[\left(1-\frac{1}{x^2}\right)\frac{y^2}{2}\right]$

(reference lost)

(76.1.4) $\sum_{k=0}^{\infty}\frac{1}{(2a)_k}C_k^{(a)}(x)H_k(y) = \exp(y^2)\ {}_1F_1(1/2;\ a+1/2;\ y^2(1/x^2-1)]$

(reference lost)

(79.2.10.1) $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2-b^2}(k^2+x^2)^{-a/2}C_n^{(a)}[x(k^2+x^2)^{-1/2}]J_{n+a}[y(k^2+x^2)^{1/2}] =$

$= -\frac{\pi}{2b}(x^2+b^2)^{-a/2}\csc(\pi b)C_n^{(a)}[x(x^2+b^2)^{-1/2}]J_{a+n}[y(x^2+b^2)^{1/2}]+$

$+\frac{(2a)_n}{2n!}b^{-2}x^{-a}J_{a+n}(xy)$

(89.8.3) $\prod_{k=0}^{\infty}\left[1-\left(\frac{x}{2k+1}\right)^n\right] = \prod_{k=1}^{n}\frac{\Gamma(1-a^k x/2)}{\Gamma(1-a^k x)}$

where $a = \exp(2\pi i/n)$.

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