NOTES ON THE LAPLACE TRANSFORM OF THE PSI FUNCTION

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Abstract. Guided by numerical experimentation, we have been able to prove that

$$\frac{8}{\pi}\int_0^{\pi/2}\frac{x^2}{x^2+\ln^2[2\cos(x)]}dx = 1-\gamma+\ln(2\pi)$$

and to establish a mysterious connection with the Laplace transform of $\psi(t+1)$.

1. Introduction

Interest in this project began with curiosity about the Laplace transform of the Digamma function

$$L(a) = \int_0^\infty e^{-as}\psi(s+1)ds \tag{1}$$

which is conspicuously absent from the extensive literature and tabulations of Euler's Gamma function. As will be seen, this can be related to the odd logarithmic integral

$$g(a) = \frac{4}{\pi}\int_0^{\pi/2}\frac{x^2}{x^2+\ln^2(2e^{-a}\cos(x))}dx. \tag{2}$$

If one plots $L(a)+\gamma/a$ and $g(a)$ the graphs coincide for $a \geq \ln(2)$, that for $g(a)$ exhibits a cusp at $a = \ln(2)$ and decreases to the finite value $g(0) = 1.13033$, whereas $L(0)$ is divergent.

The first author was quite surprised to receive an e-mail from Olivier Oloa of the University of Versailles asking about a number of integrals equivalent in form to (2) and a second note somewhat later stating, without proof, that

$$g(0) = \frac{4}{\pi}\int_0^{\pi/2}\frac{x^2}{x^2+\ln^2(2\cos(x))}dx = (1-\gamma+\ln(2\pi))/2\ . \tag{3}$$

Indeed this value had been guessed by making the reasonable assumption based on the connection with the Gamma function that the only transcendental numbers involved were Euler's constant, $\ln(2)$ and $\ln(\pi)$. By examining the expression $E(a,b,c,d) = a+b\gamma+c\ln(2)+d\ln(\pi)$ systematically for small rational values of the coefficients precisely the above value was obtained. This identity also appears in [5].

The aim of this note is to provide the details of the relation between $L(a)$ and $g(a)$ and to derive the value of $g(0)$. We present two proofs of the evaluation $g(0)$, as well as a sum formula for the integral. The general closed form of (1) for real parameter a, however, remains elusive.

To evaluate this integral, we recognize the integrand itself as a Laplace transform. Therefore it is a double integral, and we expand the innermost integrand with respect to the original variable. This gives us, formally, the representation

$$L(a) = \pi e^{-a}\int_0^1 dt\frac{te^{-at}}{\Gamma(1-t)}\sum_{k-l\neq 1}\frac{\Gamma(l-t)\Gamma(l-k-1)}{l!\Gamma(l-k)}(1-e^a)^k\ . \tag{4}$$

When $a=0$, the hypergeometric sum reduces to a ${}_3F_2$, and we conclude the evaluation by appealing to known evaluations of the ψ function.

In a second evaluation, we start by simplifying the integrand via a partial fractions decomposition and change of variables. This results in a much simpler-looking integrand:

$$\int_0^\pi \frac{y^2\ dy}{y^2+4\ln^2(2\cos(\frac{y}{2}))} = \frac{i}{4}\int_{-\pi}^{\pi} \frac{y\ dy}{\log(e^{iy}+1)}\ . \tag{5}$$

The transformation $e^{iy} \mapsto z$ in the right hand integral would be the next logical step, because it transforms the interval $(-\pi,\pi)$ to a simple, closed contour in $\mathbb{C}$ that encircles the origin. However, doing so would introduce logarithms in the integrand, which render an integral not amenable to computation by residues.

We compensate for this fact by substituting, in place of the integrand: an integral with respect to a fresh parameter t, differentiated with respect to second parameter s, which when evaluated produces something equal to the original integrand. The result is that the logarithms are eliminated from the inner form. This form is analytic with respect to the two new parameters, and thus we reverse the orders of integration and differentiation in order to perform, at last, a sensible residue calculation. The result is an equivalent but more famous integral whose evaluation proves (3).

We will use well-known identities of the Log Gamma function (which is the same as $\ln\Gamma$ for real arguments) such as

$$\int_0^1 \ln\Gamma(t)\ dt = \ln(\sqrt{2\pi}) \text{ and } (\ln\Gamma)'(2) \equiv \psi(2) = 1-\gamma\ , \tag{6}$$

each of which appear very often in the literature; see [2] for details. Throughout we exchange infinite sums and integrals and employ the principle of analytic continuation; we refer the reader to [6] for background.

From (5) we can alternatively take a series expansion of the integrand. We integrate this series termwise, as the individual summands are done quite easily. We arrive at the formula

$$\int_0^\pi \frac{y^2\ dy}{y^2+4\log^2(2\cos(\frac{y}{2}))} = \frac{\pi}{2}\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m(m+1)!}\sum_{s=0}^{m+1}\frac{S_1(m+1,s)}{s+1}\right)\ . \tag{7}$$

The $S_1(m,s)$ are signed Stirling numbers of the first kind, which appear when one expands $1/\log(1+x)$. Thus the inner sum We thus have, in view of (3), a new evaluation of the above series in terms of the same mathematical constants.

2. Derivation

We begin by proving that the integrals $L(a)+\gamma/a$ and $g(a)$ agree for $a > \ln(2)$. The former diverges for all other real values of a , so that the latter is seen as its analytic continuation. To start, formula (1.6.27) in [3] reads

$$B(y) = \int_0^{\pi/2}\cos(xy)\cos^s(x)dx = 2^{-s-1}\pi\frac{\Gamma(s+1)}{\Gamma(1+\frac{1}{2}s+\frac{1}{2}y)\Gamma(1+\frac{1}{2}s-\frac{1}{2}y)}\ . \tag{8}$$

Hence

$$\int_0^{\pi/2} x\sin(xy)\cos^s(x)dx = -B'(y) \tag{9}$$

$$= \frac{\pi}{4}\frac{2^{-s}\Gamma(s+1)}{\Gamma(1+\frac{1}{2}s+\frac{1}{2}y)\Gamma(1+\frac{1}{2}s-\frac{1}{2}y)}\left[\psi\left(1+\frac{s}{2}+\frac{y}{2}\right)-\psi\left(1+\frac{s}{2}-\frac{y}{2}\right)\right]\ .$$

This generalises Entry 33(i) in [1]. Therefore, evaluating at $y=s$, we obtain the integral representation

$$\psi(s+1) = \frac{2^{s+2}}{\pi}\int_0^{\pi/2} x\sin(sx)\cos^s(x)dx - \gamma\ . \tag{10}$$

Substituting this into the previous expression, we obtain the double integral

$$L(a)+\frac{\gamma}{a}=\frac{4}{\pi}\int_0^\infty e^{-(a-\ln\ 2)s}\int_0^{\pi/2} x\sin(sx)\cos^s(x)\ dx\ ds. \tag{11}$$

Writing $\sin(sx) = -\operatorname{Im}(e^{-isx})$, this becomes

$$L(a)+\frac{\gamma}{a}=-\frac{4}{\pi}\operatorname{Im}\int_0^\infty\int_0^{\pi/2} xe^{s(\ln[2e^{-a}\cos(x)]-ix)}\ dx\ ds. \tag{12}$$

Now the integral

$$\int_0^\infty e^{s(\ln[2e^{-a}\cos(x)]-ix)}\ ds \qquad \text{for } 0<x<\pi/2 \tag{13}$$

is equal to $1/(ix - \ln[2e^{-a}\cos(x)])$ when $a > \ln 2$ and does not converge for any other values of a . Thus, with this restriction in place, we may reverse the order of integration, yielding

$$L(a)-\frac{\gamma}{a}= \quad -\frac{4}{\pi}\operatorname{Im}\int_0^{\pi/2}\frac{x}{ix-\ln[2e^{-a}\cos(x)]}\ dx \quad = g(a)\ . \tag{14}$$

Now we retrace the our steps slightly differently, starting from (11). For fixed $s > 0$, the integrand with respect to x is even; we write

$$g(a)=\frac{2}{\pi}\operatorname{Im}\int_0^\infty\int_{-\pi/2}^{\pi/2} xe^{s\ln[2e^{-a}\cos(x)]}e^{isx}\ dx\ ds \tag{15}$$

$$-\frac{2}{\pi}Im\int_0^\infty ds\ e^{s\ln[e^{-a}]}\int_{-\pi/2}^{\pi/2} dx x(1+e^{2ix})^s =$$

$$-\frac{1}{2\pi}Im\int_{-\pi}^{\pi} dx\ x\int_0^\infty ds\ e^{s\ln[e^{-a}(1+e^{ix})]} = -\frac{1}{2\pi}Im\int_{-\pi}^{\pi}\frac{xdx}{\ln[e^{-a}(1+e^{ix})]} =$$

$$-\frac{1}{2\pi}e^a Im\int_0^1 dt\ e^{-at}\int_{-\pi}^{\pi} dx\frac{xe^{-ix}(1+e^{ix})^t}{1-(e^a-1)e^{-ix}}. \tag{7}$$

Next we expand in powers of e^{ix} and take the imaginary part to obtain

$$g(a)=-\frac{1}{\pi}e^a\int_0^1 dt\ e^{-at}\sum_{k,l=0}^\infty (e^a-1)^k\begin{pmatrix} t \\ l\end{pmatrix}\int_0^\pi dx\ x\sin(l-k-1)x. \tag{8}$$

The $x-$ integral is easily worked out and vanishes if $l - k = 1$. The binomial coefficient can be expressed as $-(-1)^l t\Gamma(l-t)/\Gamma(1-t)$ leading to

$$g(a)=e^a\int_0^1\frac{te^{-at}}{\Gamma(1-t)}\sum{}'\frac{\Gamma(l-t)\Gamma(l-k-1)}{l!\Gamma(l-k)}(e^a-1)^k, \tag{9}$$

where the prime on the sum denotes that terms with $l - k = 1$ are excluded. The sum represents a hypergeometric function of two variables, which strongly suggests that for general values of a no further progress is possible. However, for $a = 0$ only terms with $k = 0$ contribute. Hence,

$$g(0)=\int_0^1 dt\frac{t}{\Gamma(1-t)}\left[\sum_{l=2}^\infty\frac{\Gamma(l-t)\Gamma(l-1)}{l!\Gamma(l)}-\Gamma(-t)\right]. \tag{10}$$

Finally, the sum can be evaluated in terms of a generalized hypergeometric function leading to

$$g(0)=\frac{1}{2}\int_0^1 dt\ t(1-t)\ {}_3F_2(1,1,2-t;2,3;1)+1. \tag{11}$$

The hypergeometric function does not appear to be tabulated, but experimenting with MATHEMATICA leads to

$${}_3F_2(1,1,2-t;2,3;1)=\frac{2}{1-t}[1-\gamma-\psi(t+1)] \tag{12}$$

and since [3] $\int_0^1 x\psi(x+1)dx = 1 - \ln\sqrt{2\pi}$ we have proven that

$$\int_0^{\pi/2} \frac{x^2}{x^2 + \ln^2[2\cos(x)]} dx = \frac{\pi}{8}[1 - \gamma + \ln(2\pi)]. \tag{13}$$

Unfortunately, since for $0 \leq a \leq \ln(2)$ the connection between $L(a)$ and $g(a)$ remains somewhat of a mystery, it is not clear that we have clarified the nature of the Laplace transform of ψ, but perhaps some our intermediate identities will shed light on this matter. We do not know Dr. Oloa's derivation of the value of $g(0)$ as this is written.

3. A Proof of the Evaluation

One begins by finding, in a table of Laplace transforms, the formula

$$\mathcal{L}\left(\frac{\sin(sx)}{\alpha^s}\right)(a) = \frac{x}{(a+\ln\alpha)^2 + x^2}\ , \tag{16}$$

or its equivalent. (Instead of searching a table one can find this formula embedded in the **[int-trans]laplace** command in maple.) Therefore, instead of (**??**), we can write (with $1/\alpha = 2\cos x$):

$$L(a) = \frac{2}{\pi}\int_0^\infty e^{-as}2^s \int_{-\pi/2}^{\pi/2} x\cos^s x \sin(sx)\ dx\ ds\ . \tag{17}$$

One expands the trigonometric integrand with respect to the x variable and integrates termwise with respect to s. A change of variables leads to

$$L(a) = \pi e^{-a}\int_0^1 dt \frac{te^{-at}}{\Gamma(1-t)} \sum_{k-l\neq 1} \frac{\Gamma(l-t)\Gamma(l-k-1)}{l!\Gamma(l-k)}(1-e^a)^k\ . \tag{18}$$

The sum in the integrand in (**??**) is a kind of Appell function and it appears that one cannot get much further in general. However, for $a = 0$, the k-sum drops out and the l-sum can be rearranged into a ${}_3F_2$, giving

$$L(0) = 1 + \frac{1}{2}\int_0^1 dt\ t(1-t){}_3F_2(2-t,1,1:2,3:1)\ . \tag{19}$$

Since, for $1 < z < 2$,

$${}_3F_2(z,1,1;2,3;1) = \frac{2}{z-1}[1 - \gamma - \psi(3-z)]\ , \tag{20}$$

and using formulas such as

$$\int_0^1 x\psi(x+1)\ dx = 1 - \ln\sqrt{2\pi}\ , \tag{21}$$

we conclude that

$$\int_0^{\pi/2} \frac{x^2}{x^2 + \ln^2(2\cos x)} dx = \frac{\pi}{8}(1 - \gamma + \ln 2\pi)\ . \tag{22}$$

This is equivalent to (3).

Erdelyi: [3]
Fichtengolz: [4]
Berndt: [1]

4. Other Evaluations

We now give details of the calculations described in the introduction, beginning with a proof of (5).

Proposition 4.1.

$$\int_0^{\pi} \frac{y^2\ dy}{y^2+4\ln^2(2\cos(\frac{y}{2}))} = \frac{i}{4}\int_{-\pi}^{\pi} \frac{y\ dy}{\log(e^{iy}+1)}\ , \tag{23}$$

where the logarithm in the denominator of the right-hand integral takes the principal value.

Proof. We move from the integral on the left to the integral on the right using a change of variables. To begin, we factor the denominator of the integrand on the left,

$$y^2+4\ln^2\left(2\cos\left(\frac{y}{2}\right)\right) = \left[2\ln\left(2\cos\left(\frac{y}{2}\right)\right)+iy\right]\left[2\ln\left(2\cos\left(\frac{y}{2}\right)\right)-iy\right] \tag{24}$$

and expand the integrand into partial fractions. Then we split into two integrals, one for each term of the decomposition:

$$\int_0^{\pi} \frac{y^2\ dy}{y^2+4\ln^2(2\cos(\frac{y}{2}))} = \int_0^{\pi} g(y)\ dy - \int_0^{\pi} g(-y)\ d(-y)\ , \tag{25}$$

where

$$g(y) := \frac{iy}{4\ln(2\cos(\frac{y}{2}))+2iy}\ . \tag{26}$$

Now we transform the second integral on the right hand side of (25) by $y \mapsto -y$ and combine with the first one to get

$$\int_0^{\pi} \frac{y^2\ dy}{y^2+4\ln^2(2\cos(\frac{y}{2}))} = \int_{-\pi}^{\pi} \frac{iy\ dy}{4\ln(2\cos(\frac{y}{2}))+2iy}\ . \tag{27}$$

Then write

$$2\ln\left(2\cos\left(\frac{y}{2}\right)\right) = \log\left[\left(e^{iy/2}+e^{-iy/2}\right)^2\right] = 2\log\left(e^{iy}+1\right)-iy \tag{28}$$

for $y \in (-\pi,\pi)$, where the logarithm function in the middle and right formulas takes the principal value. This leads directly to the desired conclusion. □

Next, we provide an evaluation of the integral in terms of an infinite sum. This is done by expanding the integrand in a power series and then integrating termwise. We first provide a proof of this expansion.

Proposition 4.2. *For all* $z \in \mathbb{C}\setminus\{0\}$,

$$\frac{1}{\log(1+z)} = \sum_{m=0}^{\infty} \frac{z^{m-1}}{m!}\sum_{s=0}^{m} \frac{S_1(m,s)}{s+1} = \sum_{m=0}^{\infty} b_m \frac{z^{m-1}}{m!}\ , \tag{29}$$

where the $S_1(m,s)$ *are signed Stirling numbers of the first kind and the* b_m *are Bernoulli numbers of the second kind.*

Proof. Begin by observing that

$$\frac{z}{\log(1+z)} = \int_0^1 (z+1)^t\ dt \tag{30}$$

for all complex $z \neq 0$. The integrand on the right can be expanded using the binomial theorem,

$$(z+1)^t = \sum_{m=0}^{\infty} \frac{z^m}{m!} \frac{\Gamma(t+1)}{\Gamma(t-m+1)} \ , \tag{31}$$

which converges uniformly in $t \in [0,1]$ for all complex z . The coefficients of this power series are polynomials in t:

$$\frac{\Gamma(t+1)}{\Gamma(t-m+1)} \equiv \sum_{s=0}^{m} S_1(m,s) t^s \ . \tag{32}$$

The integers $S_1(m,s)$ in the above formula are Stirling numbers of the first kind, which are implicitly defined for $0 \leq m \leq s$ by the above relation. Therefore

$$\frac{z}{\log(1+z)} = \sum_{m=0}^{\infty} \frac{z^m}{m!} \int_0^1 \frac{\Gamma(t+1)}{\Gamma(t-m+1)} \ dt = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{s=0}^{m} \frac{S_1(m,s)}{s+1} \ , \tag{33}$$

and we finish by dividing everywhere by z . The second identity is even more immediate, in view of the definition of Bernoulli numbers of the second kind:

$$b_n := \int_0^1 \frac{\Gamma(t+1)}{\Gamma(t-n+1)} \ dt \ , \text{ for } n \in \{0,1,2,...\} \ . \tag{34}$$

Combine this with the middle expression in (33) and then divide by z. □

Now we evaluate the original integral as an infinite series.

Proposition 4.3.

$$\int_0^\pi \frac{y^2 \ dy}{y^2+4\log^2(2\cos(\frac{y}{2}))} = \frac{\pi}{2}\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m(m+1)!}\sum_{s=0}^{m+1}\frac{S_1(m+1,s)}{s+1}\right) = \frac{\pi}{2}\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m b_{m+1}}{m(m+1)!}\right) \tag{35}$$

Proof. Using the power series (29) with $z = e^{iy}$ for $y \in [-\pi,\pi]$ and applying this to the integral in (5) gives

$$\int_0^\pi \frac{y^2 \ dy}{y^2+4\log^2(2\cos(\frac{y}{2}))} = \frac{i}{4}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{s=0}^{m}\frac{S_1(m,s)}{s+1}\int_{-\pi}^{\pi} y e^{i(m-1)y} \ dy \ . \tag{36}$$

We are justified in bringing the integral within the infinite sum, since the argument e^{iy} is within the radius of convergence of the power series. The integrals

$$I_m := \int_{-\pi}^{\pi} y e^{imy} \ dy \ , m \in \mathbb{Z} \tag{37}$$

evaluate to $(-1)^{m+1} \times \frac{2\pi i}{m}$ for all integers $m \neq 0$, and 0 in the exceptional case. The most direct way to see this is to verify that

$$\frac{d}{dy}\left(\frac{1-imy}{m^2} e^{imy}\right) = y e^{imy} \quad \text{for } m \neq 0 \tag{38}$$

and evaluate at the endpoints. To complete the proof, substitute . □

Finally we prove the numerically-confirmed result (3).

Proposition 4.4.

$$\int_0^\pi \frac{y^2 \ dy}{y^2+4\ln^2(2\cos(\frac{y}{2}))} = \frac{\pi}{4}(1-\gamma+\ln(2\pi)) \ . \tag{39}$$

Proof. Start with (5) and the integral

$$\frac{i}{4}\int_{-\pi}^{\pi}\frac{y\ dy}{\log(e^{iy}+1)}\ , \tag{40}$$

where we take the principle value of the complex logarithm. We write the integrand as

$$\frac{iy}{\log(e^{iy}+1)} = -\int_0^1 \frac{d}{ds}\left(\frac{(e^{iy}+1)^t}{e^{iys}}\right)_{s=1} dt\ . \tag{41}$$

For fixed $t, s \in \mathbb{N}$ such that $t > s$, we integrate

$$F(s,t) := -\frac{1}{4}\int_{-\pi}^{\pi}\frac{(e^{iy}+1)^t}{e^{iys}}\ dy = -\frac{1}{4i}\int_C \frac{(z+1)^t}{z^{s+1}}\ dz \tag{42}$$

where C is the simple, closed counterclockwise path around the unit circle in the complex plane. The integrand is clearly analytic in a neighbornood of this path, as its only singularity lies at $z = 0$. By the residue theorem, the value of this contour integral is $2\pi i$ times the integrand's residue at $z = 0$. The Laurent expansion of the integrand is

$$\frac{(z+1)^t}{z^{s+1}} = \sum_{m=0}^{\infty}\frac{\Gamma(t+1)}{\Gamma(m+1)\Gamma(t-m+1)}\ z^{m-s-1}\ , \tag{43}$$

using a similar binomial expansion to the one in (31). In this expansion we use the gamma function, the extension of the factorial which is defined for all complex arguments with positive real part by the integral

$$\Gamma(n) := \int_0^{\infty} x^{n-1}e^{-x}\ dx\ ,\ \Gamma(n+1) = n! \text{ for } n \in \mathbb{N}\ . \tag{44}$$

From the series above we draw the evaluation

$$F(s,t) = -\frac{\pi}{2}\frac{\Gamma(t+1)}{\Gamma(s+1)\Gamma(t-s+1)}\ . \tag{45}$$

In light of (44) we see the right hand side as an analytic continuation of the integral for all complex parameters s and t (excluding negative integer values). Thus it is allowable to differentiate F with respect to the parameter s . This gives us

$$\frac{\partial F}{\partial s} = -\frac{\pi}{2}\left(\frac{(\psi(t-s+1)-\psi(s+1))\Gamma(t+1)}{\Gamma(s+1)\Gamma(t-s+1)}\right)\ , \tag{46}$$

where

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log(\Gamma(z)) \quad \text{for } z \in \mathbb{C}\ . \tag{47}$$

Among well-known identities ([2], p. 212) for this classical function are

$$\psi(2) = 1-\gamma\ , \quad \text{where } \gamma := \lim_{n\to\infty}\left(\sum_{j=0}^{n}\frac{1}{j} - \ln(n)\right)\ , \tag{48}$$

so that we have

$$\frac{\partial F}{\partial s}(1,t) = -\frac{\pi}{2}\left(\frac{(\psi(t)-(1-\gamma))\Gamma(t+1)}{\Gamma(t)}\right) = -\frac{\pi t}{2}(\psi(t)-(1-\gamma))\ . \tag{49}$$

Finally, we integrate both sides on $[0,1]$:

$$\int_0^1 \frac{\partial F}{\partial s}(1,t)\ dt = -\frac{\pi}{2}\int_0^1 t\psi(t)\ dt + \frac{\pi}{2}\int_0^1 (1-\gamma)t\ dt\ . \tag{50}$$

Using integration by parts to compute the right hand side, we get

$$\frac{\pi}{2}\int_0^1 \log\Gamma(t)\ dt + \frac{\pi}{4}(1-\gamma)\ . \tag{51}$$

The remaining integral is famously ([2], p. 203 and on the cover!) known to be $\ln(\sqrt{2\pi})$.

This gives us

$$-\int_0^1 \frac{d}{ds}\left(-\frac{1}{4}\int_{-\pi}^{\pi}\frac{(e^{iy}+1)^t}{e^{iys}}\ dy\right)_{s=1}\ dt\ = \frac{\pi}{4}\left(1-\gamma+\ln(2\pi)\right)\ . \tag{52}$$

To finish, we notice that the integral in the variable y that represents $F(s,t)$ is uniformly convergent in neighborhoods of $s=1$ and $t\in[0,1]$. Thus the integral can be viewed as a holomorphic function of the parameters in that region. Therefore, we can change the order of the differentiation by s and the integration over y, and switch the order of integration in the variables t and y. With (41) the result is

$$\frac{i}{4}\int_{-\pi}^{\pi}\frac{y\ dy}{\log(e^{iy}+1)} = \frac{\pi}{4}\left(1-\gamma+\ln(2\pi)\right)\ , \tag{53}$$

and in view of (5) this gives the desired identity. □

Finally, we have a sum evaluation as a corollary.

Corollary 4.5.

$$\sum_{m=1}^{\infty}\frac{(-1)^m b_{m+1}}{m(m+1)!} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m(m+1)!}\sum_{s=0}^{m+1}\frac{S_1(m+1,s)}{s+1}\ =\ \frac{1}{2}\Big(\ln(2\pi)-1-\gamma\Big)\ . \tag{54}$$

Proof. This follows from combining identities (3) and (7). □

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