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THE ROLE AND FUNCTION OF PROOF WITH SKETCHPAD*

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Introduction

The problems that students have with perceiving a need for proof is well-known to all high school teachers and is identified without exception in all educational research as a major problem in the teaching of proof. Who has not yet experienced frustration when confronted by students asking "*why do we have to prove this?*" The following conclusion by Gonobolin (1954:61) exemplifies the problem:

> "*... the pupils ... do not ... recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically*."

According to Afanasjewa in Freudenthal (1958:29) students' problems with proof should not simply be attributed to slow cognitive development (for example, an inability to reason logically), but also that students may not see the **function** (meaning, purpose and usefulness) of proof. In fact, several recent studies in opposition to Piaget have shown that very young children are quite capable of logical reasoning in situations that are real and meaningful to them (Wason & Johnson-Laird, 1972; Wallington, 1974; Hewson, 1977; Donaldson, 1979). Furthermore, attempts by researchers to teach logic to students have frequently provided no statistically significant differences in students' performance and appreciation of proof (Deer, 1969; Walter, 1972; Mueller, 1975). More than anything else, it seems that the fundamental issue at hand is the appropriate motivation of the various functions of proof to students.

The question is, however, "*what functions does proof have within mathematics itself which can potentially be utilized in the mathematics classroom to*

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* This section is a revised version of an earlier article by the author titled "The role and function of proof in mathematics," *Pythagoras*, Nov 1990, 24, 17-24. It is reproduced here with permission of the Association for Mathematics Education of South Africa (AMESA).

make proof a more meaningful activity?" The purpose of this section is to describe some important functions of proof, and briefly discuss some implications for the teaching of proof.

The functions of proof in mathematics

Traditionally the function of proof has been seen almost exclusively as being to *verify* the correctness of mathematical statements. The idea is that proof is used mainly to remove either personal doubt or the doubt of skeptics, an idea that has one-sidedly dominated teaching practice and most discussions and research on the teaching of proof. For instance, consider the following two quotes:

> "*a proof is only meaningful when it answers the student's* ***doubts****, when it proves what is not obvious.*" (bold added) - Kline (1973:151)
>
> "*the necessity, the functionality, of proof can only surface in situations in which the students meet* ***uncertainty*** *about the truth of mathematical propositions.*" (bold added) - Alibert (1988:31)

Hanna (1989) and Volmink (1990) also appear to define proof only in terms of its verification function as follows:

> "*a proof is an argument needed to* ***validate*** *a statement, an argument that may assume several different forms as long as it is convincing.*" (bold added) - Hanna (1989:20)
>
> "*Why do we bother to prove theorems? I make the claim here that the answer is: so that we may* ***convince*** *people (including ourselves) ... we may regard a* ***proof as an argument sufficient to convince a reasonable skeptic****.*" - Volmink (1990:8; 10)

Although many authors (e.g.Van Dormolen (1977), Van Hiele (1973) and Freudenthal (1973) and others) have argued that one's need for deductive rigour may undergo change and become more sophisticated with time, this is also argued from the viewpoint that the function of proof is mainly that of verification. For example:

> "*... to progress in rigour, the first step is to* ***doubt*** *the rigour one believes in at*

*this moment. Without this **doubt** there is no letting other people prescribe oneself new criteria of rigour.*" (bold added) - Freudenthal (1973:151)

Many authors have also proposed specific stages in the development of rigour, e.g. Tall (1989:30) proposes three stages in the putting up of a convincing argument, namely the convincing of oneself, the convincing of a friend and the convincing of an enemy. Although extremely useful distinctions, it considers only the verification function of proof.

However, as pointed out by Bell (1976:24) this view of verification / conviction being the main function of proof "*avoids consideration of the real nature of proof*", since conviction in mathematics is often obtained "*by quite other means than that of following a logical proof.*" Therefore the actual practice of modern mathematical research calls for a more complete analysis of the various functions and roles of proof. Although I lay claim to neither completeness nor uniqueness, I have found the following model for the functions of proof useful in my research over the past few years. It is a slight expansion of Bell's (1976) original distinction between the functions of verification, illumination and systematization. The model is presented here (in no specific order of importance) and discussed further on:

* *verification* (concerned with the truth of a statement)
* *explanation* (providing insight into why it is true)
* *systematisation* (the organization of various results into a deductive system of axioms, major concepts and theorems)
* *discovery* (the discovery or invention of new results)
* *communication* (the transmission of mathematical knowledge)
* *intellectual challenge* (the *self-realization/fulfillment* derived from constructing a proof)

Proof as a means of verification/conviction

With very few exceptions, mathematics teachers seem to believe that only proof provides certainty for the mathematician and that it is therefore the only authority for establishing the validity of a conjecture. However, proof is not necessarily a prerequisite for conviction—to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof. (For what other weird and obscure reasons would we then sometimes spend months or years trying to prove certain conjectures, if we weren't already convinced of their truth?)

The well-known George Polya (1954:83-84) writes:

> "*... having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong* ***confidence*** *in the theorem. Without such* ***confidence*** *we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is* ***true****, you start* ***proving*** *it.*" (bold added)

In situations like the above where conviction prior to proof provides the motivation for a proof, the function of the proof clearly must be something other than verification/conviction.

In real mathematical research, personal conviction usually depends on a combination of intuition, quasi-empirical verification and the existence of a logical (but not necessarily rigourous) proof. In fact, a very high level of conviction may sometimes be reached even in the absence of a proof. For instance, in their discussion of the "heuristic evidence" in support of the still unproved twin prime pair theorem and the famous Riemann Hypothesis, Davis & Hersh (1983:369) conclude that this evidence is "*so strong that it carries conviction even without rigorous proof.*"

That conviction for mathematicians is not reached by proof alone is also strikingly borne out by the remark of a previous editor of the Mathematical Reviews that approximately one half of the proofs published in it were incomplete and/or contained errors, although the theorems they were purported to prove were essentially true (Hanna, 1983:71). Research mathematicians, for instance, seldom scrutinize the published proofs of results in detail, but are rather led by the established authority of the author, the testing of special cases and an informal evaluation whether "*the methods and result fit in, seem reasonable...*" (Davis & Hersh, 1986:67). Also according to Hanna (1989) the reasonableness of results often enjoy priority over the existence of a completely rigourous proof.

When investigating the validity of a new, unknown conjecture, mathematicians usually do not only look for proofs, but also try to construct counter-examples at the same time by means of quasi-empirical testing, since such testing may expose hidden contradictions, errors or unstated assumptions. In this way counter-examples are sometimes produced, requiring mathematicians to reconstruct old proofs and construct new ones. In the attaining conviction, the failure to disprove conjectures emprically plays just as important a role as the process of deductive justification. It appears that there is a logical, as well as a psychological, dimension to attaining certainty. Logically, we require some form of deductive proof, but psychologically it seems we need some experimental exploration or intuitive understanding as well.

Of course, in view of the well-known limitations of intuition and quasi-empirical methods themselves, the above arguments are definitely not meant to disregard the importance of proof as an indispensable means of verification, especially in the case of surprising non-intuitive or doubtful results. Rather it is intended to place proof in a more proper perspective in opposition to a distorted idolization of proof as the only (and absolute) means of verification / conviction.

Proof as a means of explanation

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (for example, accurate constructions and measurement, numerical substitution, and so on), this generally provides no satisfactory explanation why the conjecture may be true.

It merely confirms that it is true, and even though considering more and more examples may increase one's confidence even more, it gives no psychological satisfactory sense of illumination—no insight or understanding into how the conjecture is the consequence of other familiar results. For instance, despite the convincing heuristic evidence in support of the earlier mentioned Riemann Hypothesis, one may still have a burning need for explanation as stated by Davis & Hersh (1983:368):

> "*It is interesting to ask, in a context such as this, why we still feel the need for a proof ... It seems clear that we want a proof because ... if something is true and we can't deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding* ***why*** *the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it* ***is*** *true.*"

Gale (1990:4) also clearly emphasizes as follows, with reference to Feigenbaum's experimental discoveries in fractal geometry, that the function of their eventual proofs was that of explanation and not that of verification at all:

> "*Lanford and other mathematicians were not trying to validate Feigenbaum's results any more than, say, Newton was trying to* ***validate*** *the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the* ***explanation****. Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining.*" (bold added))

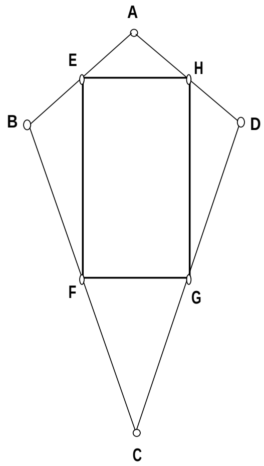
Thus, in most cases when the results concerned are intuitively self-evident and/or they are supported by convincing quasi-empirical evidence, the function of proof for mathematicians is not that of verification, but rather that of explanation (or the other functions of proof described further on).

In fact, for many mathematicians the clarification/explanation aspect of a proof is of greater importance than the aspect of verification. For instance, the well-known Paul Halmos stated some time ago that although the computer-assisted proof of the four colour theorem by Appel & Haken convinced him that it was true, he would still personally prefer a proof which also gives an "*understanding*" (Albers, 1982:239-240). Also to Manin (1981:107) and Bell (1976:24), explanation is a criterion for a "good" proof when stating respectively that it is "*one which makes us wiser*" and that it is expected "*to convey an insight into why the proposition is true.*"

Proof as a means of discovery

It is often said that theorems are most often first discovered by means of intuition and/or quasi-empirical methods, before they are verified by the production of proofs. However, there are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner; in fact, it is completely unlikely that some results (for example, the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using quasi-empirical methods. Even within the context of such formal deductive processes as axiomatization and defining, proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of verifying an already-discovered result, but often also a means of exploring, analyzing, discovering and inventing new results (compare Schoenfeld, 1986 & De Jager, 1990).

For instance, consider the following example. Suppose we have constructed a dynamic kite with *Sketchpad* and connected the midpoints of the sides as shown in Figure 1 to form a quadrilateral EFGH. Visually, EFGH clearly appears to be a rectangle, which can easily be confirmed by measuring the angles. By grabbing any vertex of the kite ABCD, we could now drag it to a new position to verify that EFGH remains a rectangle. We could also drag vertex A downwards until ABCD becomes concave to check whether it remains true. Although such continuous variation can easily convince us, it provides no satisfactory explanation why the midpoint quadrilateral of a kite is a rectangle. However, if we produce a deductive proof for this conjecture, we immediately notice that the perpendicularity of the diagonals is the essential characteristic upon which it depends, and that the property of equal adjacent sides is therefore not required. (The proof is left to the reader).



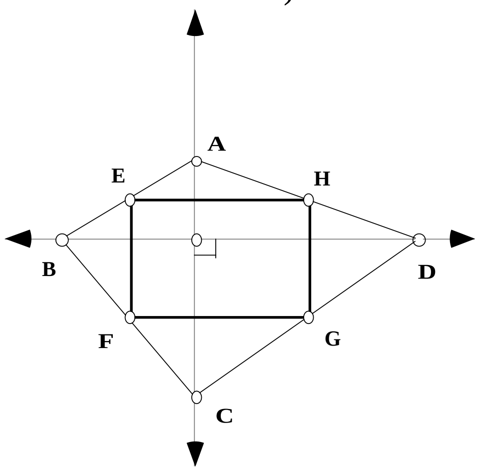


Figure 1

In other words, we can immediately generalize the result to any quadrilateral with perpendicular diagonals (a perpendicular quadrilateral) as shown by the second figure in Figure 1. In contrast, the general result is not at all suggested by the purely empirical verification of the original hypothesis. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since we would probably have restricted our investigation to the familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and isosceles trapezoids.

The Theorem of Ceva (1678) was probably discovered in a similar deductive fashion by generalizing from an "areas" proof for the concurrency of the medians of a triangle, and not by actual construction and measurement (see De Villiers, 1988). However, new results can also be discovered *a priori* by simply deductively analysing the properties of given objects. For example, without resorting to actual construction and measurement it is possible to quickly deduce that AB + CD = BC + DA for the quadrilateral ABCD circumscribed around a circle as shown in Figure 2 by using the theorem that the tangents from a point outside a circle to the circle are equal.

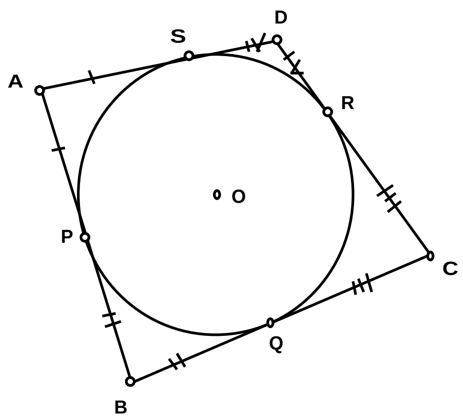


Figure 2

Proof as a means of systematisation

Proof exposes the underlying logical relationships between statements in ways no amount of quasi-empirical testing nor pure intuition can. Proof is therefore an indispensable tool for systematizing various known results into a deductive system of axioms, definitions and theorems. Some of the most important functions of a deductive systematization of known results are given as follows by De Villiers (1986):

* It helps identify inconsistencies, circular arguments and hidden or not explicitly stated assumptions.
* It unifies and simplifies mathematical theories by integrating unrelated statements, theorems, and concepts with one another, thus leading to an economical presentation of results.
* It provides a useful global perspective or bird's eyeview of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived.
* It is helpful for applications both within and outside mathematics, since it makes it possible to check the applicability of a whole complex structure or theory by simply evaluating the suitability of its axioms and definitions.
* It often leads to alternative deductive systems that provide new

perspectives and/or are more economical, elegant, and powerful than existing ones.

Although some elements of verification are obviously also present here, the main objective clearly is not "*to check whether certain statements are really true*", but to organize logically unrelated individual statements that are already known to be true into *a coherent unified whole*. Due to the global perspective provided by such simplification and unification, there is of course also a distinct element of illumination present when proof is used as a means of systematization. In this case, however, the focus falls on global rather than local illumination. Thus, it is in reality false to say at school when proving self-evident statements such as that the opposite angles of two intersecting lines are equal, that we are "making sure". Mathematicians are actually far less concerned about the truth of such theorems, than with their systematization into a deductive system.

Proof as a means of communication

Several authors have stressed the importance of the communicative function of proof, for example:

"*... it appears that proof is a form of **discourse**, a means of communication among people doing mathematics.*" (bold added) - Volmink (1990:8)

"*... we recognize that mathematical argument is addressed to a human audience, which possesses a background knowledge enabling it to understand the intentions of the speaker or author. In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is ... namely, a **human interchange** based on shared meanings, not all of which are verbal or formulaic.*" (bold added) - Davis & Hersh (1986:73).

Similarly, Davis (1976) has also mentioned that one of the real values of proof is that it creates a forum for critical debate. According to this view, proof is a unique way of communicating mathematical results between professional mathematicians, between teachers and students, and among students themselves. The emphasis thus falls on the social process of reporting and disseminating mathematical knowledge in society. Proof as a form of social interaction therefore also involves subjectively negotiating not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument. In turn, such a social filtration of a proof in various communications contributes to its refinement and the identification of errors, as well as sometimes to its rejection by the discovery of a counter-example.

Proof as a means of intellectual challenge

To mathematicians proof is an intellectual challenge that they find as appealing as other people may find puzzles or other creative hobbies or endeavours. Most people have sufficient experience, if only in attempting to solve a crossword or jigzaw puzzle, to enable them to understand the exuberance with which Pythagoras and Archimedes are said to have celebrated the discovery of their proofs. Doing proofs could also be compared to the physical challenge of completing an arduous marathon or triathlon, and the satisfaction that comes afterwards. In this sense, proof serves the function of **self-realization** and **fulfillment**. Proof is therefore a testing ground for the intellectual stamina and ingenuity of the mathematician (compare Davis & Hersh, 1983:369). To paraphrase Mallory's famous comment on his reason for climbing Mount Everest: *We prove our results because they're there.* Pushing this analogy even further: it is often not the existence of the mountain that is in doubt (the truth of the result), but whether (and how) one can conquer (prove) it!

Finally, although the six functions of proof above can be distinguished from one another, they are often all interwoven in specific cases. In some cases certain functions may dominate others, while in some cases certain functions may not feature at all. Furthermore, this list of functions is by no means complete. For instance, we could easily add an *aesthetic* function or that of *memorization* and *algorithmization* (Renz, 1981 & Van Asch, 1993).

Teaching proof with Sketchpad

When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like *Sketchpad*, they have little need for further conviction or verification. So verification serves as little or no motivation for doing a proof. However, I have found it relatively easy to solicit further curiosity by asking students *why* they think a particular result is true; that is to challenge them to try and *explain* it. Students quickly admit that inductive verification merely confirms; it gives no satisfactory sense of illumination, insight, or understanding into how the conjecture is a consequence of other familiar results. Students therefore find it quite satisfactory to then view a deductive argument as an attempt at explanation, rather than verification.

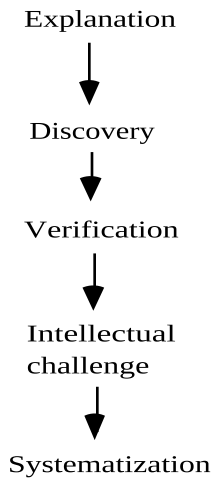


Figure 3

It is also advisable to introduce students early on to the discovery function of proof and to give attention to the communicative aspects throughout by negotiating and clarifying with your students the criteria for acceptable evidence, the underlying heuristics and logic of proof. The verification function of proof should be reserved for results where students genuinely exhibit doubts. Although some students may not experience proof as an intellectual challenge for themselves, they are able to appreciate that others can experience it in this way. Furthermore, in real mathematics, as anyone with a bit of experience will testify, the purely systematization function of proof comes to the fore only at an advanced stage, and should therefore be with-held in an introductory course to proof. It seems meaningful to initially introduce students to the various functions of proof more or less in the sequence given in Figure 3, although not in purely linear fashion as shown, but in a kind of spiral approach where other earlier introduced functions are revisited and expanded. The chapters of this book are organized according to this sequence, and a few approaches to spiraling through the sequence are suggested in the Foreword.

References

Albers, D.J. (1982). Paul Halmos: Maverick Mathologist. *The Two-year College Mathematics Journal*, 13(4), 234-241.

Alibert, D. (1988). Towards New Customs in the Classroom. *For the Learning of Mathematics*, 8(2), 31-35;43.

Bell, A.W. (1976). A study of pupils' proof-explanations in Mathematical situations. *Educational Studies in Mathematics*, 7, 23-40.

Davis, P.J. (1976). The nature of proof. In Carss, M. (Ed). *Proceedings of the fifth international congress on mathematical education*. Boston: Birkhauser.

Davis, P.J. & Hersh, R. (1983). *The Mathematical Experience*. Great Britain: Pelican Books.

Davis, P.J. & Hersh, R. (1986). *Descartes' Dream*. New York: HBJ Publishers.

Deer, G.W. (1969). The effects of teaching an explicit unit in logic on students' ability to prove theorems in geometry. Unpublished doctoral dissertation: Florida State University. Dissertation Abstracts International, 30, 387-399.

De Jager, C.J. (1990). When should we use pattern? *Pythagoras*, 23, 11-14.

De Villiers, M.D. (1986). *The role of axiomatization in mathematics and mathematics teaching.* (RUMEUS Studies in Mathematics Education No.2). University of Stellenbosch.

De Villiers, M.D. (1988). What happens if? Why? *Pythagoras*, 18, 45-47.

Donaldson, M. (1979). *Children's Minds.* New York: W..W. Norton.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics*, 3(2), 9-18.

Freudenthal, H. (Ed). (1958). *Report on Methods of Initiation into Geometry*. Groningen: Wolters.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: D.Reidel.

Gale, D. (1990). Proof as explanation. *The Mathematical Intelligencer*, 12(1), 4.

Gonobolin, F.N. (1954). Pupils' Comprehension of Geometric Proofs. In Wilson, J.W. (Ed). (1975). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*. Vol. XII, Problems of Instruction. Chicago: University of Chicago.

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto: OISE Press.

Hanna, G. (1989b). More than formal proof. *For the Learning of Mathematics*, 9(1), 20-23.

Hewson, S.N.P. (1977). Inferential problem solving in young children. Unpublished doctoral dissertation: Oxford University.

Human, P.G. (1978). Wiskundige werkwyses in Wiskunde-onderwys. Unpublished doctoral dissertation: University of Stellenbosch.

Kline, M. (1973). *Why Johnny can't add: the failure of the new math.* New York: St. Martin's Press.

Krygowska, A.Z. (1971). Treatment of the Axiomatic Method in Class. In Servais, W. & Varga, T. *Teaching school mathematics*. Penguin-Unesco, London, 124-150.

Manin, Y.I. (1981). A Digression on Proof. *The Two-year college Mathematics Journal*, 12(2), 104-107.

Mueller, D.J. (1975). Logic and the ability to prove theorems in geometry. Unpublished doctoral dissertation: Florida State University. Dissertation Abstracts International, 36, 851A.

Polya, G. (1954). *Mathematics and Plausible Reasoning. Induction and Analogy in*

Mathematics. Vol 1. Princeton: Princeton University Press.

Renz, P. (1981). Mathematical Proof: What it is and what it ought to be. *The Two-year College Mathematics Journal*, 12(2), 83-103.

Schoenfeld, A.H. (1986). On Having and Using Geometric Knowledge. In Hiebert, J. (Ed). *Conceptual and Procedural Knowledge: The Case of Mathematics*. Hillsdale: Lawrence Erlbaum.

Tall, D. (1989). The Nature of Mathematical proof. *Mathematics Teaching,* 127, June, 28-32.

Van Asch, A.G. (1993). To prove, why and how? *Int. J. Math. Educ. Sci. Technol.*, 24(2), 301-313.

Van Dormolen, J. (1977). Learning to understand what giving a proof really means. *Educational Studies in Mathematics,* 8, 27-34.

Van Hiele, P.M. (1973). *Begrip en Inzicht*. Purmerend: Muusses.

Volmink, J.D. (1990). The Nature and Role of Proof in Mathematics Education. *Pythagoras*, 23, 7-10.

Wallington, B.A. (1974). Some aspects of the development of reasoning in preschool children. Unpublished doctoral dissertation: University of Edinburgh.

Wason, P.C. & Johnson-Laird, P.N. (1972). *Psychology of reasoning: Structure and Content.* London: Batsford.

Walter, R.L. (1972). The effect of knowledge of logic in proving mathematical theorems in the context of mathematical induction. Unpublished doctoral dissertation: Florida State University. Dissertation Abstracts International, 33, 262A.

Wilder, R.L. (1944). The nature of mathematical proof. *American Mathematical Monthly*, 51, 309-323.

THE ROLE AND FUNCTION OF PROOF IN MATHEMATICS*

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Abstract

Traditionally the function of proof has been seen almost exclusively in terms of the verification of the correctness of mathematical statements. This paper strongly criticizes this view as one-sided, and instead proposes a model which distinguishes between five different functions of proof in mathematics. This analysis is based on epistemological considerations, as well as the personal testimonies of practising mathematicians. Verification being only one of the five functions, is then shown in some situations to be of far less importance than some of the other functions. Finally, it is argued that the meaningful presentation of proof at school may be highly dependent on the appropriate negotiation of these functions of proof to pupils.

1. Introduction

The problems that pupils have with perceiving a **need** for proof is well-known to all high school teachers and is identified without exception in all educational research as a major problem in the teaching of proof. Who has not yet experienced frustration when confronted by pupils asking "**why** *do we have to prove this?*" The following conclusion by Gonobolin (1954:61) exemplifies the problem:

> *"... the pupils ... do not ... recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically."*

According to Afanasjewa in Freudenthal (1958:29) pupils' problems with proof should not simply be attributed to slow cognitive development (e.g. an inability to reason logically), but also that they may not see the **function** (meaning, purpose and usefulness) of proof. In fact, several recent studies in opposition to Piaget have shown that very young children are quite capable of logical reasoning in situations which are real and **meaningful** to them (Wason & Johnson-Laird, 1972; Wallington, 1974; Hewson, 1977; Donaldson, 1979). Furthermore, attempts by researchers to teach logic to pupils have frequently provided no statistically significant differences in pupils' performance and appreciation of proof (e.g. Deer, 1969; Walter, 1972; Mueller, 1975). More than anything else, it seems that the fundamental issue at hand is the appropriate **negotiation** of the various functions of proof to pupils.

The question is, however, "*what* **functions** *does proof have within mathematics itself which can potentially be utilized in the mathematics classroom to make proof a more meaningful activity?*" The purpose of this article is to provide a meta-analysis of some important functions of proof, and briefly discuss some implications for the teaching of proof.

2. The functions of proof in mathematics

Traditionally the function of proof has been seen almost exclusively in terms of the **verification** (conviction or justification) of the correctness of mathematical statements. The idea is that proof is used mainly to remove either personal doubt and/or those of skeptics; an idea which has one-sidedly dominated teaching practice and most discussions and research on the teaching of proof. For instance, according to Wilder (1944:318) a proof is

> *"only a* **testing** *process that we apply to these suggestions of our intuitions"* (bold added).

> *"a proof is only meaningful when it answers the student's* **doubts**, *when it proves what is not obvious."* (bold added) — Kline (1973:151)

> *"... the necessity, the functionality, of proof can only surface in situations in which the students meet* **uncertainty** *about the truth of mathematical propositions."* (bold added) — Alibert (1988:31)

> *"... a proof is an argument needed to* **validate** *a statement, an argument that may assume several different forms as long as it is* **convincing**." (bold added) — Hanna (1989:20)

Recently in an article in **Pythagoras**, Volmink (1990:8; 10) also distinguished conviction (verification) as the most important function of proof by defining it as follows:

> *"Why do we bother to prove theorems? I make the claim here that the answer is: so that we may* **convince people** *(including ourselves) ... we may regard a* **proof as an argument sufficient to convince a reasonable skeptic.**"

Although many authors (e.g. Van Dormolen (1977), Van Hiele (1973) and Freudenthal (1973) and others) have argued that one's need for deductive rigour may undergo

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change and become more sophisticated with time, this point is also argued from the viewpoint that the function of proof is mainly that of verification, for example:

> "... *to progress in rigour, the first step is to* **doubt** *the rigour one believes in at this moment. Without this* **doubt** *there is no letting other people prescribe oneself new criteria of rigour.*" (bold added) – Freudenthal (1973:151)

Many authors have also proposed specific stages in the development of rigour, e.g. Tall (1989:30) proposes three stages in the putting up of a **convincing** argument, namely the convincing of oneself, the convincing of a friend and the convincing of an enemy. Although extremely useful distinctions, it clearly also falls into the same category as the above.

It is furthermore suspected that most secondary school teachers of mathematics almost exclusively hold this rather **formalistic** view on the verification/conviction function of proof in mathematics. In fact, during 1984 in a country-wide survey at 11 South African universities it was found that more than half the H.E.D. (postgraduate) students in mathematics education agreed with the statement that the **only** function of proof was that of "*making sure*", i.e. the verification of the truth of results (De Villiers, 1987:38).

However, as pointed out by Bell (1976:24) this view of verification/conviction being the main function of proof "*avoids consideration of the real nature of proof*", since conviction in mathematics is often obtained "*by quite other means than that of following a logical proof.*" Research mathematicians for instance seldom scrutinize the published proofs of results in detail, but are rather led by the established **authority** of the author, the testing of special cases and an informal evaluation whether "*the methods and result fit in, seem reasonable ...*" (Davis & Hersh, 1986:67). It is therefore within the context of the actual practice of modern mathematical research that a more complete analysis of the various functions and roles of proof is called for. Although laying claim to neither completeness nor uniqueness, the author and his colleagues have found the following model for the functions of proof useful in their research over the past few years. As will be noted, it is a slight expansion of Bell's (1976) original distinction between the functions of verification, illumination and systematisation. The model is now presented (in no specific order of importance) and discussed further on:

- *verification* (concerned with the **truth** of a statement)
- *explanation* (providing insight into **why** it is true)
- *systematisation* (the **organisation** of various results into a deductive system of axioms, major concepts and theorems)
- *discovery* (the discovery or invention of **new** results)
- *communication* (the **transmission** of mathematical knowledge)

2.1 Proof as a means of verification/conviction

With very few exceptions, teachers of mathematics seem to believe that a proof for the mathematician provides **absolute certainty** and that it is therefore the **absolute** authority in the establishment of the validity of a conjecture. They seem to hold the naïve view described by Davis & Hersh (1986:65) that behind each theorem in the mathematical literature there stands a sequence of logical transformations moving from hypothesis to conclusion, absolutely comprehensible, and irrefutably guaranteeing truth. However, this view is completely false. Proof is not necessarily a prerequisite for conviction – to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof (*For what other, weird and obscure reasons, would we then sometimes spend months or years to prove certain conjectures, if we weren't already convinced of their truth?*).

> *Proof is not necessarily a prerequisite for conviction – conviction is far more frequently a prerequisite for proof*

The well-known George Polya (1954:83-84) writes for example in this regard:

> "... *having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial* **suspicion** *and gave us a strong* **confidence** *in the theorem. Without such* **confidence** *we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is* **true**, *you start* **proving** *it.*" (bold added)

In situations like the above where conviction provides the **motivation** for a proof, the function of such a proof for the mathematician clearly cannot be that of verification/conviction, but has to be looked for in terms of the other functions of proof (see further on).

Absolute certainty also does not exist in real mathematical research, and personal conviction usually depends on a **combination** of intuition, quasi-empirical verification and the existence of a logical (but not necessarily rigourous) proof. In fact, a very high level of conviction may sometimes be reached even in the absence of a proof. For instance, in their discussion of the "*heuristic evidence*" in support of the still unproved twin prime pair theorem and the famous Riemann Hypothesis[1], Davis & Hersh (1983:369) conclude that this evidence is "*so strong that it carries conviction even without rigorous proof.*"

[1]Although Hideya Matsumoto advanced a proof for the Riemann Hypothesis in 1984, the proof has not yet been studied nor accepted by the whole mathematical community (Devlin, 1985).

That conviction for mathematicians is not reached by proof alone is also strikingly borne out by the remark of a previous editor of the **Mathematical Reviews** that approximately one half of the proofs published in it were incomplete and/or contained errors, although the theorems they were purported to prove were essentially true (Hanna, 1983:71). It therefore seems that the **reasonableness** of results often enjoy priority over the existence of a completely rigourous proof. It is furthermore a commonly held view among today's mathematicians that there is no such thing as a rigourously complete proof (e.g. consult Hanna, 1983 & 1989b; Kline, 1982). Firstly, there is the problem that no absolute standards exist for the evaluation of the logical correctness of a proof nor for its acceptance by the mathematical community as a whole. Secondly, as Davis & Hersh (1986:66) point out, mathematicians usually only publish those parts of their arguments which they deem important for the sake of conviction, thus leaving out all routine calculations and manipulations which can be done by the reader. Therefore a *"complete proof"* according to them *"simply means proof in sufficient detail to convince the intended audience"* (Davis & Hersh, 1986:73).

In addition, attempts to construct rigourously complete proofs lead to such long complicated proofs that an evaluative overview becomes impossible, while the probability of errors becomes dangerously high at the same time. For example, Manin (1981:105) estimates that *"rigourous"* proofs of the two Burnside conjectures would consist of about five hundred pages each, while a *"complete"* proof for Ramanujan's conjecture would consist of about two thousand pages. Even the well-known, but relatively simple Theorem of Pythagoras would take up at least eighty pages according to Renz (1981:85).

Limitative theorems by Gödel, Tarski and others during the early part of this century have high-lighted the inadequacy of deductive proof. Lakatos (1976, 1978) has also argued from an epistemological analysis of some examples from the history of mathematics, that proof is by nature **fallible**, and that it provides no absolute guarantee for the attainment of certainty. For instance:

> *"There have been considerable and partly successful efforts to simplify Russell's* **Principia** *and similar logistic systems. But while the results were mathematically interesting and important they could not retrieve the lost philosophical position. The* **grandes logiques** *cannot be proved true — nor even consistent; they can only be proved false — or even inconsistent."* — Lakatos (1978:31)

When investigating the validity of an unknown conjecture, mathematicians do not (should not) only look for proofs, but also try to construct counter-examples at the same time by means of **quasi-empirical testing**, since such testing may expose hidden contradictions, errors or unstated assumptions. (For example, see the letter *"A counterexample to Kendal's theorem"* in this issue). In this way counter-examples are frequently produced which necessitate a reconsideration of old proofs, and the construction of new ones. Actually the majority of today's mathematicians are only too aware of the inadequacies of deductive proof, not only from the host of examples of false and incomplete proofs from the history of mathematics, but also from personal experience.[2] Personal certainty consequently also depends on the continued absence of counter-examples in the face of quasi-empirical evaluation. In the attainment of conviction, the quasi-empirical process of failed **falsification** therefore plays just as an important a role as the process of (deductive) justification.

Of course, in view of the well-known limitations of intuition and quasi-empirical methods themselves, the above arguments are definitely not meant to disregard the importance of proof as an extremely useful means of verification, ***especially in the case of surprising non-intuitive or doubtful results***. **Rather it is intended to place proof in a more proper perspective in opposition to a presently somewhat distorted idolization of proof as the only (and absolute) means of verification/conviction.**

2.2 Proof as a means of explanation

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (e.g. accurate constructions and measurement; numerical substitution, etc.), this generally provides no satisfactory *explanation* **why** *it may be true*.[3] It merely confirms that it is true, and even though the consideration of more and more examples may increase one's confidence even more, it gives no psychological satisfactory sense of illumination, i.e. an **insight** or **understanding** into how it is the consequence of other familiar results. For instance, despite the convincing heuristic evidence in support of the earlier mentioned Riemann Hypothesis, one may still have a burning need for explanation as stated by Davis & Hersh (1983:368):

[2]The author recently had two experiences where the quasi-empirical testing of theorems he had proved earlier on, necessitated in the one case a clarification of certain undefined terms (De Villiers, 1989a) and in the other case a radical reformulation of the conjecture itself (De Villiers, in press (a)).

[3]It should however, be mentioned that sometimes a heuristic argument (e.g. the use of *analogical* reasoning in the extension of certain relationships in two dimensions to higher dimensions), may actually provide sufficient **explanation** by itself. The purpose of a deductive proof in the case where one's needs for verification and explanation have both been *a priori* fulfilled, would then probably be much more that of **systematisation**, i.e. the inclusion of the result in a deductive system or to shed a different light on it (see par. 2.3), than to provide insight into **why** it is true (or to verify **that** it is true).

"It is interesting to ask, in a context such as this, why we still feel the need for a proof... It seems clear that we want a proof because ... if something is true and we can't deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding **why** *the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it* **is** *true."*

Quasi-empirical verification provides no explanation **why** results are true – it merely confirms

Recently Gale (1990:4) also clearly emphasized as follows, with reference to Feigenbaum's experimental discoveries in fractal geometry, that the function of their eventual proofs was that of explanation and not that of verification at all:

"Lanford and other mathematicians were not trying to **validate** *Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the* **explanation.** *Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining."* (bold added))

Thus, in most cases when the results concerned are intuitively self-evident and/or they are supported by convincing quasi-empirical evidence, the function of proof for mathematicians is certainly not that of verification, but rather that of explanation[4]. It is not a question of *"making sure"*, but rather a question of *"explaining why"*. Of course, not all proofs are equally explanatory, so it is possible to distinguish between proofs that *"verify"* and proofs that *"clarify"*.[5] Steiner (1978:143) as quoted by Hanna (1989a:48) characterizes an explanatory proof as follows:

"... an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object how the theorem changes in response."

Furthermore, for probably most mathematicians the clarification/explanation aspect of a proof is generally of **greater importance** than the aspect of verification. For instance, the well-known Paul Halmos stated some time ago that although the computer-assisted proof of the four colour theorem by Appel & Haken convinced him that it was true, he would still personally prefer a proof which also gives an *"understanding"* (Albers, 1982:239-240). Also to Manin (1981:107) and Bell (1976:24), explanation is a criterion for a *"good"* proof when stating respectively that it is *"one which makes us wiser"* and that it is expected *"to convey an insight into* **why** *the proposition is true."*

2.3 Proof as a means of systematisation

In contrast to proof which manages to expose the underlying logical relationships between statements, no amount of quasi-empirical testing nor pure intuition will suffice in this respect. Proof is therefore an indispensable tool in the systematisation of various known results into a deductive system of axioms, definitions and theorems. Thus, it is intricately involved in the mathematical processes of *a posteriori* **axiomatization** and **defining** (Krygowska, 1971:129-130; Human, 1978:164-165) which form the backbones of both **local** and **global systematisation** (Freudenthal, 1973: 451-461). Some of the most important functions of a deductive systematisation of known results are given as follows by De Villiers (1986):

- it helps with the **identification** of inconsistencies, circular arguments and hidden or not explicitly stated assumptions
- it **unifies** and **simplifies** mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results
- it provides a useful **global perspective** or **bird's eyeview** of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived
- it is helpful for **applications** both within and outside mathematics, since it aids checking the applicability

[4]In De Villiers (1989b) an easily accessible example in high school geometry is given where the author himself first reached conviction through quasi-empirical methods, before seeking a deductive explanation.

[5]Some examples are given in Hanna (1989a) and De Villiers (1990). It is also possible to make additional distinctions between proofs that *"systematize"* and proofs that *"discover"* (consult De Villiers, 1990). For example the traditional proof of the concurrency of the perpendicular bisectors of the sides of a triangle in terms of concruency instead of a simpler more explanatory proof by symmetry) is purely for the purpose of *systematization* (to show that it is the consequence of the concruency statements, which are accepted as axioms in our sillabus). Furthermore while a certain proof may easily lend itself to the further generalization of the result in question (e.g. see example in par. 2.4 further on), another proof (for instance by analytic geometry) may not necessarily provide sufficient insight to lead to new discoveries by means of immediate generalization (also consult De Villiers, 1989d).

of a whole complex structure or theory simply by evaluating the suitability of its axioms and definitions

- it often leads to **alternative** deductive systems which provide new perspectives and/or are more economical, elegant and powerful than existing ones

Although some elements of verification are obviously also present here, the main objective clearly is not *"to check whether certain statements are really true"*, but to organize logically unrelated individual statements which are **already known to be true**, into *"a coherent unified whole"*. Due to the global perspective provided by such simplification and unification, there is of course also a distinct element of illumination present when proof is used as a means of systematization. In this case, however, the focus falls on global rather than local illumination. Thus, it is in reality a completely false perspective to say at school when proving self-evident statements such as those found in introductory Euclidean geometry, that one is *"making sure"*. In such cases, mathematicians are actually far less concerned about their truth, than with their systematisation into a deductive system. For example, the primary function of a proof for the intermediate value theorem for continuous functions is purely that of systematization, as a simple picture combined with an informal argument is sufficient for the purposes of both verification and illumination.

It is furthermore important to note that a mathematician's need for systematization may be fulfilled in a totally different manner than the fulfillment of his/her need for explanation and verification. For instance to use a personal example again, the author (De Villiers, in press(a)) felt an additional need to construct proofs for two dual statements regarding the point and line symmetries of differentiable functions strictly in terms of analysis for the purpose of systematization, although he had already satisfied himself sufficiently of their truth by quasi-empirical testing and had also constructed a satisfactory geometric explanation.

2.4 Proof as a means of discovery

It is often popularly said by critics of the amount of deductive rigour at school level, that deduction in general (and proof in particular) is not a particularly useful heuristic device in the actual **discovery** of new mathematical results. Such people naïvely seem to believe that theorems are virtually always first discovered by means of intuition and/or quasi-empirical methods, before they are verified by the production of proofs. For example, Hanna (1983:66) claims: *"mathematical concepts and propositions are... conceived and formulated* **before** *proofs are put in place."* (bold added). Often there is in this regard also referred to mathematicians like Ramanujan (1887-1920) or Joseph Fourier (1768-1830) who did not himself prove a single theorem of the topic which he invented, namely *Fourier Analysis*. Perhaps this perception is due also in part to the stereotyped way in which proof is normally taught (first the result is presented *followed* by the proof). This view is however completely false, as there are numerous examples in the history of mathematics where new results were discovered/invented in a purely deductive manner; in fact, it is completely unlikely that some results (e.g. the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using quasi-empirical methods. Even within the context of such formal deductive processes as *a priori* axiomatization and defining, proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of *a posteriori* verification, but often also a means of exploration, analysis, discovery and invention (e.g. compare Schoenfeld, 1986 & De Jager, 1990).

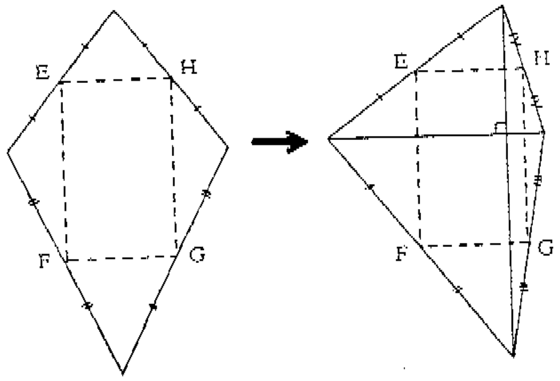

Figure 1 The generalisation of a result.

For instance, consider the following illustrative example from De Villiers (1990). Suppose we one day accidentally formulated the visually acceptable hypothesis that EFGH was always a rectangle when we connected the midpoints of the sides of a kite as shown by the first figure in Figure 1. Although we may gain a very high level of confidence in the truth of this hypothesis by the accurate construction and measurement of different kites (including concave ones), this provides no satisfactory **explanation why** it is true (as was mentioned in par.2.2). However, if we produce a deductive proof for it, we immediately notice that the perpendicularity of the diagonals is the **essential characteristic** upon which it depends, and that the property of equal adjacent sides is therefore not required. (The proof is left to the reader). In other words, we can immediately **generalize** the result to any quadrilateral with perpendicular diagonals (a *perpendicular* quadrilateral) as shown by the second figure in Figure 1. In contrast, the general result is not at all suggested by the purely empirical verification of the original hypothesis. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since such a person would probably have restricted his/her investigation to the familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and isosceles trapezia.

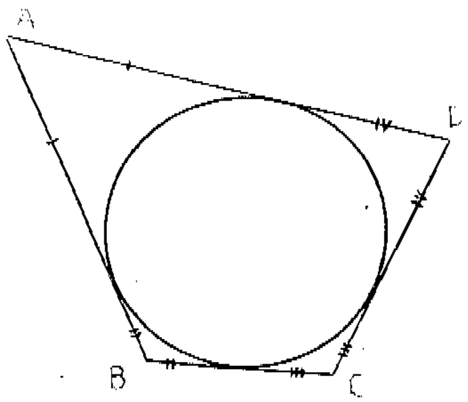

Figure 2

The deductive analysis of the properties of a mathematical object

The Theorem of Ceva (1678) was probably discovered in a similar deductive fashion by generalizing from a proof for the concurrency of the medians of a triangle, and not by actual construction and measurement (e.g. consult De Villiers, 1988). Another example where this process of deductive discovery via deductive generalization happened in precisely this way, is described in De Villiers (1989d). However, new results can also be discovered *a priori* by simply deductively analysing the properties of given objects. For example, without resorting to actual construction and measurement it is possible to quickly deduce that AB + CD = BC + DA for the *circumscribed* quadrilateral ABCD shown in Figure 2 (a proof is given in De Villiers, in press(b)).

2.5 Proof as a means of communication

Several recent authors have stressed the importance of the **communicative** function of proof, for example:

"... *it appears that proof is a form of* **discourse**, *a means of* **communication** *among people doing mathematics.*" (bold added) -- Volmink (1990:8)

"... *we recognize that mathematical argument is addressed to a* **human audience**, *which possesses a background knowledge enabling it to understand the intentions of the speaker or author. In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is — namely, a* **human interchange** *based on* **shared meanings**, *not all of which are verbal or formulaic.*" (bold added) — Davis & Hersh (1986:73).

Similarly, Davis (1976) has also mentioned that one of the real values of proof is that it creates a forum for critical debate. According to this view proof is a unique way of communicating mathematical results between professional mathematicians, between lecturers and students, between teachers and pupils, and among students and pupils themselves. The emphasis thus falls on the **social** process of reporting and disseminating mathematical knowledge in society. Proof as a form of social interaction therefore also involves the **subjective negotiation** of not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument. In turn such a social filtration of a proof in various communications contributes to its refinement and the identification of errors, as well as sometimes to its rejection by the discovery of a counter-example.

According to Thom (1971:679) it is precisely this regulatory influence of the social process that has so far ensured the avoidance of real catastrophic mistakes which could have led the whole mathematical community astray

Functions	Logical deduction	Intuition	Quasi-empirical
The verification of results	✓	✓	✓
The explanation of results	✓	✓	✗
The systematization of existing knowledge	✓	✗	✗
The discovery of new results	✓	✓	✓
The communication of results	✓	✗	✗

Table 1: A comparison between some of the functions of deduction, intuition and quasi-empirical methods.

for an indefinite time. As Hanna (1989b:20) has furthermore pointed out, this social process is usually far more important in the acceptance of a particular result and its proof by practising mathematicians than the mere application of certain formal criteria in judgement of the logical rigour of the given argument.

Finally, although the five functions of proof above can be distinguished from one another, they are often intricately interlinked in specific cases. In some cases certain functions may dominate others, while in some cases certain functions may not feature at all. In Table 1 a comparison is made between the discussed functions of proof in relation to those of intuition and quasi-empirical methods (compare De Villiers, 1990:49). From this it becomes clear that proof plays a rather unique role in the *explanation, systematisation* and *communication* of results, something which is not possible to the same degree using only intuitive and/or quasi-empirical methods. On the other hand it also shows that proof is neither a prerequisite for the discovery of new results nor a sufficient condition for their verification.

This list of functions is however by no means complete. For instance, we could easily add an **aesthetic** function or that of **personal self-realisation** where the production of a proof as such, is viewed as a *"testing ground for the stamina and ingenuity of the mathematician."* (Davis & Hersh, 1983:369). But it was felt that such aspects fell beyond the intended scope of this article.

3. Some concluding comments about the teaching of proof

Traditionally the role and function of proof in the classroom has either been completely ignored (the fact that it is in the syllabus and will be examined is considered sufficient reason for its treatment in class), or it has been presented as a means of obtaining certainty (i.e. within the context of verification/conviction). However, as pointed out in this article, mathematicians often construct proofs for quite other reasons than that of verification/conviction. The popular formalistic idea of many present mathematics teachers that conviction is a *one-to-one mapping* of deductive proof (i.e. a bijective function) should therefore be completely abandoned; *conviction is not gained exclusively from proof alone nor is the only function of proof that of verification/conviction.* Not only does such an approach represent intellectual dishonesty, but it also does not make sense to pupils, especially where self-evident or easily verifiable statements are concerned.

Conviction is not a bijective function of proof

Rather than one-sidedly focussing only on proof as a means of verification, the more fundamental function of explanation should be exploited to present proof as a meaningful activity to pupils. At the same time attention should be given to the discovery function of proof as well as the communicative aspects thereof by actually negotiating with one's pupils the criteria for acceptable evidence, explanations and/or arguments. Furthermore, as anyone with a bit of experience in real mathematics will testify, the purely systematization function of proof comes to the fore only at a very advanced stage, and should therefore be with-held in an introductory course to proof.

In conclusion this article therefore calls for a more comprehensive view and treatment of the function and role of proof than the traditional one on the basis of the following assumptions:

(1) the teaching of mathematics should (at least in part) reflect the nature of mathematics and that which is really **meaningful** to practising mathematicians
(2) as cognitive functioning human beings, pupils basically have the same **need** for meaningful activities as mathematicians, which include knowing, understanding or experiencing the functionality (usefulness) of the activities they are involved in.

References

Albers, D.J. (1982). Paul Halmos: Maverick Mathologist. **The Two-year College Mathematics Journal**, 13(4), 234-241.

Alibert, D. (1988). Towards New Customs in the Classroom. **For the Learning of Mathematics**, 8(2), 31-35;43.

Bell, A.W. (1976). A study of pupils' proof-explanations in mathematical situations. **Educational Studies in Mathematics**, 7, 23-40.

Bell, A.W. (1978). Proof. In Dörfler, W. & Fischer, R. (Eds). **Beweisen im Mathematikunterricht**. Vorträge des Internationalen Symposiums für Didaktik der Mathematik in Klagenfurt. Stuttgart: Hölder-Pichler-Tempsky.

Davis, P.J. (1976). The nature of proof. In Carss, M. (Ed). Proceedings of the fifth international congress on mathematical education. Boston: Birkhauser.

Davis, P.J. & Hersh, R. (1983). The Mathematical Experience. Great Britain: Pelican Books.

Davis, P.J. & Hersh, R. (1986). Descartes' Dream. New York: HBJ Publishers.

Deer, G.W. (1969). The effects of teaching an explicit unit in logic on students' ability to prove theorems in geometry. Unpublished doctoral dissertation: Florida State University. Dissertation Abstracts International, 30, 387-399.

De Jager, C.J. (1990). When should we use pattern? **Pythagoras**, 23, 11-14.

De Villiers, M.D. (1986). The role of axiomatization in mathematics and mathematics teaching (RUMEUS Studies in Mathematics Education No.2). University of Stellenbosch.

De Villiers, M.D. (1987). Algemene beheersingsvlakke van sekere wiskundige begrippe en werkwyses deur voornemende wiskunde-onderwysers. **S.A. Journal of Education**, 7(1), 34-41.

De Villiers, M.D. (1988). What happens if? Why? **Pythagoras**, 18, 45-47.

De Villiers, M.D. (1989a). From "TO POLY" to generalised Polyfigures and their classification: A learning experience. **The International Journal for Mathematical Education in Science and Technology**, 20(4), 585-603.

De Villiers, M.D. (1989b). Meetkunde, meting en intuïsie. **Pythagoras**, 20, 44-45.

De Villiers, M.D. (1989c). 'n Klassifiserings-en definiëringsperiode uit die klaskamer. **Spectrum**, 27(2), May, 45-49.

De Villiers, M.D. (1989d). Meetkunde, verklaring en insig. **Pythagoras**, 21, 33-38.

De Villiers, M.D. (1990). Leerlinge se betekenisgewing aan en beheersing van deduksie en verwante wiskundige werkwyses in die konteks van meetkunde. (Pupils' sense-making and mastery of deduction and associated mathematical processes in the context of Geometry). Unpublished doctoral dissertation: University of Stellenbosch.

De Villiers, M.D. (In press (a)). Vertical line and point symmetries of differentiable functions. **Pythagoras**.

De Villiers, M.D. (In press (b)). Inductive and deductive reasoning: Logic and proof. In Moodley, M. & Njisane, R.M. (Eds). **Teaching mathematics better**. Pietermaritzburg: Shuter & Shooter.

Devlin, K. (1985). The Golden Age of Mathematics. **New Scientist**.

Donaldson, M. (1979). **Children's Minds**. New York: W..W. Norton.

Fischbein, E. (1982). Intuition and Proof. **For the Learning of Mathematics**, 3(2), 9-18.

Freudenthal, H. (Ed). (1958) **Report on Methods of Initiation into Geometry**. Groningen: Wolters.

Freudenthal, H. (1973). **Mathematics as an educational task**. Dordrecht: D.Reidel.

Gale, D. (1990). Proof as explanation. **The Mathematical Intelligencer**, 12(1), 4.

Gonobolin, F.N. (1954). Pupils' Comprehension of Geometric Proofs. In Wilson, J.W. (Ed). (1975). **Soviet Studies in the Psychology of Learning and Teaching Mathematics**. vol.XII, **Problems of Instruction**. Chicago: University of Chicago.

Hanna, G. (1983). **Rigorous proof in mathematics education**. Toronto: OISE Press.

Hanna, G. (1989a). Proofs that Prove and Proofs that Explain. **Proceedings of the 13th International Conference on the Psychology of Mathematics Education**. Parys, 45-51.

Hanna, G. (1989b). More than formal proof. **For the Learning of Mathematics**, 9(1), 20-23.

Hewson, S.N.P. (1977). Inferential problem solving in young children. Unpublished doctoral dissertation: Oxford University.

Human, P.G. (1978). Wiskundige werkwyses in Wiskunde-onderwys. Unpublished doctoral dissertation: University of Stellenbosch.

Kline, M. (1973). **Why Johnny can't add: the failure of the new math**. New York: St. Martin's Press.

Kline, M. (1982). **Mathematics. The Loss of Certainty**. Oxford: Oxford University Press.

Krygowska, A.Z. (1971). Treatment of the Axiomatic Method in Class. In Servais, W. & Varga, T. **Teaching school mathematics**. Penguin-Unesco, London, 124-150.

Lakatos, I. (1976). **Proofs and Refutations**. Great Britain: Cambridge University Press.

Lakatos, I. (1978). **Mathematics, Science and Epistemology**. Cambridge: Cambridge University Press.

Manin, Y.I. (1981). A Digression on Proof. **The Two-year college Mathematics Journal**, 12(2), 104-107.

Mueller, D.J. (1975). Logic and the ability to prove theorems in geometry. Unpublished doctoral dissertation: Florida State University. **Dissertation Abstracts International**, 36, 851A.

Polya, G. (1954). **Mathematics and Plausible Reasoning. Induction and Analogy in Mathematics**. Vol 1. Princeton: Princeton University Press.

Renz, P. (1981). Mathematical Proof: What it is and what it ought to be. **The Two-year College Mathematics Journal**, 12(2), 83-103.

Schoenfeld, A.H. (1986). On Having and Using Geometric Knowledge. In Hiebert, J. (Ed). **Conceptual and Procedural Knowledge: The Case of Mathematics**. Hillsdale: Lawrence Erlbaum.

Steiner, M. (1978). Mathematical explanation. **Philosophical Studies**, 34, 135-151.

Tall, D. (1989). The Nature of Mathematical proof. **Mathematics Teaching**, 127, June, 28-32.

Thom, R. (1971). 'Modern' Mathematics: An Educational and Philosophical Error? **American Scientist**, Vol 159, 695-699.

Van Dormolen, J. (1977). Learning to understand what giving a proof really means. **Educational Studies in Mathematics**, 8, 27-34.

Van Hiele, P.M. (1973). **Begrip en Inzicht**. Purmerend: Muusses.

Volmink, J.D. (1990). The Nature and Role of Proof in Mathematics Education. **Pythagoras**, 23, 7-10.

Wallington, B.A. (1974). Some aspects of the development of reasoning in preschool children. Unpublished doctoral dissertation: University of Edinburgh.

Wason, P.C. & Johnson-Laird, P.N. (1972). **Psychology of reasoning: Structure and Content**. London: Batsford.

Walter, R.L. (1972). The effect of knowledge of logic in proving mathematical theorems in the context of mathematical induction. Unpublished doctoral dissertation: Florida State University. **Dissertation Abstracts International**, 33, 262A.

Wilder, R.L. (1944). The nature of mathematical proof. **American Mathematical Monthly**, 51, 309-323.

"The myth of totally rigorous, totally formalized mathematics is indeed a myth. Mathematics in real life is a form of social interaction where "proof" is a complex of the formal and the informal, of calculations and casual comments, of convincing argument and appeals to the imagination. The competent professional knows what are the crucial points of his argument — the points where his audience should focus their skepticism. Those are the points where he will take care to supply sufficient detail. The rest of the proof will be abbreviated. This is not a matter of the author's laziness. On the contrary, to make a proof too detailed would be more damaging to its readability than to make it too brief. Complete mathematical proof does not mean reduction to a computer program. Complete proof simply means proof in sufficient detail to convince the intended audience — a group of professionals with training and mode of thought comparable to that of the author. Consequently, our confidence in the correctness of our results is not absolute, nor is it fundamentally different in kind from our confidence in our judgements of the physical reality of ordinary life."

— David & Hersh (1986:73) in **Descartes' Dream**. New York : HBJ Publishers