

# Three-Step and Four-Step Random Walk Integrals

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## Abstract

We investigate the moments of 3-step and 4-step uniform random walk in the plane. In particular, we further analyse a formula conjectured in [BNSW09] expressing 4-step moments in terms of 3-step moments. Diverse related results including hypergeometric and elliptic closed forms for  $W_4(\pm 1)$  are given and two new conjectures are recorded.

## 1 Introduction and Preliminaries

Continuing research commenced in [BNSW09], for complex  $s$ , we consider the  $n$ -dimensional integral

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s d\mathbf{x} \quad (1)$$

which occurs in the theory of uniform random walk integrals in the plane, where at each step a unit-step is taken in a random direction. As such, the integral (1) expresses the  $s$ -th moment of the distance to the origin after  $n$  steps. The study of such walks largely originated with Pearson more than a century ago [Pea1905, Pea1905b]. In his honor we call such integrals *ramble integrals*, as he posed such questions for a walker or Rambler. As discussed in [BNSW09], and illustrated further herein, such ramble integrals are approachable by a mixture of analytic, combinatoric, algebraic and probabilistic methods. They provide interesting numeric and symbolic computation challenges. Indeed, nearly all of our results were discovered experimentally.

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For  $n \geq 3$ , the integral (1) is well-defined and analytic for  $\text{Re } s > -2$ , and admits an interesting analytic continuation to the complex plane with poles at certain negative integers, see [BNSW09]. We shall also write  $W_n$  for these continuations. In Figure 1 we show the continuations of  $W_3$  and  $W_4$  on the negative real axis. Observe the poles of  $W_3$  at negative even integers (but note that neither function has zeroes at negative odd integers even though the graphs shown in Figure 1 may suggest otherwise).

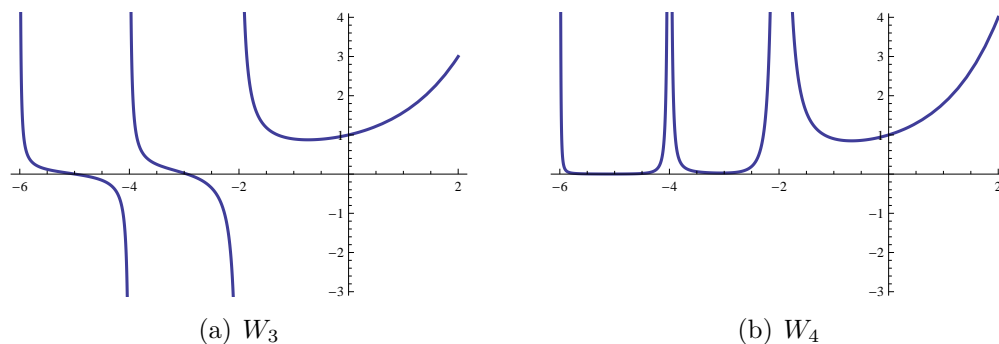


Figure 1:  $W_3, W_4$  analytically continued to the real line.

It is easy to determine that  $W_1(s) = 1$ , and  $W_2(s) = \binom{s}{s/2}$ . Furthermore, it is proven in [BNSW09] that, for  $k$  a nonnegative integer, in terms of the generalized hypergeometric function, we have

$$W_3(k) = \text{Re } {}_3F_2 \left( \begin{matrix} \frac{1}{2}, -\frac{k}{2}, -\frac{k}{2} \\ 1, 1 \end{matrix} \middle| 4 \right). \quad (2)$$

From here, the following expressions for  $W_3(1)$  can be established:

$$W_3(1) = \frac{4\sqrt{3}}{3} \left( {}_3F_2 \left( \begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1 \end{matrix} \middle| \frac{1}{4} \right) - \frac{1}{\pi} \right) + \frac{\sqrt{3}}{24} {}_3F_2 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 2, 2 \end{matrix} \middle| \frac{1}{4} \right) \quad (3)$$

$$= 2\sqrt{3} \frac{K^2(k_3)}{\pi^2} + \sqrt{3} \frac{1}{K^2(k_3)} \quad (4)$$

$$= \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6 \left( \frac{2}{3} \right) \quad (5)$$

$$= \frac{1}{\pi^2} \left( \frac{2^{1/3}}{4} \beta^2 \left( \frac{1}{3} \right) + 2^{2/3} \beta^2 \left( \frac{2}{3} \right) \right), \quad (6)$$

where  $K$  is the complete elliptic integral of the first kind,  $k_3 := \frac{\sqrt{3}-1}{2\sqrt{2}}$  is the *third singular value* as in [BB87], and  $\beta(x) := B(x, x)$  is a central Beta-function value. More simply but similarly,

$$W_3(-1) = 2\sqrt{3} \frac{K^2(k_3)}{\pi^2} = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) = \frac{2^{1/3}}{4\pi^2} \beta^2\left(\frac{1}{3}\right), \quad (7)$$

and, using the two-term recurrence for  $W_3(n)$  given in [BNSW09], it follows that similar expressions can be given for  $W_3$  evaluated at odd integers. It is one of the goals of this paper to give similar evaluations for a 4-step walk.

For  $s$  an even positive integer, the moments  $W_n(s)$  take explicit integer values. In fact, for integers  $k \geq 0$ ,

$$W_n(2k) = \sum_{a_1+\dots+a_n=k} \binom{k}{a_1, \dots, a_n}^2. \quad (8)$$

Based on the combinatorial properties of this evaluation, the following conjecture was made in [BNSW09]. Note that the case  $n = 1$  is easily resolved.

**Conjecture 1.** *For positive integers  $n$  and complex  $s$ ,*

$$W_{2n}(s) \stackrel{?[1]}{=} \sum_{j \geq 0} \binom{s/2}{j}^2 W_{2n-1}(s - 2j). \quad (9)$$

We investigate this conjecture in some detail in Section 4 below. For  $n = 2$ , in conjunction with (3) this leads to a very efficient computation of  $W_4$  at integers yielding roughly a digit per term.

## 2 Bessel integral representations

We start with the result of Kluyver [Klu1906], amplified in [Watson1932, §31.48] and exploited in [BNSW09], to the effect that the probability that an  $n$ -step walk ends up within a disc of radius  $\alpha$  is

$$P_n(\alpha) = \alpha \int_0^\infty J_1(\alpha x) J_0^n(x) dx. \quad (10)$$

From this, David Broadhurst [Bro09] obtains

$$W_n(s) = 2^{s+1-k} \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(k - \frac{s}{2})} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x} \frac{d}{dx}\right)^k J_0^n(x) dx \quad (11)$$

valid as long as  $2k > s > -\frac{n}{2}$ . Here and below  $J_\nu(z)$  denotes the *Bessel function* of the first kind.

**Example 1** ( $W_n(\pm 1)$ ). In particular, from (11), for  $n \geq 2$ , we can write:

$$W_n(-1) = \int_0^\infty J_0^n(x) dx, \quad W_n(1) = n \int_0^\infty J_1(x) J_0(x)^{n-1} \frac{dx}{x}. \quad (12)$$

◇

Equation (11) enabled Broadhurst to verify Conjecture 1 for  $n = 2, 3, 4, 5$  and odd positive  $s < 50$  to a precision of 50 digits. A different proof of (11) is outlined in Remark 1 below. In particular, for  $0 < s < n/2$ , we have

$$W_n(-s) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^n(x) dx, \quad (13)$$

so that  $W_n(-s)$  essentially is the Mellin transform of (the analytic continuation of) the  $n$ th power of the Bessel function  $J_0$ .

**Example 2.** Using (13), the evaluations  $W_1(s) = 1$  and  $W_2(s) = \binom{s}{s/2}$  translate into

$$\begin{aligned} \int_0^\infty x^{s-1} J_0(x) dx &= 2^{s-1} \frac{\Gamma(s/2)}{\Gamma(1-s/2)}, \\ \int_0^\infty x^{s-1} J_0^2(x) dx &= \frac{1}{2\Gamma(1/2)} \frac{\Gamma(s/2)\Gamma(1/2-s/2)}{\Gamma(1-s/2)^2} \end{aligned}$$

in the region where the left-hand side converges.

The Mellin transforms of  $J_0^3$  and  $J_0^4$  in terms of Meijer  $G$ -functions appear in the proofs of Theorems 2 and 3. ◇

**Remark 1** (Ramanujan’s master theorem). Here, we demonstrate how Ramanujan’s “master theorem” may be applied to find the Bessel integral representation (11) in a natural way; this and more applications of Ramanujan’s master theorem will appear in [RMT10]. For an alternative proof see [Bro09].

Ramanujan’s master theorem [Har78] states that, under certain conditions on the analytic function  $\varphi$ ,

$$\int_0^\infty x^{\nu-1} \left( \sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) x^k \right) = \Gamma(\nu) \varphi(-\nu). \quad (14)$$

Based on the evaluation (8), we have, as noted (but not used) in [BNSW09], the generating function

$$\sum_{k \geq 0} W_n(2k) \frac{(-x)^k}{(k!)^2} = \left( \sum_{k \geq 0} \frac{(-x)^k}{(k!)^2} \right)^n = J_0(2\sqrt{x})^n \quad (15)$$

for the even moments. Applying Ramanujan's master theorem (14) to  $\varphi(k) := W_n(2k)/k!$ , we find

$$\Gamma(\nu)\varphi(-\nu) = \int_0^\infty x^{\nu-1} J_0^n(2\sqrt{x}) \, dx. \quad (16)$$

Upon a change of variables and setting  $s = 2\nu$ ,

$$W_n(-s) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^n(x) \, dx. \quad (17)$$

This is the case  $k = 0$  of (11). The general case follows from the general fact that if  $F(s)$  is the Mellin transform of  $f(x)$  then  $(s-2)(s-4)\cdots(s-2k)F(s-2k)$  is the Mellin transform of  $(-\frac{1}{x} \frac{d}{dx})^k f(x)$ .  $\diamond$

## 2.1 Pole structure

A very useful consequence of equation (13) is the following proposition.

**Proposition 1** (Poles). *The structure of the poles of  $W_n$  is as follows:*

(a) (Reflection) *For  $n = 3$ , we have explicitly for  $k = 0, 1, 2, \dots$  that*

$$\text{Res}_{(-2k-2)}(W_3) = \frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}} > 0,$$

*and the corresponding poles are simple.*

(b) *For each integer  $n \geq 5$ , the function  $W_n(s)$  has a simple pole at  $-2k - 2$  for integers  $0 \leq k < (n-1)/4$  with residue given by*

$$\text{Res}_{(-2k-2)}(W_n) = \frac{(-1)^k}{2^{2k}(k!)^2} \int_0^\infty x^{2k+1} J_0^n(x) \, dx. \quad (18)$$

(c) *Moreover, for odd  $n \geq 5$ , all poles of  $W_n(s)$  are simple as soon as the first  $(n-1)/2$  are.*

In fact, we believe that for odd  $n$ , all poles of  $W_n(s)$  are simple as stated in Conjecture 2. For individual  $n$  this may be verified as in Example 3.

*Proof.* (a) For  $n = 3$  it was shown in [BNSW09] that  $\text{Res}_{-2}(W_3) = 2/(\sqrt{3}\pi)$ . This also follows from (30) of Corollary 1. We remark that from [Watson1932, (4) p. 412] this is also the value of the conditional integral  $\int_0^\infty x J_0^3(x) dx$  in accordance with (18). Letting  $r_3(k) := \text{Res}_{(-2k)}(W_n)$ , the explicit residue equation is

$$r_3(k) = \frac{(10k^2 - 30k + 23)r_3(k-1) - (k-2)^2 r_3(k-2)}{9(k-1)^2},$$

which has the asserted solution, when compared to the recursion for  $W_3(s)$ :

$$(s+4)^2 W_3(s+4) - 2(5s^2 + 30s + 46)W_3(s+2) + 9(s+2)^2 W_3(s) = 0. \quad (19)$$

We give another derivation in Example 8 in Section 3.

(b) For  $n \geq 5$  we note that the integral in (18) is absolutely convergent since  $|J_0(x)| \leq 1$  on the real axis and  $J_0(x) \approx \sqrt{2/(\pi x)} \cos(x - \pi/4)$  (see [AS72, (9.2.1)]). Since

$$\lim_{s \rightarrow 2k} (s - 2k)\Gamma(1 - s/2) = 2 \frac{(-1)^k}{(k-1)!}$$

the residue is as claimed.

(c) As shown in [BNSW09]  $W_n$ , for odd  $n$ , satisfies a recursion of the form

$$(-1)^\lambda (n!!)^2 \prod_{j=1}^{\lambda-1} (s+2j)^2 W_n(s) + c_1(s)W_n(s+2) + \dots + (s+2\lambda)^{n-1} W_n(s+2\lambda) = 0,$$

with polynomial coefficients of degree  $n-1$  where  $\lambda := (n+1)/2$ . From this, on multiplying by  $(s+2k)(s+2k-2) \cdots (s-2k+2\lambda)$  one may derive a corresponding recursion for  $\text{Res}_{(-2k)}(W_n)$  for  $k = 1, 2, \dots$ . Inductively, this lets us establish that the poles are simple. The argument breaks down if one of the initial values is infinite as it is when  $4|n$ . □

**Example 3** (Poles of  $W_5$ ). We illustrate Proposition 1 in the case  $n = 5$ . In particular, we demonstrate how to show that all poles are indeed simple. To this end, we start with the recursion:

$$(s+6)^4 W_5(s+6) - (35(s+5)^4 + 42(s+5)^2 + 3)W_5(s+4) + (s+4)^2(259(s+4)^2 + 104)W_5(s+2) = 225(s+4)^2(s+2)^2 W_5(s).$$

From here,

$$\lim_{s \rightarrow -2} (s+2)^2 W_5(s) = \frac{4}{225} (285W_5(0) - 201W_5(2) + 16W_5(4)) = 0$$

which shows that the first pole is indeed simple as is also guaranteed by Proposition 1b. Similarly,

$$\lim_{s \rightarrow -4} (s+4)^2 W_5(s) = -\frac{4}{225} (5W_5(0) - W_5(2)) = 0$$

showing that the second pole is simple as well. It follows from Proposition 1c that all poles of  $W_5$  are simple. More specifically, let  $r_5(k) := \text{Res}_{(-2k)}(W_5)$ . With initial values  $r_5(0) = 0, r_5(1)$  and  $r_5(2)$ , we derive that

$$\begin{aligned} r_5(k+3) &= \frac{k^4 r_5(k) - (5 + 28k + 63k^2 + 70k^3 + 35k^4) r_5(k+1)}{225(k+1)^2(k+2)^2} \\ &\quad + \frac{(285 + 518k + 259k^2) r_5(k+2)}{225(k+2)^2}. \end{aligned}$$

Moreover, in light of Example 5 the approximations  $r_5(1) \approx 0.329933801060064059$  as well as  $r_5(2) \approx 0.00661673$  may be obtained.  $\diamond$

**Example 4** (Poles of  $W_4$ ). Let  $r_4(k) := \lim_{s \rightarrow -2k} (s+2k)^2 W_4(s)$ , then the recursion for  $W_4(s)$

$$(s+4)^3 W_4(s+4) - 4(s+3)(5s^2 + 30s + 48)W_4(s+2) + 64(s+2)^3 W_4(s) = 0$$

gives us

$$r_4(k+2) = \frac{1}{32} \frac{(2k+1)(5k^2 + 5k + 2)}{(k+1)^3} r_4(k+1) - \frac{1}{64} \frac{k^3}{(k+1)^3} r_4(k).$$

We also compute that

$$\frac{3}{2\pi^2} = r_4(1) = \lim_{s \rightarrow -2} (s+2)^2 W_4(s) = \frac{3 + 4W_4'(0) - W_4'(2)}{8}.$$

The first equality was recorded in [BNSW09], and is obtainable from (31). Further, L'Hôpital's rule shows that the residue of the double pole of  $W_4$  at  $s = -2$  is

$$\lim_{s \rightarrow -2} \frac{d}{ds} ((s+2)^2 W_4(s)) = \frac{9 + 18W_4'(0) - 3W_4'(2) + 4W_4''(0) - W_4''(2)}{16}$$

with a numerical value of  $0.316037\dots$  which we were able to identify as  $\frac{9}{2\pi^2} \log(2)$ . Similarly, the second residue is found to be  $\frac{9}{128\pi^2} (4 \log(2) - 1)$  with similar formulas for the other residues.

We finally record a remarkable identity related to the pole of  $W_4$  at  $-2$  that was established in [Watson1932, (10) p. 415]. It is

$$\int_0^\infty J_\nu^4(x) x^{1-2\nu} dx = \frac{1}{2\pi} \frac{\Gamma(2\nu)\Gamma(\nu)}{\Gamma(3\nu)\Gamma(\nu + 1/2)} \quad (20)$$

for  $\text{Re } \nu > 0$ . Hence  $\int_0^\infty J_\nu^4(x) x^{1-2\nu} dx \approx \frac{3/\nu}{4\pi^{3/2}}$  as  $\nu \rightarrow 0$ .  $\diamond$

**Example 5** (Derivatives of  $W_n$ ). We may reverse the process of obtaining the residues to determine auxiliary derivative information.

$\mathbf{n} = 3$ . Using (19) and L'Hôpital's rule, we obtain

$$\text{Res}_{-2}(W_3) = \frac{8 + 12W_3'(0) - 4W_3'(2)}{9}. \quad (21)$$

Differentiating the double integral for  $W_3(s)$  under the integral sign, we have

$$W_3'(0) = \frac{1}{2} \int_0^1 \int_0^1 \log(4 \sin(\pi y) \cos(2\pi x) + 3 - 2 \cos(2\pi y)) dx dy.$$

Then, using  $\int_0^1 \log(a + b \cos(2\pi x)) dx = \log(\frac{1}{2}(a + \sqrt{a^2 - b^2}))$  for  $a > b > 0$  we deduce

$$W_3'(0) = \int_{1/6}^{5/6} \log(2 \sin(\pi y)) dy = \frac{1}{\pi} \text{Cl}\left(\frac{\pi}{3}\right), \quad (22)$$

where  $\text{Cl}$  denotes the *Clausen* function. Knowing as we do that the residue is  $\frac{2}{\sqrt{3}\pi}$ , we may obtain  $W_3'(2) = 2 + \frac{3}{\pi} \text{Cl}(\frac{\pi}{3}) - \frac{3\sqrt{3}}{2\pi}$  from (21).

$\mathbf{n} = 4$ . In like fashion,

$$\begin{aligned} W_4'(0) &= \frac{3}{8\pi^2} \int_0^\pi \int_0^\pi \log(3 + 2 \cos x + 2 \cos y + 2 \cos(x - y)) dx dy \\ &= \frac{7 \zeta(3)}{2 \pi^2}. \end{aligned} \quad (23)$$

We give a complete alternative derivation of (22) and (23) in Example 6. Note that we may also write

$$W_3'(0) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log(3 + 2 \cos x + 2 \cos y + 2 \cos(x - y)) dx dy.$$



The similarity between  $W_3'(0)$  and  $W_4'(0)$  is not coincidental, but comes from applying  $\int_0^1 \log((a + \cos 2\pi x)^2 + (b + \sin 2\pi x)^2) dx = \log(a^2 + b^2)$  if  $a^2 + b^2 > 1$  and 0 otherwise to the triple integral of  $W_4'(0)$ . As this reduction breaks the symmetry, we cannot apply it to  $W_5'(0)$  to get a similar one-dimensional integral.

$\mathbf{n = 5}$ . As in the  $n = 3$  case, we obtain

$$\text{Res}_{-2}(W_5) = \frac{16 + 1140W_5'(0) - 804W_5'(2) + 64W_5'(4)}{225} \approx 0.329933801060064059,$$

and

$$\text{Res}_{-4}(W_5) = \frac{26 \text{Res}_{-2}(W_5) - 16 - 20W_5'(0) + 4W_5'(2)}{225} \approx 0.00661673.$$

In this case the three derivatives of  $W_5$  can be computed from (11) with  $k \leq 3$ . Thence,

$$W_5'(0) = 5 \int_0^\infty \left( \log\left(\frac{2}{t}\right) - \gamma \right) J_0^4(t) J_1(t) dt \approx 0.54441256$$

with similar but more elaborate formulae for  $W_5'(2)$ , and  $W_5'(4)$ . Note that the above can also be written as

$$W_5'(0) = \log(2) - \gamma - \int_0^1 (J_0^5(x) - 1) \frac{dx}{x} - \int_1^\infty J_0^5(x) \frac{dx}{x}.$$

$\mathbf{n \geq 3}$ . In general, differentiating equation (11) under the integral sign gives

$$\begin{aligned} W_n'(0) &= n \int_0^\infty \left( \log\left(\frac{2}{x}\right) - \gamma \right) J_0^{n-1}(x) J_1(x) dx \\ &= \log(2) - \gamma - n \int_0^\infty \log(x) J_0^{n-1}(x) J_1(x) dx, \end{aligned} \quad (24)$$

(where  $\gamma$  is the Euler-Mascheroni constant) and

$$W_n''(0) = n \int_0^\infty \left( \log\left(\frac{2}{x}\right) - \gamma \right)^2 J_0^{n-1}(x) J_1(x) dx.$$

Likewise

$$W_n'(-1) = (\log 2 - \gamma) W_n(-1) - \int_0^\infty \log(x) J_0^n(x) dx,$$

and

$$W_n'(1) = \int_0^\infty \frac{n}{x} J_0^{n-1}(x) J_1(x) (1 - \gamma - \log(2x)) dx.$$

We may therefore obtain many identities by comparing the above equations to known values. For instance,

$$3 \int_0^\infty \log(x) J_0^2(x) J_1(x) dx = \log(2) - \gamma - \frac{1}{\pi} \text{Cl} \left( \frac{\pi}{3} \right),$$

and so on. ◇

## 2.2 Meijer G-function representations

We recall that the *Meijer G-function*—introduced in 1936 by the Dutch mathematician Cornelis Simon Meijer (1904-1974)—is defined, for parameter vectors  $\mathbf{a}$  and  $\mathbf{b}$  [AAR99], by

$$G_{p,q}^{m,n} \left( \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| x \right) = G_{p,q}^{m,n} \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right) \quad (25)$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{k=1}^m \Gamma(b_k - t) \prod_{k=1}^n \Gamma(1 - a_k + t)}{\prod_{k=m+1}^q \Gamma(1 - b_k + t) \prod_{k=n+1}^p \Gamma(a_k - t)} x^t dt. \quad (26)$$

In the case  $|x| < 1$  and  $p = q$  the contour  $L$  is a loop that starts at infinity on a line parallel to the positive real axis, encircles the poles of the  $\Gamma(b_k - t)$  once in the negative sense and returns to infinity on another line parallel to the positive real axis; with a similar contour when  $|x| > 1$ . Moreover  $G_{m,n}^{p,q}$  is analytic in each parameter; in consequence so are the compositions arising below.

Our main tool below is the following special case of Parseval's formula giving the Mellin transform of a product.

**Theorem 1** (Mellin transform). *Let  $G(s)$  and  $H(s)$  be the Mellin transforms of  $g(x)$  and  $h(x)$  respectively. Then*

$$\int_0^\infty x^{s-1} g(x) h(x) dx = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} G(z) H(s-z) dz \quad (27)$$

for any real number  $\delta$  in the common region of analyticity.

This leads to:

**Theorem 2** (Meijer form for  $W_3$ ). *For all complex  $s$*

$$W_3(s) = \frac{\Gamma(1+s/2)}{\Gamma(1/2)\Gamma(-s/2)} G_{3,3}^{2,1} \left( \begin{matrix} 1, 1, 1 \\ 1/2, -s/2, -s/2 \end{matrix} \middle| \frac{1}{4} \right). \quad (28)$$

*Proof.* We apply Theorem 1 to  $J_0^3 = J_0^2 \cdot J_0$  for  $s$  in a vertical strip. Using Example 2 we then obtain

$$\begin{aligned} \int_0^\infty x^{s-1} J_0^3(x) dx &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{2^{s-z-2}}{\Gamma(1/2)} \frac{\Gamma(z/2)\Gamma(1/2-z/2)}{\Gamma(1-z/2)^2} \frac{\Gamma(s/2-z/2)}{\Gamma(1-s/2+z/2)} dz \\ &= \frac{2^s}{2\Gamma(1/2)} \frac{1}{2\pi i} \int_{\delta/2-i\infty}^{\delta/2+i\infty} 4^{-t} \frac{\Gamma(t)\Gamma(1/2-t)\Gamma(s/2-t)}{\Gamma(1-t)^2\Gamma(1-s/2+t)} dt \\ &= \frac{2^s}{2\Gamma(1/2)} G_{3,3}^{2,1} \left( \begin{matrix} 1, 1, 1 \\ 1/2, s/2, s/2 \end{matrix} \middle| \frac{1}{4} \right) \end{aligned}$$

where  $0 < \delta < 1$ . The claim follows from (17) by analytic continuation.  $\square$

Similarly we obtain:

**Theorem 3** (Meijer form for  $W_4$ ). *For all complex  $s$  with  $\operatorname{Re} s > -2$*

$$W_4(s) = \frac{2^s \Gamma(1+s/2)}{\pi \Gamma(-s/2)} G_{4,4}^{2,2} \left( \begin{matrix} 1, (1-s)/2, 1, 1 \\ 1/2, -s/2, -s/2, -s/2 \end{matrix} \middle| 1 \right). \quad (29)$$

*Proof.* We now apply Theorem 1 to  $J_0^4 = J_0^2 \cdot J_0^2$ , again for  $s$  in a vertical strip. Using once more Example 2 we then obtain

$$\begin{aligned} \int_0^\infty x^{s-1} J_0^4(x) dx &= \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{4\pi} \frac{\Gamma(z/2)\Gamma(1/2-z/2)}{\Gamma(1-z/2)^2} \frac{\Gamma(s/2-z/2)\Gamma(1/2-s/2+z/2)}{\Gamma(1-s/2+z/2)^2} dz \\ &= \frac{1}{2\pi} G_{4,4}^{2,2} \left( \begin{matrix} 1, (1+s)/2, 1, 1 \\ 1/2, s/2, s/2, s/2 \end{matrix} \middle| 1 \right) \end{aligned}$$

where  $0 < \delta < 1$ . The claim again follows from (17).  $\square$

We illustrate with graphs of  $W_3, W_4$  in the complex plane in Figure 2. Note the poles and removable singularities. These graphs were produced employing the Meijer forms in their hypergeometric form as presented in the next section. In the case  $n = 4$ , the functional equation is employed for  $s$  with  $\operatorname{Re} s \leq -2$ .

## 2.3 Hypergeometric representations

By Slater's theorem [Mar83, p. 57], the Meijer  $G$ -function representations for  $W_3(s)$  and  $W_4(s)$  given in Theorems 2 and 3 can be expanded in terms of generalized hypergeometric functions.

For  $n = 3, 4$  we obtain the following:

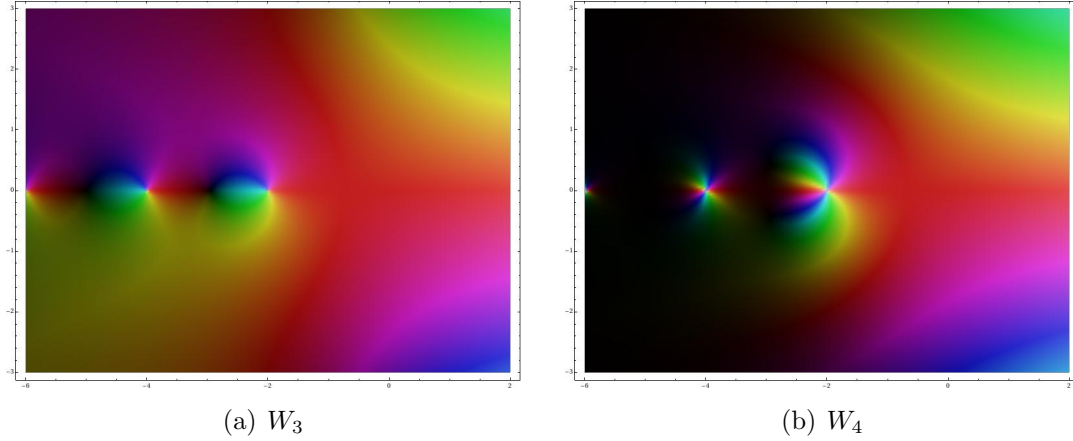


Figure 2:  $W_3$  via (28) and  $W_4$  via (29) in the complex plane.

**Corollary 1** (Hypergeometric forms). *For  $s$  not an odd integer, we have*

$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| \frac{1}{4}\right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, -\frac{s-1}{2} \end{matrix} \middle| \frac{1}{4}\right), \quad (30)$$

and, if also  $\operatorname{Re} s > -2$ , we have

$$W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^3 {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s}{2} + 1 \\ \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| 1\right) + \binom{s}{\frac{s}{2}} {}_4F_3\left(\begin{matrix} \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, 1, -\frac{s-1}{2} \end{matrix} \middle| 1\right). \quad (31)$$

These lovely analytic continuations of  $W_3$  and  $W_4$ , first found in [Cra09] using *Mathematica*, can also be obtained by symbolic integration of (11) in *Mathematica*.

**Example 6** (Derivatives revisited). If we write (30) or (31) as  $W_n(s) = f_1(s)F_1(s) + f_2(s)F_2(s)$ , where  $F_1, F_2$  are the corresponding hypergeometric functions, then it can be readily verified that  $f_1(0) = f_2(0) = F_1'(0) = F_2'(0) = 0$ . Thus, differentiating (30) by appealing to the product rule we get:

$$W_3'(0) = \frac{1}{\pi} {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{1}{4}\right) = \frac{1}{\pi} \operatorname{Cl}\left(\frac{\pi}{3}\right).$$

The last equality follows from setting  $\theta = \pi/6$  in the identity

$$2 \sin(\theta) {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \sin^2 \theta\right) = \operatorname{Cl}(2\theta) + 2\theta \log(2 \sin \theta).$$

Likewise, differentiating (31) gives

$$W_4'(0) = \frac{4}{\pi^2} {}_4F_3 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \frac{7\zeta(3)}{2\pi^2},$$

again confirming our previous result. In this case the hypergeometric evaluation

$${}_4F_3 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| 1 \right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} = \frac{7}{8}\zeta(3),$$

is elementary.

Differentiating (30) at  $s = 2$  leads to the evaluation

$${}_3F_2 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{2}, \frac{5}{2} \end{matrix} \middle| \frac{1}{4} \right) = \frac{27}{4} \left( \text{Cl} \left( \frac{\pi}{3} \right) - \frac{\sqrt{3}}{2} \right),$$

while from (31) at  $s = 2$  we obtain

$$W_4'(2) = 3 + \frac{14\zeta(3) - 12}{\pi^2}.$$

Note that with two starting values, all derivatives of  $W_3(s)$  or  $W_4(s)$  at even  $s$  may be computed recursively.

We also note here that the same technique yields

$$\begin{aligned} W_3''(0) &= \frac{\pi^2}{12} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^{2n}} \frac{H_{n+\frac{1}{2}}}{(2n+1)^2} \\ &= \frac{\pi^2}{12} + \frac{4 \log 2}{\pi} \text{Cl} \left( \frac{\pi}{3} \right) - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^{2n}} \frac{\sum_{k=0}^n \frac{1}{2k+1}}{(2n+1)^2}, \end{aligned} \quad (32)$$

and, quite remarkably,

$$\begin{aligned} W_4''(0) &= \frac{\pi^2}{12} + \frac{7\zeta(3) \log 2}{\pi^2} + \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{H_n - 3H_{n+\frac{1}{2}}}{(2n+1)^3} \\ &= \frac{24\text{Li}_4\left(\frac{1}{2}\right) - 18\zeta(4) + 21\zeta(3) \log 2 - 6\zeta(2) \log^2 2 + \log^4 2}{\pi^2}, \end{aligned} \quad (33)$$

where the very final evaluation is obtained from results in [BZB08, §5]. Here  $\text{Li}_4(1/2)$  is the *polylogarithm* of order 4, while  $H_n := \gamma + \Psi(n+1)$  denotes the  $n$ -th harmonic number, where  $\Psi$  is the *digamma* function; so that  $H_0 = 0$  and for positive integers  $n$ , we have explicitly  $H_n = \sum_{k=1}^n 1/k$  and  $H_{n+1/2} = 2 \sum_{k=1}^{n+1} 1/(2k-1) - 2 \log 2$ .  $\diamond$

We note that while Corollary 1 makes it easy to analyse the poles, the provably removable singularities at odd integers are much harder to resolve explicitly [Cra09]. For  $W_4(-1)$  we proceed as follows:

**Theorem 4** (Hypergeometric form for  $W_4(-1)$ ).

$$\begin{aligned} W_4(-1) &= \frac{\pi}{4} {}_7F_6 \left( \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right) \\ &= \frac{\pi}{4} {}_6F_5 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right) + \frac{\pi}{64} {}_6F_5 \left( \begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2 \end{matrix} \middle| 1 \right). \end{aligned} \quad (34)$$

*Proof.* Using Theorem 3 we write

$$W_4(-1) = \frac{1}{2\pi} G_{4,4}^{2,2} \left( \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| 1 \right).$$

Using the definition (25) of the Meijer G-function as a contour-integral, we see that the corresponding integrand is

$$\frac{\Gamma(\frac{1}{2} - t)^2 \Gamma(t)^2}{\Gamma(\frac{1}{2} + t)^2 \Gamma(1 - t)^2} x^t = \frac{\Gamma(\frac{1}{2} - t)^2 \Gamma(t)^4}{\Gamma(\frac{1}{2} + t)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2} x^t, \quad (35)$$

where we have used  $\Gamma(t)\Gamma(1 - t) = \frac{\pi}{\sin(\pi t)}$ . We choose the contour of integration to enclose the poles of  $\Gamma(\frac{1}{2} - t)$ . Note then that the presence of  $\sin^2(\pi t)$  does not interfere with the contour or the residues (for  $\sin^2(\pi t) = 1$  at half integers). Hence we may ignore  $\sin^2(\pi t)$  in the integrand altogether. Then the right-hand side of (35) is the integrand of another Meijer G-function; thus we have shown that

$$G_{4,4}^{2,2} \left( \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| 1 \right) = \frac{1}{\pi^2} G_{4,4}^{2,4} \left( \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| 1 \right). \quad (36)$$

The same argument shows that the factor of  $\frac{1}{\pi^2}$  applies to all  $W_4(s)$  when we change from  $G_{4,4}^{2,2}$  to  $G_{4,4}^{2,4}$ .

Now, using the transformation

$$x^\alpha G_{p,q}^{m,n} \left( \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| x \right) = G_{p,q}^{m,n} \left( \begin{matrix} \mathbf{a} + \alpha \\ \mathbf{b} + \alpha \end{matrix} \middle| x \right) \quad (37)$$

we deduce that

$$W_4(-1) = \frac{1}{2\pi^3} G_{4,4}^{2,4} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0, 0 \end{matrix} \middle| 1 \right).$$

Finally, we appeal to Bailey's identity [Bai32, Formula (3.4)] that when the series on the left-hand side converges,

$$\begin{aligned} & {}_7F_6 \left( \begin{matrix} a, 1 + \frac{a}{2}, b, c, d, e, f \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a - f \end{matrix} \middle| 1 \right) \\ &= \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)\Gamma(1 + a - e)\Gamma(1 + a - f)}{\Gamma(1 + a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1 + a - b - c)\Gamma(1 + a - b - d)\Gamma(1 + a - c - d)\Gamma(1 + a - e - f)} \\ & \times G_{4,4}^{2,4} \left( \begin{matrix} e + f - a, 1 - b, 1 - c, 1 - d \\ 0, 1 + a - b - c - d, e - a, f - a \end{matrix} \middle| 1 \right). \end{aligned} \quad (38)$$

The claim follows upon setting all parameters to  $1/2$ .  $\square$

An attempt to analogously apply Bailey's identity for  $W_4(1)$  fails, since its Meijer G representation as obtained from Theorem 3 does not meet the precise form required in the formula. Nevertheless, a combination of Nesterenko's theorem ([Nest]) and Zudilin's theorem ([Zudilin02]) gives the following result:

**Theorem 5** (Hypergeometric form for  $W_4(1)$ ).

$$W_4(1) = \frac{3\pi}{4} {}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{matrix} \middle| 1 \right) - \frac{3\pi}{8} {}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 2, 1 \end{matrix} \middle| 1 \right). \quad (39)$$

*Proof.* We first prove a result that will allow us to use Nesterenko's theorem. It converts the Meijer G form of  $W_4(1)$  to a triple integral. We need the following identities which can be readily verified:

$$\frac{d}{dz} \left( z^{-b_1} G_{4,4}^{2,2} \left( \begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{matrix} \middle| z \right) \right) = -z^{-1-b_1} G_{4,4}^{2,2} \left( \begin{matrix} a_1, a_2, a_3, a_4 \\ b_1 + 1, b_2, b_3, b_4 \end{matrix} \middle| z \right) \quad (40)$$

$$\frac{d}{dz} \left( z^{1-a_1} G_{4,4}^{2,2} \left( \begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{matrix} \middle| z \right) \right) = z^{-a_1} G_{4,4}^{2,2} \left( \begin{matrix} a_1 - 1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{matrix} \middle| z \right) \quad (41)$$

Let  $a(z) := G_{4,4}^{2,2} \left( \begin{matrix} 0, 1, 1, 1 \\ -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{matrix} \middle| z \right) = G_{4,4}^{2,2} \left( \begin{matrix} -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ -1, 0, -1, -1 \end{matrix} \middle| z \right)$  where the final equality holds because of (37). Note that  $a(1) = -2\pi W_4(1)$  by Theorem 3. Applying identity (40) to the first form of  $a(z)$  and using the product rule, we get  $\frac{1}{2}a(1) + a'(1) = c_1$ , where  $c_1 := -G_{4,4}^{2,2} \left( \begin{matrix} 0, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{matrix} \middle| 1 \right)$ . Applying identity (41) to the second form of  $a(z)$ , we obtain  $a'(1) = b_1$  where  $b_1 := G_{4,4}^{2,2} \left( \begin{matrix} -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, -1, -1, -1 \end{matrix} \middle| 1 \right)$ . Appealing to equation (77), we see that  $b_1 = -c_1$ . Hence  $a(1) = 4c_1$ . Converting  $c_1$  to a  $G_{4,4}^{2,4}$  as in (36), which finally satisfies the conditions of Nesterenko's theorem, we obtain:

$$W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} dx dy dz.$$

We now make a change of variable  $z' = 1 - z$ . Writing

$$(z')^{\frac{1}{2}} = (z')^{-\frac{1}{2}}(1 - (1 - z')) = (z')^{-\frac{1}{2}} - (z')^{-\frac{1}{2}}(1 - z')$$

splits the previous triple integral into two terms. Each term satisfies Zudilin's theorem and so can be written as a  ${}_7F_6$ . We thence obtain the result as claimed.  $\square$

Indeed, the following alternative relation, which led us to Theorem 5, was first predicted by the *integer relation algorithm* PSLQ in a computational hunt for results similar to that in Theorem 4:

**Theorem 6** (Alternative hypergeometric form for  $W_4(1)$ ).

$$W_4(1) = \frac{9\pi}{4} {}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{matrix} \middle| 1 \right) - 2\pi {}_7F_6 \left( \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right). \quad (42)$$

*Proof.* For notational convenience, let

$$\begin{aligned} A &:= \frac{3\pi^4}{128} {}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{matrix} \middle| 1 \right), \\ B &:= \frac{3\pi^4}{256} {}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 2, 1 \end{matrix} \middle| 1 \right), \\ C &:= \frac{\pi^4}{16} {}_7F_6 \left( \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right). \end{aligned}$$

By (39),  $W_4(1) = (32/\pi^3)(A - B)$ , and the truth of (42) is equivalent to the evaluation  $W_4(1) = (32/\pi^3)(3A - C)$ . Thus, we only need to show  $2A + B - C = 0$ .

The triple integral for  $A$  encountered in the application of Zudilin's theorem is

$$A = \frac{1}{8} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)}{(1-x)yz(1-z)(1-x(1-yz))}} dx dy dz,$$

and can be reduced to a one dimensional integral:

$$A = A_1 := \int_0^1 \frac{(K'(k) - E'(k))^2}{1 - k^2} dk,$$

Here, as usual,  $K'(k) := K(\sqrt{1 - k^2})$  and  $E'(k) := E(\sqrt{1 - k^2})$ .



Happily, we may apply a non-trivial action on the exponents of  $x, y, z$  and leave the value of the integral unchanged (see [Zudilin04], remark after lemma 8). We obtain:

$$\begin{aligned} A &= \frac{1}{8} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{1 - x(1 - yz)}{xyz(1 - x)(1 - y)(1 - z)}} dx dy dz \\ &= A_2 := \int_0^1 K'(k)E'(k) dk. \end{aligned}$$

The like integral for  $B$  can also be reduced to a one dimensional integral,

$$B = B_2 := \int_0^1 k^2 K'(k)^2 dk.$$

But  $B$  also satisfies the conditions of Bailey's identity and Nesterenko's theorem, from which we are able to produce an alternative triple integral, and reduce it to:

$$B = B_1 := \int_0^1 (K'(k) - E'(k)) (E'(k) - k^2 K'(k)) \frac{dk}{1 - k^2}.$$

This more complicated looking form will be useful.

As for  $C$ , equation (69) details its evaluation, which we also record here:

$$C = \int_0^1 K'(k)^2 dk.$$

Now  $2A + B - C = A_1 + A_2 + B_1 - C = 0$ , because the integrand of the later expression is zero.  $\square$

Note that the theorem gives the identity

$$2 \int_0^1 K'(k)E'(k) dk = \int_0^1 (1 - k^2)K'(k)^2 dk, \quad (43)$$

among others.

**Remark 2.** Note that each of the  ${}_7F_6$ 's involved in Theorems 5 and 6 can also be easily written as a sum of two  ${}_6F_5$ 's.

Also note that the first  ${}_7F_6$  term in either theorem satisfies the conditions of Bailey's identity (38) (with  $a = e = f = \frac{3}{2}, b = c = d = \frac{1}{2}$ ):

$${}_7F_6 \left( \begin{matrix} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{matrix} \middle| 1 \right) = -\frac{16}{3\pi^4} G_{4,4}^{2,4} \left( \begin{matrix} \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 0, 0, 0 \end{matrix} \middle| 1 \right). \quad (44)$$

We can thence convert the right-hand side to the Meijer G form. On the other hand,

$$W_4(1) = -\frac{1}{2\pi^3} G_{4,4}^{2,4} \left( \begin{matrix} 0, 1, 1, 1 \\ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{matrix} \middle| 1 \right).$$

We thus have the non-trivial identity:

$$G_{4,4}^{2,4} \left( \begin{matrix} \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 1, 0, 0, 0 \end{matrix} \middle| 1 \right) = 24 G_{4,4}^{2,4} \left( \begin{matrix} \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 0, 0, 0 \end{matrix} \middle| 1 \right) + 8 G_{4,4}^{2,4} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0, 0 \end{matrix} \middle| 1 \right). \quad (45)$$

◇

**Corollary 2** (Elliptic integral representation for  $W_4(1)$ ). *We have*

$$W_4(1) = \frac{16}{\pi^3} \int_0^1 (1 - 3k^2) K'(k)^2 dk. \quad (46)$$

*Proof.* The conclusion of Theorem 6 implies  $(\pi^3/16) W_4(1) = C - 3B = C - 3B_2$ . □

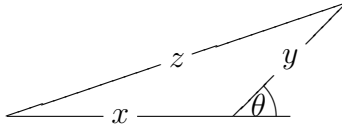
### 3 Probabilistically inspired representations

In this section, we build on the probabilistic approach taken in Section 6 of [BNSW09]. We may profitably view a  $(m+n)$ -step walk as a composition of an  $m$ -step and  $n$ -step walk for  $m, n \geq 1$ . Different decompositions make different structures apparent.

We express the distance  $z$  of an  $(n+m)$ -step walk conditioned on a given distance  $x$  of the first  $n$  steps as well as the distance  $y$  of the remaining  $m$  steps. Then, by the cosine rule,

$$z^2 = x^2 + y^2 + 2xy \cos(\theta),$$

where  $\theta$  is the outside angle of the triangle with sides of lengths  $x$ ,  $y$ , and  $z$ :



It follows that for  $s > 0$ , the  $s$ -th moment of an  $(n+m)$ -step walk conditioned on the distance  $x$  of the first  $n$  steps and the distance  $y$  of the remaining  $m$  steps is

$$g_s(x, y) := \frac{1}{\pi} \int_0^\pi z^s d\theta = |x - y|^s {}_2F_1 \left( \begin{matrix} \frac{1}{2}, -\frac{s}{2} \\ 1 \end{matrix} \middle| -\frac{4xy}{(x-y)^2} \right). \quad (47)$$

Here we appealed to symmetry to restrict the angle to  $\theta \in [0, \pi)$ . We then evaluated the integral in hypergeometric form which, for instance, can be done with the help of *Mathematica* or *Maple*.

**Remark 3** (Alternate forms for  $g_s$ ). Using Kummer's quadratic transformation [AAR99], we obtain

$$g_s(x, y) = y^s {}_2F_1 \left( \begin{matrix} -\frac{s}{2}, -\frac{s}{2} \\ 1 \end{matrix} \middle| \frac{x^2}{y^2} \right) \quad (48)$$

for  $0 < x < y$ . In fact,

$$g_s(x, y) = \operatorname{Re} y^s {}_2F_1 \left( \begin{matrix} -\frac{s}{2}, -\frac{s}{2} \\ 1 \end{matrix} \middle| \frac{x^2}{y^2} \right) \quad (49)$$

for general positive  $x, y$ . This provides an analytic continuation of  $s \mapsto g_s(x, y)$ . In particular, we have

$$g_{-1}(x, y) = \frac{2}{\pi} \operatorname{Re} \frac{1}{y} K \left( \frac{x}{y} \right) \quad (50)$$

and, with  $E$  the complete elliptic integral of the second kind, we have

$$g_1(x, y) = \frac{2}{\pi} \operatorname{Re} y \left\{ 2E \left( \frac{x}{y} \right) - \left( 1 - \frac{x^2}{y^2} \right) K \left( \frac{x}{y} \right) \right\}. \quad (51)$$

This later form has various re-expressions. ◇

Denote by  $p_n(x)$  the density of the distance  $x$  for an  $n$ -step walk. Since  $W_{n+m}(s)$  is the  $s$ -th moment of the distance of an  $(n + m)$ -step walk, we obtain

$$W_{n+m}(s) = \int_0^n \int_0^m g_s(x, y) p_n(x) p_m(y) dy dx, \quad (52)$$

for  $s \geq 0$ . Since for the 1-step walk we have  $p_1(x) = \delta_1(x)$ , this generalizes the corresponding formula given for  $W_{n+1}(s)$  in [BNSW09].

In (52), if  $n = 0$ , then we may take  $p_0(x) = \delta_0(x)$ , and regard the limits of integration as from  $-\epsilon$  and  $+\epsilon$ ,  $\epsilon \rightarrow 0$ . Then  $g_s = y^s$  as the hypergeometric collapses to 1, and we recover the basic form

$$W_m(s) = \int_0^m y^s p_m(y) dy. \quad (53)$$

It is also easily shown that the probability density for a 2-step walk is given by

$$p_2(x) = \frac{2}{\pi \sqrt{4 - x^2}}$$

for  $0 \leq x \leq 2$  and 0 otherwise.

The density  $p_3(x)$  for  $0 \leq x \leq 3$  can be expressed by

$$p_3(x) = \operatorname{Re} \left( \frac{\sqrt{x}}{\pi^2} K \left( \sqrt{\frac{(x+1)^3(3-x)}{16x}} \right) \right), \quad (54)$$

see, e.g., [Pea1906]. To make (54) more accessible we need the following cubic identity.

**Proposition 2.** *For all  $0 \leq x \leq 1$  we have*

$$K \left( \sqrt{\frac{16x^3}{(3-x)^3(1+x)}} \right) = \frac{3-x}{3+3x} K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right).$$

*Proof.* We take the second-order differential equation satisfied by  $K(x)$ , and use substitution and the chain rule to derive the differential equation

$$4x^2(x+3)^2 f(x) + (x-3)(x+1)^2 ((x^3-9x^2-9x+9)f'(x) + x(x^3-x^2-9x+9)f''(x)) = 0,$$

which is satisfied by both functions above, as is readily verified by CAS. Moreover, both function values and derivative values agree at the origin.  $\square$

We note here that for  $0 \leq x < 3$  we can use Proposition 2 to deduce the more symmetrical formula

$$p_3(x) = \frac{2}{\pi} \operatorname{Re} \left( \frac{x}{\operatorname{AGM} \left( \sqrt{(3+x)(1-x)^3}, \sqrt{(3-x)(1+x)^3} \right)} \right) \quad (55)$$

since  $\operatorname{AGM}(1, \sqrt{1-k^2}) = \frac{\pi}{2K(k)}$ . Here AGM denotes the Gaussian arithmetic-geometric mean.

To use Proposition 2, we first apply Jacobi's imaginary transform (see [BB87])

$$\operatorname{Re} K(x) = \frac{1}{x} K \left( \frac{1}{x} \right) \quad \text{for } x > 1$$

to express  $p_3(x)$  as a real function over the intervals  $[0, 1]$  and  $[1, 3]$ . This leads to

$$\begin{aligned} W_3(-1) &= \int_0^3 \frac{p_3(x)}{x} dx \\ &= \frac{4}{\pi^2} \int_0^1 \frac{K \left( \sqrt{\frac{16x}{(3-x)(1+x)^3}} \right)}{\sqrt{(3-x)(1+x)^3}} dx + \frac{1}{\pi^2} \int_1^3 \frac{K \left( \sqrt{\frac{(3-x)(1+x)^3}{16x}} \right)}{\sqrt{x}} dx. \end{aligned}$$

Now in the second integral, we make the change of variable  $x \rightarrow \frac{3-t}{1+t}$ , and after some algebra it transforms exactly into the first integral. Therefore,

$$W_3(-1) = 2 \int_0^1 \frac{p_3(x)}{x} dx, \quad (56)$$

where now we need integrate only from 0 to 1.

To make sense of what has happened more abstractly, let

$$\sigma(x) := \frac{3-x}{1+x}, \quad \lambda(x) := \frac{(1+x)^3(3-x)}{16x}. \quad (57)$$

Then for  $0 < x < 3$  we have  $\sigma^2(x) = x$  and  $\lambda(x)\lambda(\sigma(x)) = 1$ . In consequence  $\sigma$  is an involution that sends  $[0, 1]$  to  $[1, 3]$  and

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)) \quad (58)$$

for  $0 < x < 3$ , because  $\sqrt{\sigma(x)\lambda(x)/x} = (3-x)(x+1)/(4x)$ . Equation (54) now lets us deduce that  $\frac{3}{4}p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$ .

**Example 7** (Series for  $p_3$  and  $W_3(-1)$ ). We know that

$$W_3(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j}$$

is the sum of squares of trinomials (see (8) and [BNSW09]). Using Proposition 2, we may now apply equation (184) in [BBBG08, Section 5.10] to obtain

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}, \quad (59)$$

with radius of convergence one. For  $1 < x < 3$ , on using (58) we obtain

$$p_3(x) = \frac{8x}{\pi\sqrt{3}(x+1)^2} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{3-x}{3+3x}\right)^{2k}. \quad (60)$$

From (59) and (56) we deduce that

$$W_3(-1) = \frac{4}{\pi\sqrt{3}} \sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(2k+1)}$$

as a type of reflection formula. ◇

For  $p_4$  we know of no corresponding entirely closed form but we can use (10) to deduce for all  $n \geq 4$  that

$$p_n(\alpha) = \alpha \int_0^\infty J_0(t)^n J_0(\alpha t) t dt. \quad (61)$$

Alternatively, [Hug95] provides: setting  $\phi_n(r) := p_n(r)/(2\pi r)$ , we have that for integers  $n \geq 2$

$$\phi_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi_{n-1} \left( \sqrt{r^2 + 1 - 2r \cos t} \right) dt. \quad (62)$$

The densities  $p_3$  and  $p_4$  are shown in Figure 3. Note that  $p_3$  has a singularity at 1 as follows from (54). We remark that the derivative of  $p_4$  has singularities at 0 and 4. Also,  $p_4(2) \approx 0.4942337$  is slightly less than 0.5.

Further, computed from (61), the densities  $p_5$ ,  $p_6$ ,  $p_7$ , and  $p_8$  are shown in Figure 4 to illustrate that the case of small  $n$  is very different from the case of large  $n$ . In particular, it should be noted that, for  $n \geq 7$ , the density  $p_n$  is already quite well approximated by the limiting  $\frac{2x}{n} e^{-x^2/n}$ . Also notice that, as the pictures suggest,  $p_n$  is continuously differentiable for  $n \geq 6$  while  $p_5$  is not. This, as well as similar statements about higher-order differentiability, may be proven using dominated convergence and equation (61).

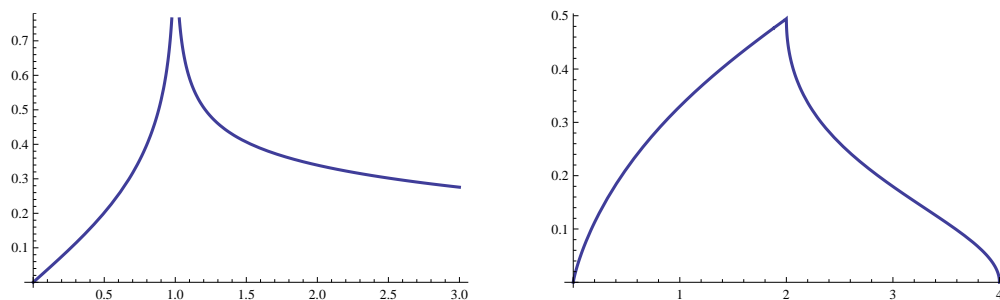


Figure 3: The densities  $p_3$  (L) and  $p_4$  (R).

**Example 8** (Poles of  $W_3$ ). From here we may efficiently recover the explicit form for the residues of  $W_3$  given in Proposition 1a. Fix integers  $N > 2k > 0$  and  $0 < \alpha < 1$ .

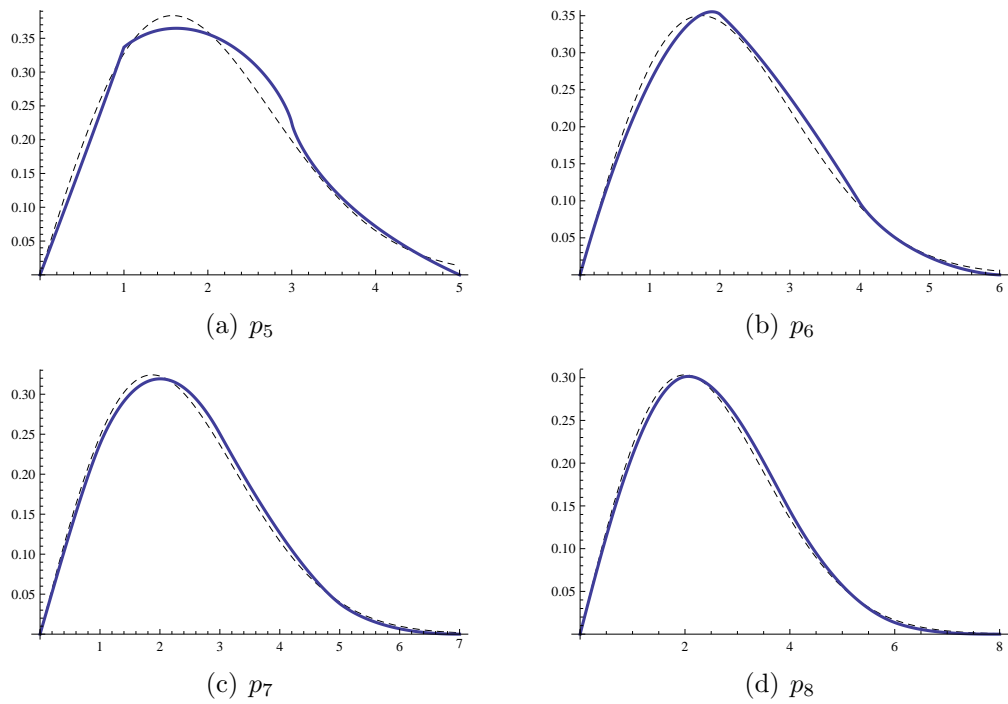


Figure 4: Densities  $p_n$  with the limiting behaviour superimposed.

Use the series  $p_3(x) = \sum_{j \geq 0} a_j x^{2j+1}$  in (59) to write

$$\begin{aligned} W_3(s) - \int_{\alpha}^3 p_3(x) x^s dx - \int_0^{\alpha} \sum_{j=N}^{\infty} a_j x^{2j+1+s} dx &= \int_0^{\alpha} \sum_{j=0}^{N-1} a_j x^{2j+1+s} dx \\ &= \sum_{j=1}^N a_{j-1} \frac{\alpha^{2j+s}}{2j+s}, \end{aligned} \quad (63)$$

and observe that both sides are holomorphic and so (63) holds in a neighborhood of  $s = -2k$ . Since only the first term on the left has a pole at  $-2k$  we may deduce that  $\text{Res}_{(-2k)}(W_3) = a_{k-1}$ . Equivalently,

$$\text{Res}_{(-2k-2)}(W_3) = \frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}},$$

which exposes an elegant reflection property, a generalization of which we have been unable to fully recapture when  $n = 5$ .

Similarly, the analysis in Example 4 suggests that  $p_4$  should have a density of the form  $p_4(x) = a(x) + b(x) \log x$  where  $a, b$  are odd and analytic in a neighbourhood of the origin. The log term is sufficient to fit the double poles. Working as above, we discover that the first two approximations are

$$\left( \frac{9}{2} \frac{\log 2}{\pi^2} - \frac{3}{2} \frac{\log x}{\pi^2} \right) x$$

and

$$\left( \frac{9}{2} \frac{\log 2}{\pi^2} - \frac{3}{2} \frac{\log x}{\pi^2} \right) x + \left( \frac{9}{128} \frac{4 \log 2 - 1}{\pi^2} - \frac{3}{32} \frac{\log x}{\pi^2} \right) x^3$$

The later is already an excellent approximation on  $[0, 1]$  and the approximations appear to be converging for  $0 < x < 2$ . Indeed the fifth approximation at  $x = 2$  yields 0.488946... We also record that  $p_4^-(2) \approx .144687$  while  $p_4^+(2) = -\infty$ . This can be proven by using the large-order asymptotic expansion for  $J_\nu$  to estimate  $p_4'(s)$  for  $s$  near 2 as a combination of Fresnel integrals.  $\diamond$

**Remark 4** ( $W_5$ ). Using (52) we may express  $W_5(s)$ —or  $W_6(s)$ —as double integrals, for example,

$$W_5(-1) = \frac{4}{\pi^4} \int_0^3 \int_0^2 \frac{\sqrt{x}}{y\sqrt{4-y^2}} \text{Re} \left( K \left( \frac{x}{y} \right) \right) \text{Re} \left( K \left( \sqrt{\frac{(x+1)^3(3-x)}{16x}} \right) \right) dy dx.$$



We also have an expression based on taking two two-step walks and a one-step walk, namely:

$$\begin{aligned} W_5(-1) &= \frac{8}{\pi^4} \int_0^2 \int_0^2 \int_0^\pi \frac{\operatorname{Re} K(\sqrt{x^2 + y^2 + 2xy \cos z})}{\sqrt{(4-x^2)(4-y^2)}} dz dx dy \\ &= \frac{8}{\pi^4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\pi \operatorname{Re} K\left(2\sqrt{\sin^2 a + \sin^2 b + 2\sin a \sin b \cos c}\right) dc da db. \end{aligned}$$

But we have been unable to make further progress with these forms.  $\diamond$

### 3.1 Elliptic integral representations

From (52), we derive

$$\begin{aligned} W_4(s) &= \frac{2^{s+2}}{\pi^2} \int_0^1 \int_0^1 \frac{g_s(x, y)}{\sqrt{(1-x^2)(1-y^2)}} dx dy \\ &= \frac{2^{s+2}}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} g_s(\sin u, \sin v) du dv. \end{aligned}$$

where  $s > -2$ . In particular, for  $s = -1$ , again using Jacobi's imaginary transformation, we have:

$$W_4(-1) = \frac{4}{\pi^3} \operatorname{Re} \int_0^1 \int_0^1 \frac{K(x/y)}{y\sqrt{(1-x^2)(1-y^2)}} dx dy \quad (64)$$

$$\begin{aligned} &= \frac{8}{\pi^3} \int_0^1 \int_0^1 \frac{K(t)}{\sqrt{(1-t^2y^2)(1-y^2)}} dy dt \\ &= \frac{8}{\pi^3} \int_0^1 K^2(k) dk. \end{aligned} \quad (65)$$

Also, the following Fourier series [BBBG08, Eqn (70)] allows one to apply the Parseval-Bessel formula

$$K(\sin \theta) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + n)^2}{\Gamma(n+1)^2} \sin((4n+1)\theta), \quad (66)$$

from which we may obtain

$$W_4(-1) = \frac{4}{\pi} \sum_{n, m \geq 0} \left( \frac{\binom{1}{2}_n \binom{1}{2}_m}{n!m!} \right)^2 \left( \frac{1}{1 - (4(m-n))^2} + \frac{1}{1 - (4(m+n)+2)^2} \right), \quad (67)$$

in terms of the rising factorial  $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ .

Starting instead with Nesterenko's theorem [Nest] we have the following:

$$W_4(-1) = \frac{1}{2\pi^3} \int_{[0,1]^3} \frac{dx dy dz}{\sqrt{xyz(1-x)(1-y)(1-z)(1-x(1-yz))}}. \quad (68)$$

[Such integrals are related to Beukers integrals, which were used in the elementary derivation of the irrationality of  $\zeta(3)$ .] Upon computing the  $dx$  integral, followed by the change of variable  $k^2 = yz$ , we have:

$$W_4(-1) = \frac{1}{\pi^3} \int_0^1 \int_0^1 \frac{K(\sqrt{1-yz})}{\sqrt{yz(1-y)(1-z)}} dy dz \quad (69)$$

$$\begin{aligned} &= \frac{2}{\pi^3} \int_0^1 \int_{k^2}^1 \frac{K(\sqrt{1-k^2})}{\sqrt{y(1-y)(y-k^2)}} dy dk \\ &= \frac{4}{\pi^3} \int_0^1 K'(k)^2 dk. \end{aligned} \quad (70)$$

Compare this with the corresponding (64). In particular, appealing to Theorem 4 we derive the closed forms:

$$2 \int_0^1 K(k)^2 dk = \int_0^1 K'(k)^2 dk = \left(\frac{\pi}{2}\right)^4 {}_7F_6 \left( \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right). \quad (71)$$

Recalling Corollary 2 and equation (43) we also deduce that

$$W_4(1) = \frac{96}{\pi^3} \int_0^1 E'(k)K'(k) dk - 8W_4(-1). \quad (72)$$

If we make a trigonometric change of variables in (69), we obtain

$$W_4(-1) = \frac{4}{\pi^3} \int_0^{\pi/2} \int_0^{\pi/2} K \left( \sqrt{1 - \sin^2 x \sin^2 y} \right) dx dy. \quad (73)$$

We may rewrite the integrand as a sum, and then interchange integration and summation to arrive at a slowly convergent representation of the same general form as in Conjecture 1:

$$W_4(-1) = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 {}_3F_2 \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, -n \\ 1, 1 \end{matrix} \middle| 1 \right). \quad (74)$$

**Remark 5** (Relation to Watson integrals). From the evaluation (7) we note that  $W_3(-1)$  equals twice the second of three triple integrals considered by Watson in [Watson1939]:

$$W_3(-1) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dudvdw}{3 - \cos v \cos w - \cos w \cos u - \cos u \cos v}. \quad (75)$$

This is derived in [BBG05] and various related extensions are to be found in [BBBG08]. It is not clear how to generalize this to  $W_4(-1)$ .

Watson's second integral (75) also gives the alternative representation:

$$W_3(-1) = \pi^{-5/2} G_{3,3}^{3,2} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0 \end{matrix} \middle| 4 \right). \quad (76)$$

The equivalence of this and the Meijer G representation coming from Theorem 2 can be established similarly to the proof of Theorem 4 upon using the Meijer G transformation

$$G_{p,q}^{m,n} \left( \begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| x \right) = G_{q,p}^{n,m} \left( \begin{matrix} 1 - \mathbf{b} \\ 1 - \mathbf{a} \end{matrix} \middle| \frac{1}{x} \right). \quad (77)$$

◇

**Remark 6** (Probability of return to the unit disk). By a simple geometric argument, there is a  $\frac{1}{3}$  chance of returning to the unit disk in a 2-step walk. Similarly, for a 3-step walk, if the second step makes an angle  $\theta$  with the first step, then the third step can only vary over a range of  $\theta$  to return to the unit disk (it can be parallel to the first step, to the second step, or anywhere in between). Thus the probability of returning to the unit disk in three steps is

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} |\theta| d\theta = \frac{1}{4} = \int_0^1 p_3(x) dx.$$

Appealing to (59) we deduce that

$$\sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(k+1)} = \frac{\sqrt{3}\pi}{4}.$$

In fact, as Kluyver showed [Klu1906], the probability of an  $n$ -step walk ending in the unit disk is  $1/(n+1)$ . This is easily obtained by setting  $\alpha = 1$  in (10) and computing the integral. ◇

## 4 Partial resolution of Conjecture 1

We may now investigate Conjecture 1 which is restated below for convenience.

**Conjecture.** For positive integers  $n$  and complex  $s$ ,

$$W_{2n}(s) \stackrel{?[1]}{=} \sum_{j \geq 0} \binom{s/2}{j}^2 W_{2n-1}(s-2j). \quad (78)$$

We can resolve this conjecture modulo a conjectured technical estimate given in Conjecture 3. The proof outline below certainly explains Conjecture 1 by identifying the terms of the infinite sum as natural residues.

*Proof.* Using (17) we write  $W_{2n}$  as a Bessel integral

$$W_{2n}(-s) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^{2n}(x) dx.$$

Then we apply Theorem 1 to  $J_0^{2n} = J_0^{2n-1} \cdot J_0$  for  $s$  in a vertical strip. Since, again by (17), we have

$$\int_0^\infty x^{s-1} J_0^{2n}(x) dx = 2^{s-1} \frac{\Gamma(s/2)}{\Gamma(1-s/2)} W_n(-s)$$

we obtain

$$\begin{aligned} W_{2n}(-s) &= 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^{2n-1}(x) \cdot J_0(x) dx \\ &= \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{2} \frac{\Gamma(z/2)\Gamma(s/2-z/2)}{\Gamma(1-z/2)\Gamma(1-s/2+z/2)} W_{2n-1}(-z) dz \end{aligned}$$

where  $0 < \delta < 1$ .

Observe that the integrand has poles at  $z = s, s+2, s+4, \dots$  coming from  $\Gamma(s/2-z/2)$  as well as (irrelevant for current purposes) poles at  $z = 0, -2, -4, \dots$  coming from  $\Gamma(z/2)$ . On the other hand, the term  $W_{2n-1}(-z)$  has at most simple poles at  $z = 2, 4, 6, \dots$  which are cancelled by the corresponding zeros of  $\Gamma(1-z/2)$ . This asserted pole structure of  $W_{2n-1}$  was shown in Proposition 1 for  $n = 3, 4$  and that argument can with work be extended to  $7, 9, \dots$ , but this will not resolve the general case. (See Conjecture 2 below.)

Next, we determine the residue of the integrand at  $z = s + 2j$ . Since  $\Gamma(s/2 - z/2)$  has a residue of  $-2(-1)^j/j!$  at  $z = s + 2j$ , the residue of the integrand is

$$-\frac{(-1)^j \Gamma(s/2 + j)}{(j!)^2 \Gamma(1 - s/2 - j)} W_{2n-1}(-(2j + s)) = -\frac{\Gamma(s/2)}{\Gamma(1 - s/2)} \binom{-s/2}{j}^2 W_{2n-1}(-s - 2j).$$

Thus it follows that

$$W_{2n}(-s) = \sum_{j \geq 0} \binom{-s/2}{j}^2 W_{2n-1}(-s - 2j), \quad (79)$$

which is what we want to prove; provided that we can close the contour in the right half-plane so as to show that

$$\liminf_{\alpha \rightarrow \infty} \int_{\gamma_\alpha} \frac{\Gamma(z/2) \Gamma(s/2 - z/2)}{\Gamma(1 - z/2) \Gamma(1 - s/2 + z/2)} W_{2n-1}(-z) dz = 0. \quad (80)$$

Here,  $\gamma_\alpha$  is a right half-circle of radius  $r_\alpha$  around  $\delta$ . This follows from Conjecture 3 below.  $\square$

To make this proof rigorous we therefore need to show that the next two conjectures hold.

**Conjecture 2** (Poles of  $W_{2n-1}$ ). *For each  $n \geq 1$  all poles of  $W_{2n-1}$  are simple.*

**Conjecture 3** (Growth of  $W_{2n-1}$ ). *For given  $s$ , the maximum modulus of*

$$\frac{\Gamma(z/2) \Gamma(s/2 - z/2)}{\Gamma(1 - z/2) \Gamma(1 - s/2 + z/2)} W_{2n-1}(-z)$$

*over the half-circle  $\gamma_{\alpha_m}$ , is achieved on the real axis at a point  $a_m$ ; and the value  $W_{2n-1}(a_m)$  tends to zero for properly chosen  $r_{\alpha_m} \rightarrow \infty$ .*

**Remark 7** (Other approaches to Conjecture 1). We restrict ourself to the core case with  $n = 2$ . One can prove that both sides of the needed identity satisfy the recursion for  $W_4$ . Hence, it suffices to show that the conjecture is correct for  $s = \pm 1$ . Working entirely formally with (11) and ignoring the restriction on  $s$  we have:

$$\begin{aligned} \sum_{j \geq 0} \binom{-1/2}{j}^2 W_3(-1 - 2j) &= \sum_{j=0}^{\infty} \binom{-1/2}{j}^2 2^{-2j} \frac{\Gamma(\frac{1}{2} - j)}{\Gamma(\frac{1}{2} + j)} \int_0^{\infty} x^{2j} J_0^3(x) dx \\ &= \int_0^{\infty} J_0^3(x) \sum_{j=0}^{\infty} \binom{-1/2}{j}^2 \frac{\Gamma(\frac{1}{2} - j)}{\Gamma(\frac{1}{2} + j)} \left(\frac{x}{2}\right)^{2j} dx \\ &= \int_0^{\infty} J_0^4(x) dx \\ &= W_4(-1), \end{aligned}$$

on appealing to Example 1, since

$$\sum_{j=0}^{\infty} \binom{-1/2}{j}^2 \frac{\Gamma(\frac{1}{2} - j)}{\Gamma(\frac{1}{2} + j)} x^{2j} = J_0(2x)$$

for  $x > 0$ . There is a corresponding manipulation for  $s = 1$  but we have been unable to legitimate the steps in either case.  $\diamond$

**Example 9** (Other residue evaluations). Similar arguments yield evaluations for  $W_3$  such as:

$$W_3(-1) = \frac{16}{\pi^3} K^2 \left( \frac{\sqrt{3} - 1}{2\sqrt{2}} \right) \log 2 + \frac{3}{\pi} \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{\sum_{k=1}^{2n} \frac{(-1)^k}{k}}{4^{4n}}. \quad (81)$$

In conjunction with (7) we obtain

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{\sum_{k=1}^{2n} \frac{(-1)^k}{k}}{4^{4n}} = \left( \frac{2}{\sqrt{3}\pi} - \frac{16}{3\pi^2} \log 2 \right) K^2 \left( \frac{\sqrt{3} - 1}{2\sqrt{2}} \right). \quad (82)$$

For comparison, we record

$$W_4(-1) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4}{4^{4n}} \sum_{k=2n+1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

which follows from (31) using L'Hôpital's rule.  $\diamond$

## Conclusion

In addition to the two new conjectures made explicitly above, it would be fascinating to obtain closed forms for any of the residues in Proposition 1 with  $n \geq 5$ . It would likewise be very informative to obtain a closed form for  $W_5(\pm 1)$ , or for the residues at the poles of  $W_5$ . It would also be instructive to determine a closed form for other derivative values such as  $W_3'(1)$ .

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