

An Introduction to Knot Theory

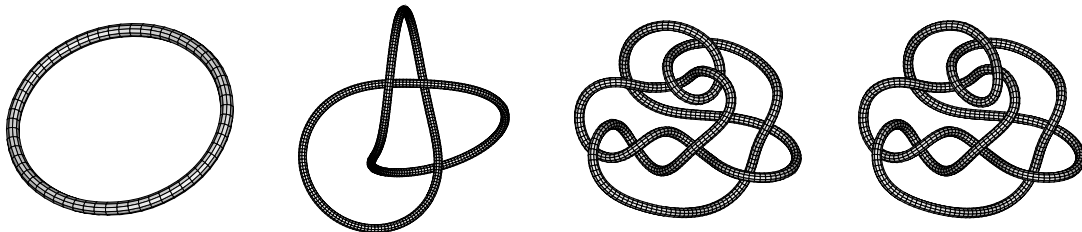
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1 Introduction

A knot, mathematically speaking, is a closed curve sitting in three dimensional space that does not intersect itself. Intuitively if we were to take a piece of string, cord, or the like, tie a knot in it and then glue the loose ends together, we would have a knot. It *should* be impossible to untangle the knot without cutting the string somewhere. The use of the word ‘should’ here is quite deliberate, however. It is possible, if we have not tangled the string very well, that we will be able to untangle the mess we created and end up with just a circle of string. Of course, this circle of string still fits the definition of a closed curve sitting in three dimensional space and so is still a knot, but it’s not very interesting. We call such a knot the *trivial knot* or the *unknot*.

Knots come in many shapes and sizes from small and simple like the unknot through to large and tangled and messy, and beyond (and everything in between). The biggest questions to a knot theorist are “are these two knots the same or different” or even more importantly “is there an easy way to tell if two knots are the same or different”. This is the heart of knot theory. Merely looking at two tangled messes is almost never sufficient to tell them apart, at least not in any interesting cases. Consider the following four images for example.



It might be surprising to find out that the four images only show *two* different knots. The question is which ones are the same? Are three of them the same and one different, or are there two pairs of identical knots? A better question is how can we work out which are the same and which are different? Unless we are particularly imaginative and/or good at visualisation simply looking at the images won’t help. Even if visualisation could give us the answer, it wouldn’t help us to impart our knowledge to another person, we’d need something external to ourselves for that, and preferably something more concrete than just saying “this is the answer”. We could perhaps try using string to create the knots shown in the images, and manipulate them to see which we could make the same. This method is very intuitive (not to mention a not unpleasant way of spending time) although it requires time, patience and tenacity (particularly in the case of very tangled up messes of knots).

The biggest problem however, with both these methods and others as well, is that they are subject to circumstance. Maybe we can see the answer, or maybe we can’t. Maybe we’re clever or lucky enough with the string to get an answer, or maybe we’re not. There is no mathematical pattern to this, just circumstance. Furthermore, if we have two knots and, using whatever techniques we find appropriate, not succeed in turning one into the other (or in turning them both into something something identical) then we have no useful information. We have not shown that the two knots are

different, but only that we cannot make them be the same. This puts a new light on our previous questions of telling knots apart. We at least have a chance of showing that two knots are the same, but how to we conclusively show that two tangled messes can *never* be manipulated to be the same?

This is what knot theory is about, and this is what we will be discussing in the following pages. We will formalise how we describe and present (ie draw) knots and how we manipulate them. We will look briefly at joining knots together, and at knots that have multiple strings. But most importantly, we will introduce the tools that allows us to show that knots are different, the *knot invariants*, and will examine and discuss several of these.

1.1 Knot Projections

Three dimensional space is difficult to model effectively on paper, so when looking at knots, we usually use a two dimensional image of the knot. Since knots themselves are 1 dimensional objects, they are not difficult to draw on a plane, so long as we take care to note, when strings cross, which string is above and which string is below. We call this a *projection* of a knot. Two such projections are shown below in Table 1, along with a corresponding “three dimensional” representation of the same knot. The first is a projection of the unknot, the second is a projection of what is called the trefoil knot.



Figure 1: The Unknot and the Trefoil

Note the break in some of the lines in the trefoil projection. These are not, in fact, breaks in the string making up the knot, but a way of showing that the string goes under another string. This is how we differentiate between the string that is above and the string that is below. Such a point on the knot is known as a crossing. So the trefoil projection shown above has, as the name might have implied, three crossings.

Each entire unbroken line of a projection is known as a *strand* of the knot projection. See Figure 2 for some examples of strands. The strands are marked in grey, but they are not the only strands of the projection in question.



Figure 2: Examples of a strands

If we were to trace our finger around the projection in a single direction, we would pass through every crossing twice, once by tracing the uppermost string (the *overstrand*) and once, later on, when tracing the undermost string (the *understrand*). When tracing the knot this way, we call a crossing an *overcrossing* if we are traveling on the overstrand, and an *undercrossing* if we are traveling on the understrand. We can give the projection (or, indeed, the knot) an *orientation* by choosing the direction in which we trace our finger (or travel) around it. There are only 2 choices. We then draw

the projection with arrows indicating the direction of travel. We call a knot with an orientation an *oriented* knot. An oriented trefoil is shown below in Figure 3.

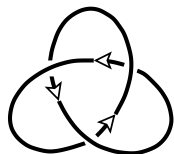


Figure 3: An Oriented Trefoil Knot

We now have a way of describing knots on paper. The bad news is that a knot projection is not unique to its knot. Any knot may have many different projections. If we consider our string analogue, we could look at the knotted string from any direction (above, behind, besides, or some combination) and draw a projection of what we saw. Also, we could manipulate the string, pulling it here, and there, twisting it around itself etc and draw a projection of the result. So long as we do not cut the string, any manipulation of the string we do can be undone, returning us to wherever we started from, so we have not changed the knot at all, only made it look different. To complicate matters even further, we consider the “string” of a mathematical knot to be indefinitely shrinkable or stretchable, so our analogue of a knotted string is a little inaccurate; knotted rubber tube might be a little better. Some examples of different projections of the trefoil knot are shown in Figure 4, below, to illustrate this idea.

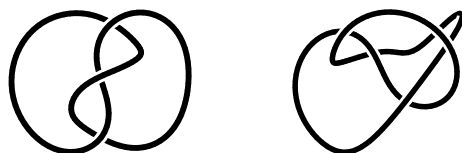


Figure 4: Two Projections of the Trefoil Knot

Since we are usually dealing with projections of knots, and not three dimensional objects, we need some rules to formalise these concepts for projections.

1.2 Projection Deformations and Reidemeister Moves

If we deform our now rubber knot in such a way as to not add or remove any crossings, then it should be easy to see that the knot itself has not changed significantly. Such a deformation is called a *planar isotopy*. It should be pointed out here that although the “rubber” making up the knot is considered to be indefinitely shrinkable, we still cannot remove part of the knot by infinitely shrinking string, so that the knot becomes a point. This fits our rubber or string analogy, as we could not remove part of a knot by pulling the strings of the knot tighter and tighter. Some planar isotopies of the trefoil are shown below.



Figure 5: Planar isotopies of the Trefoil

Planar isotopy only gets us so far. We haven’t accounted for much of what we could do with a physical knot. So far we have allowed only deformations that do not add or remove crossings. This is

clearly unreasonable, as we can twist the string of the knot around itself in any number of imaginable ways. So we need some way of modeling this for a projection.

Enter the Reidemeister moves, a set of three transformations that add or remove links to a projection by moving the strings around to produce another projection of the same knot. They are named after their creator, Kurt Reidemeister. The first such move, called a Type I Reidemeister move, allows us to “twist” a single piece of string thereby adding a crossing to the projection, or to remove such a twist.



Figure 6: Type I Reidemeister Move

The second Reidemeister move, called a Type II Reidemeister move, allows us to pull part of a string either over or under another string. This will either add, or remove, two crossings.



Figure 7: Type II Reidemeister Move

The last Reidemeister move allows us to move a string that is either above or below another crossing to the other side of that crossing.

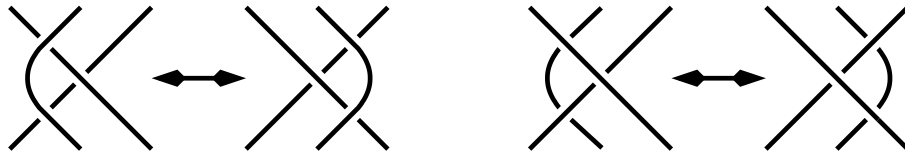


Figure 8: Type III Reidemeister Move

In each of these examples, it is assumed that the knot projection presented is only a small part of a larger projection, and that the rest of that projection remains unchanged. It was proven by Reidemeister that if two knot projections have a sequence of Reidemeister moves and planar isotopies that transform one into the other, then those two projections are of the same knot.

Unfortunately, while this gives us a nice way to show that two knot projections are the same, it does not give us a very satisfactory method of telling when two knot projections are different. Any attempt to show that two projections are distinct would need to show that there is no possible sequence of Reidemeister moves¹ to transform one projection into the other. Such proofs tend to be difficult.² We will discuss more about identification of distinct knots in Section 2.

1.3 Prime and Composite Knots

If we take two knots, K_1 and K_2 say, we may produce a new knot by composing the two knots together. We do this by taking a projection of each knot, making sure they don't overlap at all, and then remove

¹and planar isotopies, but it's simpler to take the planar isotopies as read and only consider the sequences of Reidemeister moves

²In much the same way that mountains tend to be tall

a small arc from a each projection. We then create two new arcs, each of which connect one of the endpoints of the break in K_1 with one of the endpoints of the break in K_2 . We denote the new knot $K_1 \# K_2$. The arcs to be removed must be removed from the outside of the projection of the knot they represent. Also the new arcs may not cross each other, nor may they cross either of the original knot projections. See Figure 9 to see how two trefoil knots can be composed.

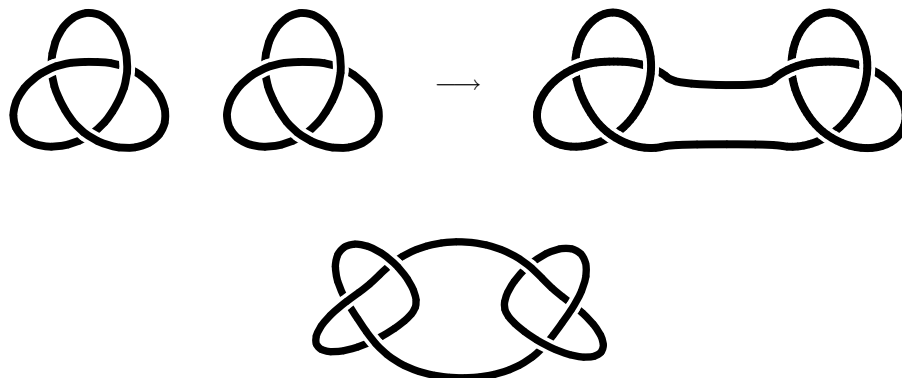


Figure 9: Knot Composition

A knot created in such a way from *nontrivial* knots is called a *composite* knot. Indeed any knot which *may* be constructed in such a way is a composite knot. The knots that compose to make the knot are called the *factor knots* of the composite knot. This is very similar to the idea of composite integers, and their prime factors. In fact, A knot that cannot be be constructed by composition of nontrivial knots is known as a *prime* knot.

We have already shown the existence of at least one composite knot in Figure 9, above. However it remains to be shown whether or not prime knots actually exist; for all we know at the moment, every knot may be able to be constructed by composition of other knots. As it happens prime knots most certainly do exist. In fact the trefoil knot presented earlier is a prime knot and although it is by no means obvious, we will not prove this fact here.

The similarity to the integers continues, for if we compose any knot with the unknot, what we end up with is the original knot. This is similar to the integer 1, and is the reason for requiring that a composite knot is made up of nontrivial factors (for otherwise every knot would be composite with itself and the unknot as factors). And to extend the similarity still, it was shown in 1949 by a man named Schubert that the factors of a composite knot are unique.

Much work has gone into tabulating prime knots, and all the prime knots having up to 10 crossings are known³. The number of prime knots of n crossings is tabulated as follows for $n = 1, \dots, 10$

Number of Crossings (n)	1	2	3	4	5	6	7	8	9	10
Number of Knots of n crossings	0	0	1	1	2	3	7	21	49	165

The collection of all prime knots of up to seven crossings is presented in Figure 10.

As a final note, it should be pointed out that (unlike the integers) it is possible to obtain different knots by composing the same two knots in different ways. This does not always happen, but is possible. Of course, even if it does happen the different resultant composite knots will both have the same factor knots, even though the composite knots are different.

³The author is of the understanding that all prime knots of up to n crossings are known for some $n > 10$, which is probably either 13 or 16. Unfortunately the author cannot find a source to validate exactly which n

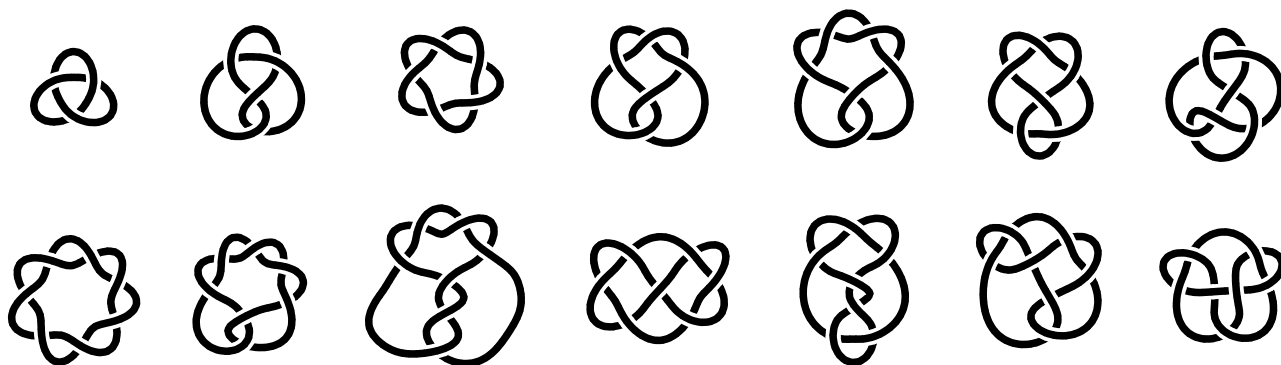


Figure 10: All Prime Knots of up to 7 crossings

1.4 Links

So far we have only concerned ourselves with knots consisting of one knotted ‘string’. This is fine as far as it goes, but there is absolutely no reason not to consider knots comprised of multiple, distinct ‘strings’ all tangled together. Such a thing is called a *link*. Each “string” of a link is called a *component* of the link, and is a closed curve in three dimensional space, just as a knot is. A link with n strings is called a *link of n components*. Everything previously stated for knots holds for links, in fact a knot is really nothing more than a link of one component.

The simplest link (known as the *unlink* or *trivial link* of two components) is nothing more than two unknots sitting next to each other, and not touching. The next simplest link is what is known as the hopf link.

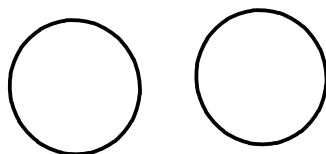


Figure 11: The unlink of two components

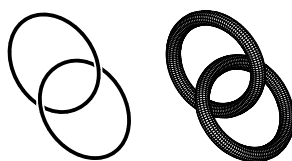


Figure 12: The hopf link

The unlink of two components is also an example of a particular type of link; the *splittable* link. A link is called splittable if each component can be separated with a plane between them (in three dimensional space) as is the case with the unlink of two components above. However, it may not be immediately obvious, when looking at a projection of a link, whether or not that link is splittable.

2 Knot Invariants

One large concern in knot theory is to be able to tell different knots apart. Since there are a potentially large number of projections for a given knot, simply looking at two knot projections is not a good method for determining whether those projections are of the same knot or not. Indeed, the information

presented thus far does not even conclusively show the existence of any knots apart from the unknot. For all we know at the moment, any knot projection we see is just a messy, distorted, projection of the unknot.

A small glance at the simple projection of the trefoil knot would seem to indicate that it cannot be unknotted by applying Reidemeister moves. If we were to try a number of such moves we would not be successful in unknotting the trefoil. However trying a number of times and failing is not a proof. What we must do is find some way to show that no sequence of Reidemeister moves exists that will unknot the trefoil knot.

In general, once we are happy that nontrivial knots DO exist, we want to have some method of being able to determine the existence or not of a sequence of Reidemeister moves to transform one knot projection into another.

To this end we have the concept of a *knot invariant*. A knot invariant is a property of a knot that does not change (is invariant) as the knot is deformed. Such a property, therefore, is independent of a choice of knot projection. Furthermore, if two knot projections have a different such property, then they *must* be different knots. However, this property does not necessarily help us in showing that two knots are the same. It is not impossible for two different knots to have the same invariant property. That property will not change as either knot is deformed, but the knots remain distinct. We say that an invariant is a *complete* invariant if it is an invariant that gives every distinct knot has a distinct invariant property.

A knot invariant can be thought of as a value assigned to a knot. Be careful with this, however, as the “value” need not be numerical at all, as will be shown in the first example below. We will now examine several invariants.

2.1 Tricolourability

The first knot invariant we will look at is that of Tricolourability. We colour a knot by assigning a colour to each strand of a projection of the knot. We say a knot projection is *tricolourable* if it can be coloured with up to three colours in such a way that the strands at each crossing have either three distinct colours, or only one colour. This is illustrated below in Figure 13 Any colouring of a knot

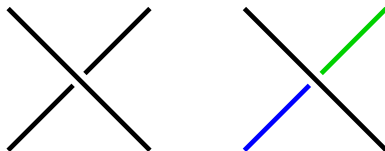


Figure 13: Two possibilities for crossings in a valid tricolouring

projection that meets these criteria is called a *valid tricolouring* or just a *tricolouring*. An example of a tricolouring of the trefoil knot is shown below.

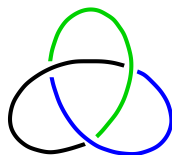


Figure 14: A Tricolouring of the Trefoil

Some thought should lead us to the conclusion that the definition given above allows any knot at all to be tricolourable, since we can colour every strand of a given knot the same colour and have a valid tricolouring. In order to prevent this, an extra stipulation is added that at least two colours must be used for a knot to be called tricolourable.

Before we show that tricolourability is, indeed, a knot invariant, we should show that the definition is a good one. This is not immediately obvious, since the definition gives rules for small parts of a knot (the crossings and strands) and mentions nothing about the knot as a whole. Formally, we say the definition only gives local rules. We need to verify that such a thing is, in fact, possible. This is clear, in at least one case, since we have shown a tricolouring of a trefoil knot projection in Figure 14, above. We also need to know that this definition is not a trivial one (ie one that applies to all knots). This is achieved by observing that the projection of the unknot shown above in Figure 1 has only one strand, and so (with our extra stipulation of at least 2 colours having to be used) can never have a valid tricolouring.

The discussion of tricolouring has only dealt with projections so far. We have shown a projection of the trefoil knot that is tricolourable, but this does not mean that another projection of the trefoil is necessarily tricolourable. Similarly we know that a projection of the unknot is not tricolourable, but for all we know now, a different projection might be. Remember, that we are aiming to show that tricolourability is a knot invariant. To do this we need to show that tricolourability is a property of the knot itself, and not just a property of some of its projections. We will do this by showing that if a knot has a tricolourable projection, then every projection of that knot is tricolourable.⁴ If we can show this, then we have shown that the unknot and trefoil knots are different knots.

We need to show that no deformation of the knot will change a projection with a valid tricolouring into a projection without a valid tricolouring (or vice versa). This means showing that planar isotopies and Reidemeister moves, when applied to a projection, do not produce a new projection that is tricolourable where the original was not (and vice versa). We should notice first that given any projection of a knot, all planar isotopies of that projection will have the same tricolourability (or lack thereof).

What remains for us to do is to show that the Reidemeister moves do not alter the tricolourability of a projection. We will do this by looking at each Reidemeister move in turn. For each Reidemeister move we will examine all the possible valid tricolourings for the particular move, and show that in all cases after the move is applied we still have a valid tricolouring. Recall from above, we only need to show that a valid tricolouring is never invalidated, and this in turn implies that an untricolourable projection will never produce a tricolourable projection. Also, when doing this we must be careful. As previously stated, the projections of the Reidemeister moves are merely a small part of a larger projection. Because of this we also have to make sure that our valid tricolouring preserves the colours of the strands that do not end at a crossing, for they are where the Reidemeister move projection meets the rest of the larger projection.

Let us look at the effect of a Type I Reidemeister move on a knot. Such a move allows us to either add or remove a twist to/from a strand of a projection. If we add a twist to a strand, then we have not effected tricolourability, since we can allow the 2 new strands to remain the same colour, and any previously tricolourable projection is still tricolourable. But what if we remove a twist instead? Observe that such a twist can be coloured with either 1 or 2 colours (as there are only 2 strands in the twist). So a valid tricolouring of a projection with such a twist will have the twist coloured with only one colour. Removing that twist will yield a single strand of only one colour, and so valid tricolouring is preserved. So applying Type I Reidemeister moves does not invalidate a valid tricolouring.

Now for Type II Reidemeister moves. Such a move allows us to either add two crossings to a projection by pulling part of a strand under (or over) another strand, or to remove two crossings by straightening out a strand that has been pulled over or under another strand. If we are adding crossings, then there are two possibilities; either the two strands were different colours, or they were the same colour. In the first case we leave the newly created strand the same colour, and in the latter case, we colour the newly created strand the remaining colour that the original strands were not coloured. If we are removing crossings, then the same happens, only in reverse. Notice, in the case

⁴And in showing this, then we will have also shown that if a knot has a projection that is not tricolourable, then no projection of that knot is tricolourable.

of removing crossings, that the topmost and bottommost strand of the understring *must* be the same colour in order to have a valid tricolouring. Verification of this fact is left to the reader. In all cases, if the original projection was colourable with at least two colours, then so is the resulting projection, so the application of the Type II Reidemeister move will not invalidate a valid tricolouring, as Figure 15 demonstrates.

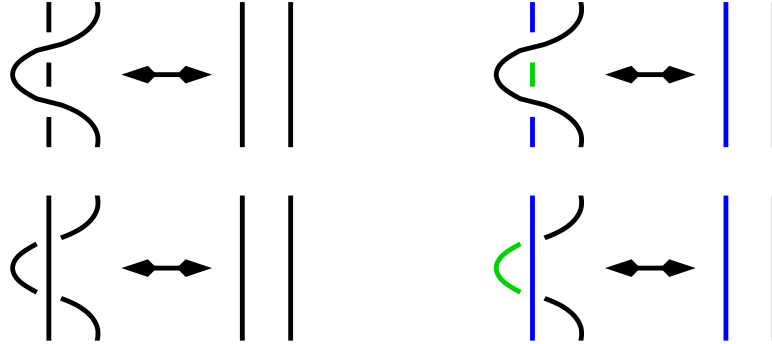


Figure 15: Reidemeister II Moves do not effect Tricolourability

Finally, Type III Reidemeister moves. Such moves allow us to shift a strand from one side of a crossing, to another. The position of the strand being moved, ie whether it is above or below the crossing, is important in this case. First we will consider the cases where the strand being moved is above the crossing. Looking at Figure 16 we can see there are 6 strands in all with three possible

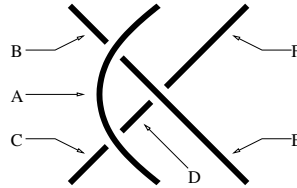


Figure 16:

colours per strand, so we have $3^6 = 729$ possible strand colouring combinations. It looks like we're in for a big job, even when we consider that not all of these combinations are valid tricolourings. Fortunately we can very greatly reduce this number with a little cunning.

Let's start with strand A, for no better reason than it is the topmost strand, and choose a colour for it. We need not worry which particular colour is chosen, it is only important that A *has* a colour. Now we will choose a colour for B, which will either be the same colour as that of A, or it will be a different colour to A. Notice that now we have done this, there is only one possible colour for the strand labeled E, if we are to have a valid tricolouring. If B has the same colour as A then E must also have that same colour, otherwise E must be the remaining colour that is A and B are not. Finally we choose a colour for C, which will either be the same colour as A, the same colour as B, or different to both A and B. Once this is chosen then, just as for strand E, there is only one possible colour that strand D can be and since the colour of D is fixed there can also only be one possible colour for strand F by the same reasoning. We need not worry about that actual colours used, only in their relationship to each other (ie which strands have the same colour, and which have different colours). We now have only 5 options (2 options for strand C when strands A and B are the same colour, and 3 options for strand C when strands A and B are different colours). In all cases, if the knot was tricolourable to start with, then it is still tricolourable after a Type III Reidemeister move is performed, as shown below in Figure 17.

Now we can consider the cases where the strand being moved is below the crossing. See Figure

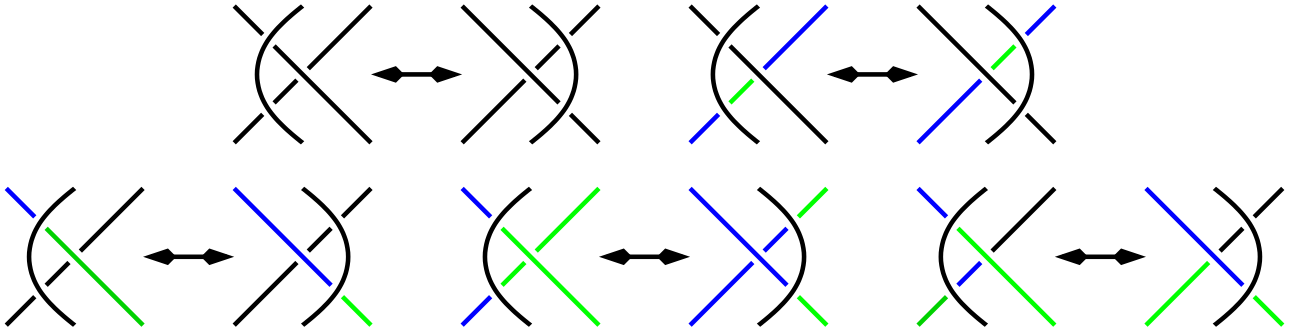


Figure 17: Reidemeister III Moves do not effect Tricolourability - Part 1

18. Fortunately, all our previous observations still hold. The labellings have been changed around

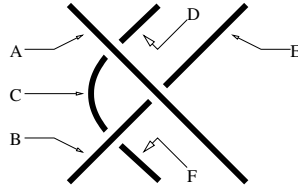


Figure 18:

a little to make the previous observations a little more obviously relevant. We still have 5 options to consider and we still choose colours for strands A, B and C. Strand E's colour is still uniquely determined from the colours of strands A and B, strand D's colour is still uniquely determined from the colours of strands A and C. The only difference is that in Figure 18, strand F's colour is now uniquely determined from the colours of strands B and C (whereas in Figure 16, the colour of strand F was determined by the colours of the strands D and E). In every possible case, we have shown that

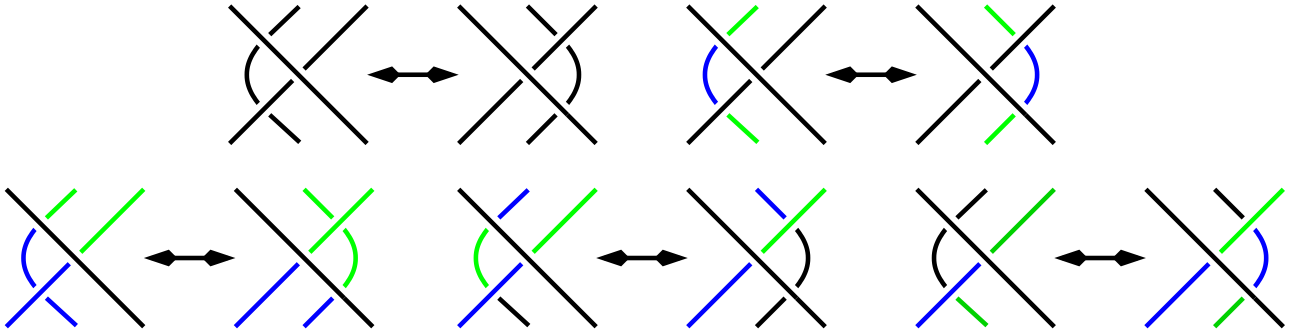


Figure 19: Reidemeister III Moves do not effect Tricolourability - Part 2

applying a Type III Reidemeister moves does not invalidate a valid tricolourability of a projection as shown in Figure 19

We have now shown that applying the Reidemeister moves to a projection will not invalidate an otherwise valid tricolouring. We can conclude from this (as stated before) that Reidemeister moves do not effect the tricolourability of a knot projection, and so tricolourability is a knot invariant.

Unfortunately this is not a very useful invariant to have. Either a knot is tricolourable or it isn't, but two different tricolourable knots are still tricolourable, and as such are not able to be distinguished between by this invariant. It has served, at least, to convince us that the unknot and the trefoil are different knots. It can also be shown that the number of different tricolourings of a knot projection is

also a knot invariant. This is a potentially more fine grained invariant, but even so it is not a great deal more usable. Instead we will look at other invariants.

2.2 Crossing Number

The crossing number is a remarkably simple invariant. It is quite simply, for a knot K , the smallest possible number of crossings that a projection of K can be drawn with. Another way to say this is that if there is a projection of K that can be drawn with n crossings and there is no possible projection of K that can be drawn with less than n crossings, then K has a crossing number of n . We denote the crossing number of the knot K as $c(K)$.

To see that this is an invariant should be quite easy, since the definition depends on the existence of a particular projection. If a knot, K , has crossing number $c(K) = n$ then there is a projection of K that has n crossings and there is no projection with less than n crossings. Applying Reidemeister moves to any projection of K will never change this fact. Neither will performing planar isotopies on any projection of K . So this property of a knot is clearly invariant.

The big problem with this invariant is calculating it for a particular knot. Given a projection of a knot, how do we know whether or not there is another projection with less crossings? We can deform the knot until the cows come home and not succeed in producing a projection with less crossings, but this in no way means that there is no such crossing.

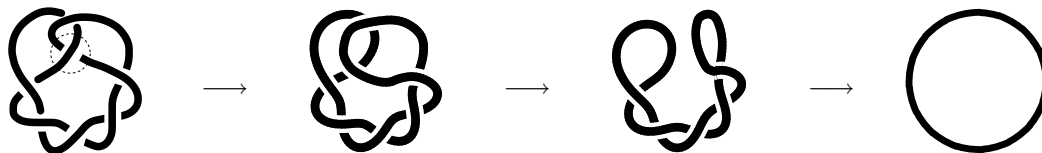
Luckily there are two saving graces for this invariant. The first is that if we have a knot K with n crossings, and we know what every knot with less than n crossings looks like, then we look at all of the known knots and if K is not one of those knots then it must have a crossing number $c(K) = n$. This isn't really much of a saving grace, since we still have the same potential problem of knowing whether or not a given projection can be deformed into one with less crossings. Finding a deformation from the given projection to one of the knots with less crossings may be difficult. Worse still showing that there is no possible deformation to any of the knots with less projections is even more difficult (after all, the problems inherent in trying to distinguish knots by their projections alone was the motivation for introducing invariants in the first place).

The second saving grace of the crossing number is somewhat more useful, but at the expense of only being usable for a particular type of knot. The knot type in question is an *alternating* knot. An alternating knot is a knot which, when you trace the knot from a starting point all the way round the curve of the knot back to the starting point in one direction, each crossing you pass alternates between being an overcrossing and an undercrossing. It was shown in 1986 that an alternating knot, A , which has a projection with n crossings and no easily removed crossings has crossing number $c(A) = n$. Such a projection (one that has no easily removed crossings) is said to be *reduced*. This result has proven to be very useful to the point that the crossing number is easily found for any alternating knot, as it is very easy to see whether a projection of an alternating knot is reduced.

2.3 Unknotting Number

Like the crossing number, above, the unknotting number is an integer value associated with a knot. For a knot K the unknotting number is denoted $u(K)$, and is the smallest number of crossing changes required, for any projection of K , to turn K into the unknot. By "crossing change" we mean swapping the over and under strands at the crossing so that the strand that was originally the understrand becomes the overstrand. As an example the knot below, which is one of the prime knots of seven crossings, has unknotting number 1 as shown by changing the circled crossing.

We were fortunate in this case. Since we started with a prime knot, we know it is not the unknot. As such, the smallest unknotting number we could get was 1 (as 0 would mean the knot was the unknot, which it wasn't). Since we found a projection where removing one crossing was sufficient to unknot the knot, then it must have unknotting number 1.



Before we proceed any further, we still have not shown that the unknotting number is indeed an invariant. First we must convince ourselves that for any projection of any knot, only a finite number of crossings need to be changed to turn that knot into the unknot. A proof of this is given in [1] pp.58-59. From this fact we know that there is an unknotting number for every projection of the knot. If we consider all projections of the knot, and look at the unknotting number of each of those projections, then there will be a smallest one (since the crossing number can never be negative). No amount of deformation of the knot will change the existence of that smallest unknotting number. So unknotting number must be an invariant.

Unfortunately, the unknotting number tends to be as difficult to find (if not more so) than the crossing number. Finding crossings to change that will let us turn a projection into the unknot is easy, especially when we have read the proof cited above. But how do we know that there isn't a projection that allows us to change less crossings to end up with the unknot? How do we show that, for a particular projection, no other possible projection of the same knot will allow us to obtain the unknot by with less crossing changes? This is not an easy thing. especially when we take into account the fact that the projection of a knot that has the minimum unknotting number is not always a minimal projection of the knot (ie a projection that has a number of crossings equal to the crossing number of the knot). We at least have invariants to help us with the problem of telling knots apart. There does not appear to be any such machinery for the unknotting number.

However, there have been some interesting developments with the unknotting number over the years. It has been shown that a knot with unknotting number 1 is prime, thus also showing that no composite knot can have unknotting number 1 ([1]).

3 Polynomial Invariants

So far our discussion of invariants has dealt with invariants that are either an abstract property of the knot (eg tricolourable) or a numeric property of the knot (eg unknotting or crossing number). We now present the idea of a polynomial invariant. A polynomial invariant is a knot invariant that associates a polynomial with a knot (rather than a numeric value). The polynomial is computed from a projection of the knot in question, although since the calculated polynomial is an invariant, the same polynomial will be calculated regardless of the choice of projection.

The polynomials used in this section are *Laurent* polynomials. These are polynomials which can have both positive and negative powers of the variable in the polynomial (eg $x^2 + 3 + x^{-2} + x^{-5}$). Three such invariants are presented.

3.1 The Kauffman Polynomial X

We will begin our discussion with the Kauffman Polynomial X, which is not to be confused with the Kauffman Polynomial E which will not be discussed here. The Kauffman Polynomial X, denoted $X(K)$ for a knot K , is a one variable polynomial invariant for knots and links. That is, it is a polynomial invariant, and the polynomials calculated are in one variable (A in this case).

For the remainder of our discussion we will refer to the Kauffman Polynomial X simply as the X polynomial. We calculate the X polynomial in three steps. The first is to calculate the Bracket Polynomial which we introduce in Section 3.1.1. The second is to calculate the writhe of the knot which we show in Section 3.1.2. The final step is to calculate the X polynomial itself by relating the

write and the bracket polynomial, which we deal with in Section 3.1.3.

3.1.1 The Bracket Polynomial

The first step in calculating the X polynomial is to calculate the Bracket polynomial. The Bracket polynomial is not actually an invariant itself, although we will show that it is invariant under the Type II and III Reidemeister moves only (ie applying these moves to a projection will not affect it's Bracket polynomial). Because of this, we must be careful with the projection during the calculation process, or we may produce an incorrect result. We denote the bracket polynomial as $\langle K \rangle$ for a knot K .

To calculate the Bracket polynomial for a knot K , we choose a crossing in the projection of the knot we have, and apply the rules below. The crossing will match one of the two crossings on the left hand side of (2), below. We create projections of two simpler knots K', K'' by removing the crossing we chose from the projection of K , and connecting the loose strands in the only two possible ways that do not result in another crossing (which correspond to the two projections on the right hand sides of the 2). We now have a formula that is something like $\langle K \rangle = A \langle K' \rangle + B \langle K'' \rangle$ and we have two new knots to calculate the Bracket polynomial for. We repeat this step for each new projection created.

Since a knot only has a finite number of crossings, and we are removing a crossing at each step, we will eventually end up with a knot projection with no crossings. This will either be an unknot, or an unconnected combination of unknots and knots. In either case rule (1) or (3) is appropriate. Sooner or later we have a polynomial in 3 variables, A, B and C .

The rules for calculating the bracket polynomial are:

$$\langle \bigcirc \rangle = 1 \quad (1)$$

$$\langle \times \rangle = A \langle \rangle + B \langle \rangle \quad (2)$$

$$\langle \bigcirc \cup L \rangle = C \langle L \rangle \quad (3)$$

It should be mentioned here that these rules, and consequently the polynomial they calculate, are in fact not at all invariant for knots. We will show soon the conditions under which these rules are invariant for type II and III Reidemeister moves, but for the time being the projections of K' and K'' created by this process must not be changed before we go and calculate their Bracket polynomial. They must be calculated in exactly the same projection they were created in, we may not deform them at all apart from with planar isotopies. This also means that the (1) rule applies only to a projection that is an unbroken circle (up to planar isotopies) and not to any projection of the unknot. Once we have calculated their Bracket polynomials, we substitute the resultant polynomial in for $\langle K' \rangle$ and $\langle K'' \rangle$

We will now calculate the bracket polynomial for the trefoil, using the rules above.

$$\begin{aligned} \langle \text{trefoil} \rangle &= A \langle \text{trefoil}_1 \rangle + B \langle \text{trefoil}_2 \rangle \\ &= A \left(A \langle \text{trefoil}_1 \rangle + B \langle \text{trefoil}_1 \rangle \right) + B \left(A \langle \text{trefoil}_2 \rangle + B \langle \text{trefoil}_2 \rangle \right) \\ &= A^2 \langle \text{trefoil}_1 \rangle + AB \langle \text{trefoil}_1 \rangle + AB \langle \text{trefoil}_2 \rangle + B^2 \langle \text{trefoil}_2 \rangle \\ &= A^2 \left(A \langle \text{trefoil}_1 \rangle + B \langle \text{trefoil}_1 \rangle \right) + AB \left(A \langle \text{trefoil}_2 \rangle + B \langle \text{trefoil}_2 \rangle \right) + \\ &\quad AB \left(A \langle \text{trefoil}_2 \rangle + B \langle \text{trefoil}_2 \rangle \right) + B^2 \left(A \langle \text{trefoil}_2 \rangle + B \langle \text{trefoil}_2 \rangle \right) \end{aligned}$$

$$\begin{aligned}
& \text{And since } \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} \text{ and } \text{Diagram 4} = \text{Diagram 5} = \text{Diagram 6} = \text{Diagram 7} \\
& = A^2B + A^2B + A^2B \left\langle \text{Diagram 1} \right\rangle + A^3 + AB^2 + AB^2 + AB^2 \left\langle \text{Diagram 2} \right\rangle + B^3 \left\langle \text{Diagram 3} \right\rangle \\
& = 3A^2B + (A^3 + 3AB^2)C + B^3C^2
\end{aligned}$$

We are not quite yet done, however. As stated above, the 3 variable polynomial we calculate using the rules above is not at all invariant yet. We need to make some substitutions so that it is invariant under the type II and III Reidemeister moves. These substitutions are $B = A^{-1}$ and $C = (-A^2 - A^{-2})$. The rules for calculating the Bracket polynomial are now:

$$\begin{aligned}
\langle \bigcirc \rangle &= 1 \\
\langle \times \rangle &= A \langle \rangle \langle \rangle + A^{-1} \langle \times \rangle \\
\langle \times \rangle &= A \langle \times \rangle + A^{-1} \langle \rangle \langle \rangle \\
\langle \bigcirc \cup L \rangle &= (-A^2 - A^{-2}) \langle L \rangle
\end{aligned}$$

Note that it is perfectly reasonable to calculate the Bracket polynomial directly with these rules, and in doing so we may deform the knot projections between steps by type II and III Reidemeister moves (since the polynomial is invariant under those moves) as well as by planar isotopies. Be careful though, the invariance does not extend to type I Reidemeister moves although it might be easy to forget this fact when we are merrily deforming knot projections to make our calculations easier.

Of course, before we can take any advantage of this whatsoever we must first show that polynomials created in this way are indeed invariant for the type II and III Reidemeister moves. We will start with the type II moves.

$$\begin{aligned}
\langle \text{Diagram 1} \rangle &= A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle \\
&= A \left(A \langle \text{Diagram 4} \rangle + A^{-1} \langle \text{Diagram 5} \rangle \right) + A^{-1} \left(A \langle \text{Diagram 6} \rangle + A^{-1} \langle \text{Diagram 7} \rangle \right) \\
&= A \left(A \langle \text{Diagram 4} \rangle + A^{-1} \langle \text{Diagram 5} \rangle \right) + A^{-1} \left(A \left((-A^2 - A^{-2}) \langle \text{Diagram 8} \rangle \right) + A^{-1} \langle \text{Diagram 7} \rangle \right) \\
&= A^2 \langle \text{Diagram 4} \rangle + \langle \text{Diagram 5} \rangle - A^2 \langle \text{Diagram 8} \rangle - A^{-2} \langle \text{Diagram 8} \rangle + A^{-2} \langle \text{Diagram 7} \rangle \\
&= \langle \text{Diagram 5} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \text{Diagram 1} \rangle &= A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle \\
&= A \left(A \langle \text{Diagram 4} \rangle + A^{-1} \langle \text{Diagram 5} \rangle \right) + A^{-1} \left(A \langle \text{Diagram 6} \rangle + A^{-1} \langle \text{Diagram 7} \rangle \right) \\
&= A \left(A \langle \text{Diagram 4} \rangle + A^{-1} \left((-A^2 - A^{-2}) \langle \text{Diagram 8} \rangle \right) \right) + A^{-1} \left(A \langle \text{Diagram 6} \rangle + A^{-1} \langle \text{Diagram 7} \rangle \right) \\
&= A^2 \langle \text{Diagram 4} \rangle - A^2 \langle \text{Diagram 8} \rangle - A^{-2} \langle \text{Diagram 8} \rangle + \langle \text{Diagram 6} \rangle + A^{-2} \langle \text{Diagram 7} \rangle
\end{aligned}$$

$$= \langle \rangle$$

And so we now know that the Bracket polynomial is invariant under type II Reidemeister moves. Armed with this, the type III moves are much easier.

$$\begin{aligned} \langle \text{crossing} \rangle &= A \langle \text{crossing} \rangle + A^{-1} \langle \text{crossing} \rangle \\ &= A \langle \text{crossing} \rangle + A^{-1} \langle \text{crossing} \rangle = \langle \text{crossing} \rangle \end{aligned}$$

$$\begin{aligned} \langle \text{crossing} \rangle &= A \langle \text{crossing} \rangle + A^{-1} \langle \text{crossing} \rangle \\ &= A \langle \text{crossing} \rangle + A^{-1} \langle \text{crossing} \rangle = \langle \text{crossing} \rangle \end{aligned}$$

And this shows invariance under the type III Reidemeister moves.

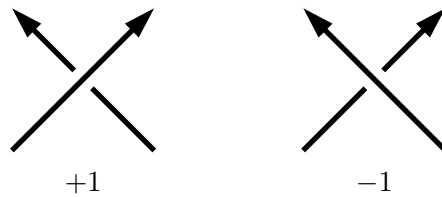
Since we have already stated that the bracket polynomial is not invariant under a Type I Reidemeister move, we will not show it here. Instead it is left as an exercise to the reader to confirm this. The reader may also check [1]. Instead we will go on to show how we can achieve invariance under Type I moves. But first, let's finish calculating the the bracket polynomial for the trefoil knot. We know from before that the bracket polynomial of the trefoil is $3A^2B + (A^3 + 3AB^2)C + B^3C^2$, so we may apply the substitutions directly.

$$\begin{aligned} \langle \text{trefoil} \rangle &= 3A^2A^{-1} + (A^3 + 3A(A^{-1})^2)(-A^2 - A^{-2}) + (A^{-1})^3(-A^2 - A^{-2})^2 \\ &= 3A + (A^3 + 3A^{-1})(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 \\ &= 3A - A^5 - A - 3A - 3A^{-3} + A^{-3} + A + 2A^{-3} + A^{-7} \\ &= -A^5 - A^{-3} + A^{-7} \end{aligned}$$

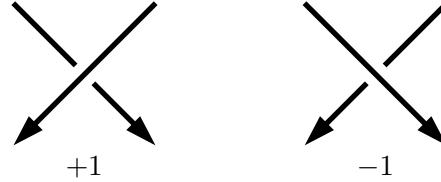
And there we have it. We could also have calculated directly from the second set of rules, although if we look at the calculation we did above, there are no places where a Type II or III Reidemeister move would have benefited us, so there would be little benefit. In practice, calculating from the second set of rules is probably easier, as we can use the Type II and III Reidemeister moves to simplify the projections we create, and hopefully make our calculations quicker and simpler.

3.1.2 Writhe

Having calculated the bracket polynomial for a knot, we will now the writhe of that knot. We need to give an orientation (see Section 1.1) to the knot projection we used to calculate the bracket polynomial. Once this is done, we look at every crossing of that projection. Each crossing will look like one of the following crossings, only rotated.



We mark each crossing on the knot as either a $+1$ crossing or a -1 crossing (as depicted above), but we think of each markings as the value of the crossing. The *writhe* of the knot is simply the sum of the value of every crossing. We denote the writhe as $w(K)$ for a knot K . But hold on, what happens if we choose the opposite orientation for our knot? Well, first realise that if we change the orientation of the knot projection then *every* arrow is reversed. So let's look at the different crossings with the arrows reversed.



Now, if we look closely at the two diagrams, we see that they're the same, only rotated through 180 degrees. That is a good thing to know. It means that regardless of the orientation we choose for a projection, the writhe of that projection will be unchanged.

Since we've been working with the trefoil so far, let's calculate its writhe. As can be seen from

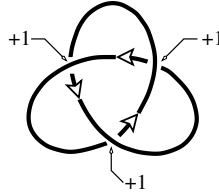


Figure 20: The Trefoil has Writhe of 3

Figure 20 the trefoil has three $+1$ crossings, giving it a writhe of 3.

3.1.3 $X(K)$

Once we have the writhe of and the bracket polynomial for the projection we are working with of our knot (which we'll call K), we define the X polynomial as

$$X(K) = (-A^3)^{-w(K)} \langle K \rangle$$

and claim that it is a knot invariant. We will now prove this claim.

First we will deal with the Type II and Type III Reidemeister moves. We will show that neither of these moves change the writhe of the polynomial.

For the Type II Reidemeister move. There are but two possibilities. Either the strands have the same orientation, or they have opposite orientation. In either case, using the move either adds two crossings whose total writhe is 0, or it removes two such crossings, as is shown in Figure 21. So the writhe of the projection remains unaltered.

For the Type III Reidemeister move, we also have but two possibilities. The particular crossing used is irrelevant, as it is not altered by the Reidemeister move, and so its writhe will not change. We only need to look at the crossings made by the strand we are moving. As such, we only have two options, that being the orientation of the strand to be moved (either up the page, or down the page). In every case, the Reidemeister moves shift one -1 crossing to be a $+1$ crossing, and shift one $+1$ crossing to be a -1 crossing, as is shown in Figure 22. So the writhe of the projection remains unaltered.

Now that we have shown this, it is easy to show that the X polynomial is invariant under the Type II and Type III Reidemeister moves. We know that both the writhe and the Bracket polynomial

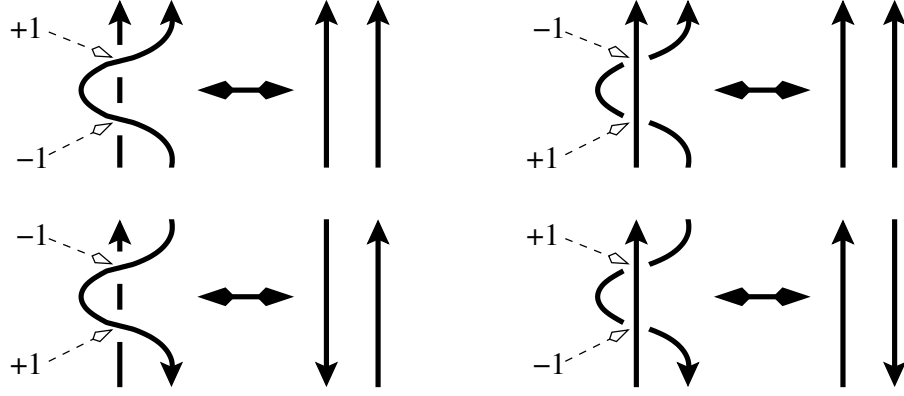


Figure 21: Type II Reidemeister moves do not change the writhe of a link

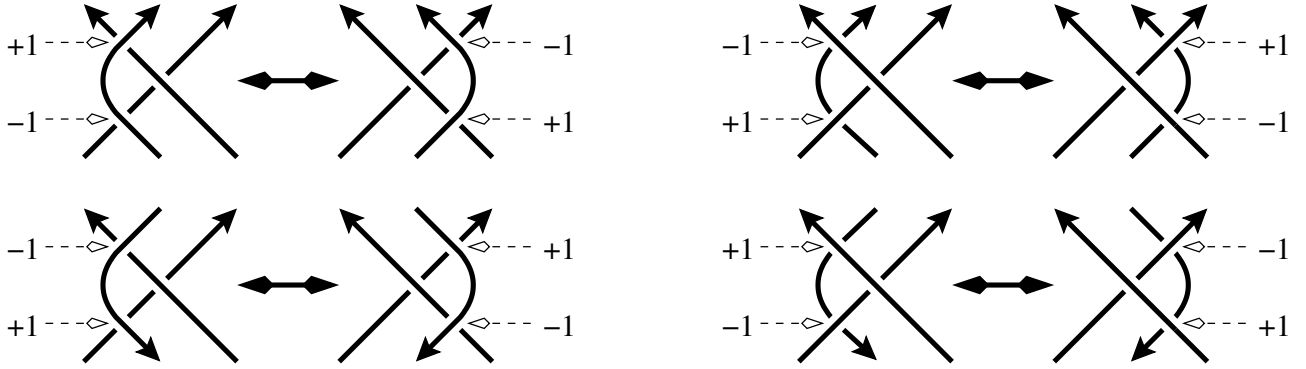


Figure 22: Type III Reidemeister moves do not change the writhe of a link

is invariant under these moves, and $-A^3$ is constant (as far as determining the particular polynomial goes). So $(A^3)^{-w(K)} \langle K \rangle$ will never change if we only perform Type II and Type III Reidemeister moves and so must be invariant under those moves.

Now we will show that the X polynomial is invariant under Type I Reidemeister moves.

$$\begin{aligned}
 X \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) &= (-A^3)^{-w(\begin{array}{c} \nearrow \\ \searrow \end{array})} \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \\
 &= (-A^3) \left(A \left\langle \begin{array}{c} \nearrow \\ \circ \\ \searrow \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \right) \\
 &= (-A^3) \left(A \left((-A^2 - A^{-2}) \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \right) + A^{-1} \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \right) \\
 &= (-A^3) \left((-A^3 - A^{-1}) \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \right) \\
 &= (-A^3) \left(-A^3 \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle \right) = \left\langle \begin{array}{c} \nearrow \\ \searrow \end{array} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 X \left(\begin{array}{c} \searrow \\ \nearrow \end{array} \right) &= (-A^3)^{-w(\begin{array}{c} \searrow \\ \nearrow \end{array})} \left\langle \begin{array}{c} \searrow \\ \nearrow \end{array} \right\rangle \\
 &= (-A^3) \left(A \left\langle \begin{array}{c} \searrow \\ \circ \\ \nearrow \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \searrow \\ \nearrow \end{array} \right\rangle \right)
 \end{aligned}$$

$$\begin{aligned}
&= (-A^3) \left(A \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + A^{-1} \left((-A^2 - A^{-2}) \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \right) \right) \\
&= (-A^3) \left(A \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + (-A - A^{-3}) \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \right) \\
&= (-A^3) \left(-A^{-3} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \right) = \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle
\end{aligned}$$

So we have

$$X \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = X \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle = (-A^3)^0 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle = (-A^3)^{-w(\emptyset)} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle = X \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right)$$

And we have shown that the X polynomial is invariant under the Type I Reidemeister moves. In turn we have shown that the X polynomial is a knot invariant.

We can now calculate the X polynomial for the trefoil knot.

$$\begin{aligned}
X \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) &= (-A^3)^{-w(\text{trefoil})} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle \\
&= -A^{-9}(-A^5 - A^{-3} + A^{-7}) \\
&= A^{-4} + A^{-12} - A^{-16}
\end{aligned}$$

3.2 The Alexander Polynomial

The obvious connection between the name Alexander and knots is the well known historical tale of Alexander the Great cutting the Gordian knot in 333BC or thereabouts. Apart from the inclusion of a knot, this tale has nothing to do with knot theory, and especially not the Alexander polynomial which is named after James W. Alexander of the Princeton Institute for Advanced Study, and a contemporary of Albert Einstein, John von Neumann, and Oswald Veblen as part of the original IAS mathematics faculty. James Alexander discovered this polynomial, which was the first ever knot polynomial, in 1928. It saw much use over the next 56 until the Jones polynomial in 1984.

To calculate the Alexander polynomial we apply the following skein relation. This defines a relationship for the Alexander Polynomials of knots with similar projections. By this we mean that if three knot projections are identical except for *one* crossing where each projection has exactly one of the three following crossings from Figure 23

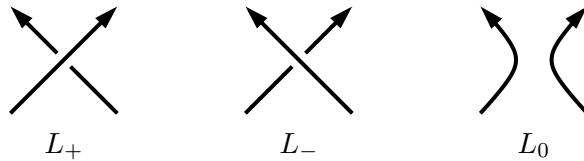


Figure 23: The three crossings of a skein relation

Then the following relationship holds

$$\Delta(\bigcirc) = 1 \tag{4}$$

$$\Delta(L_+) - \Delta(L_-) = \left(t^{\frac{1}{2}} - t^{(-\frac{1}{2})} \right) \Delta(L_0) \tag{5}$$

The skein relationship we are referring to is (5). We included the rule (4) for completeness since it is also needed to calculate the Alexander Polynomial. It simply states that the Alexander Polynomial

of the unknot is 1. This is the case for *any* projection of the unknot, since the Alexander Polynomial is a knot invariant.

We will not prove the invariance of the Alexander polynomial in this document. We will consider the Alexander, Jones (Section 3.3) and HOMFLY (Section 3.4) polynomials to be sufficiently well known and documented, and accept that they have already been proven invariant. The interested reader is encouraged to read the reference material for such proofs. Of interest is the fact that this is not the original formulation of the Alexander polynomial. The skein relation above was shown in 1969, by a man named John Conway, to calculate the Alexander polynomial.[1].

It is not immediately obvious that this skein relation actually does calculate anything. We don't know if we will ever stop calculating. However, recall that we used a result from [1] that stated any knot projection only had a finite number of crossing changes needed to change that knot into the unknot. From this fact then, we know that calculations using the rules above will eventually end up with a projection of the unknot, and our process will terminate. (Since for any knot, we change a crossing, and remove a crossing). We only need to be a little clever in which crossings we change.

We will calculate the Alexander polynomial of the trefoil knot now.

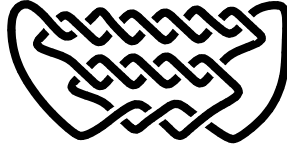
$$\begin{aligned}\Delta\left(\text{trefoil with crossing at top}\right) - \Delta\left(\text{trefoil with crossing at bottom}\right) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at top}\right) \\ \Delta\left(\text{trefoil with crossing at top}\right) - \Delta\left(\text{trefoil with crossing at bottom}\right) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at top}\right)\end{aligned}$$

From which we may deduce

$$\begin{aligned}\Delta\left(\text{trefoil with crossing at top}\right) &= \Delta\left(\text{trefoil with crossing at bottom}\right) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at top}\right) \\ \Delta\left(\text{trefoil with crossing at bottom}\right) &= \Delta\left(\text{trefoil with crossing at top}\right) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at bottom}\right) \\ \text{However } \Delta\left(\text{trefoil with crossing at top}\right) - \Delta\left(\text{trefoil with crossing at bottom}\right) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at top}\right) \\ &\Rightarrow \Delta\left(\text{trefoil with crossing at top}\right) = 0\end{aligned}$$

$$\begin{aligned}\text{So } \Delta\left(\text{trefoil with crossing at top}\right) &= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \\ \Delta\left(\text{trefoil with crossing at top}\right) &= 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta\left(\text{trefoil with crossing at top}\right) \\ &= 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 \\ &= 1 + t - 1 + t^{-1} \\ &= t - 1 + t^{-1}\end{aligned}$$

The Alexander polynomial is a very good invariant for distinguishing knots, however it is not without its problems. Although we have not discussed this at all, the mirror image of a knot is sometimes different from the knot itself. In fact the trefoil is an example of this. Unfortunately, the Alexander polynomial cannot distinguish knots from their mirror image. Also the Alexander polynomial does not distinguish all knots from the unknot. Indeed, the following knot has the same Alexander Polynomial as the unknot. These problems aside, the Alexander polynomial is still a very useful knot invariant, as evidenced by it's usage over many years.



3.3 The Jones Polynomial

The Jones polynomial is named after it's creator, Vaughn Jones from New Zealand. He discovered this invariant in 1984 whilst working on operator algebras, something completely unrelated (at least at the time) to knot theory.

The Jones polynomial, denoted V_K for a knot K , is simply the Kauffman Polynomial X with the substitution $A = t^{-\frac{1}{4}}$ into the X polynomial. That's it. Really. It is clearly an invariant, since it is the same invariant as the X polynomial which we put so much work into showing invariance for.

Since we have been working with the trefoil so much in this section, we will continue doing so now by calculating it's Jones polynomial.

$$V_{\text{trefoil}} = \left(t^{-\frac{1}{4}}\right)^{-4} + \left(t^{-\frac{1}{4}}\right)^{-12} - \left(t^{-\frac{1}{4}}\right)^{-16} = t + t^3 - t^4$$

This is not the usual way of coming by the Jones polynomial, however. The Jones polynomial is usually arrived at by a set of rules that includes a skein relation like that of the Alexander polynomial. Those rules are:

$$V_{\text{O}} = 1 \tag{6}$$

$$t^{-1}V_{L_+} - tV_{L_-} = \left(t^{\frac{1}{2}} - t^{(-\frac{1}{2})}\right)V_{L_0} \tag{7}$$

Since we know that the Jones polynomial is an invariant, we know that (6) applies to any projection of the unknot. The second rule (7), is a skein relation that describes a relation between the Jones polynomial of any three almost identical projections that differ only at one crossing according to Figure 23 above.

The Jones polynomial is capable of distinguishing more knots than the Alexander polynomial. In particular it can distinguish *all* knots of nine crossings or less. The first pair of knots that it cannot distinguish are an eight crossing prime knot and a ten crossing prime knot [6]. These knots are shown below in Figure 24. It is clear then that the Jones polynomial is not a complete invariant. What is

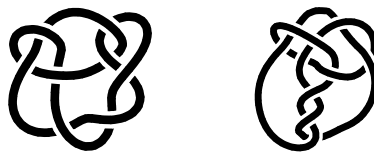


Figure 24: These two knots have the same Jones polynomial

unclear is whether or not the Jones polynomial is capable of distinguishing the unknot from every other knot. This would be a very useful attribute to have in a knot invariant, but remains an open question.

3.4 The HOMFLY Polynomial

The HOMFLY Polynomial came about four months after Vaughn Jones discovered the Jones Polynomial. The name HOMFLY is derived from the names of those who discovered it; Hoste, Ocneanu, Millett, Freyd, Lickorish and Yetter. The polynomial itself, is a Laurent polynomial in *two* variables (α, z) . We denote the HOMFLY polynomial of a link L to be $P(L)$

To calculate the HOMFLY polynomial, another skein relation is used. The rules look very similar to those of the Alexander polynomial, and of the skein relationship we showed for the Jones polynomial. This is no accident, the creation of the HOMFLY was a direct result of attempts to generalise the Alexander and Jones polynomials. We will show later that the Alexander and Jones polynomials are both special cases of the HOMFLY polynomial.

To calculate the HOMFLY polynomial, the following rules are used:

$$P(\bigcirc) = 1 \quad (8)$$

$$\alpha P(L_+) - \alpha^{-1} P(L_-) = z P(L_0) \quad (9)$$

where (8) holds for any projection of the unknot, and L_+, L_-, L_0 are three identical links differing only by one crossing as with the other skein relations and Figure 23.

We have already stated that the HOMFLY polynomial is a generalisation of both the Alexander and Jones polynomials. Let us examine this a little more closely. Look at (9). If we let $\alpha = t^{-1}$ and $z = t^{\frac{1}{2}} - t^{(-\frac{1}{2})}$ then our equation becomes

$$t^{-1} P(L_+) - t P(L_-) = \left(t^{\frac{1}{2}} - t^{(-\frac{1}{2})}\right) P(L_0)$$

which is the skein relation for the Jones polynomial. Also if we let $\alpha = 1$ and $z = t^{\frac{1}{2}} - t^{(-\frac{1}{2})}$ then the equation becomes

$$P(L_+) - P(L_-) = \left(t^{\frac{1}{2}} - t^{(-\frac{1}{2})}\right) P(L_0)$$

which is the skein relation for the Alexander polynomial.

Let us calculate the HOMFLY polynomial of the trefoil knot. The skein relation tells us that

$$\begin{aligned} \alpha P \left(\text{trefoil with crossing} \right) - \alpha^{-1} P \left(\text{trefoil with crossing} \right) &= z P \left(\text{trefoil with crossing} \right) \\ \alpha P \left(\text{trefoil with crossing} \right) - \alpha^{-1} P \left(\text{trefoil with crossing} \right) &= z P \left(\text{trefoil with crossing} \right) \end{aligned}$$

From which we may deduce

$$\begin{aligned} P \left(\text{trefoil with crossing} \right) &= \alpha^{-1} \left(\alpha^{-1} P \left(\text{trefoil with crossing} \right) + z P \left(\text{trefoil with crossing} \right) \right) \\ &= \alpha^{-2} + \alpha^{-1} z P \left(\text{trefoil with crossing} \right) \end{aligned}$$

$$\begin{aligned} P \left(\text{trefoil with crossing} \right) &= \alpha^{-1} \left(\alpha^{-1} P \left(\text{trefoil with crossing} \right) + z P \left(\text{trefoil with crossing} \right) \right) \\ &= \alpha^{-2} P \left(\text{trefoil with crossing} \right) + \alpha^{-1} z \end{aligned}$$

$$\begin{aligned} \text{However } \alpha P \left(\text{trefoil with crossing} \right) - \alpha^{-1} P \left(\text{trefoil with crossing} \right) &= z P \left(\text{trefoil with crossing} \right) \\ \Rightarrow P \left(\text{trefoil with crossing} \right) &= z^{-1} (\alpha - \alpha^{-1}) \end{aligned}$$

$$\text{So } P \left(\text{trefoil with crossing} \right) = \alpha^{-2} P \left(\text{trefoil with crossing} \right) + \alpha^{-1} z$$

$$\begin{aligned}
&= \alpha^{-2} (z^{-1}\alpha - z^{-1}\alpha^{-1}) + \alpha^{-1}z \\
&= z^{-1}\alpha^{-1} - z^{-1}\alpha^{-3} + \alpha^{-1}z \\
P\left(\text{trefoil}\right) &= \alpha^{-2} + \alpha^{-1}zP\left(\text{trefoil}\right) \\
&= \alpha^{-2} + \alpha^{-1}z(z^{-1}\alpha^{-1} - z^{-1}\alpha^{-3} + \alpha^{-1}z) \\
&= \alpha^{-2} + \alpha^{-2} - \alpha^{-4} + \alpha^{-2}z^2 \\
&= 2\alpha^{-2} - \alpha^{-4} + \alpha^{-2}z^2
\end{aligned}$$

Now if we substitute in $\alpha = t^{-1}$ and $z = t^{\frac{1}{2}} - t^{(-\frac{1}{2})}$

$$\begin{aligned}
V_{\otimes} &= 2(t^{-1})^{-2} - (t^{-1})^{-4} + (t^{-1})^{-2} (t^{\frac{1}{2}} - t^{(-\frac{1}{2})})^2 \\
&= 2t^2 - t^4 + t^2(t - 2 + t^{-1}) \\
&= 2t^2 - t^4 + t^3 - 2t^2 + t \\
&= t + t^3 - t^4
\end{aligned}$$

Which is exactly the polynomial we calculated for the Jones polynomial of the trefoil, above. We may also substitute $\alpha = 1$ and $z = t^{\frac{1}{2}} - t^{(-\frac{1}{2})}$ in order to calculate the Alexander polynomial of the trefoil.

$$\begin{aligned}
\Delta\left(\text{trefoil}\right) &= 2 \cdot 1^{-2} - 1^{-4} + 1^{-2} (t^{\frac{1}{2}} - t^{(-\frac{1}{2})})^2 \\
&= 1 + (t - 2 + t^{-1}) \\
&= t - 1 + t^{-1}
\end{aligned}$$

The HOMFLY polynomial is the most powerful yet of all the polynomial invariants. It is not complete, however, [1] shows two knots that have the same HOMFLY polynomial. The HOMFLY polynomial also has some interesting properties. For two knots J and K , the following hold the composition of the two knots has HOMFLY polynomial $P(J\#K) = P(J)P(K)$ which is the product of the polynomials of the factor knots. Also the splittable link of J and K has HOMFLY polynomial $P(J \cup K) = z^{-1}(\alpha - \alpha^{-1})P(J)P(K)$. We saw a particular case of this above with the trivial link of two components when we were calculating the HOMFLY polynomial for the trefoil knot.

4 Concluding Remarks

We have really only scratched the surface of knot theory in this document. There is much we have not discussed such as types of knots, notations for describing knots, braids, the list goes on. We have not dealt with the topological significance of knots, nor have we discussed knots in dimensions higher than three. The interested reader is encouraged to pursue any and all of these topics, and any other related topics as well.

What we *have* discussed is a good introduction to the concepts and definitions involved with knot theory and, motivated by the desire to be able to distinguish different knots, we have discussed and examined the idea of knot invariants. We have by no means exhausted the list of invariants, although we have looked at most of the “big players” of the knot invariants. The following two open questions, and many others remain.

- Do the Jones or HOMFLY polynomials distinguish the unknot from all other knots?
- Find a complete knot invariant.

One can find many other open problems in [1] as well as [10].

The reader is now encouraged, armed with the information discussed in these pages, to continue their study of knots through the reference material or any other reference material at their disposal.

References

- [1] Colin C. Adams: *The Knot Book - An Elementary Introduction to the Mathematical Theory of Knots*. W. H. Freeman and Company. New York, 1994
- [2] John L. Casti: *Five More Golden Rules - Knots, Codes, Chaos and Other Great Theories of 20th-Century Mathematics*. John Wiley & Sons, Inc, 2000
- [3] Arun Ram: *Quantum Groups*. at the Workshop on Algebra, Geometry, and Topology 22 January to 9 February Australian National University Canberra, 1996
- [4] V.F.R. Jones: *Milnor's Work and Knot Polynomials*. From Topological Methods in Modern Mathematics (Stony Brook, New York, 1992) pp195-202
- [5] Vaughan F. R. Jones: *Von Neumann Algebras in Mathematics and Physics*. A plenary address presented at the International Congress of Mathematicians held in Kyoto, August 1990.
- [6] Vaughan F. R. Jones: *Subfactors and Knots*. American Mathematical Society. Providence, Rhode Island, 1991
- [7] J. Scott Carter *How Surfaces Intersect in Space - An introduction to topology*. World Scientific. second edition. 1995
- [8] Michael Atiyah *The geometry of Physics and Knots*. Cambridge University Press. 1998
- [9] *Eric Weisstein's World of Mathematics*. A free service for the mathematical community provided by Wolfram Research, makers of Mathematica, with additional support from the National Science Foundation. <http://mathworld.wolfram.com/>
- [10] *A List of Approachable Open Problems in Knot Theory*.
<http://www.williams.edu/Mathematics/cadams/knotproblems.html>