# APPENDIX I. ANOTHER CASE STUDY

## LOG-CONCAVITY

Consider the *unsolved* **Problem 10738** in the 1999 *American Mathematical Monthly*:

**Problem:** For $t > 0$ let

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t) \frac{t^k}{k!}$$

be the $nth$ moment of a *Poisson distribution* with parameter $t$. Let $\mathbf{c_n(t) = m_n(t)/n!}$ . Show

a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex* for all $t > 0$.

b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.

c*) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geq 1$.

*A sequence $\{a_n\}$ is *log-convex* if $a_{n+1}a_{n-1} \geq a_n^2$, for $n \geq 1$ and log-concave when the sign is reversed.

**Solution.** (a) Neglecting the factor of $\exp(-t)$ as we may, this reduces to

$$\sum_{k,j\geq 0} \frac{(jk)^{n+1}t^{k+j}}{k!j!} \leq \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!\,j!}\,k^2 = \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!j!}\,\frac{k^2+j^2}{2},$$

and this now follows from $2jk \leq k^2 + j^2$.

(b) As

$$m_{n+1}(t) = t\sum_{k=0}^{\infty}(k+1)^n \exp(-t)\,\frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that $m_n(t)$ satisfies the recurrence

$$\boxed{m_{n+1}(t) = t\sum_{k=0}^{n}\binom{n}{k} m_k(t), \qquad m_0(t) = 1.}$$

In particular for $t = 1$, we obtain the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140\,\ldots\,.$$

- These are the *Bell numbers* as was discovered by consulting *Sloane's Encyclopedia*.

  `www.research.att.com/personal/njas/sequences/index.html`

- Sloane can also tell us that, for $t = 2$, we have the *generalized Bell numbers*, and gives the exponential generating functions.*

▶ Inter alia, an explicit computation shows that

$$t\,\frac{1+t}{2} = c_0(t)\, c_2(t) \le c_1(t)^2 = t^2$$

exactly if $t \ge 1$, which completes (b).

Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n\ge 0} c_n u^n = \exp\left(t(e^u - 1)\right). \tag{8}$$

*The Bell numbers were known earlier to Ramanujan — an example of *Stigler's Law of Eponymy*!

(c*)* We appeal to a recent theorem due to E. Rodney Canfield,[†] which proves the lovely and quite difficult result below.

**Theorem 1** *If a sequence* $1, b_1, b_2, \cdots$ *is non-negative and log-concave then so is the sequence* $1, c_1, c_2, \cdots$ *determined by the generating function equation*

$$\sum_{n \geq 0} c_n u^n = \exp\left(\sum_{j \geq 1} b_j \frac{u^j}{j}\right).$$

Using equation (8) above, we apply this to the sequence $\mathbf{b_j = t/(j-1)!}$ which is log-concave exactly for $t \geq 1$. **QED**

*The '*' indicates this was the unsolved component.

[†]A search in 2001 on *MathSciNet* for "Bell numbers" since 1995 turned up 18 items. This paper showed up as number 10. Later, *Google* found it immediately!

- It transpired that the given solution to (c) was the only one received by the *Monthly*

  ▶ This is quite unusual

- The reason might well be that it relied on the following sequence of steps:

**(??)** $\Rightarrow$ Computer Algebra System $\Rightarrow$ Interface

$\Rightarrow$ Search Engine $\Rightarrow$ Digital Library

$\Rightarrow$ Hard New Paper $\Rightarrow$ **Answer**

★ Now if only we could automate this!