BOOLE SUMMATION AND ASYMPTOTIC EXPANSIONS OF SOME SPECIAL SERIES

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1. Introduction.

In this contribution we report on work which was recently published in [3]. However, most of Sections 4 and 5 is new.

If we truncate Gregory's series for π after 50,000 terms we get, evaluated to 50 digits,

$$\frac{\pi}{2} - 2\sum_{k=1}^{50,000} \frac{(-1)^{k-1}}{2k-1} = 1.5707\underset{1}{\underline{8}}632679489\underset{-1}{\underline{7}}61923132\underset{5}{\underline{1}}1916397\underset{-61}{\underline{205}}209858\underline{33147388}.$$

A related example is

$$\log 2 \sim \sum_{k=1}^{50,000} \frac{(-1)^{k+1}}{k} = .6931\underset{1}{\underline{3}}7180\underset{-1}{\underline{6}}599453093\underset{2}{\underline{39}}72321214\ \underset{-16}{\underline{74}}1765680\underset{272}{\underline{483}}00134\underline{46572}.$$

In both cases all but the underlined digits are correct. The numbers under the underlined digits are the numbers that must be added to correct these.

It is the purpose to this note to draw attention to a relatively little known summation formula, and to how it can be used to explain this (on first sight unlikely) phenomenon.

2. The Boole Summation Formula.

The Euler-Maclaurin summation formula is an important tool in number theory and numerical analysis. Here we are going to use a less known analogue, the Boole summation formula (see, for example, [5, p.34]).

While the Euler-Maclaurin formula uses Bernoulli polynomials, the

Boole formula is based on the Euler polynomials $E_n(x)$, defined by

$$(2.1) \qquad \frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (|t| < \pi);$$

(see [1, p. 804]). Each $E_n(x)$ is a polynomial of degree n with leading coefficient 1. We also define the periodic Euler function $\bar{E}_n(x)$ by

$\bar{E}_n(x+1) = -\bar{E}_n(x)$ for all x, and $\bar{E}_n(x) = E_n(x)$ for $0 \le x < 1$. It can be shown that $\bar{E}_n(x)$ has continuous derivatives up to and including the (n-1)st order.

LEMMA 1. Let f (t) be a function with m continuous derivatives, defined on the interval $x \le t \le x + \omega$. Then for $0 \le h \le 1$,

$$f(x + h\,\omega) = \sum_{k=0}^{m-1} \frac{\omega^k}{k!} E_k(h) \cdot \frac{1}{2}\left(f^{(k)}(x+\omega) + f^{(k)}(x)\right) + R_m ,$$

where

$$R_m = \frac{1}{2}\omega^m \int_0^1 \frac{\bar{E}_{m-1}(h-t)}{(m-1)!} f^{(m)}(x + \omega t)dt \quad .$$

This summation formula is easy to establish by repeated integration by parts of the above integral. To derive a convenient version of Lemma 1 for the applications we have in mind, we set $\omega = 1$ and impose further restrictions on f.

LEMMA 2. Let f be a function with m continuous derivatives, defined on $t \ge x$. Suppose that $f^{(k)}(t) \to 0$ as $t \to \infty$ for all $k = 0, 1, \ldots, m$. Then for $0 \le h \le 1$,

$$\sum_{v=0}^{\infty} (-1)^v f(x+h+v) = \sum_{k=0}^{m-1} \frac{E_k(h)}{2k!} f^{(k)}(x) + R_m ,$$

where

$$R_m = \frac{1}{2}\int_0^{\infty} \frac{\overline{E}_{m-1}(h-t)}{(m-1)!} f^{(m)}(x+t)dt .$$

3. The Remainder for Gregory's Series.

The Euler numbers E_n may be defined by the generating function

$$(3.1) \qquad \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} .$$

Comparing (3.1) with (2.1), we see that

$$(3.2) \qquad E_n = 2^n E_n\left(\frac{1}{2}\right).$$

The first non-zero Euler numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1385$.

The phenomenon mentioned in the introduction is entirely explained by the next proposition - if we set $n = 50{,}000$. It is also clear that we will get similar patterns for $n = 10^m/2$ with any positive integer m.

PROPOSITION 1. For positive integers n and M we have

$$(3.3) \qquad 4\sum_{k=n}^{\infty} \frac{(-1)^k}{2k+1} = (-1)^n \sum_{k=0}^{M} \frac{2E_{2k}}{(2n)^{2k+1}} + R_1(M) ,$$

where

$$|R_1(M)| \le \frac{2\,|E_{2M}|}{(2n)^{2M+1}} .$$

PROOF. Apply Lemma 2 with $f(x) = 1/x$; then set $x = n$ and $h = 1/2$. We get

$$(3.4) \qquad \sum_{v=0}^{\infty} \frac{(-1)^v}{n+v+1/2} = \sum_{k=0}^{m-1} \frac{E_k(1/2)}{2k!} \frac{(-1)^k k!}{n^{k+1}} + R_m ,$$

with

$$R_m = \frac{1}{2} \int_0^{\infty} \frac{\bar{E}_{m-1}(h-t)}{(m-1)!} \frac{(-1)^m m!}{(x+t)^{m+1}} dt .$$

We multiply both sides of (3.4) by $2(-1)^n$. Then the left-hand side is seen to be identical with the left-hand side of (3.3). After replacing m by $2M+1$ and taking into account (3.2) and the fact that odd-index Euler numbers vanish, we see that the first terms on the right-hand sides of (3.3) and (3.4) agree. To estimate the error term, we use the following inequality

$$|E_{2M}(x)| \le 2^{-2M} |E_{2M}| \qquad \text{for } 0 \le x \le 1$$

(see, e.g., [1, p. 805]). Carrying out the integration now leads to the error estimate given in Proposition 1. □

4. Other Examples.

a. If we proceed as in the proof of Proposition 1, only with $h = 1$, we obtain

$$(4.1) \qquad \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k} = (-1)^{n+1} \left\{ \frac{1}{2n} + \sum_{k=1}^{M} \frac{T_{2k-1}}{(2n)^{2k}} \right\} + R_2(M),$$

where the remainder term $R_2(M)$ is easy to estimate (as in Section 3). Here T_n is the n-th <u>tangent number</u>, related with the Euler polynomials via

$$(4.2) \qquad T_n = (-1)^n \, 2^n \, E_n(1)$$

(see, e.g., [5, p.28]). The first nonzero tangent numbers are $T_0 = 1$, $T_1 = -1$, $T_3 = 2$, $T_5 = -16$, $T_7 = 272$. We see now that (4.1) explains the second example in the introduction.

<u>b</u>. <u>Catalan's constant</u> is defined by

$$C = \sum_{v=1}^{\infty} \frac{(-1)^{v-1}}{(2v-1)^2}.$$

To deal with this case, we set $f(x) = x^{-2}$ in Lemma 2, and then $x = n$, $h = 1/2$. Then we obtain

$$(4.3) \qquad 2\sum_{v=n+1}^{\infty} \frac{(-1)^{v-1}}{(2v-1)^2} = (-1)^n \sum_{k=0}^{M} \frac{(2k+1)\,E_{2k}}{(2n)^{2k+2}} + R_3(M),$$

with a remainder term $R_3(M)$ that is again easy to estimate. This case is illustrated by the following calculation to 50 digits:

$$2\sum_{v=1}^{5,000} \frac{(-1)^{v-1}}{(2v-1)^2} = 1.8319311\underset{1}{\underline{7}}8354438\underset{-3}{\underline{3}}30109\underset{25}{\underline{182}}02986\underset{-427}{\underline{903}}822\underset{12465}{\underline{03017}}\underline{993042984}.$$

Here, again, the numbers under the underlined digits (i.e., $(2k+1)E_{2k}$) must be added to correct these.

<u>c</u>. As should be clear by now from the previous examples, we can apply

Lemma 2 to any $f(x) = x^{-s}$ with positive real s. While in general we cannot expect the phenomenon of the agreeing digits to be "visible", in many cases it will still work. To illustrate this, we take $s = 1/2$ and $h = 1$. Then Lemma 2 with (4.2) gives

$$(4.4) \qquad \sum_{v=0}^{\infty} \frac{(-1)^v}{(n+1+v)^{1/2}} = \sum_{k=0}^{m-1} \frac{T_k}{2^{k+1}k!} \frac{(-1)^k(1/2)_k}{n^{k+1/2}} + R_4(m),$$

where $(a)_k$ is the Pochhammer symbol $(a)_0 = 1$, $(a)_k = a(a+1) \dots (a+k-1)$ $(k \geq 1)$, with a remainder term $R_4(m)$ that is again easy to estimate. Now we use the Genocchi numbers G_k (see, e.g., [4, pp.525, 548]) which are related to the tangent numbers via

$$(4.5) \qquad G_{k+1} = (k+1)2^{-k}\, T_k,$$

and observe that

$$(4.6) \qquad (1/2)_k = 2^{-2k}\, k! \binom{2k}{k} = 2^{-2k}(k+1)!\, C_k,$$

where $C_k = \binom{2k}{k}/(k+1)$ is the k-th Catalan number (see, e.g., [4, p. 203]). Now with (4.5) and (4.6) we can rewrite (4.4) as

$$(4.7) \qquad \sum_{v=n+1}^{\infty} \frac{(-1)^{v-1}}{\sqrt{v}} = (-1)^n \sum_{k=0}^{m-1} \frac{G_{k+1}C_k}{(2\sqrt{n})^{2k+1}} + R_4(m).$$

We note that both the Genocchi and the Catalan numbers are integers; the first few values of $G_{k+1}\, C_k$ are 1, -1, 0, 5, 0, -126, 0, 7293. Finally, if we choose $n = 25 \times 10^{2k}$ for any integer $k \geq 0$, we see from (4.7) that the phenomenon in question is "visible". In fact, we have

$$\sum_{v=1}^{2,500} \frac{(-1)^{v-1}}{\sqrt{v}} = .\underline{59}489\underline{9}643421\underline{58}037025\underline{988}659\underline{069429}\underline{630358475}\,.$$

1 -1 5 -126 7293

5. Concluding Remarks.

We remark that expressions similar to the ones in this note can be given for numerous related constants, in particular for $\zeta(k)$ and $\zeta(k-1/2)$, $k = 1, 2, \ldots,$ where $\zeta(s)$ is the Riemann zeta function.

A generalization of the Euler-Maclaurin and Boole summation formulas was derived by Berndt [2]. This can be applied to character analogues of the series in this note, such as Dirichlet L-series. The roles of the E_n and T_n in Sections 3 and 4 are then played by generalized Bernoulli numbers or by related numbers.

It is obvious from the formulas (3.3), (4.1), (4.3), and (4.7) that what we did here in decimal expansions could be done in any other even base. Moreover, it is not necessary to add as many terms as we did in the numerical examples in order to get meaningful results. In fact, if we use (3.3) with $n = 5$ and $M = 3$, we get the approximation

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) - 2(1.10^{-1} - 1.10^{-3} + 5.10^{-5} - 61.10^{-7}) \equiv 3.14159473\ ,$$

which agrees with π to 5 digits after the decimal point. With $n = 50$ and $M = 15$ we get π to 35 digits, with a calculation that would easily be feasible "by hand". It is interesting to note that it took Ludolph van Ceulen (1540-1610) most of his life to calculate that many digits.

We also remark that (4.7) can be used to find the sum of the corresponding (extremely slowly convergent) series to 11 decimal places, with only $n = 25$ and using the first four nonzero values $1, -1, 5, -126$ of $G_{k+1}C_k$.

Finally, we would like to mention that the phenomenon in question was brought to our attention by Mr. R.D. North of Colorado Springs. The book [6] proved to be a useful tool for identifying the number sequences that arose. The computations involved in this work were done with the symbolic manipulation package MAPLE.

References

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