ON DIFFERENCE CONVEXITY OF LOCALLY LIPSCHITZ FUNCTIONS (PREPARED FOR *OPTIMIZATION* IN HONOUR OF ALFREDO IUSEM'S SIXTIETH BIRTHDAY)

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ABSTRACT. We survey and enhance salient parts of the literature about difference convex functions with specific regard to current knowledge and applications of DC functions.

1. Introduction and Preliminaries

There is a large if somewhat scattered literature on *difference convex*, (= delta-convex, or DC), functions—functions which are the difference of two continuous convex functions. It is our goal in this note to survey and enhance salient parts of this literature with specific regard to the current state of knowledge and interesting examples of DC functions. Throughout, our "assertions" are often formulated for real-valued functions even if some of them were originally proved for more general mappings.

The class of DC functions is a remarkable subclass of locally Lipschitz functions that is of interest both in analysis and optimization. It appears very naturally as the smallest vector space containing all continuous convex functions on a given set.

Let X be a normed linear space, X^* its dual and S_X the unit sphere in X. Unless stated otherwise, all spaces are *real*. The duality between X and X^* is denoted $\langle \cdot, \cdot \rangle$, that is, $\langle x^*, x \rangle = \langle x, x^* \rangle := x^*(x)$, for all $x \in X$ and $x^* \in X^*$. In the Hilbert space context, we will use $\langle \cdot, \cdot \rangle$ for the inner product. The *distance function* $d_C : X \to \mathbb{R}$ of a closed set $C \subset X$ is defined by $d_C(x) = \inf_{c \in C} \|x - c\|$ for all $x \in X$.

If $f : X \to \mathbb{R}$ is a continuous convex function, its *subdifferential* at $x \in X$ is the set

$$\partial f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^* \rangle \text{ for all } y \in X\}.$$

A locally Lipschitz function $f : X \to \mathbb{R}$ is *strictly Gâteaux differentiable* at $x \in X$ if it is Gâteaux differentiable at x (where $\nabla f(x)$ denotes the Gâteaux differential at x) and for each $y \in S_X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(z + ty) - f(z)}{t} - \nabla f(x)(y) \right| < \varepsilon,$$

whenever $0 < t < \delta$ and $\|z - x\| < \delta$. If this holds uniformly over $y \in S_X$, we say that f is *strictly (Fréchet) differentiable* at $x \in X$. Note that in finite dimensions these two notions agree.

2000 *Mathematics Subject Classification.* 46B99.

Key words and phrases. Delta-convex function, continuous convex function, locally Lipschitz function.

Research supported in part by the Australian Research Council.

Given a real-valued function f on X, we say that f is *weak Hadamard differentiable* at a point $x \in X$ if there exists $x^* \in X^*$ such that, for any weakly compact set $C \subset X$, the limit

$$\lim_{t \downarrow 0} \frac{f(x+th) - f(x) - x^*(th)}{t} = 0$$

uniformly in $h \in C$. If a function $f : X \to \mathbb{R}$ is Fréchet differentiable and the derivative is a Lipschitz mapping, we shall say that f is of the class $C^{1,1}$.

Let $f : X \to \mathbb{R}$ be locally Lipschitz around $x \in X$. Since f is locally Lipschitz the *Clarke directional derivative* of f at x in the direction $u \in X$, denoted by $f^\circ(x; u)$, may be defined as follows:

$$f^\circ(x; u) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+tu) - f(y)}{t},$$

where $y \in X$ and $t > 0$. The *Clarke subdifferential* of f at x is

$$\partial_{\mathrm{C}} f(x) := \{z \in X : \langle z, u \rangle \leq f^\circ(x; u) \text{ for all } u \in X\}.$$

In particular $\partial_{\mathrm{C}} f(x) = \partial f(x)$ for a convex function f continuous around x.

Let $C \subset X$ be a convex set. We say that a function $f : C \to \mathbb{R}$ is *DC* (*delta convex* or *difference convex* [14]) on C if it is expressible as the difference of two continuous convex functions on C, or equivalently, if there exists a continuous convex function $g : C \to \mathbb{R}$ such that the functions $f + g$ and $-f + g$ are both convex.

When Y is another normed linear space, a mapping $F : C \to Y$ is said to be *DC* when there exists a continuous convex function $g : C \to \mathbb{R}$ such that for all $y^* \in S_{Y^*}$ the function $y^* \circ F + g$ is convex. In this case, g is called a *control function.* Note that when Y is finite-dimensional this is equivalent to each component of F being DC. This definition of a DC mapping is due to L. Veselý and L. Zajíček [29] and is far from being obvious. In the same paper, the authors show why this definition is to be preferred to various other possibilities. We consider one other option in Section 3.7. Lastly, a function or a mapping is *locally DC* if each point of its domain has a convex neighborhood wherein it is DC.

Standard lattice notation is used. The pointwise maximum, resp. minimum of $f, g : X \to \mathbb{R}$ is denoted $f \vee g$, resp. $f \wedge g$, and we put $f^+ := 0 \vee f$ along with $f^- := -(0 \wedge f)$. Finally, $\mathbb{N}$ stands for the set of positive integers.

2. Positive results

Let us start with well-known properties of DC functions, most of which can be traced back to [14] and in some cases further. When practicable, we give more direct proofs.

2.1. **Lattice and ring structures.** First observe that each DC function is a difference of two *nonnegative* convex continuous functions. Indeed, if X is a normed linear space and $f : X \to \mathbb{R}$ is DC, then

$$f = f_1 - f_2 + h = f_1 - f_2 + h^+ - h^-$$

where f_1, f_2 are nonnegative convex and h is affine —since any lower semicontinuous proper convex function has an affine minorant. The functions h^+, h^- are then also nonnegative and convex.

The equality $(f-g)\vee 0=(f\vee g)-g$ for any $f,g:X\to\mathbb{R}$ implies that if a function h is DC so are $h^+,h^-,|h|$. Hence the class of DC functions on X (or on a subset A) forms a *vector lattice.*

We now give a direct proof that the product of two DC functions is DC (and hence that squares of DC functions are DC). Let $f,g:X\to\mathbb{R}$ be DC function, that is,

$$f=f_1-f_2,\qquad g=g_1-g_2$$

where f_1,f_2,g_1,g_2 are nonnegative, continuous, and convex. Since

$$2f_1f_2=(f_1+f_2)^2-f_1^2-f_2^2$$

is DC, so is

$$f^2=f_1^2+f_2^2-2f_1f_2.$$

And consequently, fg is DC because

$$2fg=(f+g)^2-f^2-g^2.$$

The latter result also follows from [32, Theorem 4.1].

We conclude that the class of DC functions on X (or on a subset A) is also an *algebra.*

In finite dimensions the reciprocal of a strictly positive DC function is DC, see [14, Corollary]. We shall see in Theorem 4.4 that this fails generally in infinite dimensional Banach space.

2.2. **Mixing property.** What follows is a broad generalization of the fact that the class of DC functions is closed under taking maxima of finitely many functions, [29, Lemma 4.8].

Proposition 2.1 (Veselý, Zajíček)**.** *Let X be a normed linear space and $A\subset X$ a convex open set. Suppose $f_1,\dots,f_n$ are DC functions on A. If $f:A\to\mathbb{R}$ is continuous and*

$$f(x)\in\{f_1(x),\dots,f_n(x)\}\quad\text{for all } x\in A,$$

then f is DC on A.

2.3. **Approximation of continuous functions.** As a corollary of the result in subsection 2.1 we obtain the following approximation result via the Stone-Weierstrass theorem (since DC functions contain constants and separate points of the underlying space).

Proposition 2.2. *Let X be a normed linear space, $K\subset X$ a compact convex set, and $f:K\to\mathbb{R}$ a continuous function. Then there exists a sequence $\{f_n\}_n$ of DC functions on K which converges to f uniformly on K.*

In Euclidean spaces this also follows from the fact that polynomials are DC, see subsection 3.3. In consequence, there are in some sense too many DC functions for the class to preserve many structurally useful properties.

2.4. **Differentiability properties of DC functions.** Some differentiability properties are inherited from convex functions, but not all, see Example 4.7. We first recall some of the positive results from [15]. Let $f:\mathbb{R}^n\to\mathbb{R}$ be DC with a decomposition $f=f_1-f_2$. Then:

- $\partial_{\mathrm{C}}f(x)=\partial f_1(x)-\partial f_2(x)$ for all $x\in\mathbb{R}^n$;
- $\partial_{\mathrm{C}}f$ reduces to ∇f a.e. on $\mathbb{R}^n$;

- $\partial_{\mathrm{C}} f$ is differentiable a.e. on $\mathbb{R}^n$;
- f has a second-order Taylor expansion a.e. on $\mathbb{R}^n$;
- f is strictly Fréchet differentiable a.e. on $\mathbb{R}^n$.

Unlike the case of convex functions, it is easy to see that $\partial_{\mathrm{C}} f(x)$ need not reduce to a singleton when f is differentiable at $x \in \mathbb{R}^n$. See Example 5.3.

Observe also that $f : [0,1] \to \mathbb{R}$ is DC if and only if f is absolutely continuous (AC) and f' has bounded variation. Indeed, just recall that a function of bounded variation (a BV function) is precisely a difference of two nondecreasing functions and conversely.

We next present a portion of [29, Theorem 3.10] and [29, Proposition 3.1].

Theorem 2.3 (Veselý, Zajíček)**.** *Let X be a Banach space and $A \subset X$ an open convex subset. Suppose $f : A \to \mathbb{R}$ is locally DC.*

(a) *All one-sided directional derivatives of f exist on A.*
(b) *If X is Asplund, then f is strictly Fréchet differentiable everywhere on A excepting a set of the first category.*
(c) *If X is weak Asplund, then f is Gâteaux differentiable everywhere on A excepting a set of the first category.*

Finally, recall [29, Proposition 3.9] and compare it with Theorems 4.8 and 4.9.

Proposition 2.4. *Let X be a normed linear space and $A \subset X$ open and convex. Suppose $f : A \to \mathbb{R}$ is DC on A with a control function $\widetilde{f}$.*

(a) *If $\widetilde{f}$ is Fréchet differentiable at $x \in A$, then f is strictly Fréchet differentiable at x.*
(b) *If $\widetilde{f}$ is Gâteaux differentiable at $x \in A$, then f is Gâteax differentiable at x.*

2.5. **Composition of DC mappings.** We first recall a classical result on composition of DC functions and mappings due to P. Hartman [14].

Theorem 2.5 (Hartman)**.** *Let $A \subset \mathbb{R}^m$ be convex and either open or closed. Let $B \subset \mathbb{R}^n$ be convex and open. If $F : A \to B$ and $g : B \to \mathbb{R}$ are DC, then $g \circ F$ is a locally DC function on A.*

Hartman also proved that a function on $A = \mathbb{R}^m$ is DC if and only if it is locally DC. This fails broadly for infinite dimensional Banach spaces.

We now give a generalization of Hartman's theorem for Banach spaces. The ingenious proof technique developed by L. Veselý also applies to this slightly modified version of a result of his in [33]. We chose to give the proof because it so well exemplifies the virtues of the notion of a control function.

Theorem 2.6 (Veselý)**.** *Let X be a Banach space, Y a normed linear space, and $A \subset X$, $B \subset Y$ be open convex sets. If $F : A \to B$ and $g : B \to \mathbb{R}$ are locally DC, then $g \circ F$ is a locally DC function on A.*

Proof. Choose $a \in A$ and then $V \subset B$ a convex open neighborhood of $F(a)$ such g is DC on V with a control function $\widetilde{g} : V \to \mathbb{R}$ and both g and $\widetilde{g}$ are Lipschitz on V with some Lipschitz constants L_1 and L_2, respectively. Further, choose $U \subset A$ a convex open neighborhood of a such that $F(U) \subset V$ and F is DC on U with a control function $\widetilde{F}$. Put $L := L_1 + L_2$.

We will show that $g \circ F$ is DC on U with a control function $\widetilde{g} \circ F + L\widetilde{F}$. Let $x_0 \in U$ and $x^* \in \partial \widetilde{F}(x_0)$.

(i) Let $y^* \in \partial\widetilde{g}\,(F(x_0))$ and $y_0^* \in Y^*$ such that $\|y_0^*\| = 1$ and $y^* = \|y^*\| \cdot y_0^*$. Let $u^* \in \partial\left(y_0^* \circ F + \widetilde{F}\right)(x_0)$.

Then, for any $x \in U$,

$$\begin{aligned} & \widetilde{g}\,(F(x)) + L\widetilde{F}(x) - \widetilde{g}\,(F(x_0)) - L\widetilde{F}(x_0) \\ \geq\; & \langle F(x) - F(x_0), y^*\rangle + L\left(\widetilde{F}(x) - \widetilde{F}(x_0)\right) \\ =\; & \|y^*\|\left(\langle F(x) - F(x_0), y_0^*\rangle + \widetilde{F}(x) - \widetilde{F}(x_0)\right) + (L - \|y^*\|)\left(\widetilde{F}(x) - \widetilde{F}(x_0)\right) \\ \geq\; & \|y^*\|\langle x - x_0, u^*\rangle + (L - \|y^*\|)\,\langle x - x_0, x^*\rangle. \end{aligned}$$

Thus, the function $\widetilde{g} \circ F + L\widetilde{F}$ is supported by a continuous affine function at any $x_0 \in U$. By [27, Theorem 43C] it is continuous and convex on U.

(ii) Let $v_\pm^* \in \partial(\pm g + \widetilde{g})\,(F(x_0))$ and $\overline{v_\pm^*} \in Y^*$ such that $\|\overline{v_\pm^*}\| = 1$ and $v_\pm^* = \|v_\pm^*\| \cdot \overline{v_\pm^*}$. Let $w^* \in \partial\left(\overline{v_\pm^*} \circ F + \widetilde{F}\right)(x_0)$.

Then for every $x \in U$ we have

$$\begin{aligned} & \pm g\,(F(x)) + \widetilde{g}\,(F(x)) + L\widetilde{F}(x) - \left(\pm g\,(F(x_0)) + \widetilde{g}\,(F(x_0)) + L\widetilde{F}(x_0)\right) \\ \geq\; & \left\langle F(x) - F(x_0), v_\pm^*\right\rangle + \|v_\pm^*\|\left(\widetilde{F}(x) - \widetilde{F}(x_0)\right) + \left(L - \|v_\pm^*\|\right)\left(\widetilde{F}(x) - \widetilde{F}(x_0)\right) \\ \geq\; & \|v_\pm^*\|\langle x - x_0, w^*\rangle + \left(L - \|v_\pm^*\|\right)\langle x - x_0, x^*\rangle. \end{aligned}$$

By the same argument as above, the functions $\pm g \circ F + \widetilde{g} \circ F + L\widetilde{F}$ are continuous and convex on U. This finishes the proof. □

An even more general version of a composition theorem follows. For its proof see [32, Proposition 3.1].

Theorem 2.7 (Veselý, Zajíček)**.** *Let X, Y be normed linear spaces, $A \subset X$ a convex set, and $B \subset Y$ open and convex. If $F : A \to B$ and $g : B \to \mathbb{R}$ are locally DC, then $g \circ F$ is locally DC on A.*

Some limitations for composition results are described in Theorem 4.2. We finish this subsection by recording a global composition theorem.

There are several known conditions which ensure that composition of two DC mappings is a DC mapping, see [32, Sections 3,4]. We formulate a part of [32, Proposition 3.3].

Proposition 2.8 (Veselý, Zajíček)**.** *Let X, Y be Banach spaces, $A \subset X$ open convex, and $B \subset Y$ convex. Let $F : A \to B$ be DC and $g : B \to \mathbb{R}$ when restricted on any convex bounded subset of B is Lipschitz and DC with a Lipschitz control function. Then $g \circ F$ is DC on A.*

2.6. **Difference convexity is an absolute property.** It follows easily from Proposition 2.8 that if a mapping F is DC then $\|\cdot\| \circ F$ is a DC function. Quite pleasantly, we observe that the converse is true for Lipschitz *functions.* This follows fairly directly from the mixing property of Theorem 2.1. We are indebted to Scott Sciffer for the explicit proof.

Theorem 2.9 (Absoluteness of difference convexity)**.** *Let X be a Banach space and $A \subset X$ open convex set. Suppose $f : A \to \mathbb{R}$ is Lipschitz on A. Then f is DC on A if and only if $|f|$ is.*

Proof. It remains to show the 'if' part. Suppose $|f|$ is DC, that is $|f| = g - h$, where g and h are continuous convex functions on A. Define functions $r : A \to \mathbb{R}$ and $s : A \to \mathbb{R}$, for $x \in A$, by

$$r(x) := \begin{cases} g(x) & \text{if } f(x) \geq 0 \\ \\ h(x) & \text{if } f(x) < 0 \end{cases} \qquad \text{and} \qquad s(x) := \begin{cases} h(x) & \text{if } f(x) \geq 0 \\ \\ g(x) & \text{if } f(x) < 0 \end{cases}$$

It is immediate that r and s are Lipschitz and that $f = r - s$. What remains to show is that r and s are convex. Around any point $x \in A$ where $f(x) \neq 0$ there is a neighborhood where r (or s) is equal to g (or h), and hence it has a local subgradient. At a point $x \in A$ where $f(x) = 0$ we have $g(x) = h(x)$, but also $g \geq h$ (since $g - h = |f| \geq 0$), which shows that any subgradient of h is also a subgradient of r and of s. But then r and s have (local) subgradients at every point, which certainly makes them convex [7]. □

On the other hand, in the next section we show that the previous statement fails with modulus replaced by norm for Lipschitz *mappings*, even when the range is just two-dimensional (Example 4.11), and for order-convex mappings (Example 4.10).

2.7. **Toland duality.** In this subsection we reproduce some results from [12, 16]. The proof for Euclidean spaces appeared in [12] and holds without modification in the Banach space context, however, the latter was explicitly given in [16].

We work with *extended-valued* functions, that is, functions with values in $(-\infty, +\infty]$. For such a function $f : X \to (-\infty, +\infty]$ define its *domain* as $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. Given any function $f : X \to (-\infty, +\infty]$ on a Banach space X we define its (convex) *conjugate function* by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \qquad x^* \in X^*.$$

The definition immediately yields

$$\inf_{x \in X} f(x) = -f^*(0) \tag{1}$$

The following theorem states *Toland duality,* see [12, 16], and [28, Section 3.1].

Theorem 2.10 (Ellaia, Hiriart-Urruty)**.** *Let X be a Banach space, $h : X \to \mathbb{R}$ be convex continuous, and $g : X \to (-\infty, +\infty]$ any function. Then*

$$(g - h)^*(x^*) = \sup_{y^* \in \operatorname{dom} h^*} \{g^*(x^* + y^*) - h^*(y^*)\}$$

for any $x^* \in \operatorname{dom} g^*$.

Proof. Pick $x^* \in \operatorname{dom} g^*$. By direct calculation,

$$\begin{aligned} (g - h)^*(x^*) &= \sup_{x \in X} \{\langle x^*, x\rangle - (g - h)(x)\} \\ &\geq \langle x^* + y^*, x\rangle - g(x) + h(x) - \langle y^*, x\rangle \qquad \text{for any } x \in X, y^* \in X^*. \end{aligned}$$

Consequently,

$$(g - h)^*(x^*) + h^*(y^*) \geq g^*(x^* + y^*)$$

for any $y^* \in X^*$, and

$$(g - h)^*(x^*) \geq \sup_{y^* \in \operatorname{dom} h^*} \{g^*(x^* + y^*) - h^*(y^*)\}.$$

Note that we have not used convexity of h yet. To prove the converse inequality it suffices, for a given $x \in \operatorname{dom} g$, to find some $y_x^* \in \operatorname{dom} h^*$ such that

$$\langle x^*, x\rangle - (g-h)(x) \le g^*(x^* + y_x^*) - h^*(y_x^*).$$

Given $x \in \operatorname{dom} g$ we choose $y_x^* \in \partial h(x)$, and get

$$h(x) + h^*(y_x^*) = \langle y_x^*, x\rangle$$

by the Fenchel-Young inequality, see [7, Proposition 4.4.1]. Having $y_x^* \in \operatorname{dom} h^*$ we conclude

$$\begin{aligned}\langle x^*, x\rangle - (g-h)(x) &= \langle x^* + y_x^*, x\rangle - g(x) + h(x) - \langle y_x^*, x\rangle \\ &= \langle x^* + y_x^*, x\rangle - g(x) - h^*(y_x^*) \\ &\le \sup_{z\in X} \{\langle x^* + y_x^*, z\rangle - g(z)\} - h^*(y_x^*) = g^*(x^* + y_x^*) - h^*(y_x^*).\end{aligned}$$

□

Corollary 2.11. *By Theorem 2.10 and (1) one gets*

$$\inf_{x\in X} (g(x) - h(x)) = \inf_{x^*\in \operatorname{dom} h^*} (h^*(x^*) - g^*(x^*)) . \tag{2}$$

If we assume that both g, h are continuous convex, hence $g - h$ is DC on X, we arrive at (2) along with a similar results for suprema.

$$\sup_{x\in X} (g(x) - h(x)) = \sup_{x^*\in \operatorname{dom} g^*} (h^*(x^*) - g^*(x^*)) .$$

2.8. **Formula for the ε-subdifferential.** In connection with Toland duality of Section 2.7, we mention a formula for the ε-subdifferential of a DC function due to Martínez-Legaz and Seeger, [23]. Recall that, for a lsc function $f : X \to (-\infty, \infty]$ and $\varepsilon \ge 0$, the *ε-subdifferential* of f at $x \in X$ is the set

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) \ge f(x) + \langle y - x, x^*\rangle - \varepsilon \text{ for all } y \in X\} .$$

We can now state the main result of [23] for DC functions on Banach spaces. We use the notation

$$A \ominus B := \{x^* \in X^* : x^* + B \subset A\}$$

where $A, B \subset X^*$.

Theorem 2.12 (Martínez-Legaz, Seeger)**.** *Let f, g be continuous convex functions on a Banach space X. Suppose $x \in X$ and $\varepsilon \ge 0$. Then*

$$\partial_\varepsilon (f-g)(x) = \bigcap_{\lambda\ge 0} \partial_{\varepsilon+\lambda} f(x) \ominus \partial_\lambda g(x). \tag{3}$$

In [23], the authors also show that (3) is equivalent to a result by Hiriart-Urruty, [17]:

$$x \text{ is an } \varepsilon\text{-minimum of } f-g \quad \text{if and only if} \quad \partial_\lambda f(x) \subset \partial_{\varepsilon+\lambda} g(x) \text{ for all } \lambda \ge 0.$$

For further details, we refer the reader to [17].

3. DC functions in analysis

In this section we present a few examples of DC functions from various parts of analysis.

3.1. **Variational analysis.** A function $f : X \to \mathbb{R}$ is *paraconvex* if there exists $\lambda \geq 0$ such that the function

$$f + \frac{\lambda}{2}\|\cdot\|^2$$

is continuous and convex. A function g is *paraconcave* if $-g$ is paraconvex. Obviously, paraconvex and paraconcave functions are 'very' DC. Recall that lower-$\mathcal{C}^2$ functions on Hilbert spaces coincide with locally paraconvex functions, [26, Theorem 3.2].

3.2. **Game theory.** Nash's celebrated proof of the existence of equilibrium points in finite non-cooperative games applies Brouwer's fixed-point theorem to a mapping which happens to be DC, [24, Theorem 1]. We recall the basic framework of this we work.

One considers a finite game with n players. Each player has finitely many *pure strategies,* say $(\pi_{i\alpha})_\alpha$ is the set of pure strategies of player i. This set is identified with the canonical basis of a Euclidean space. Define *mixed strategies* of player i as convex combinations of pure strategies $(\pi_{i\alpha})_\alpha$, and denote the set by S_i. Further, player i has a *pay-off function* p_i, which is a real-valued function defined on n-tuples $(\pi_{1\alpha_1},\dots,\pi_{n\alpha_n})$ of pure strategies. Clearly, it can be affinely extended to the n-tuples of mixed strategies. We denote $(s,t_i) = (s_1,\dots,s_{i-1},t_i,s_{i+1},\dots,s_n)$ if player i chooses a strategy t_i and $s_j \in S_j$ for $j \neq i$. An n-tuple $s = (s_1,\dots,s_n)$, where $s_i \in S_i$, is an *equilibrium point* of the game if

$$p_i(s) = \max_{t_i\in S_i} p_i(s,t_i),$$

for all $i = 1,\dots,n$. The maximum can be equivalently taken just over pure strategies of player i.

Theorem 3.1 (Nash)**.** *Every such non-cooperative n-person game admits an equilibrium point.*

Sketch of proof. Denote $p_{i\alpha}(s) := p_i(s,\pi_{i\alpha})$, and

$$\varphi_{i\alpha}(s) := \max\{0, p_{i\alpha}(s) - p_\alpha(s)\} \quad i = 1,\dots,n,$$

which are obviously convex functions. Define the mapping $T : s \mapsto s'$ componentwise by

$$s_i' := \frac{s_i + \sum_\alpha \varphi_{i\alpha}\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}\pi_{i\alpha}}.$$

Observe that the equilibrium points coincide with fixed points of T, which exist by Brouwer's theorem. □

We observe that the components of T are quotients of convex functions, and hence T is DC, see Section 2.1.

3.3. **Polynomials on $\mathbb{R}^N$.** We next show that polynomials on $\mathbb{R}^N$ are DC functions—and *inter alia* give an explicit decomposition in the one-dimensional case. Consider a polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad x \in \mathbb{R}$$

with $n \in \mathbb{N}$ and $a_n,\dots,a_0 \in \mathbb{R}$. Denote $I_- = \{k \in \mathbb{N} : 2k-1 \leq N, a_{2k-1} < 0\}$, $I_+ = \{k \in \mathbb{N} : 2k-1 \leq N, a_{2k-1} > 0\}$, $J_- = \{k \in \mathbb{N} : 2k \leq N, a_{2k} < 0\}$ and

$J_+ = \{k \in \mathbb{N} : 2k \le N, a_{2k} > 0\}$. Then, for $x \in \mathbb{R}$,

$$p(x) = f_1(x) - f_2(x) \quad \text{where}$$
$$f_1(x) := \sum_{k\in I_+} a_{2k-1} \max\{0, x^{2k-1}\} + \sum_{k\in I_-} a_{2k-1} \min\{0, x^{2k-1}\} + \sum_{k\in J_+} a_{2k} x^{2k}$$
$$f_2(x) := -\sum_{k\in I_+} a_{2k-1} \min\{0, x^{2k-1}\} - \sum_{k\in I_-} a_{2k-1} \max\{0, x^{2k-1}\} - \sum_{k\in J_-} a_{2k} x^{2k},$$

and f_1, f_2 are clearly continuous convex functions.

Having proved the one-dimensional case, observe that polynomials on $\mathbb{R}^N$ are DC. Indeed, the function $(x, y) \mapsto xy$ from $\mathbb{R}^2$ to $\mathbb{R}$ is DC since $2xy = (x + y)^2 - x^2 - y^2$, and hence the claim follows easily by induction.

For a different approach to difference convexity of polynomials, we refer the interested reader to [13] and the references therein.

3.4. **Functions with Lipschitz gradient.** If the underlaying space is sufficiently nice, for instance Hilbert, then every $C^{1,1}$ function (that is, differentiable with Lipschitz gradient) is DC. More generally, we have the following.

Theorem 3.2 (Duda, Veselý, Zajíček)**.** *Let X be a normed linear space. Then, for any normed linear space Y and open convex set $A \subset X$, each $F \in C^{1,1}(A, Y)$ is DC if and only if X admits an equivalent norm with modulus of convexity of power type 2.*

Proof. See [11, Theorem 11]. □

3.5. **Spectral theory.** Spectral functions are often DC. Denote $\mathcal{S}_N$ the set of real symmetric matrices N by N. Let $A \in \mathcal{S}_N$ and $\lambda(A) = (\lambda_1(A), \ldots, \lambda_N(A))$ denote its eigenvalues ranked in descending order. Then, for example, whenever $f\colon \mathbb{R}^N \to (-\infty, \infty]$ is convex and rearrangement invariant one has that $f \circ \lambda$ is convex (as a *spectral* function) and

$$(f \circ \lambda)^* = f^* \circ \lambda.$$

This relies on earlier work by von Neumann, Fan and Davis among others. See [9, Corollary 7.2.9, Example 7.3.34], and also [4, 7]. In particular, we deduce that the sum of the k largest eigenvalues is a locally Lipschitz convex function. As an immediate corollary:

Theorem 3.3. *The k-th largest eigenvalue function $\lambda_k : A \to \lambda_k(A)$ is DC on the space of symmetric matrices $\mathcal{S}_N$. Indeed, $\lambda_k = \sigma_k - \sigma_{k-1}$ where σ_k, the sum of the k largest eigenvalues, is convex for each $1 \le k \le n$.*

The same ideas show that many other spectral functions are DC. For instance, let us say that a function f on $\mathbb{R}^m$ is *symmetric DC* if it is both rearrangement invariant and DC. In this case, $f \circ \lambda$ is a DC function on the m-by-m symmetric matrices. To see this, we observe that we may replace g, h in any decomposition by their averages over all permutations.

Example 3.4. An explicit example of a DC spectral function which is neither convex nor concave occurs in $\mathbb{R}^3$. Indeed, consider the set $\mathcal{S}_3$ of real symmetric matrices 3×3. Whereas the largest-eigenvalue function λ_1 is convex (being supremum of linear functions and the smallest-eigenvalue function λ_3 is concave (being infimum of linear functions), the middle-eigenvalue function λ_2 is in general neither convex nor concave. In fact,

$$\lambda_2 : A \mapsto \operatorname{Trace} A - \lambda_1(A) - \lambda_3(A) = -\lambda_3(A) - (\lambda_1(A) - \operatorname{Trace} A)$$

for each $A \in \mathcal{S}_3$. Figure 1 shows the function λ_2 when restricted on an affine subspace of $\mathcal{S}_3$ that consists of diagonal matrices with diagonal entries $(1, a, -1)$ for $a \in \mathbb{R}$. □

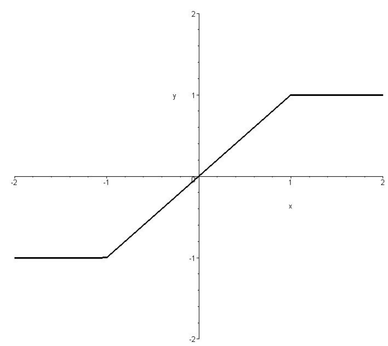

FIG 1. The middle-eigenvalue function.

A similar statement to Theorem 3.3 extends to infinite dimensions (see [9, 7]) but we are forced to consider only positive operators. Denote $\mathcal{B}_{\mathrm{sa}}$ the set of self-adjoint bounded linear operators on the complex separable Hilbert space ℓ^2_{c}. Note that $\mathcal{B}_{\mathrm{sa}}$ is a real Banach space and contains Schatten classes $\mathcal{B}_p$ as subspaces. Recall that an operator $A \in \mathcal{B}_{\mathrm{sa}}$ belongs to the *Schatten class* $\mathcal{B}_0$ if it is compact, and belongs to the Schatten class $\mathcal{B}_p$, for $p \in [1, +\infty)$, if

$$\|A\|_p := (\operatorname{Trace}(|A|^p))^{1/p} < \infty,$$

where $|A| = (A^*A)^{1/2}$. We say that $A \in \mathcal{B}_{\mathrm{sa}}$ is positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \ell^2_{\mathrm{c}}$.

Theorem 3.5. *For $p \in \{0\} \cup [1, +\infty)$ the k-th largest eigenvalue function $\lambda_k : A \to \lambda_k(A)$ is DC on the set of positive $\mathcal{B}_p$-operators.*

3.6. **Further operator theory.** Let X be a Banach space. Each symmetric bounded linear operator $T : X \to X^*$ generates a quadratic form on X. In [20, Theorem 2.12], the authors give a necessary and sufficient condition on the operator which assures that the corresponding quadratic form is DC.

Before stating the result we need some definitions. A finite sequence $(f_0, \ldots, f_n)$ of X-valued functions on $\{-1, 1\}^n$ is called a *Walsh-Paley martingale* if each f_k depends only on the first k coordinates and

$$f_k(\omega) = \frac{1}{2}\left[f_{k+1}(\omega, -1) + f_{k+1}(\omega, 1)\right]$$

whenever $0 \leq k < n$ and $\omega \in \{-1, 1\}^k$.

Given an integer $n \geq 1$, and a function $f : \{-1, 1\}^n \to X$, the *expectation* of f is defined as

$$\mathbb{E}f := 2^{-n} \sum_{\eta \in \{-1,1\}^n} f(\eta) = \int_{\{-1,1\}^n} f \, \mathrm{d}\mathbb{P}$$

where $\mathbb{P} = \mathbb{P}_n$ is the uniformly distributed probability measure on $\{-1, 1\}^n$.

A bounded linear operator $T : X \to Y$, where X, Y are Banach spaces, is a *UMD-operator* if there exists $C > 0$ such that

$$\mathbb{E}\left\|\sum_{k=1}^{n} \varepsilon_k T(f_k - f_{k-1})\right\|^2 \leq C\, \mathbb{E}\|f_n\|^2$$

whenever $(f_0, \ldots, f_n)$ is an X-valued Walsh-Paley martingale and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$.

Theorem 3.6 (Kalton, Konyagin, Veselý). *Let X be a Banach space and T a symmetric bounded linear operator from X to X^*. Then the quadratic form $x \mapsto \langle Tx, x\rangle$ is DC if and only if T is a UMD operator.*

Theorem 3.6 embraces the following special case.

Proposition 3.7. *Let T be a symmetric bounded linear operator on a Hilbert space. Then the function $x \mapsto \langle Tx, x\rangle$ is DC on X.*

Proof. An elementary argument can be found in [29, Corollary 1.12]: it is easy to see that the function $\langle T\cdot, \cdot\rangle$ is $C^{1,1}$, which in Hilbert spaces implies DC. Another approach is to use [20, Theorem 1.2], which moreover yields a stronger result: the function $\langle T\cdot, \cdot\rangle$ is a difference of two nonnegative quadratic forms (which are convex functions). □

We finish this section by making connections between difference-convex operators with values in ordered vector spaces and DC mappings.

3.7. **Difference of two convex operators.** Let X, Y be real normed linear spaces and $S \subset Y$ a closed convex cone inducing an ordering on Y by $y_1 \geq_S y_2$ iff $y_1 - y_2 \in S$. We can then naturally define S-convex mappings between X and Y by saying that $H : X \to Y$ is *S-convex* (or *order-convex*) [3] if for all $x, y \in X$ and $\lambda \in [0, 1]$ we have

$$H(\lambda x + (1-\lambda)y) \leq_S \lambda H(x) + (1-\lambda)H(y).$$

Further, we say that a mapping $F : X \to Y$ is *S-DC*, or *order-DC* (with respect to S), if it is the difference of two continuous S-convex mappings. Before stating when such a mapping is DC let us recall that the *dual cone* of Y is by definition the set

$$S^+ := \{\varphi \in Y^* : \varphi(s) \geq 0 \text{ for all } s \in S\}.$$

Proposition 3.8. *Suppose S^+ has an order unit (equivalently S^+ has nonempty interior) and $F : X \to Y$ is an S-DC mapping. Then F is DC.*

Proof. It suffices to show that an S-convex F operator is DC as the DC functions form a vector space. Let $u^* \in S^+$ be an order unit and fix $n \in \mathbb{N}$ such that $y^* \leq_S nu^*$ for all $y^* \in S_{Y^*}$. So for any $y^* \in S_{Y^*}$ we have that $y^* + nu^* \in S^+$ and hence

$$(y^* + nu^*) \circ F$$

is a continuous convex function on X. Denote $f := nu^* \circ F$ which is a continuous convex function. Then

$$y^* \circ F + f$$

is continuous and convex for any $y^* \in S_{Y^*}$. Hence F is a DC mapping with f as its control function. □

We refer the interested reader to [30, 31] for other results on order-convex and order-DC operators. In particular, there are assumptions under which DC implies order-DC.

4. Negative results

We turn now to provide a variety of counterexamples to accompany the positive results from the previous section and to show their limitations.

4.1. **Composition of DC functions.** Violating an assumption in Theorem 2.5 one can easily produce counterexamples on composition of DC functions.

Example 4.1. [14, p. 708] We first show that the composition of two DC functions need not be DC even in one-dimensional spaces. Indeed, let

$$f : (0,1) \to [0,1) : x \mapsto |x - 1/2|,$$

and

$$g : [0,1) \to \mathbb{R} : y \mapsto 1 - \sqrt{y}.$$

Then $g \circ f$ is not DC as it has both left and right derivatives infinite at $1/2$, see the picture in Figure 1. Note that the assumption of openness of B in Theorem 2.5 was not fulfilled in this case, and g is not Lipschitz at zero. □

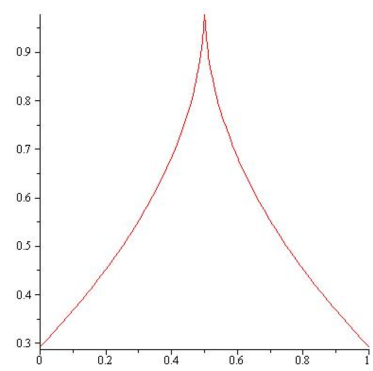

Fig 2. Function $g \circ f = 1 - \sqrt{|\cdot - 1/2|}$ is not DC.

In [32, Theorem 5.5], the authors give a very general construction of counterexamples to difference convex composition theorems.

Theorem 4.2 (Veselý, Zajíček)**.** *Let X, Y be normed linear spaces, X infinite-dimensional. Let $A \subset X$ be open convex and $B \subset Y$ convex. Let $g : B \to \mathbb{R}$ unbounded on a bounded subset of B. Then there exists a DC mapping $F : A \to B$ such that $g \circ F$ is not DC on A.*

The proof of the above theorem is, however, rather abstract. One can get a more explicit example:

Example 4.3. In Theorem 4.2 set $X = Y = A = B = \ell_2$. Denote the standard basis in ℓ_2 by $\{e_n\}_{n\in\mathbb{N}}$. Define $g : \ell_2 \to \mathbb{R}$ as

$$g(x) := \sum_1^\infty \langle x, e_n\rangle^{2n}, \qquad \text{for } x \in \ell_2.$$

Then, as discussed more generally in [7, Chapter 9], g is a convex continuous function which is unbounded on $2B_X$. Using Theorem 4.2, one gets a DC mapping $F : \ell_2 \to \ell_2$ such that $g \circ F$ is not DC. □

4.2. **Infinite vs finite dimensions.** The following characterization is from [32, Corollary 5.6].

Theorem 4.4 (Veselý, Zajíček)**.** *Let X be a normed linear space and $A \subset X$ open convex set. Then the following are equivalent.*

(a) *X is infinite-dimensional.*
(b) *There is a positive DC function f on A such that $1/f$ is not DC on A.*
(c) *There is a locally DC function on A which is not DC on A.*

The following explicit example appeared in [21, Theorem 14].

Theorem 4.5 (Kopecká, Malý)**.** *There exists an explicit function on ℓ_2 which is DC on each bounded convex subset of ℓ_2 but is not DC on ℓ_2.*

Recently, an interesting result appeared, [18]:

Theorem 4.6 (Holický, Kalenda, Veselý, Zajíček)**.** *A Banach space X is nonreflexive if and only if there is a positive convex continuous function f on X such that $1/f$ is not DC.*

4.3. **Differentiability of DC functions.** Let X be a Banach space. Then X does not contain ℓ_1 if and only if weak Hadamard and Fréchet differentiability coincide for continuous convex (resp. concave) functions on X, see [8, Theorem 2] or [7, Chapter 9]. This is, however, not the case of DC functions, as the following example shows.

Example 4.7. [8, Theorem 1b] Every nonreflexive Banach space X admits equivalent norms p_1, p_2 such that $p_1 - p_2$ is weakly Hadamard differentiable and not Fréchet differentiable at some point. □

The next two examples shed some light on how differentiability of a DC function is related to differentiability of its control function. The first example comes from [21, Theorem 7].

Theorem 4.8 (Kopecká, Malý)**.** *There exists a DC function on $\mathbb{R}^2$ which is strictly Fréchet differentiable at the origin but which does not admit a control function that is Fréchet differentiable at the origin.*

Pavlica [25] complemented this result with the following example.

Theorem 4.9 (Pavlica)**.** *There exists a DC function on $\mathbb{R}^2$ which belongs to the class $\mathcal{C}^1$ but does not admit a control function that is Fréchet differentiable at the origin.*

The same author constructed a DC function on $\mathbb{R}^2$ which is of class $\mathcal{C}^1$ on $\mathbb{R}\setminus\{0\}$, and is Fréchet differentiable at the origin, but is not strictly Fréchet differentiable at the origin, [25].

Note also that $\sqrt{\cdot}$ does not preserve the (local) DC property, see Section 5.

4.4. **Difference convexity in norm.** The following two examples show that Theorem 2.9 cannot be extended to mappings.

Example 4.10. Consider $F : \ell_2 \to \ell_2$ defined as

$$F(x_1, x_2, \dots) = (|x_1|, |x_2|, \dots), \qquad (x_1, x_2, \dots) \in \ell_2.$$

Then $\|\cdot\| \circ F = \|\cdot\|$, so it is DC, and clearly F is Lipschitz. But F is not DC as it is nowhere Fréchet differentiable, cf. [29, Theorem 3.10].

Note that F is ℓ_2^+-convex. □

This example thus shows that an order-convex operator need not be DC in our sense; and so also illustrates the need for the dual cone to have nonempty interior in Proposition 3.8.

Example 4.11. Let $g(t) := t^2 \sin(1/t),\ t \in \mathbb{R}$, and define $h(t) := \exp(\mathrm{i}\, g(t)),\ t \in \mathbb{R}$, viewed as a function from $\mathbb{R}$ to $\mathbb{R}^2$. Then h is Lipschitz but is not DC (as its first coordinate is not) while $\|h(t)\| \equiv 1$. □

5. Distance functions

Recall, that, given a closed subset C of a Banach space X, the distance function is denoted d_C. Edgar Asplund observed that, in Hilbert spaces, d_C^2 is DC on the whole space [4, 9, 7]. Indeed,

$$d_C^2(x) = \inf_{c \in C} \|x - c\|^2 = -\sup_{c \in C} \left(-\|x - c\|^2\right) = \|x\|^2 - \sup_{c \in C} \left(2\langle x, c\rangle - \|c\|^2\right).$$

On the other hand the following example shows that this needn't be true for d_C. To be more explicit, even in $\mathbb{R}^2$ there is a closed set C such that d_C is not (locally) DC on $\mathbb{R}^2$. In particular, the operation $\sqrt{\cdot}$ does not preserve DC.

Example 5.1 (Borwein, Moors)**.** Let $C_1 \subset [0, 1]$ be a Cantor set of positive measure and $C := C_1 \times C_1 \subset \mathbb{R}^2$. It is shown in [5, Example 9.2] that the distance function d_C is not strictly differentiable at any point of $\mathrm{bd}(C) = C$. Consequently, d_C cannot be locally DC on $\mathbb{R}^2$ since locally DC functions in finite dimensions are almost everywhere strictly Fréchet differentiable—as noted in Section 2.4. □

In [1, Introduction], the authors note that the distance function to a nonempty set in a Hilbert space is the difference of two convex functions. The above example, however, contradicts this statement even in $\mathbb{R}^2$. On the other hand a positive result does hold true *on the complement* of the set, as the next theorem establishes.

Theorem 5.2. [9, Theorem 5.3.2] *Let X be a Hilbert space and $C \subset X$ a closed set. Then d_C is locally DC on $X \setminus C$.*

Finally, we present an example of a distance function witnessing that Clarke subdifferential of a differentiable DC function needn't be a singleton.

Example 5.3. Let $A \subset \mathbb{R}^2$ and $B = -A \subset \mathbb{R}^2$ be Euclidean unit balls centered at $(-1, 0)$ and $(1, 0)$, respectively. Put $f := d_{A \cup B}$ which is a DC function since

$$d_{A \cup B} = d_A \wedge d_B.$$

One can observe that f is Fréchet differentiable at the origin whereas $\partial_{\mathrm{C}} f(0) = \mathrm{conv}\{\partial_{\mathrm{C}} d_A(0), \partial_{\mathrm{C}} d_B(0)\}$ is not a singleton. □

As an open problem we ask whether for all closed sets C in a Banach space the function d_C^2 is DC (locally) if the norm is sufficiently nice?

Acknowledgements. We wish to thank Libor Veselý for providing a copy of [29], Giles Auchmuty for precipitating this work by asking some pertinent questions, and Joydeep Dutta for his suggestions. We are grateful to Scott Sciffer for his ingenious proof of Theorem 2.9 by which he replaced an earlier and clumsy one. Finally we are indebted to the referees for pointing out various inaccuracies in the manuscript and for many useful suggestions.

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