Meetings with Computer Algebra and Special Functions

A Ramanujan Style Talk

Jonathan M. Borwein FRSC FAA FAAAS

Laureate Professor & Director of CARMA, Univ. of Newcastle

this talk: `http://carma.newcastle.edu.au/jon/evims.pdf`

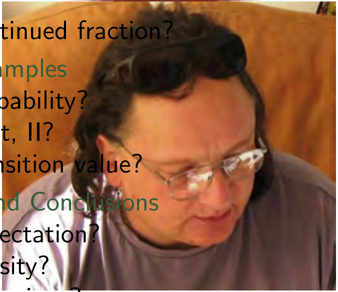
Prepared for

JonFest DownUnder, Nov 29, **30** and Dec 1, 2011

Revised Nov 20, 2012 for eViMS (23-25 November, 2012)

Companion paper and software: `http://carma.newcastle.edu.au/jon/wmi-paper.pdf`

Contents. We will cover some of the following:



Abstract

toc

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
  - In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic, numeric and graphic computation. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as gfun.

My intention is to show off the interplay between symbolic, numeric and graphic computing while exploring the various topics in my title.

Abstract

toc

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
  - In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic, numeric and graphic computation. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as `gfun`.

My intention is to show off the interplay between symbolic, numeric and graphic computing while exploring the various topics in my title.

Abstract

toc

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
  - In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic, numeric and graphic computation. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as gfun.

My intention is to show off the interplay between symbolic, numeric and graphic computing while exploring the various topics in my title.

Abstract

toc

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
  - In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic, numeric and graphic computation. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as `gfun`.

My intention is to show off the interplay between symbolic, numeric and graphic computing while exploring the various topics in my title.

Abstract

toc

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; (Carl Friedrich Gauss, 1777-1855)

- I display roughly a dozen examples where computational experimentation, computer algebra and special function theory have lead to pleasing or surprising results.
  - In the style of Ramanujan, very few proofs are given but may be found in the references.
- Much of this work requires extensive symbolic, numeric and graphic computation. It makes frequent use of the new NIST Handbook of Mathematical Functions and related tools such as `gfun`.

My intention is to show off the interplay between symbolic, numeric and graphic computing while exploring the various topics in my title.

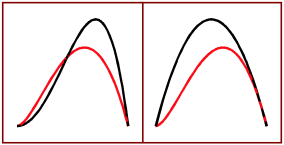
Mathodology

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

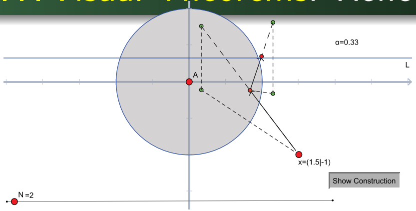

MATH LAB

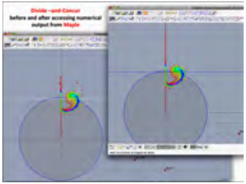
Computer experiments are transforming mathematics

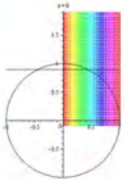


Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

. . . Visual Theorems: Reflect-Reflect-Average





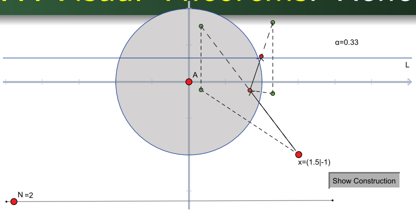


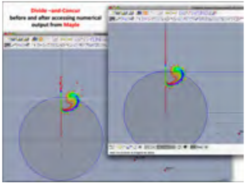
To find a point on a sphere and in an affine subspace

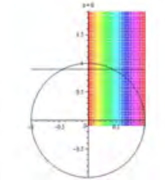
Briefly, a visual theorem is the graphical or visual output from a computer program — usually one of a family of such outputs — which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation.
— Chandler Davis, 1993, p. 333.

CARMA

. . . Visual Theorems: Reflect-Reflect-Average







To find a point on a sphere and in an affine subspace

> *Briefly, a visual theorem is the graphical or visual output from a computer program — usually one of a family of such outputs — which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation.*
> — *Chandler Davis, 1993, p. 333.*

CARMA

Congratulations to NIST

http://dlmf.nist.gov/

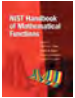
"What's fire?" "What's walking?"

DLMF: NIST is still a 19C handbook in 21C dress.

DDMF: INRIA's way of the future?

Special Functions in the 21st Century: Theory & Applications

April 6–8, 2011
Washington, DC



Objectives. The conference will provide a forum for the exchange of expertise, experience and insights among world leaders in the subject of special functions. Participants will include expert authors, editors and validators of the recently published NIST Handbook of Mathematical Functions and Digital Library of Mathematical Functions (DLMF). It will also provide an opportunity for DLMF users to interact with its creators and to explore potential areas of fruitful future developments.

Special Recognition of Professor Frank W. J. Olver. This conference is dedicated to Professor Olver in light of his seminal contributions to the advancement of special functions, especially in the area of asymptotic analysis and as Mathematics Editor of the DLMF.

F.W.J. Olver

Plenary Speakers
Richard Askey, University of Wisconsin
Michael Berry, University of Bristol
Nalini Joshi, University of Sydney, Australia
Leonard Maximon, George Washington University
William Reinhardt, University of Washington
Roderick Wong, City University of Hong Kong

Call for Contributed Talks (25 Minutes)
Abstracts may be submitted to Daniel.Lozier@nist.gov until March 15, 2011.

Registration and Financial Assistance. Registration fee: $120. Courtesy of SIAM, limited travel support is available for US-based postdoc and early career researchers. Courtesy of City University of Hong Kong and NIST, partial support is available for others in cases of need. Submit all requests for financial assistance to Daniel.Lozier@nist.gov.

Venue. Renaissance Washington Dupont Circle Hotel, 1143 New Hampshire Avenue NW, Washington, DC, 20037 USA. The conference rate is $259, available until March 15. Refreshments are supplied courtesy of University of Maryland.

Organizing Committee. Daniel Lozier, NIST, Gaithersburg, Maryland; Adri Olde Daalhuis, University of Edinburgh; Nico Temme, CWI, Amsterdam; Roderick Wong, City University of Hong Kong

To register online for the conference, and reserve a room at the conference hotel, see http://math.nist.gov/~DLozier/SF21

NIST siam CARMA

Congratulations to NIST

http://dlmf.nist.gov/

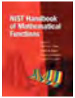
"What's fire?" "What's walking?"

DLMF: NIST is still a 19C handbook in 21C dress.

DDMF: INRIA's way of the future?

Special Functions in the 21st Century: Theory & Applications

April 6–8, 2011
Washington, DC



Objectives. The conference will provide a forum for the exchange of expertise, experience and insights among world leaders in the subject of special functions. Participants will include expert authors, editors and validators of the recently published NIST Handbook of Mathematical Functions and Digital Library of Mathematical Functions (DLMF). It will also provide an opportunity for DLMF users to interact with its creators and to explore potential areas of fruitful future developments.

Special Recognition of Professor Frank W. J. Olver. This conference is dedicated to Professor Olver in light of his seminal contributions to the advancement of special functions, especially in the area of asymptotic analysis and as Mathematics Editor of the DLMF.

F.W.J. Olver

Plenary Speakers
Richard Askey, University of Wisconsin
Michael Berry, University of Bristol
Nalini Joshi, University of Sydney, Australia
Leonard Maximon, George Washington University
William Reinhardt, University of Washington
Roderick Wong, City University of Hong Kong

Call for Contributed Talks (25 Minutes)
Abstracts may be submitted to Daniel.Lozier@nist.gov until March 15, 2011.

Registration and Financial Assistance. Registration fee: $120. Courtesy of SIAM, limited travel support is available for US-based postdoc and early career researchers. Courtesy of City University of Hong Kong and NIST, partial support is available for others in cases of need. Submit all requests for financial assistance to Daniel.Lozier@nist.gov.

Venue. Renaissance Washington Dupont Circle Hotel, 1143 New Hampshire Avenue NW, Washington, DC, 20037 USA. The conference rate is $259, available until March 15. Refreshments are supplied courtesy of University of Maryland.

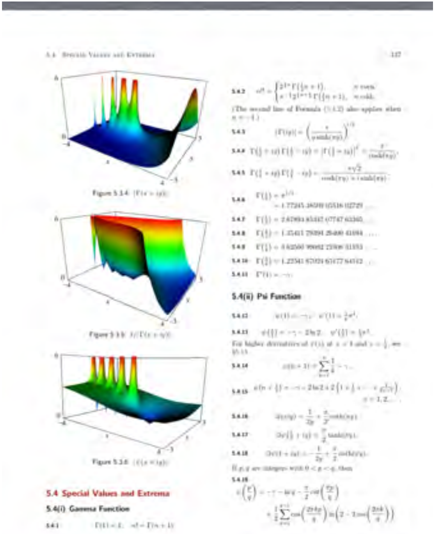
Organizing Committee. Daniel Lozier, NIST, Gaithersburg, Maryland; Adri Olde Daalhuis, University of Edinburgh; Nico Temme, CWI, Amsterdam; Roderick Wong, City University of Hong Kong

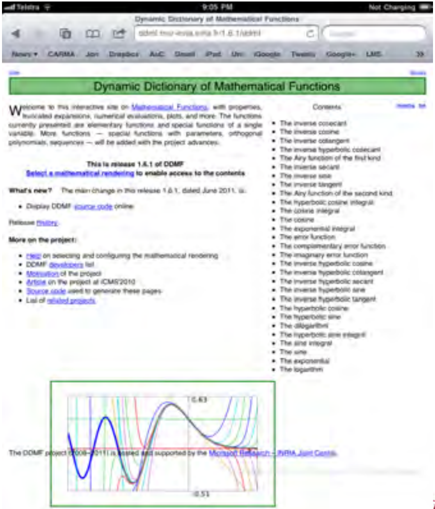
To register online for the conference, and reserve a room at the conference hotel, see http://math.nist.gov/~DLozier/SF21

NIST siam CARMA

DLMF and DDMF

http://ddmf.msr-inria.inria.fr/





Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.
2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)
   - *The Computer as Crucible* with Keith Devlin (2008).

   www.carma.newcastle.edu.au/~jb616/papers.html#BOOKS.
3. Recent results surveyed with AS in *Theor. Comp Sci* 2012:
   - http://carma.newcastle.edu.au/jon/wmi-paper.pdf
4. Exploratory experimentation: with DHB, AMS Notices Nov11
   - http://carma.newcastle.edu.au/jon/expexp.pdf

   What are closed forms: with REC, AMS Notices Jan13
   - http://carma.newcastle.edu.au/jon/closed.pdf
5. This talk and related talks are housed at www.carma.newcastle.edu.au/~jb616/papers.html#TALKS

Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.
2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)
   - *The Computer as Crucible* with Keith Devlin (2008).

   `www.carma.newcastle.edu.au/~jb616/papers.html#BOOKS`.
3. Recent results surveyed with AS in *Theor. Comp Sci* 2012:
   - `http://carma.newcastle.edu.au/jon/wmi-paper.pdf`
4. Exploratory experimentation: with DHB, AMS Notices Nov11
   - `http://carma.newcastle.edu.au/jon/expexp.pdf`

   What are closed forms: with REC, AMS Notices Jan13
   - `http://carma.newcastle.edu.au/jon/closed.pdf`
5. This talk and related talks are housed at `www.carma.newcastle.edu.au/~jb616/papers.html#TALKS`

Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.
2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)
   - *The Computer as Crucible* with Keith Devlin (2008).

   www.carma.newcastle.edu.au/~jb616/papers.html#BOOKS.
3. Recent results surveyed with AS in *Theor. Comp Sci* 2012:
   - http://carma.newcastle.edu.au/jon/wmi-paper.pdf
4. Exploratory experimentation: with DHB, AMS Notices Nov11
   - http://carma.newcastle.edu.au/jon/expexp.pdf

   What are closed forms: with REC, AMS Notices Jan13
   - http://carma.newcastle.edu.au/jon/closed.pdf
5. This talk and related talks are housed at www.carma.newcastle.edu.au/~jb616/papers.html#TALKS

CARMA

Related Work and References

1. This describes joint research with many collaborators over many years – especially DHB and REC.
2. Earlier results are to be found in the books:
   - *Mathematics by Experiment* with DHB (2004-08) and *Experimentation in Mathematics* with DHB & RG (2005)
   - *The Computer as Crucible* with Keith Devlin (2008).

   www.carma.newcastle.edu.au/~jb616/papers.html#BOOKS.
3. Recent results surveyed with AS in *Theor. Comp Sci* 2012:
   - http://carma.newcastle.edu.au/jon/wmi-paper.pdf
4. Exploratory experimentation: with DHB, AMS Notices Nov11
   - http://carma.newcastle.edu.au/jon/expexp.pdf

   What are closed forms: with REC, AMS Notices Jan13
   - http://carma.newcastle.edu.au/jon/closed.pdf
5. This talk and related talks are housed at www.carma.newcastle.edu.au/~jb616/papers.html#TALKS

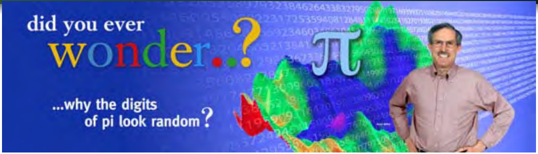
Some of my Current Collaborators (Straub, Borwein and Wan)

CARMA

La plus ça change, I

1. What is that Integral? (Bailey and Crandall)

toc



Question

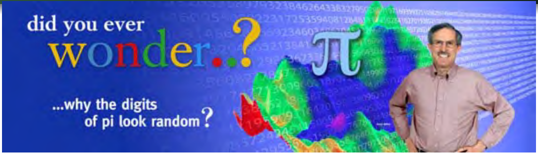
$$\int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = ??? \tag{1}$$

Remark (Kondo-Yee, 2011.)

Pi now computed to ten trillion decimal places. First four trillion hex digits appear very normal base 16 (Exp. Maths, in press). See http://carma.newcastle.edu.au/jon/normality.pdf.

1. What is that Integral? (Bailey and Crandall)

toc



Question

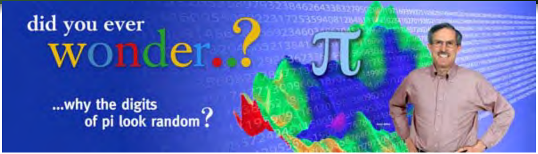
$$\int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = ??? \tag{1}$$

Remark (Kondo-Yee, 2011.)

Pi now computed to ten trillion decimal places. First four trillion hex digits appear very normal base 16 (Exp. Maths, in press). See http://carma.newcastle.edu.au/jon/normality.pdf.

1. What is that Integral? (Bailey and Crandall)

toc



Question

$$\int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx \;=\; ??? \tag{1}$$

Remark (Kondo-Yee, 2011.)

Pi now computed to ten trillion decimal places. First four trillion hex digits appear very normal base 16 (Exp. Maths, in press). See `http://carma.newcastle.edu.au/jon/normality.pdf`.

Let's be Clear: π Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* 'knows' this since

$$0 \quad < \quad \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx \;=\; \frac{22}{7} - \pi, \tag{2}$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (2) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to π. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

CARMA

Let's be Clear: π Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* 'knows' this since

$$0 \; < \; \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx \; = \; \frac{22}{7} - \pi, \tag{2}$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (2) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to π. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

Let's be Clear: π Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* 'knows' this since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \tag{2}$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (2) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to π. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad (H)$$

$$b_{n+1} = \sqrt{a_{n+1} b_n} \qquad (G)$$

These tend to π, error decreasing by a *factor of four* at each step.

- The greatest mathematician (scientist) to live before the *Enlightenment*. To compute π Archimedes had to *invent many subjects* — including numerical and interval analysis.

CARMA

Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad (H)$$

$$b_{n+1} = \sqrt{a_{n+1} b_n} \qquad (G)$$

These tend to π, error decreasing by a *factor of four* at each step.

- The greatest mathematician (scientist) to live before the *Enlightenment*. To compute π Archimedes had to *invent many subjects* — including numerical and interval analysis.

CARMA

Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_nb_n}{a_n+b_n} \qquad (H)$$

$$b_{n+1} = \sqrt{a_{n+1}b_n} \qquad (G)$$

These tend to π, error decreasing by a *factor of four* at each step.

- The greatest mathematician (scientist) to live before the *Enlightenment*. To compute π Archimedes had to *invent many subjects* — including numerical and interval analysis.

Archimedes Method circa 1800 CE

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad (H)$$

$$b_{n+1} = \sqrt{a_{n+1} b_n} \qquad (G)$$

These tend to π, error decreasing by a *factor of four* at each step.

- The greatest mathematician (scientist) to live before the *Enlightenment*. To compute π Archimedes had to *invent many subjects* — including numerical and interval analysis.

CARMA

Proving π is not $\frac{22}{7}$

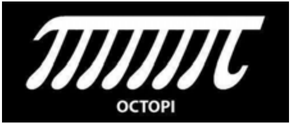
In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^{t} \frac{x^4(1-x)^4}{1+x^2}\,dx \quad = \quad \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus **proves** (2). **QED**

One can take this idea a bit further. Note that

$$\int_0^1 x^4(1-x)^4\,dx \quad = \quad \frac{1}{630}. \tag{3}$$



Proving π is not $\frac{22}{7}$

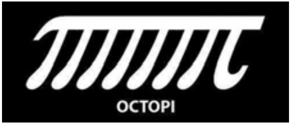
In this case, the indefinite integral provides immediate reassurance. We obtain

$$\int_0^{t} \frac{x^4\,(1-x)^4}{1+x^2}\,dx \quad = \quad \frac{1}{7}\,t^7 - \frac{2}{3}\,t^6 + t^5 - \frac{4}{3}\,t^3 + 4\,t - 4\,\arctan\,(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus **proves** (2). **QED**

One can take this idea a bit further. Note that

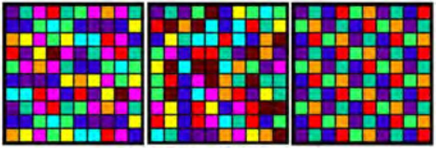
$$\int_0^1 x^4\,(1-x)^4\,dx \quad = \quad \frac{1}{630}. \qquad (3)$$



... Going Further

Hence

$$\frac{1}{2}\int_0^1 x^4\,(1-x)^4\,dx < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx < \int_0^1 x^4\,(1-x)^4\,dx.$$



Archimedes: 223/71 < π < 22/7

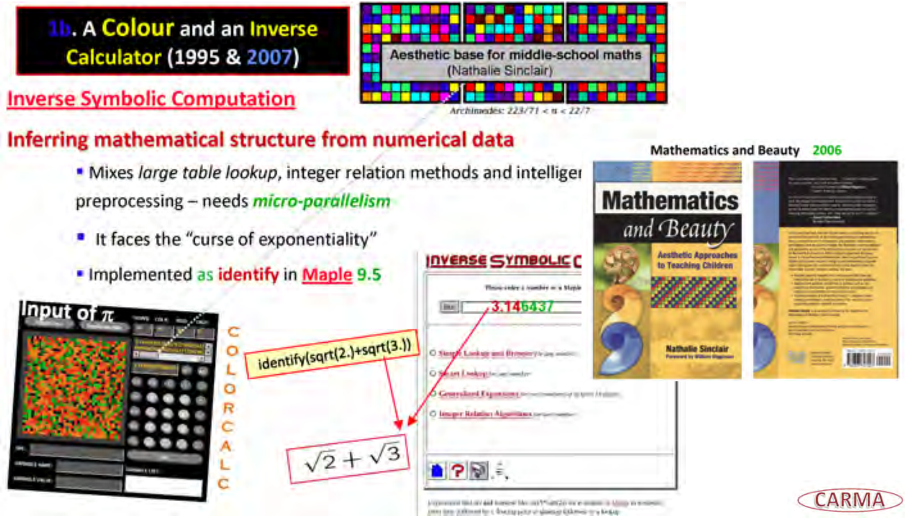
Combine this with (2) and (3) to derive:

$$223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$$

and so re-obtain Archimedes' famous

$$\mathbf{3\frac{10}{71}} < \pi < \mathbf{3\frac{10}{70}}. \tag{4}$$

Aesthetics and the Colour Calculator



Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60's Sydney exam, and traces back to **1944** (Dalziel).



Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

> *I have no satisfaction in formulas unless I feel their arithmetical magnitude.*—Baron William Thomson Kelvin
>
> In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.

Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60's Sydney exam, and traces back to **1944** (Dalziel).



Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

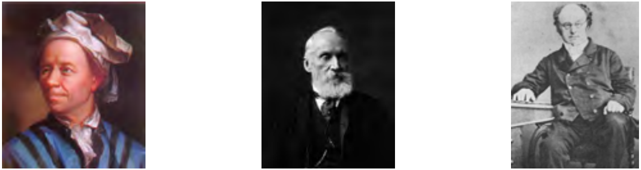
I have no satisfaction in formulas unless I feel their arithmetical magnitude.—Baron William Thomson Kelvin

In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.

Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60's Sydney exam, and traces back to **1944** (Dalziel).



Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

> *I have no satisfaction in formulas unless I feel their arithmetical magnitude.*—Baron William Thomson Kelvin
>
> In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.

CARMA

Never Trust Secondary References

- See Dalziel in *Eureka* (1971), a Cambridge student journal.
- Integral (2) was on the 1968 *Putnam*, an early 60's Sydney exam, and traces back to **1944** (Dalziel).



Leonhard Euler (1737-1787), William Kelvin (1824-1907) and Augustus De Morgan (1806-1871)

> *I have no satisfaction in formulas unless I feel their arithmetical magnitude.*—Baron William Thomson Kelvin
>
> In Lecture 7 (7 Oct 1884), of his Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light.

– Archimedes, Huygens, Riemann, De Morgan, and many others had similar sentiments.

2. BBP Digit Extraction Formulas

toc

Algorithm (What We Did, January to March 2011)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I obtained and confirmed on a 4-rack BlueGene/P system at IBM's Benchmarking Centre in Rochester, Minn, USA:

1. **106** digits of π^2 base **2** at the **ten trillion**th place base **64**
2. **94** digits of π^2 base **3** at the **ten trillion**th place base **729**
3. **141** digits of G base **2** at the **ten trillion**th place base **4096**

– G is Catalan's constant. The full computation suite took about **1500** cpu years.

– *Notices of the AMS*, in Press: `http://www.carma.newcastle.edu.au/~jb616/bbp-bluegene.pdf`

2. BBP Digit Extraction Formulas

toc

Algorithm (What We Did, January to March 2011)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I obtained and confirmed on a 4-rack BlueGene/P system at IBM's Benchmarking Centre in Rochester, Minn, USA:

1. **106** digits of π^2 base **2** at the **ten trillion**th place base **64**
2. **94** digits of π^2 base **3** at the **ten trillion**th place base **729**
3. **141** digits of G base **2** at the **ten trillion**th place base **4096**

– G is Catalan's constant. The full computation suite took about **1500** cpu years.

– *Notices of the AMS*, in Press: `http://www.carma.newcastle.edu.au/~jb616/bbp-bluegene.pdf`

What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:

- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.

What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:
- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.

What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:
- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.

CARMA

What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:

- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.

CARMA

What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:

- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
  2. requires no multiple precision software;
  3. requires very little memory; and has
  4. a computational cost growing only slightly faster than the digit position.

CARMA

What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{5}$$

- The millionth hex digit (four millionth binary digit) of π can be found in under **30** secs on a fairly new computer in Maple (not C++) and the billionth in **10** hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, **CECM**. It arrived in the coded form:

$$\pi = 4\,{}_2F_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where ${}_2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.

What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{5}$$

- The millionth hex digit (four millionth binary digit) of π can be found in under **30** secs on a fairly new computer in Maple (not C++) and the billionth in **10** hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, CECM. It arrived in the coded form:

$$\pi = 4\,{}_2\mathrm{F}_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where ${}_2\mathrm{F}_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.

CARMA

What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) \tag{5}$$

- The millionth hex digit (four millionth binary digit) of π can be found in under **30** secs on a fairly new computer in Maple (not C++) and the billionth in **10** hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, **CECM**. It arrived in the coded form:

$$\pi = 4\,{}_2F_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where ${}_2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.

CARMA

What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{5}$$

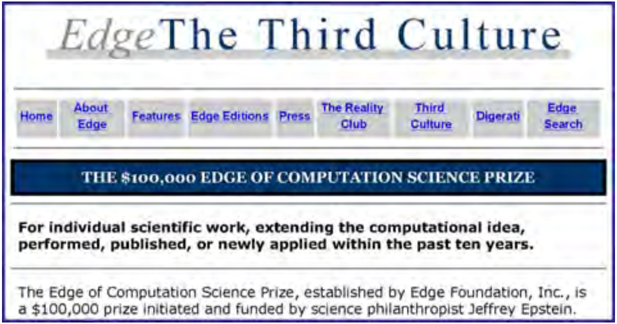
- The millionth hex digit (four millionth binary digit) of π can be found in under **30** secs on a fairly new computer in Maple (not C++) and the billionth in **10** hrs.

Equation (5) was discovered numerically using integer relation methods over months in our Vancouver lab, **CECM**. It arrived in the coded form:

$$\pi = 4\,{}_2\mathrm{F}_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

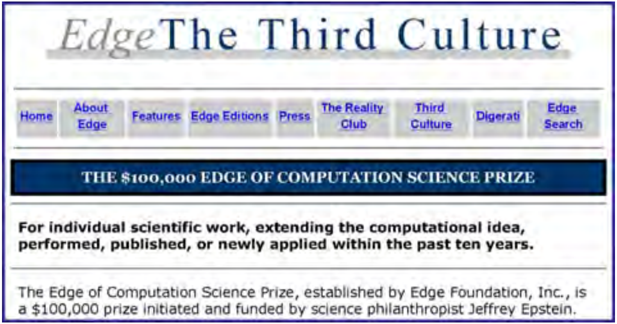
where ${}_2\mathrm{F}_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gauss hypergeometric function.

Edge of Computation Prize Finalist



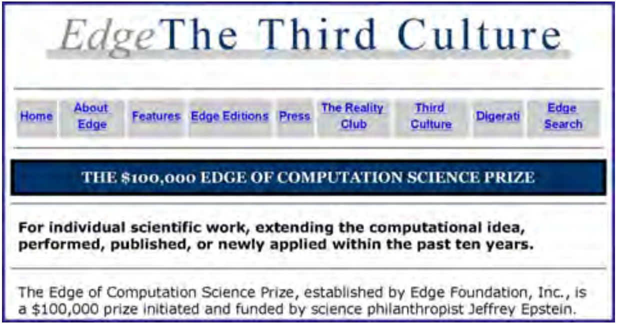
- BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
  - Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...
- Won by David Deutsch — discoverer of Quantum Computing.

Edge of Computation Prize Finalist



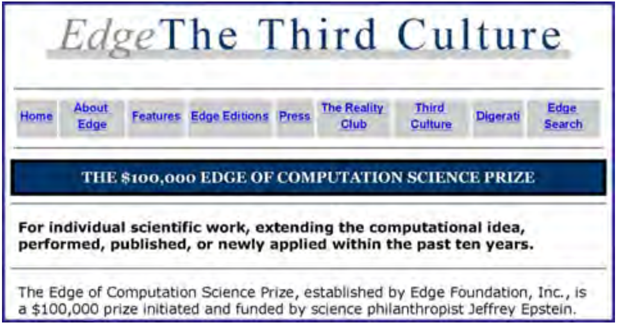
- BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
  - Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...
- Won by David Deutsch — discoverer of Quantum Computing.

Edge of Computation Prize Finalist



- BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
  - Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...
- Won by David Deutsch — discoverer of Quantum Computing.

Edge of Computation Prize Finalist



- BBP was the only mathematical finalist (of about 40) for the first **Edge of Computation Science Prize**
  - Along with founders of Google, Netscape, Celera and many brilliant thinkers, ...
- Won by David Deutsch — discoverer of Quantum Computing.

π^2 base 2 or base 3

Remarkably, both formulas below have the needed digit-extraction properties:

$$\pi^2 = \frac{9}{8}\sum_{k=0}^{\infty}\frac{1}{2^{6k}} \times \left\{\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right\}$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{6k}} \times \left\{\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right\}$$

π^2 base 2 or base 3

Remarkably, both formulas below have the needed digit-extraction properties:

$$\pi^2 = \frac{9}{8}\sum_{k=0}^{\infty}\frac{1}{2^{6k}} \times \left\{\frac{16}{(6k+1)^2}-\frac{24}{(6k+2)^2}-\frac{8}{(6k+3)^2}-\frac{6}{(6k+4)^2}+\frac{1}{(6k+5)^2}\right\}$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{6k}} \times \left\{\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}-\frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2}-\frac{72}{(12k+6)^2}-\frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right\}$$

π^2 base 2 (with DHB & IBM, 2011)

Base-64 digits of π^2 beginning at position 10 trillion.

The first run produced base-64 digits from position $10^{12} - 1$. It required an average of 253,529 secs per thread, divided into seven partitions of 2048 threads. The total cost was

$$7 \cdot 2048 \cdot 253529 = 3.6 \times 10^9 \quad \text{CPU-secs.}$$

Each IBM Blue Gene P system rack features 4096 cores, so the total cost is **10.3** "rack-days." The second run, producing digits starting from position 10^{12}, took the same time (within a few minutes).

The two resulting base-8 digit strings are

$$75|601145053032364757245000057432627545303630524163506341|573227604$$

$$xx|601145053032364757245000057432627545303630524163506341|220210566$$

(each pair of base-8 digits corresponds to a base-64 digit).

Digits in agreement are delimited by |. Note that 53 consecutive base-8 digits (159 binary digits) agree.

π^2 base 2 (with DHB & IBM, 2011)

toc

Base-64 digits of π^2 beginning at position 10 trillion.

The first run produced base-64 digits from position $10^{12} - 1$. It required an average of 253,529 secs per thread, divided into seven partitions of 2048 threads. The total cost was

$$7 \cdot 2048 \cdot 253529 = 3.6 \times 10^9 \text{ CPU-secs.}$$

Each IBM Blue Gene P system rack features 4096 cores, so the total cost is **10.3** "rack-days." The second run, producing digits starting from position 10^{12}, took the same time (within a few minutes).
The two resulting base-8 digit strings are

$$75|60114505303236475724500005743262754530363052416350634|573227604$$

$$xx|60114505303236475724500005743262754530363052416350634|220210566$$

(each pair of base-8 digits corresponds to a base-64 digit).
Digits in agreement are delimited by $|$. Note that 53 consecutive base-8 digits (159 binary digits) agree.

CARMA

π^2 base three

Base-729 digits of π^2 beginning at position 10 trillion.

Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

$$6.5 \times 10^9 \text{CPU-secs}$$

or **18.4** "rack-days" for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

$$001|12264485064548583177111135210162856048323453468|10565567635862$$

$$xxx|12264485064548583177111135210162856048323453468|04744867134524$$

(each triplet of base-9 digits corresponds to one base-729 digit).

Note that 47 consecutive base-9 digits (94 base-3 digits) agree.

π^2 base three

Base-729 digits of π^2 beginning at position 10 trillion.

Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

$$6.5 \times 10^9 \text{CPU-secs}$$

or **18.4** "rack-days" for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

$$001|12264485064548583177111135210162856048323453468|10565567635862$$

$$xxx|12264485064548583177111135210162856048323453468|04744867134524$$

(each triplet of base-9 digits corresponds to one base-729 digit).
Note that 47 consecutive base-9 digits (94 base-3 digits) agree.

π^2 base three

Base-729 digits of π^2 beginning at position 10 trillion.

Now the two runs each required an average of 795,773 seconds per thread, similarly subdivided as above, so that the total cost was

$$6.5 \times 10^9 \text{CPU-secs}$$

or **18.4** "rack-days" for each run.

- Each rack-day is approximately 11.25 years of serial computing time on one core.

The two resulting base-9 digit strings are

$$001|12264485064548583177111135210162856048323453468|10565567635862$$

$$xxx|12264485064548583177111135210162856048323453468|04744867134524$$

(each triplet of base-9 digits corresponds to one base-729 digit). Note that 47 consecutive base-9 digits (94 base-3 digits) agree.

But not π^2 base 10 or π base 3: Trojan horses

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{\left(532\,n^2 + 126\,n + 9\right)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{12n} \times \left\{ \frac{177147}{24n+1} + \frac{118098}{24n+2} + \frac{78732}{24n+5} + \frac{104976}{24n+6} + \frac{52488}{24n+7} + \frac{23328}{24n+10} + \frac{23328}{24n+11} - \frac{15552}{24n+13} - \frac{10368}{24n+14} - \frac{6912}{24n+17} - \frac{9216}{24n+18} - \frac{4608}{24n+19} - \frac{2048}{24n+22} - \frac{2048}{4n+23} \right\}.$$

- It will not work base-3 because of the **2**.

CARMA

But not π^2 base 10 or π base 3: Trojan horses

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{\left(532\,n^2 + 126\,n + 9\right)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{12n} \times \left\{ \frac{177147}{24n+1} + \frac{118098}{24n+2} + \frac{78732}{24n+5} + \frac{104976}{24n+6} + \frac{52488}{24n+7} + \frac{23328}{24n+10} + \frac{23328}{24n+11} - \frac{15552}{24n+13} - \frac{10368}{24n+14} - \frac{6912}{24n+17} - \frac{9216}{24n+18} - \frac{4608}{24n+19} - \frac{2048}{24n+22} - \frac{2048}{4n+23} \right\}.$$

- It will not work base-3 because of the **2**.

CARMA

But not π^2 base 10 or π base 3: Trojan horses

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{\left(532\,n^2 + 126\,n + 9\right)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{12n} \times \left\{ \frac{177147}{24n+1} + \frac{118098}{24n+2} + \frac{78732}{24n+5} + \frac{104976}{24n+6} + \frac{52488}{24n+7} + \frac{23328}{24n+10} + \frac{23328}{24n+11} - \frac{15552}{24n+13} - \frac{10368}{24n+14} - \frac{6912}{24n+17} - \frac{9216}{24n+18} - \frac{4608}{24n+19} - \frac{2048}{24n+22} - \frac{2048}{4n+23} \right\}.$$

- It will not work base-3 because of the **2**.

CARMA

But not π^2 base 10 or π base 3: Trojan horses

Be skeptical. Almqvist-Guillera (2011) discovered:

$$\frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{\left(532\,n^2 + 126\,n + 9\right)}{10^{6n+3}}.$$

- It will not work base-10 because of the factorial term.

Zhang (2011) discovered and proved:

$$\pi = \frac{2}{177147} \sum_{n=0}^{\infty} \left(\frac{\mathbf{2}}{3}\right)^{12n} \times \left\{ \frac{177147}{24n+1} + \frac{118098}{24n+2} + \frac{78732}{24n+5} + \frac{104976}{24n+6} + \frac{52488}{24n+7} + \frac{23328}{24n+10} + \frac{23328}{24n+11} - \frac{15552}{24n+13} - \frac{10368}{24n+14} - \frac{6912}{24n+17} - \frac{9216}{24n+18} - \frac{4608}{24n+19} - \frac{2048}{24n+22} - \frac{2048}{4n+23} \right\}.$$

- It will not work base-3 because of the **2**.

Two Sporadic Rational Gems — PSLQ, I

Gourevich **2001**

$$\frac{2^5}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{2}\right)^{6n}$$

where $a_n := a(a+1)\cdots(a+n-1)$ so that $(1)_n = n!$

Cullen **2010**

$$\frac{2^{11}}{\pi^4} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n^7 \left(\frac{3}{4}\right)_n}{(1)_n^9} (21 + 466n + 4340n^2 + 20632n^3 + 43680n^4) \left(\frac{1}{2}\right)^{12n}$$

I rediscovered and confirmed both to **10,000** digits while preparing the slide! As follows....

CARMA

Two Sporadic Rational Gems

PSLQ, I

Gourevich **2001**

$$\frac{2^5}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7}(1+14n+76n^2+168n^3)\left(\frac{1}{2}\right)^{6n}$$

where $a_n := a(a+1)\cdots(a+n-1)$ so that $(1)_n = n!$

Cullen **2010**

$$\frac{2^{11}}{\pi^4} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n^7\left(\frac{3}{4}\right)_n}{(1)_n^9}(21+466n+4340n^2+20632n^3+43680n^4)\left(\frac{1}{2}\right)^{12n}$$

I rediscovered and confirmed both to **10,000** digits while preparing the slide! As follows....

Two Sporadic Rational Gems

PSLQ, II

Discovering and validating Cullen's formula in *Maple*:

```
> Digits:=100:r:=n->p(1/4,n)*p(3/4,n)*p(1/2,n)^(7)/n!^(9);
> S4:=k-> Sum(r(n)*n^k/2^(12*n),n=0..infinity);
> normal(combine(Pslq(1/Pi^4,[seq(S4(k),k=0..4)],50)));
```

$$r := n \to \frac{p\left(\frac{1}{4}, n\right) p\left(\frac{3}{4}, n\right) p\left(\frac{1}{2}, n\right)^7}{n!^9}$$

$$S4 := k \to \sum_{n=0}^{\infty} \frac{r(n)\, n^k}{2^{12 n}}$$

[2048, 21, 466, 4340, 20632, 43680], "Error is", $-1.987\ 10^{-58}$, "checking to", 60, *places*

$$\frac{1}{\pi^4} = \sum_{n=0}^{\infty} \frac{1}{2048} \frac{p\left(\frac{1}{2}, n\right)^7 p\left(\frac{1}{4}, n\right) p\left(\frac{3}{4}, n\right) 2^{-12 n} \left(466 n + 4340 n^2 + 20632 n^3 + 43680 n^4 + 21\right)}{n!^9}$$

- Confirming the value of the sum to 10,000 places is near instant and 100,000 places took **21.35** secs.

CARMA

Two Sporadic Rational Gems

PSLQ, II

Discovering and validating Cullen's formula in *Maple*:

```
> Digits:=100:r:=n->p(1/4,n)*p(3/4,n)*p(1/2,n)^(7)/n!^(9);
> S4:=k-> Sum(r(n)*n^k/2^(12*n),n=0..infinity);
> normal(combine(Pslq(1/Pi^4,[seq(S4(k),k=0..4)],50)));
```

$$r := n \to \frac{p\left(\frac{1}{4}, n\right) p\left(\frac{3}{4}, n\right) p\left(\frac{1}{2}, n\right)^7}{n!^9}$$

$$S4 := k \to \sum_{n=0}^{\infty} \frac{r(n)\, n^k}{2^{12n}}$$

[2048, 21, 466, 4340, 20632, 43680], "Error is", $-1.987\ 10^{-58}$, "checking to", 60, *places*

$$\frac{1}{\pi^4} = \sum_{n=0}^{\infty} \frac{1}{2048} \frac{p\left(\frac{1}{2}, n\right)^7 p\left(\frac{1}{4}, n\right) p\left(\frac{3}{4}, n\right) 2^{-12n} \left(466\, n + 4340\, n^2 + 20632\, n^3 + 43680\, n^4 + 21\right)}{n!^9}$$

- Confirming the value of the sum to 10,000 places is near instant and 100,000 places took **21.35** secs.

3. What is that Sequence? $\left(\operatorname{sinc}(x) := \frac{\sin x}{x}\right)$.

For $n = 0, 1, 2, \ldots$ set

$$J_n := \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{2n+1}\right) \, \mathrm{d}x.$$

Then — as *Maple* and *Mathematica* confirm — we have:

$$J_0 = \int_{-\infty}^{\infty} \operatorname{sinc} x \, \mathrm{d}x = \pi,$$

$$J_1 = \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \, \mathrm{d}x = \pi,$$

$$\vdots$$

$$J_6 = \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{13}\right) \, \mathrm{d}x = \pi.$$

$\pi, \pi, \pi, \pi, \pi, \pi, \pi, ?$

The really obvious pattern — see Corollary below — is confounded by

$$\begin{aligned} J_7 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) \mathrm{d}x \\ &= \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi < \pi, \end{aligned}$$

where the fraction is approximately $0.99999999998529\ldots$.

1912 G. Pólya showed that given the slab

$$S_k(\theta) := \{x \in R^n \ : \ |\langle k, x\rangle| \le \theta/2,\ x \in C^n\}$$

inside the hypercube $C^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ cut off by the hyperplanes $\langle k, x\rangle = \pm\theta/2$, then

$$\mathrm{Vol}_n(S_k(\theta)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\theta x)}{x} \prod_{j=1}^{n} \frac{\sin(k_j x)}{k_j x}\, \mathrm{d}x. \qquad (6)$$

CARMA

$\pi, \pi, \pi, \pi, \pi, \pi, \pi, ?$

The really obvious pattern — see Corollary below — is confounded by

$$\begin{aligned} J_7 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) \mathrm{d}x \\ &= \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi < \pi, \end{aligned}$$

where the fraction is approximately $0.99999999998529\ldots$.

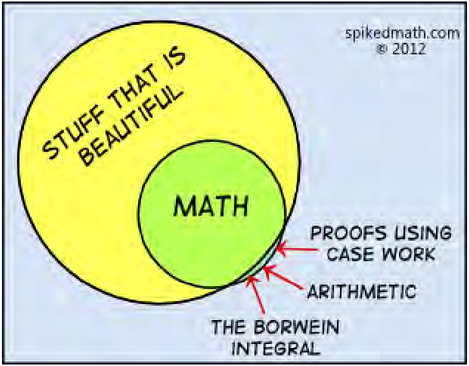
1912 G. Pólya showed that given the slab

$$S_k(\theta) := \{x \in R^n \ : \ |\langle k, x\rangle| \le \theta/2,\ x \in C^n\}$$

inside the hypercube $C^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ cut off by the hyperplanes $\langle k, x\rangle = \pm\theta/2$, then

$$\operatorname{Vol}_n(S_k(\theta)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\theta x)}{x} \prod_{j=1}^{n} \frac{\sin(k_j x)}{k_j x}\, \mathrm{d}x. \tag{6}$$

$\pi, \pi, \pi, \pi, \pi, \pi, \pi, ?$ has gone viral



- Also http://www.tumblr.com/tagged/the-borwein-integral-is-the-troll-of-calculus
- There is even a movie: http://www.qwiki.com/embed/Borwein_integral.

Mathematics is becoming Hybrid: and none to soon

5529 [1967, 1015; 1968, 914]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\text{Evaluation of } \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x-a_j)}{x-a_j}\,dx,$$

with $k_j, a_j, j=1, 2, \cdots, n$ real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the k's and be symmetric in k_j, a_j. The formula obtained in the solution

$$I = \pi \prod_{j=1}^{n} \frac{\sin k_j(a_{j-1}-a_j)}{a_{j-1}-a_j}$$

does not involve k_1 and is not symmetric as required. ($k_1=0$ must imply $I=0$.)

Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

1968 A 'solved' MAA problem.
1971 Withdrawn.
May 2011 Seemed still 'open'? (JSTOR).
Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) challenge:
`http://carma.newcastle.edu.au/jon/sink.pdf` and below:

Mathematics is becoming Hybrid: and none to soon

5529 [1967, 1015; 1968, 914]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\text{Evaluation of } \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x-a_j)}{x-a_j}\, dx,$$

with $k_j, a_j, j=1, 2, \cdots, n$ real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the k's and be symmetric in k_j, a_j. The formula obtained in the solution

$$I = \pi \prod_{j=2}^{n} \frac{\sin k_j(a_{j-1}-a_j)}{a_{j-1}-a_j}$$

does not involve k_1 and is not symmetric as required. ($k_1=0$ must imply $I=0$.)

Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

1968 A 'solved' MAA problem.
1971 Withdrawn.
May 2011 Seemed still 'open'? (JSTOR).
Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) challenge:
`http://carma.newcastle.edu.au/jon/sink.pdf` and below:

Mathematics is becoming Hybrid: and none to soon

5529 [1967, 1015; 1968, 914]. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Evaluate

$$\text{Evaluation of } \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin k_j(x-a_j)}{x-a_j}\,dx,$$

with k_j, a_j, $j=1, 2, \cdots, n$ real numbers.

Note. The published solution for this problem is in error. Murray S. Klamkin remarks that it is to be expected that the given integral depend on all the k's and be symmetric in k_j, a_j. The formula obtained in the solution

$$I = \pi \prod_{j=2}^{n} \frac{\sin k_j(a_{j-1}-a_j)}{a_{j-1}-a_j}$$

does not involve k_1 and is not symmetric as required. ($k_1=0$ must imply $I=0$.)

Accordingly the solution is withdrawn and we urge our readers to reconsider the problem.

1968 A 'solved' MAA problem.
1971 Withdrawn.
May 2011 Seemed still 'open'? (JSTOR).

Oct 2011 (MAA, Aug-Sept 2012): a fine symbolic/numeric/graphic (SNaG) challenge:
`http://carma.newcastle.edu.au/jon/sink.pdf` and below:

What has happened to J_7?

The fact that $J_0 = J_1 = \cdots = J_6 = \pi$ follows from:

Corollary (Simplest Case)

Suppose $k_1, k_2, \ldots, k_n > 0$ *and there is an index* ℓ *such that*

$$k_\ell > \frac{1}{2} \sum k_i.$$

Then, the original solution to the MONTHLY *problem is valid:*

$$I_n = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\sin(k_i(x - a_i))}{x - a_i} \mathrm{d}x = \pi \prod_{i \neq \ell} \frac{\sin(k_i(a_\ell - a_i))}{a_\ell - a_i}.$$

CARMA

What has happened to J_7?

Theorem (First bite, DB-JB 1999)

Denote $K_m = k_0 + k_1 + l, \cdots + k_m$. *If* $2k_j \geq k_n > 0$ *for* $j = 0, 1, \ldots, n-1$ *and* $K_n > 2k_0 \geq K_{n-1}$ *then*

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \frac{\sin(k_j x)}{x} \,\mathrm{d}x = \pi k_1 k_2 \cdots k_n - \frac{\pi}{2^{n-1} n!} (K_n - 2k_0)^n. \quad (7)$$

But if $2k_0 > K_n$ *the integral evaluates to* $\pi k_1 k_2 \cdots k_n$.

The theorem makes it clear that the pattern that $J_n = \pi$ for $n = 0, 1, \ldots, 6$ breaks for J_7 because

$$\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{15} > 1$$

whereas all earlier partial sums are less than 1.

What has happened to J_7?

Theorem (First bite, DB-JB 1999)

Denote $K_m = k_0 + k_1 + l, \cdots + k_m$. *If* $2k_j \geq k_n > 0$ *for* $j = 0, 1, \ldots, n-1$ *and* $K_n > 2k_0 \geq K_{n-1}$ *then*

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \frac{\sin(k_j x)}{x} \, \mathrm{d}x = \pi k_1 k_2 \cdots k_n - \frac{\pi}{2^{n-1} n!} (K_n - 2k_0)^n. \quad (7)$$

But if $2k_0 > K_n$ *the integral evaluates to* $\pi k_1 k_2 \cdots k_n$.

The theorem makes it clear that the pattern that $J_n = \pi$ for $n = 0, 1, \ldots, 6$ breaks for J_7 because

$$\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{15} > 1$$

whereas all earlier partial sums are less than 1.

Other Surprises

Theorem (Baillie-Borwein-Borwein, MAA 2008)

Suppose that $k_1, k_2, \ldots, k_n > 0$. *If* $k_1 + k_2 + \ldots + k_n < 2\pi$ *then*

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j x)\, \mathrm{d}x = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j m). \tag{8}$$

As a consequence, with $k_j = \frac{1}{2j+1}$:

Corollary

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{x}{2j+1}\right) \mathrm{d}x \geq \sum_{m=-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{m}{2j+1}\right) \tag{9}$$

with equality iff $n = 1, 2, \ldots, 7, 8, \ldots, 40248$.

Other Surprises

Theorem (Baillie-Borwein-Borwein, MAA 2008)

Suppose that $k_1, k_2, \ldots, k_n > 0$. *If* $k_1 + k_2 + \ldots + k_n < 2\pi$ *then*

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j x)\, \mathrm{d}x = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j m). \tag{8}$$

As a consequence, with $k_j = \frac{1}{2j+1}$:

Corollary

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{x}{2j+1}\right) \mathrm{d}x \geq \sum_{m=-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{m}{2j+1}\right) \tag{9}$$

with equality iff $n = 1, 2, \ldots, 7, 8, \ldots, 40248$.

Other Surprises II

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds. (John Maynard Keynes, 1883-1946)

Example (What is equality?)

- An entertaining example takes the reciprocals of primes $2, 3, 5, \ldots$: using the Prime Number theorem one estimates that the sinc integrals equal the sinc sums until the number of products is about 10^{176}.
- That of course makes it rather unlikely to find by mere testing an example where the two are unequal.
- Even worse for the naive tester is the fact that the discrepancy between integral and sum is always less than $10^{-10^{86}}$ — smaller if the Riemann hypothesis is true.

Other Surprises II

The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds. (John Maynard Keynes, 1883-1946)

Example (What is equality?)

- An entertaining example takes the reciprocals of primes $2, 3, 5, \ldots$: using the Prime Number theorem one estimates that the sinc integrals equal the sinc sums until the number of products is about 10^{176}.
- That of course makes it rather unlikely to find by mere testing an example where the two are unequal.
- Even worse for the naive tester is the fact that the discrepancy between integral and sum is always less than $10^{-10^{86}}$ — smaller if the Riemann hypothesis is true.

How to Judge a new Scientific Claim

Was the problem and solution the 'GPS'

- See http://experimentalmath.info/blog/2011/11/mathematics-and-scientific-fraud/, http://experimentalmath.info/blog/2011/06/quick-tests-for-checking-whether-a-new-math-result-is-plausible/ and http://experimentalmath.info/blog/2011/06/has-the-3n1-conjecture-been-proved/

4. What is that Number?

toc

1995: Andrew Granville emailed and challenged me to identify:

$$\alpha := 1.4331274267223\ldots \tag{10}$$

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

$$\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \tag{11}$$

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}$$

where I_0 and I_1 are *Bessel functions* of the first kind.

CARMA

4. What is that Number?

toc

1995: Andrew Granville emailed and challenged me to identify:

$$\alpha := 1.4331274267223\ldots \tag{10}$$

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

$$\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \tag{11}$$

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}$$

where I_0 and I_1 are *Bessel functions* of the first kind.

4. What is that Number?

toc

1995: Andrew Granville emailed and challenged me to identify:

$$\alpha := 1.4331274267223\ldots \tag{10}$$

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

$$\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \tag{11}$$

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}$$

where I_0 and I_1 are *Bessel functions* of the first kind.

CARMA

4. What is that Number?

1995: Andrew Granville emailed and challenged me to identify:

$$\alpha := 1.4331274267223\ldots \tag{10}$$

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

$$\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots]. \tag{11}$$

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}$$

where I_0 and I_1 are *Bessel functions* of the first kind.

CARMA

4. What is that Number?

toc

1995: Andrew Granville emailed and challenged me to identify:

$$\alpha := 1.4331274267223\ldots \tag{10}$$

I think this was a test I could have failed.

- I asked *Maple* for its continued fraction.
- In conventional concise notation I was rewarded with

$$\alpha = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...]. \tag{11}$$

- Even those unfamiliar with continued fractions, will agree the representation in (11) has structure not apparent from (10)!
- I reached for a good book on continued fractions and found

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{12}$$

where I_0 and I_1 are *Bessel functions* of the first kind.

CARMA

What is that Number?

Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In **2011** there are at least three "zero-knowledge" strategies:

1. Given (11), type "arithmetic progression", "continued fraction" into *Google*.
2. Type "$1, 4, 3, 3, 1, 2, 7, 4, 2$" into *Sloane's Encyclopedia of Integer Sequences*.[1]
3. Type the decimal digits of α into the *Inverse Symbolic Calculator*.[2]

I illustrate the results of each strategy.

---

[1] See http://www.research.att.com/~njas/sequences/.

[2] The *Inverse Symbolic Calculator* http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.

What is that Number?

Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In **2011** there are at least three "zero-knowledge" strategies:

1. Given (11), type "arithmetic progression", "continued fraction" into *Google*.
2. Type "$1, 4, 3, 3, 1, 2, 7, 4, 2$" into *Sloane's Encyclopedia of Integer Sequences*.[1]
3. Type the decimal digits of α into the *Inverse Symbolic Calculator*.[2]

I illustrate the results of each strategy.

[1] See http://www.research.att.com/~njas/sequences/.

[2] The *Inverse Symbolic Calculator* http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.

What is that Number?

Actually, I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In **2011** there are at least three "zero-knowledge" strategies:

1. Given (11), type "arithmetic progression", "continued fraction" into *Google*.
2. Type "$1, 4, 3, 3, 1, 2, 7, 4, 2$" into *Sloane's Encyclopedia of Integer Sequences*.[1]
3. Type the decimal digits of α into the *Inverse Symbolic Calculator*.[2]

I illustrate the results of each strategy.

[1] See http://www.research.att.com/~njas/sequences/.

[2] The *Inverse Symbolic Calculator* http://isc.carma.newcastle.edu.au/ was newly web-accessible in the same year, 1995.

What is that Number? Strategy 1

1. On Oct 15, 2008, on typing "arithmetic progression", "continued fraction" into *Google*, the first 3 hits were:

Continued Fraction Constant -- from Wolfram MathWorld

- 3 visits - 14/09/07Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. ...
*mathworld.wolfram.com/**ContinuedFraction**Constant.html* - 31k

HAKMEM -- *CONTINUED FRACTIONS* -- DRAFT, NOT YET PROOFED

The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is I (2/D) A/D [A+D, A+2D, A+3D, ...
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

On simple *continued fractions* with partial quotients in *arithmetic* ...

0. This means that the sequence of partial quotients of the *continued fractions* under. investigation consists of finitely many *arithmetic progressions* (with ...
www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the MathWorld entry includes

$$[A+D, A+2D, A+3D, \ldots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$

CARMA

What is that Number? Strategy 2

2. Typing the first few digits into Sloane's interface results in the response shown in the Figure on the next slide.

- In this case we are even told what the series representations of the requisite Bessel functions are.
- We are given sample code (in this entry in *Mathematica*), and we are lead to many links and references.
- The site is well moderated.
- Note also that this strategy only became viable after May 14th 2001 when the sequence was added to the database which now contains in excess of $158,000$ entries.

Sloane's Online Encyclopedia (*OEIS*)

AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,4,3,3,1,2,7,4,2

Search   Help

Search: **1, 4, 3, 3, 1, 2, 7, 4, 2**
Displaying 1-1 of 1 results found.   page
Format: long | short | internal | text   Sort: relevance | references | number   Highlight: on | off

A060997   Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ...   +2

1, 4, 3, 3, 1, 2, 7, 4, 2, 6, 7, 2, 2, 3, 1, 1, 7, 5, 8, 3, 1, 7, 1, 8, 3, 4, 5, 5, 7, 7, 3, 9, 9, 1, 8, 2, 0, 4, 3, 1, 5, 1, 2, 7, 6, 7, 9, 0, 5, 9, 8, 0, 5, 2, 3, 4, 3, 4, 4, 2, 8, 6, 3, 6, 3, 9, 4, 3, 0, 9, 1, 8, 3, 2, 5, 4, 1, 7, 2, 9, 0, 0, 1, 3, 6, 5, 0, 3, 7, 2, 6, 4, 3, 5, 7, 8, 6, 1, 1, 4, 6, 5, 9, 5, 0 (list; cons; graph; listen)

OFFSET	1,2
COMMENT	The value of this continued fraction is the ratio of two Bessel functions: BesselI(0,2)/BesselI(1,2) = A070910/A096789. Or, equivalently, to the ratio of the sums: sum_{n=0..inf} 1/(n!n!) and sum_{n=0..inf} n/(n!n!). - Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003
FORMULA	1/A052119.
EXAMPLE	c=1.4331274267223117583171834557759 ...
MATHEMATICA	RealDigits[FromContinuedFraction[Range[44]], 10, 110] [[1]] (* Or *) RealDigits[BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]] (* Or *) RealDigits[Sum[1/(n!n!), {n, 0, Infinity}] / Sum[n/(n!n!), {n, 0, Infinity}], 10, 110] [[1]]
CROSSREFS	Cf. A052119, A001053. Adjacent sequences: A060994 A060995 A060996 this_sequence A060998 A060999 A061000 Sequence in context: A016699 A060373 A090280 this_sequence A129624 A019975 A073871
KEYWORD	cons,easy,nonn
AUTHOR	Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001

CARMA

What is that Number? Strategy 3

3. If one types the decimal representation of α into the Inverse Symbolic Calculator (ISC) it returns:

Best guess: BesI(0,2)/BesI(1,2)

- Most of the functionality of the ISC is built into the `identify` function in versions of *Maple* starting with version 9.5.
- For example,

```
> identify(4.45033263602792)
```

returns

$$\sqrt{3} + e.$$

- As always, the experienced user will be able to extract more from this tool than the novice for whom the ISC will often produce more.

5. What is that Limit?

MAA Problem 10832, 2000 (Donald E. Knuth): Evaluate

$$S = \sum_{k=1}^{\infty} \left(\frac{k^k}{k! e^k} - \frac{1}{\sqrt{2\pi k}} \right).$$

Solution: Using *Maple*, we easily produced the approximation

$$S \approx -0.084069508727655996 46.$$

"Smart Lookup" in the Inverse Symbolic Calculator, yielded

$$S \approx -\frac{2}{3} - \frac{1}{\sqrt{2\pi}} \zeta\left(\frac{1}{2}\right). \tag{13}$$

- Calculations to higher precision (50 decimal digits) confirmed this approximation. Thus within a few minutes we "knew" the answer.

What is that Limit? Proof 1.

Why should such an identity hold and be provable?

- One clue was provided by the surprising speed with which *Maple* was able to calculate a high-precision value of this slowly convergent infinite sum.
- Evidently, the *Maple* software knew something that we did not. Peering under the covers, we found that *Maple* was using the Lambert W function, which is the functional inverse of $w(z) = ze^z$.
- Another clue was the appearance of $\zeta(1/2)$ in the discovered identity, together with an obvious allusion to Stirling's formula in the problem.

What is that Limit? Proof 2.

This led us to

Conjecture

$$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2\pi k}}-\frac{(1/2)_{k-1}}{(k-1)!\sqrt{2}}\right)\stackrel{?}{=}\frac{1}{\sqrt{2\pi}}\,\zeta\left(\frac{1}{2}\right), \tag{14}$$

where $(x)_n := x(x+1)\cdots(x+n-1)$.

- *Maple* successfully evaluated this summation, to the RHS.

We now needed to establish that

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{(1/2)_{k-1}}{(k-1)!\sqrt{2}}\right)=-\frac{2}{3}.$$

CARMA

What is that Limit? Proof 3.

We noted the presence of the Lambert W function,

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!}. \tag{15}$$

Since

$$\sum_{k=1}^{\infty} \frac{(1/2)_{k-1} \, z^{k-1}}{(k-1)!} = \frac{1}{\sqrt{1-z}}$$

an appeal to Abel's limit theorem showed it sufficed to prove:

Conjecture

$$\lim_{z \to 1} \left(\frac{dW(-z/e)}{dz} + \frac{1}{\sqrt{2-2z}} \right) \stackrel{?}{=} \frac{2}{3}.$$

- Again, *Maple* can be coaxed to establish the identity.

CARMA

What is that Limit? Proof 3.

We noted the presence of the Lambert W function,

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!}. \tag{15}$$

Since

$$\sum_{k=1}^{\infty} \frac{(1/2)_{k-1}\, z^{k-1}}{(k-1)!} = \frac{1}{\sqrt{1-z}}$$

an appeal to Abel's limit theorem showed it sufficed to prove:

Conjecture

$$\lim_{z \to 1} \left(\frac{dW(-z/e)}{dz} + \frac{1}{\sqrt{2-2z}} \right) \stackrel{?}{=} \frac{2}{3}.$$

- Again, *Maple* can be coaxed to establish the identity.

What is that Limit? Final thoughts.

- The above manipulations took considerable human ingenuity, in addition to symbolic manipulation and numerical discovery.
- A challenge for the next generation of mathematical computing software, is to more completely automate this class of operations.
- E.g., *Maple* does not recognize W from its Maclaurin series (15).

Figure : W on the real line

6. What is that Continued fraction?

toc

The *Ramanujan AGM continued fraction*

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$

enjoys attractive algebraic properties such as a striking arithmetic-geometric mean relation & elegant links with elliptic-function theory.

- The fraction presented a serious computational challenge, which we could not resist.

5. What is that Continued fraction? The AG fraction.

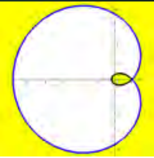


Figure : Yellow cardioid in which everything works

Theorem (AG continued fraction)

For $\eta > 0$ and complex a, b the fraction $\mathcal{R}_\eta$ converges and satisfies:

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}$$

if and only if $a/b \in \mathcal{H}$ the cardioid given by

$$\mathcal{H} := \{z \in \mathcal{C} : \left|\frac{2\sqrt{z}}{1+z}\right| < 1\}.$$

What is that Continued fraction? A hidden fractal

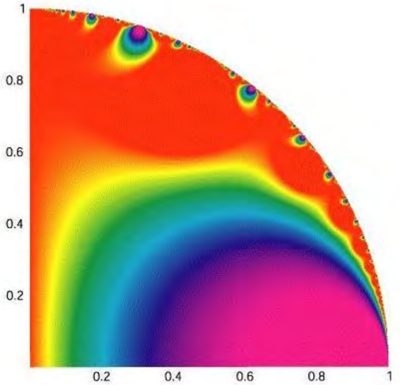


Figure : The modulus of $\theta_3(q)$

What is that Continued fraction? Closed forms, 1.

Theorem (For $a>0$)

$$\begin{aligned}
\mathcal{R}_1(a,a) &= \int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi x}{2a}\right)}{1+x^2}\,dx \\
&= 2a\sum_{k=1}^\infty \frac{(-1)^{k+1}}{1+(2k-1)a} \\
&= \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{4a}\right)-\psi\left(\frac{1}{4}+\frac{1}{4a}\right)\right) \\
&= \frac{2a}{1+a}\,F\left(\frac{1}{2a}+\frac{1}{2},1;\frac{1}{2a}+\frac{3}{2};-1\right) \quad \textit{(Gauss c.f.)} \\
&= 2\int_0^1 \frac{t^{1/a}}{1+t^2}\,dt \\
&= \int_0^\infty e^{-x/a}\operatorname{sech}(x)\,dx.
\end{aligned}$$

What is that Continued fraction? Closed forms, 2.

- This is deduced from a Riemann sum via an elliptic integral/theta-function formula.
- For $a = p/q$ rational we obtain an explicit closed form. Special cases include

$$\mathcal{R}(1) = \log 2 \qquad \text{and} \qquad \mathcal{R}\left(\frac{1}{2}\right) = 2 - \frac{\pi}{2}.$$

  - Originally, we could not compute 4 digits of these values! Now have fast methods in all of $\mathcal{C}^2$.

- For a with strictly positive (or negative) real part $\mathcal{R}(a) := \mathcal{R}_1(a)$ exists and is holomorphic.
- $\mathcal{R}(ri)$ $(r \neq 0)$ behaves chaotically with 4-fold bifurcation.
- Find a closed form for $\mathcal{R}(a, b)$ for some $a \neq b$?

CARMA

What is that Continued fraction? Closed forms, 2.

- This is deduced from a Riemann sum via an elliptic integral/theta-function formula.
- For $a = p/q$ rational we obtain an explicit closed form. Special cases include

$$\mathcal{R}(1) = \log 2 \qquad \text{and} \qquad \mathcal{R}\left(\frac{1}{2}\right) = 2 - \frac{\pi}{2}.$$

  - Originally, we could not compute 4 digits of these values! Now have fast methods in all of $\mathcal{C}^2$.

- For a with strictly positive (or negative) real part $\mathcal{R}(a) := \mathcal{R}_1(a)$ exists and is holomorphic.
- $\mathcal{R}(ri)$ $(r \neq 0)$ behaves chaotically with 4-fold bifurcation.
- Find a closed form for $\mathcal{R}(a, b)$ for some $a \neq b$?

CARMA

What is that Continued fraction? Closed forms, 3.

The first sech-integral for $\mathcal{R}(a)$ and the even Euler numbers

$$E_{2n} := (-1)^n \int_0^\infty \operatorname{sech}(\pi x/2) x^{2n}\, dx$$

yield

$$\mathcal{R}(a) \sim \sum_{n\geq 0} E_{2n}\, a^{2n+1},$$

giving an **asymptotic series of** *zero* **radius** of convergence. Here the E_{2n} commence $1, -1, 5, -61, 1385, -50521, 2702765\ldots$ Moreover, for the asymptotic error, we have:

$$\left|\mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n}\, a^{2n+1}\right| \leq |E_{2N}|\; a^{2N+1},$$

- It is a classic theorem of Borel that for every real sequence (a_n) there is a C^∞ function f on R with $f^{(n)}(0) = a_n$.
- Who knew they could be so explicit?

CARMA

What is that Continued fraction? Closed forms, 3.

The first sech-integral for $\mathcal{R}(a)$ and the even Euler numbers

$$E_{2n} := (-1)^n \int_0^\infty \operatorname{sech}(\pi x/2) x^{2n}\, dx$$

yield

$$\mathcal{R}(a) \sim \sum_{n\geq 0} E_{2n}\, a^{2n+1},$$

giving an **asymptotic series of** *zero* **radius** of convergence.
Here the E_{2n} commence $1, -1, 5, -61, 1385, -50521, 2702765 \ldots$
Moreover, for the asymptotic error, we have:

$$\left|\mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n}\, a^{2n+1}\right| \leq |E_{2N}|\; a^{2N+1},$$

- It is a classic theorem of Borel that for every real sequence (a_n) there is a C^∞ function f on R with $f^{(n)}(0) = a_n$.
- Who knew they could be so explicit?

What is that Continued fraction? Visual Dynamics

Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

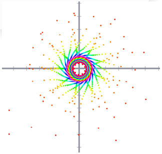
Theorem (Divergence of $\mathcal{R}$)

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd resp. (or is more general). Then $\sqrt{n}\, t_n$ is bounded $\Leftrightarrow \mathcal{R}_1(a,b)$ diverges.

Numerically all we learned is that $t_n \to 0$ slowly. Pictorially we saw more (in *Cinderella*): `http://carma.newcastle.edu.au/jon/dynamics.html` and originally in *Maple*.



What is that Continued fraction? Visual Dynamics

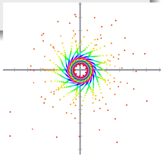
Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

Theorem (Divergence of $\mathcal{R}$)

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd resp. (or is more general). Then $\sqrt{n}\, t_n$ is bounded $\Leftrightarrow \mathcal{R}_1(a,b)$ diverges.



Numerically all we learned is that $t_n \to 0$ slowly. Pictorially we saw more (in *Cinderella*): http://carma.newcastle.edu.au/jon/dynamics.html and originally in *Maple*.

What is that Continued fraction? Visual Dynamics

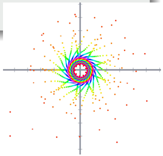
Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

Theorem (Divergence of $\mathcal{R}$)

Consider the linearised dynamical system $t_0 := t_1 := 1$*:*

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ *for* n *even, odd resp. (or is more general). Then* $\sqrt{n}\, t_n$ *is bounded* $\Leftrightarrow$ $\mathcal{R}_1(a,b)$ *diverges.*



Numerically all we learned is that $t_n \to 0$ slowly.

Pictorially we saw more (in *Cinderella*):
http://carma.newcastle.edu.au/jon/dynamics.html and originally in *Maple*.

CARMA

What is that Continued fraction? Visual Dynamics

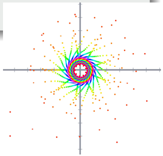
Six months after these discoveries we had a beautiful proof using genuinely new dynamical results:

Theorem (Divergence of $\mathcal{R}$)

Consider the linearised dynamical system $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd resp. (or is more general). Then $\sqrt{n}\, t_n$ is bounded $\Leftrightarrow \mathcal{R}_1(a,b)$ diverges.



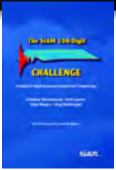
Numerically all we learned is that $t_n \to 0$ slowly. Pictorially we saw more (in *Cinderella*): `http://carma.newcastle.edu.au/jon/dynamics.html` and originally in *Maple*.

CARMA

La plus ça change, II

7. What is that Probability?

toc



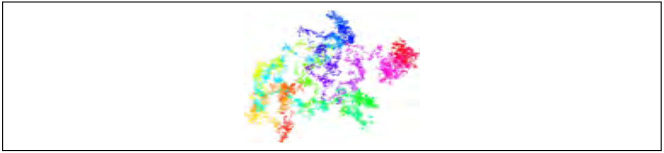
Question (SIAM 100 digit challenge, 2003)

[#10.] *A particle at the center of a* 10×1 *rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?*

- J.M. Borwein, "The SIAM 100 Digit Challenge," Extended review, *Mathematical Intelligencer*, **27** (4) (2005), 40–48. See `http://carma.newcastle.edu.au/jon/digits.pdf`.
- See also: `http://www-m3.ma.tum.de/m3old/bornemann/challengebook/index.html`.
- Image is a walk on the first two billion bits of Pi: see `http://carma.newcastle.edu.au/walks/`.

CARMA

7. What is that Probability?



Question (SIAM 100 digit challenge, 2003)

[#10.] *A particle at the center of a* 10×1 *rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?*

- J.M. Borwein, "The SIAM 100 Digit Challenge," Extended review, *Mathematical Intelligencer*, **27** (4) (2005), 40–48. See `http://carma.newcastle.edu.au/jon/digits.pdf`.
- See also: `http://www-m3.ma.tum.de/m3old/bornemann/challengebook/index.html`.
- Image is a walk on the first two billion bits of Pi: see `http://carma.newcastle.edu.au/walks/`.

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

CARMA

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

CARMA

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

What is that Probability? Bornemann's solution, 1.

Problem #10: **Hitting the Ends**.

1. Monte-Carlo methods are impracticable.
2. Reformulate *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
3. Solved with a well chosen *sparse Cholesky* solver.
4. A reliable numerical value of

$$3.837587979 \cdot 10^{-7}$$

is obtained. And the posed problem is solved numerically to the requisite ten places.

This is only the warm up.

CARMA

What is that Probability? Bornemann's solution, 2.

We develop two analytic solutions — which must agree — on a general $2a \times 2b$ rectangle:

1. Via *separation of variables* on the underlying PDE

$$p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) \tag{16}$$

   where $\rho := a/b$.

2. Using *conformal mappings*, yields

$$\operatorname{arccot} \rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \tag{17}$$

   where K is the *complete elliptic integral* of the first kind.

CARMA

What is that Probability? Bornemann's solution, 2.

We develop two analytic solutions — which must agree — on a general $2a \times 2b$ rectangle:

1. Via *separation of variables* on the underlying PDE

$$p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) \tag{16}$$

where $\rho := a/b$.

2. Using *conformal mappings*, yields

$$\operatorname{arccot} \rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \tag{17}$$

where K is the *complete elliptic integral* of the first kind.

What is that Probability? Bornemann's solution, 2.

We develop two analytic solutions — which must agree — on a general $2a \times 2b$ rectangle:

1. Via *separation of variables* on the underlying PDE

$$p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) \tag{16}$$

where $\rho := a/b$.

2. Using *conformal mappings*, yields

$$\operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \tag{17}$$

where K is the *complete elliptic integral* of the first kind.

What is that Probability? Bornemann's solution, 3.

Now (3.2.29)] in Pi&AGM shows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) = \frac{1}{2} \arcsin k_\rho \tag{18}$$

exactly when k_{ρ^2} is parameterized by *theta functions* as follows.

- As Jacobi discovered via the *nome*, $q = \exp(-\pi\rho)$:

$$k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \qquad q := e^{-\pi\rho}.$$

- Comparing (18) and (16) we see that the solution is

$$p = \frac{2}{\pi} \arcsin(k_{100}),$$
$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7}.$$

What is that Probability? Bornemann's solution, 3.

Now (3.2.29)] in Pi&AGM shows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) = \frac{1}{2} \arcsin k_\rho \qquad (18)$$

exactly when k_{ρ^2} is parameterized by *theta functions* as follows.

- As Jacobi discovered via the *nome*, $q = \exp(-\pi\rho)$:

$$k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \qquad q := e^{-\pi\rho}.$$

- Comparing (18) and (16) we see that the solution is

$$p = \frac{2}{\pi} \arcsin(k_{100}),$$

$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7}.$$

CARMA

What is that Probability? Bornemann's solution, 3.

Now (3.2.29)] in Pi&AGM shows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) = \frac{1}{2}\arcsin k_\rho \qquad (18)$$

exactly when k_{ρ^2} is parameterized by *theta functions* as follows.

- As Jacobi discovered via the *nome*, $q = \exp(-\pi\rho)$:

$$k_{\rho^2} \;=\; \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \qquad q := e^{-\pi\rho}.$$

- Comparing (18) and (16) we see that the solution is

$$p \;=\; \frac{2}{\pi}\arcsin\left(k_{100}\right),$$

$$k_{100} \;=\; 6.02806910155971082882540712292\ldots \cdot 10^{-7}.$$

CARMA

What is that Probability? Bornemann's solution, 4.

- Classical nineteenth century modular function theory tells us all rational singular values k_n are algebraic (solvable).
- Now, we can hunt in books or obtain the solution *automatically* in *Maple*: Thence

$$k_{100} := \left(\left(3-2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  - For what boundaries can one emulate this?
- In fact k_{210} was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.

What is that Probability? Bornemann's solution, 4.

- Classical nineteenth century modular function theory tells us all rational singular values k_n are algebraic (solvable).
- Now, we can hunt in books or obtain the solution *automatically* in *Maple*: Thence

$$k_{100} := \left(\left(3-2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  - For what boundaries can one emulate this?
- In fact k_{210} was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.

What is that Probability? Bornemann's solution, 4.

- Classical nineteenth century modular function theory tells us all rational singular values k_n are algebraic (solvable).
- Now, we can hunt in books or obtain the solution *automatically* in *Maple*: Thence

$$k_{100} := \left(\left(3 - 2\sqrt{2}\right)\left(2 + \sqrt{5}\right)\left(-3 + \sqrt{10}\right)\left(-\sqrt{2} + \sqrt[4]{5}\right)^2\right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  - For what boundaries can one emulate this?
- In fact k_{210} was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.

What is that Probability? Bornemann's solution, 4.

- Classical nineteenth century modular function theory tells us all rational singular values k_n are algebraic (solvable).
- Now, we can hunt in books or obtain the solution *automatically* in *Maple*: Thence

$$k_{100} := \left(\left(3-2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2$$

- No one anticipated a closed form like this, except perhaps a few harmonic analysts.
  - For what boundaries can one emulate this?
- In fact k_{210} was sent by Ramanujan to Hardy in his famous letter of introduction – if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.

CARMA

What is that Probability? A taste of Ramanujan

Srinivasa Ramanujan
(1887-1920)

MODULAR FUNCTIONS AND APPROXIMATIONS TO PI

A modular function is a function, $\lambda(q)$, that can be related through an algebraic expression called a modular equation to the same function expressed in terms of the same variable, q, raised to an integral power: $\lambda(q^p)$. The integral power, p, determines the "order" of the modular equation. An example of a modular function is

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 .$$

Its associated seventh-order modular equation, which relates $\lambda(q)$ to $\lambda(q^7)$, is given by

$$\sqrt[8]{\lambda(q)\lambda(q^7)} + \sqrt[8]{[1-\lambda(q)][1-\lambda(q^7)]} = 1 .$$

Singular values are solutions of modular equations that must also satisfy additional conditions. One class of singular values corresponds to computing a sequence of values, k_p, where

$$k_p = \sqrt{\lambda(e^{-\pi\sqrt{p}})}$$

and p takes integer values. These values have the curious property that the logarithmic expression

$$\frac{-2}{\sqrt{p}} \log\left(\frac{k_p}{4}\right)$$

coincides with many of the first digits of pi. The number of digits the expression has in common with pi increases with larger values of p.

Ramanujan was unparalleled in his ability to calculate these singular values. One of his most famous is the value when p equals 210, which was included in his original letter to G. H. Hardy. It is

$$k_{210} = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7})(\sqrt{10}-3)^2(\sqrt{15}-\sqrt{14})(4-\sqrt{15})^2(6-\sqrt{35}) .$$

This number, when plugged into the logarithmic expression, agrees with pi through the first 20 decimal places. In comparison, $k_{2^{40}}$ yields a number that agrees with pi through more than one million digits.

Applying this general approach, Ramanujan constructed a number of remarkable series for pi, including the one shown in the illustration on the preceding page. The general approach also underlies the two-step, iterative algorithms in the top illustration on the opposite page. In each iteration the first step (calculating y_n) corresponds to computing one of a sequence of singular values by solving a modular equation of the appropriate order; the second step (calculating α_n) is tantamount to taking the logarithm of the singular value.

8. What is that Limit, II?

toc

Consider:

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n := 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n} \frac{u_k-u_j}{u_k+u_j}\right)^2 dt_2\, dt_3 \cdots dt_n,$$

where (in the last line) $u_k = \prod_{i=1}^k t_i$.

- The D_n integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts)
- The C_n have tight connections to quantum field theory. Also $E_n \le D_n \le C_n$ and $E_n \sim D_n$.

CARMA

8. What is that Limit, II?

Consider:

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n := 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n} \frac{u_k - u_j}{u_k + u_j}\right)^2 dt_2\, dt_3 \cdots dt_n,$$

where (in the last line) $u_k = \prod_{i=1}^k t_i$.

- The D_n integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts)
- The C_n have tight connections to quantum field theory. Also $E_n \le D_n \le C_n$ and $E_n \sim D_n$.

8. What is that Limit, II?

toc

Consider:

$$C_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$D_n := \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

$$E_n := 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n} \frac{u_k - u_j}{u_k + u_j}\right)^2 dt_2\, dt_3 \cdots dt_n,$$

where (in the last line) $u_k = \prod_{i=1}^k t_i$.

- The D_n integrals arise in the Ising model (showing ferromagnetic temperature driven phase shifts)
- The C_n have tight connections to quantum field theory. Also $E_n \le D_n \le C_n$ and $E_n \sim D_n$.

CARMA

What is that Limit, II? — A discovery

- Fortunately, the C_n can be written as one-dim integrals:

$$C_n = \frac{2^n}{n!}\int_0^\infty pK_0^n(p)\,\mathrm{d}p,$$

where K_0 is the *modified Bessel function*.

- Computing C_n to 1000-digit (overkill) accuracy, we identified

$$C_3 = \mathrm{L}_{-3}(2) \ := \sum_{n\geq 0}\left(\frac{1}{(3n+1)^2}-\frac{1}{(3n+2)^2}\right), \quad C_4 = \frac{7}{12}\zeta(3),$$

- Here ζ is Riemann zeta. In particular

$$C_{1024} = 0.630473503374386796122040192710878904354587078712 73\ldots,$$

is the limit value to that precision. The *ISC* returned

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma},$$

where γ is *Euler's constant*. (Now proven.)

What is that Limit, II? A discovery

- Fortunately, the C_n can be written as one-dim integrals:

$$C_n = \frac{2^n}{n!} \int_0^\infty p K_0^n(p)\, \mathrm{d}p,$$

where K_0 is the *modified Bessel function*.

- Computing C_n to 1000-digit (overkill) accuracy, we identified

$$C_3 = \mathrm{L}_{-3}(2) := \sum_{n\geq 0} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right), \quad C_4 = \frac{7}{12}\zeta(3),$$

- Here ζ is Riemann zeta. In particular

$$C_{1024} = 0.6304735033743867961220401927108789043545870787127 3\ldots,$$

is the limit value to that precision. The *ISC* returned

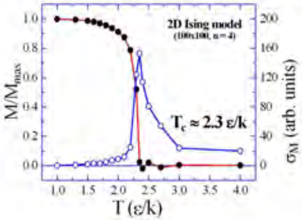
$$\lim_{n\to\infty} C_n = 2e^{-2\gamma},$$

where γ is *Euler's constant*. (Now proven.)

7. What is that Limit, II? Sterner stuff I.

For D_5, E_5, we could integrate one variable symbolically.

$$
\begin{aligned}
E_5 = {} & \int_0^1\int_0^1\int_0^1 [2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\
& (-[4(x+1)(xy+1)\log(2)(y^5z^3x^7-y^4z^2(4(y+1)z+3)x^6-y^3z((y^2+1)z^2+4(y+ \\
& 1)z+5)x^5+y^2(4y(y+1)z^3+3(y^2+1)z^2+4(y+1)z-1)x^4+y(z(z^2+4z \\
& +5)y^2+4(z^2+1)y+5z+4)x^3+((-3z^2-4z+1)y^2-4zy+1)x^2-(y(5z+4) \\
& +4)x-1)]/[(x-1)^3(xy-1)^3(xyz-1)^3]+[3(y-1)^2y^4(z-1)^2z^2(yz \\
& -1)^2x^6+2y^3z(3(z-1)^2z^3y^5+z^2(5z^3+3z^2+3z+5)y^4+(z-1)^2z \\
& (5z^2+16z+5)y^3+(3z^5+3z^4-22z^3-22z^2+3z+3)y^2+3(-2z^4+z^3+2 \\
& z^2+z-2)y+3z^3+5z^2+5z+3)x^5+y^2(7(z-1)^2z^4y^6-2z^3(z^3+15z^2 \\
& +15z+1)y^5+2z^2(-21z^4+6z^3+14z^2+6z-21)y^4-2z(z^5-6z^4-27z^3 \\
& -27z^2-6z+1)y^3+(7z^6-30z^5+28z^4+54z^3+28z^2-30z+7)y^2-2(7z^5 \\
& +15z^4-6z^3-6z^2+15z+7)y+7z^4-2z^3-42z^2-2z+7)x^4-2y(z^3(z^3 \\
& -9z^2-9z+1)y^6+z^2(7z^4-14z^3-18z^2-14z+7)y^5+z(7z^5+14z^4+3 \\
& z^3+3z^2+14z+7)y^4+(z^6-14z^5+3z^4+84z^3+3z^2-14z+1)y^3-3(3z^5 \\
& +6z^4-z^3-z^2+6z+3)y^2-(9z^4+14z^3-14z^2+14z+9)y+z^3+7z^2+7z \\
& +1)x^3+(z^2(11z^4+6z^3-66z^2+6z+11)y^6+2z(5z^5+13z^4-2z^3-2z^2 \\
& +13z+5)y^5+(11z^6+26z^5+44z^4-66z^3+44z^2+26z+11)y^4+(6z^5-4 \\
& z^4-66z^3-66z^2-4z+6)y^3-2(33z^4+2z^3-22z^2+2z+33)y^2+(6z^3+26 \\
& z^2+26z+6)y+11z^2+10z+11)x^2-2(z^2(5z^3+3z^2+3z+5)y^5+z(22z^4 \\
& +5z^3-22z^2+5z+22)y^4+(5z^5+5z^4-26z^3-26z^2+5z+5)y^3+(3z^4- \\
& 22z^3-26z^2-22z+3)y^2+(3z^3+5z^2+5z+3)y+5z^2+22z+5)x+15z^2+2z \\
& +2y(z-1)^2(z+1)+2y^3(z-1)^2z(z+1)+y^4z^2(15z^2+2z+15)+y^2(15z^4 \\
& -2z^3-90z^2-2z+15)+15]/[(x-1)^2(y-1)^2(xy-1)^2(z-1)^2(yz-1)^2 \\
& (xyz-1)^2]-[4(x+1)(y+1)(yz+1)(-z^2y^4+4z(z+1)y^3+(z^2+1)y^2 \\
& -4(z+1)y+4x(y^2-1)(y^2z^2-1)+x^2(z^2y^4-4z(z+1)y^3-(z^2+1)y^2 \\
& +4(z+1)y+1)-1)\log(x+1)]/[(x-1)^3x(y-1)^3(yz-1)^3]-[4(y+1)(xy \\
& +1)(z+1)(x^2(z^2-4z-1)y^4+4x(x+1)(z^2-1)y^3-(x^2+1)(z^2-4z-1) \\
& y^2-4(x+1)(z^2-1)y+z^2-4z-1)\log(xy+1)]/[x(y-1)^3y(xy-1)^3(z- \\
& 1)^3]-[4(z+1)(yz+1)(x^3y^5z^7+x^2y^4(4x(y+1)+5)z^6-xy^3((y^2+ \\
& 1)x^2-4(y+1)x-3)z^5-y^2(4y(y+1)x^3+5(y^2+1)x^2+4(y+1)x+1)z^4+ \\
& y(y^2x^3-4y(y+1)x^2-3(y^2+1)x-4(y+1))z^3+(5x^2y^2+y^2+4x(y+1) \\
& y+1)z^2+((3x+4)y+4)z-1)\log(xyz+1)]/[xy(z-1)^3z(yz-1)^3(xyz-1)^3])] \\
& /[(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2]\,dx\,dy\,dz
\end{aligned}
$$



What is that Limit, II? Sterner stuff, II.

- Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable D_5, E_5 hunts.
  - We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

$$
\begin{aligned}
D_2 &= 1/3, \quad D_3 = 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2), \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\
E_2 &= 6 - 8\log 2, \quad E_3 = 10 - 2\pi^2 - 8\log 2 + 32\log^2 2, \\
E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 \\
&- 22\pi^2/3.
\end{aligned}
$$

For $D_2, D_3, D_4,$ these confirmed known analytic (physics) results. Also:

$$
\begin{aligned}
E_5 &\stackrel{?}{=} 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 - 40\log 2 \\
&+ 40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2, \quad (19)
\end{aligned}
$$

where Li_4 denotes the quadra-logarithm.

What is that Limit, II? Sterner stuff, II.

- Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable D_5, E_5 hunts.
  - We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

$$\begin{aligned} D_2 &= 1/3, \quad D_3 = 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2), \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\ E_2 &= 6 - 8\log 2, \quad E_3 = 10 - 2\pi^2 - 8\log 2 + 32\log^2 2, \\ E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 \\ &- 22\pi^2/3. \end{aligned}$$

For $D_2, D_3, D_4,$ these confirmed known analytic (physics) results. Also:

$$\begin{aligned} E_5 &\stackrel{?}{=} 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 - 40\log 2 \\ &+ 40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2, \quad (19) \end{aligned}$$

where Li_4 denotes the quadra-logarithm.

CARMA

What is that Limit, II? Sterner stuff, II.

- Nonetheless, we obtained 240-digits or more on a highly parallel computer system — impossible without a dimension reduction, and needed for reliable D_5, E_5 hunts.
  - We give the integral in extenso to show the difference between a humanly accessible answer and one a computer finds useful.

In this way, we produced the following evaluations:

$$\begin{aligned} D_2 &= 1/3, \quad D_3 = 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2), \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2, \\ E_2 &= 6 - 8\log 2, \quad E_3 = 10 - 2\pi^2 - 8\log 2 + 32\log^2 2, \\ E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 \\ &- 22\pi^2/3. \end{aligned}$$

For $D_2, D_3, D_4,$ these confirmed known analytic (physics) results. Also:

$$\begin{aligned} E_5 &\stackrel{?}{=} 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 - 40\log 2 \\ &+ 40\pi^2\log^2 2 - 62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2, \quad (19) \end{aligned}$$

where Li_4 denotes the quadra-logarithm.

What is that Limit, II? — Data sets

I only understand things through examples and then gradually make them more abstract. I don't think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It's just very strange. That's the way his mind worked.
(David Mumford, 2004)

1. The form in (19) for E_5 was confirmed to 240-digit accuracy.
2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.
3. We tried but failed to recognize D_5 in terms of similar constants as described in the paper.
4. The 500-digit numerical value is accessible[3] if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

[3] http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.

What is that Limit, II? Data sets

I only understand things through examples and then gradually make them more abstract. I don't think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It's just very strange. That's the way his mind worked. (David Mumford, 2004)

1. The form in (19) for E_5 was confirmed to 240-digit accuracy.
2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.
3. We tried but failed to recognize D_5 in terms of similar constants as described in the paper.
4. The 500-digit numerical value is accessible[3] if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

[3] http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.

What is that Limit, II? — Data sets

I only understand things through examples and then gradually make them more abstract. I don't think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It's just very strange. That's the way his mind worked.
(David Mumford, 2004)

1. The form in (19) for E_5 was confirmed to 240-digit accuracy.
2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.
3. We tried but failed to recognize D_5 in terms of similar constants as described in the paper.
4. The 500-digit numerical value is accessible[3] if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

[3] http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.

CARMA

What is that Limit, II? — Data sets

I only understand things through examples and then gradually make them more abstract. I don't think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It's just very strange. That's the way his mind worked. (David Mumford, 2004)

1. The form in (19) for E_5 was confirmed to 240-digit accuracy.
2. This is 180 digits beyond the level that could be ascribed to numerical round-off; thus we are quite confident in this result.
3. We tried but failed to recognize D_5 in terms of similar constants as described in the paper.
4. The 500-digit numerical value is accessible[3] if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values.

[3] `http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf`.

9. What is that Transition value?

Example (Weakly coupling oscillators)

In an important analysis of coupled *Winfree oscillators*, Quinn, Rand, and Strogatz looked at an N-oscillator scenario whose bifurcation phase offset ϕ is implicitly defined, with a conjectured asymptotic behavior: $\sin\phi \sim 1 - c_1/N,$; and with experimental estimate $c_1 = 0.605443657\ldots$. We derived the exact value of this "QRS constant':

c_1 is the *unique zero* of the Hurwitz zeta $\zeta(1/2, z/2)$ for $z \in (0, 2)$.

- We were able to prove the conjectured behavior. Moreover, we sketched the higher-order asymptotic behavior; something that would have been impossible without discovery of an analytic formula.

9. What is that Transition value?

toc

Example (Weakly coupling oscillators)

In an important analysis of coupled *Winfree oscillators*, Quinn, Rand, and Strogatz looked at an N-oscillator scenario whose bifurcation phase offset ϕ is implicitly defined, with a conjectured asymptotic behavior: $\sin\phi \sim 1 - c_1/N,$; and with experimental estimate $c_1 = 0.605443657\ldots$. We derived the exact value of this "QRS constant':

c_1 is the *unique zero* of the Hurwitz zeta $\zeta(1/2, z/2)$ for $z \in (0, 2)$.

- We were able to prove the conjectured behavior. Moreover, we sketched the higher-order asymptotic behavior; something that would have been impossible without discovery of an analytic formula.

What is that Transition value? Chimera

PRL 101, 084103 (2008) PHYSICAL REVIEW LETTERS week ending 22 AUGUST 2008

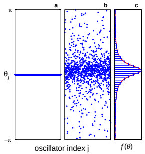


FIG. 1 (color online). Snapshot of a chimera state, obtained by numerical integration of (1) with $\beta = 0.1$, $A = 0.2$, and $N_1 = N_2 = 1024$. (a) Synchronized population. (b) Desynchronized population. (c) Density of desynchronized phases predicted by Eqs. (6) and (12) (smooth curve) agrees with observed histogram.

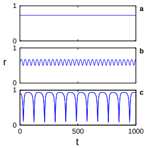


FIG. 2 (color online). Order parameter r versus time. In all three panels, $N_1 = N_2 = 128$ and $\beta = 0.1$. (a) $A = 0.2$: stable chimera; (b) $A = 0.28$: breathing chimera; (c) $A = 0.35$: long-period breather. Numerical integration began from an initial condition close to the chimera state, and plots shown begin after allowing a transient time of 2000 units.

- Does this deserve to be called a closed form?
  - Resoundingly 'yes', unless all inverse functions such as that in Bornemann's probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on *chimera* — coupled systems which self-organize in part and remain disorganized elsewhere.
  - Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.

What is that Transition value? Chimera

PRL 101, 084103 (2008) PHYSICAL REVIEW LETTERS week ending 22 AUGUST 2008

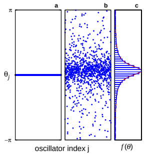


FIG. 1 (color online). Snapshot of a chimera state, obtained by numerical integration of (1) with $\beta = 0.1$, $A = 0.2$, and $N_1 = N_2 = 1024$. (a) Synchronized population. (b) Desynchronized population. (c) Density of desynchronized phases predicted by Eqs. (6) and (12) (smooth curve) agrees with observed histogram.

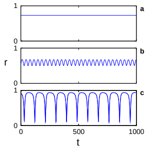


FIG. 2 (color online). Order parameter r versus time. In all three panels, $N_1 = N_2 = 128$ and $\beta = 0.1$. (a) $A = 0.2$: stable chimera; (b) $A = 0.28$: breathing chimera; (c) $A = 0.35$: long-period breather. Numerical integration began from an initial condition close to the chimera state, and plots shown begin after allowing a transient time of 2000 units.

- Does this deserve to be called a closed form?
  - Resoundingly 'yes', unless all inverse functions such as that in Bornemann's probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on *chimera* — coupled systems which self-organize in part and remain disorganized elsewhere.
  - Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.

What is that Transition value? Chimera

PRL 101, 084103 (2008) PHYSICAL REVIEW LETTERS week ending 22 AUGUST 2008

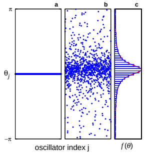


FIG. 1 (color online). Snapshot of a chimera state, obtained by numerical integration of (1) with $\beta = 0.1$, $A = 0.2$, and $N_1 = N_2 = 1024$. (a) Synchronized population. (b) Desynchronized population. (c) Density of desynchronized phases predicted by Eqs. (6) and (12) (smooth curve) agrees with observed histogram.

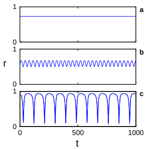


FIG. 2 (color online). Order parameter r versus time. In all three panels, $N_1 = N_2 = 128$ and $\beta = 0.1$. (a) $A = 0.2$: stable chimera; (b) $A = 0.28$: breathing chimera; (c) $A = 0.35$: long-period breather. Numerical integration began from an initial condition close to the chimera state, and plots shown begin after allowing a transient time of 2000 units.

- Does this deserve to be called a closed form?
  - Resoundingly 'yes', unless all inverse functions such as that in Bornemann's probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on *chimera* — coupled systems which self-organize in part and remain disorganized elsewhere.
  - Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.

What is that Transition value? Chimera

PRL **101**, 084103 (2008) PHYSICAL REVIEW LETTERS week ending 22 AUGUST 2008

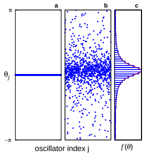


FIG. 1 (color online). Snapshot of a chimera state, obtained by numerical integration of (1) with $\beta = 0.1$, $A = 0.2$, and $N_1 = N_2 = 1024$. (a) Synchronized population. (b) Desynchronized population. (c) Density of desynchronized phases predicted by Eqs. (6) and (12) (smooth curve) agrees with observed histogram.

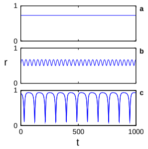


FIG. 2 (color online). Order parameter r versus time. In all three panels, $N_1 = N_2 = 128$ and $\beta = 0.1$. (a) $A = 0.2$: stable chimera; (b) $A = 0.28$: breathing chimera; (c) $A = 0.35$: long-period breather. Numerical integration began from an initial condition close to the chimera state, and plots shown begin after allowing a transient time of 2000 units.

- Does this deserve to be called a closed form?
  - Resoundingly 'yes', unless all inverse functions such as that in Bornemann's probability are to be eschewed.
- Such QRS constants are especially interesting in light of recent work by Strogatz, Lang et al on *chimera* — coupled systems which self-organize in part and remain disorganized elsewhere.
  - Now numerical limits still need a closed form.
- Often, the need for high accuracy computation drives development of effective analytic expressions which in turn shed substantial light on the subject being studied.

10. What is that Expectation? Box integrals

toc

- There is much recent research on calculation of expected distances of points inside a hypercube to the hypercube
  - or expected distances between points in a hypercube, etc.
- Some expectations $\langle|\vec{r}|\rangle$ for random $\vec{r} \in [0,1]^n$ are

Example

$n = 2$ $\frac{\sqrt{2}}{3} + \frac{1}{3}\log\left(1+\sqrt{2}\right).$

$n = 3$ $\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right).$

$n = 4$ $\frac{2}{5} - \frac{G}{10} + \frac{3}{10}\,\mathrm{Ti}_2\left(3-2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10}\arctan\left(\frac{1}{\sqrt{8}}\right).$

- Box integrals are not just a mathematician's curiosity — they are being used to assess randomness of (rat) brain synapses positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.

10. What is that Expectation? Box integrals

toc

- There is much recent research on calculation of expected distances of points inside a hypercube to the hypercube
  - or expected distances between points in a hypercube, etc.
- Some expectations $\langle|\vec{r}|\rangle$ for random $\vec{r} \in [0,1]^n$ are

Example

$$n = 2 \quad \frac{\sqrt{2}}{3} + \frac{1}{3}\log\left(1+\sqrt{2}\right).$$

$$n = 3 \quad \frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right).$$

$$n = 4 \quad \frac{2}{5} - \frac{G}{10} + \frac{3}{10}\,\mathrm{Ti}_2\left(3-2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10}\arctan\left(\frac{1}{\sqrt{8}}\right).$$

- Box integrals are not just a mathematician's curiosity — they are being used to assess randomness of (rat) brain synapses positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.

10. What is that Expectation? Box integrals

toc

- There is much recent research on calculation of expected distances of points inside a hypercube to the hypercube
  - or expected distances between points in a hypercube, etc.
- Some expectations $\langle|\vec{r}|\rangle$ for random $\vec{r} \in [0,1]^n$ are

Example

$$n = 2 \quad \frac{\sqrt{2}}{3} + \frac{1}{3}\log\left(1+\sqrt{2}\right).$$

$$n = 3 \quad \frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right).$$

$$n = 4 \quad \frac{2}{5} - \frac{G}{10} + \frac{3}{10}\operatorname{Ti}_2\left(3-2\sqrt{2}\right) + \log 3 - \frac{7\sqrt{2}}{10}\arctan\left(\frac{1}{\sqrt{8}}\right).$$

- Box integrals are not just a mathematician's curiosity — they are being used to assess randomness of (rat) brain synapses positioned within a parallelepiped. But now we (B-Crandall-Rose) wish to use Cantor Boxes.

What is that Expectation?

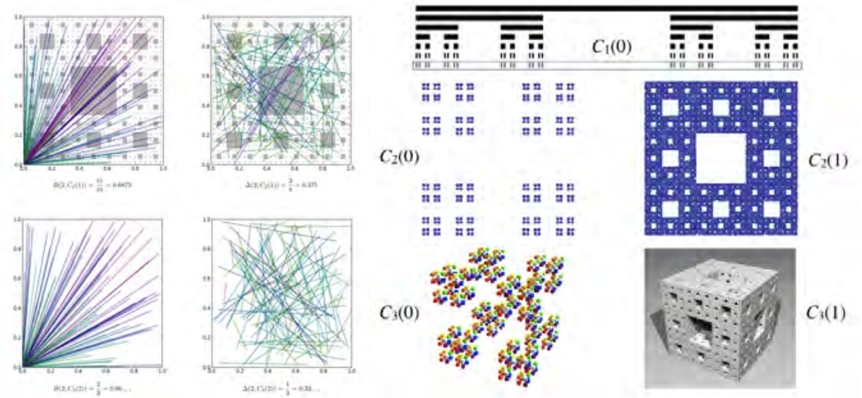


Figure : $B(2, C_2(1))$ (top-left) average squared distance of a carpet point from origin; $\Delta(2, C_1(1))$ (top-right) expected squared separation of two carpet points. Below corresponding quantities over unit square. As distance increases, colour shifts to violet end of visible spectrum)

CARMA

What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer n, and dimensions $1, 2, 3, 4, 5$ are *"hyperclosed"*.

- Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)}\, \mathrm{d}x\, \mathrm{d}y. \tag{20}$$

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0,1]^5$ can be written in explicit form involving a 10^5-character symbolic $J(3)$.
- We reduced the 5-dim box value to "only" 10^4 characters.

What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle|\vec{r}|^n\rangle$ for integer n, and dimensions $1, 2, 3, 4, 5$ are *"hyperclosed"*.

- Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)}\,\mathrm{d}x\,\mathrm{d}y. \tag{20}$$

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle|\vec{r}|^{-2}\rangle$ for $\vec{r} \in [0,1]^5$ can be written in explicit form involving a 10^5-character symbolic $J(3)$.
- We reduced the 5-dim box value to "only" 10^4 characters.

CARMA

What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle|\vec{r}|^n\rangle$ for integer n, and dimensions $1, 2, 3, 4, 5$ are *"hyperclosed"*.

- Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, dx \, dy. \tag{20}$$

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle|\vec{r}|^{-2}\rangle$ for $\vec{r} \in [0, 1]^5$ can be written in explicit form involving a 10^5-character symbolic $J(3)$.
- We reduced the 5-dim box value to "only" 10^4 characters.

What is that Dimension? Hyperclosure, 1.

A very recent result is that every box integral $\langle|\vec{r}|^n\rangle$ for integer n, and dimensions $1, 2, 3, 4, 5$ are *"hyperclosed"*.

- Five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) := \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, \mathrm{d}x \, \mathrm{d}y. \qquad (20)$$

- BCC (2011) proved hyperclosure of $J(t)$ for algebraic $t \geq 0$. Thus $\langle|\vec{r}|^{-2}\rangle$ for $\vec{r} \in [0, 1]^5$ can be written in explicit form involving a 10^5-character symbolic $J(3)$.
- We reduced the 5-dim box value to "only" 10^4 characters.

What is that Dimension? Hyperclosure, 2.

A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \quad (21)$$

— $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1}\sin(n\theta)/n^2$ a simple non-elementary Fourier series).

Thomas Clausen (**1801-1885**) learned to read at **12**.
He computed π to **247** places in **1847** using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser we are designing software to automate this process and to use before submission of any equation-rich paper: `http://www.carma.newcastle.edu.au/jon/auto.pdf`

CARMA

What is that Dimension? Hyperclosure, 2.

A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \quad (21)$$

— $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1} \sin(n\theta)/n^2$ a simple non-elementary Fourier series).

Thomas Clausen (**1801-1885**) learned to read at **12**.
He computed π to **247** places in **1847** using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser we are designing software to automate this process and to use before submission of any equation-rich paper: `http://www.carma.newcastle.edu.au/jon/auto.pdf`

What is that Dimension? Hyperclosure, 2.

A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \quad (21)$$

— $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1}\sin(n\theta)/n^2$ a simple non-elementary Fourier series).

Thomas Clausen (**1801-1885**) learned to read at **12**.
He computed π to **247** places in **1847** using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser we are designing software to automate this process and to use before submission of any equation-rich paper: http://www.carma.newcastle.edu.au/jon/auto.pdf

What is that Dimension? Hyperclosure, 2.

A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to a only a few dozen characters:

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \quad (21)$$

— $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1}\sin(n\theta)/n^2$ a simple non-elementary Fourier series).

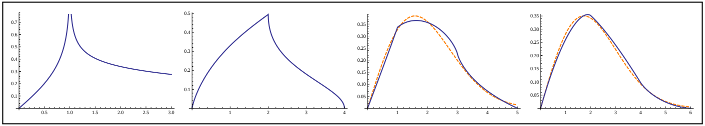
Thomas Clausen (**1801-1885**) learned to read at **12**.
He computed π to **247** places in **1847** using a Machin formula.

- Automating such reductions requires a sophisticated simplification scheme plus a very large and extensible knowledge base.
- With Alex Kaiser we are designing software to automate this process and to use before submission of any equation-rich paper: `http://www.carma.newcastle.edu.au/jon/auto.pdf`

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

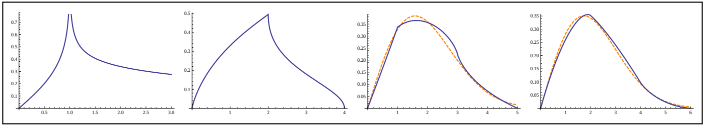
This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

CARMA

11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

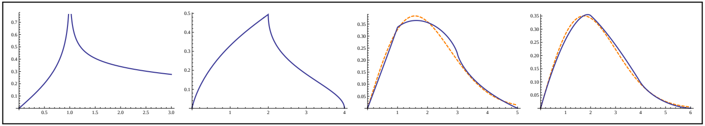
This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



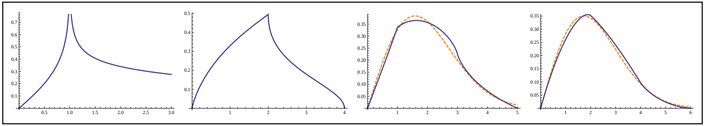
- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

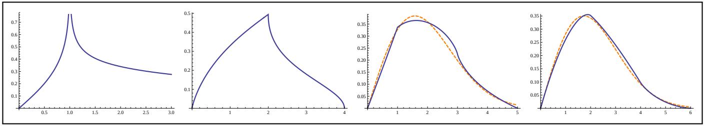
This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

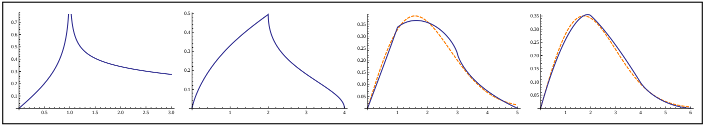
This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

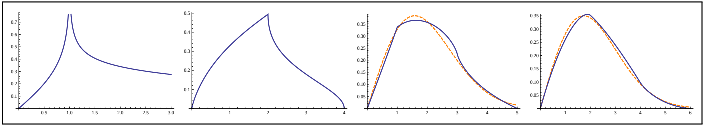
- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

CARMA

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

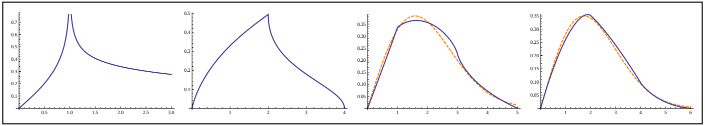
This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

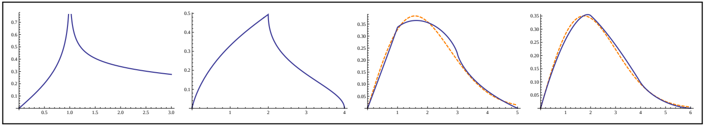
- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

CARMA

11. What is that Density?

toc

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:



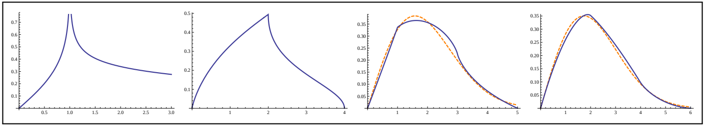
- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

11. What is that Density?

Current work with Straub, Wan and Zudilin looks at classical short uniform random walks in the plane:

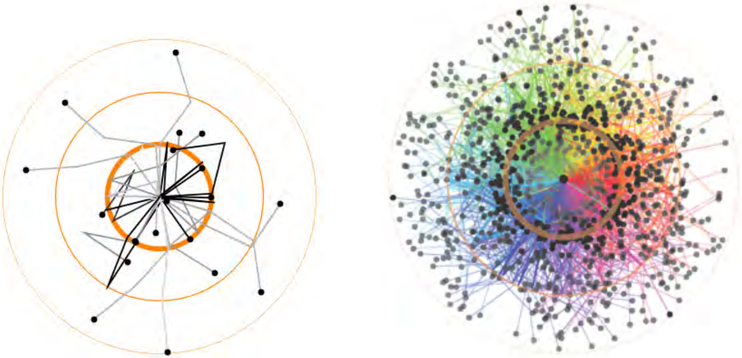


- Radial densities p_n of a random planar walk.
  - especially p_3, p_4, p_5 (as above with p_6).
- Expectations and moments $W_n(s)$.

This led Straub and JMB to make detailed study of:

- Mahler Measures $\mu(P)$ and logsin integrals
  - $\mu(1 + x_1 + \cdots x_{n-1}) = W_n'(0)$ is *known* for $n = 3, 4, 5, 6$.
- Multiple Mahler measures like $\mu_n(1 + x + y)$ and QFT.
- The next presentation describes what we know. Hidden below the surface is much use of Meijer-G functions.

Visualising Three Step Walks



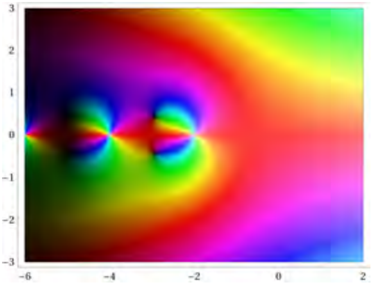
Moments of a Four Step Walk

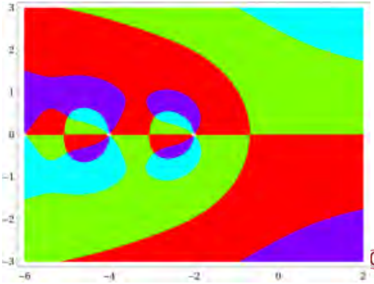
Theorem (Meijer-G form for W_4)

For $\mathrm{Re}\, s > -2$ *and* s *not an odd integer*

$$W_4(s) = \frac{2^s}{\pi} \frac{\Gamma(1+\frac{s}{2})}{\Gamma(-\frac{s}{2})} G^{22}_{44}\left(\begin{matrix} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \middle| 1 \right). \tag{22}$$

W_4 with phase colored continuously (L) and by quadrant (R)





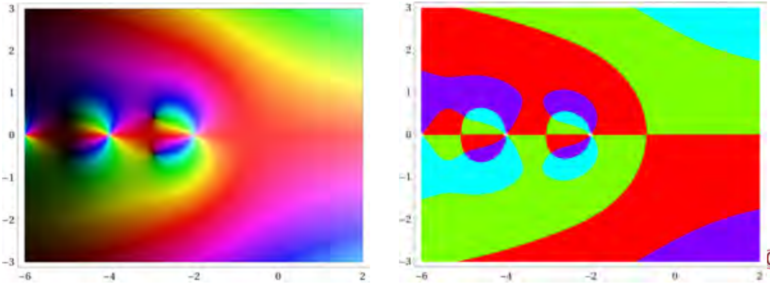
Moments of a Four Step Walk

Theorem (Meijer-G form for W_4)

For $\mathrm{Re}\, s > -2$ *and* s *not an odd integer*

$$W_4(s) = \frac{2^s}{\pi} \frac{\Gamma(1+\frac{s}{2})}{\Gamma(-\frac{s}{2})} G_{44}^{22} \left(\begin{matrix} 1, \frac{1-s}{2}, 1, 1 \\ \frac{1}{2} - \frac{s}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \middle| 1 \right). \quad (22)$$

W_4 with phase colored continuously (L) and by quadrant (R)



CARMA

Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1}, 1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

Part II (as time permits) and Conclusions

toc

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1},1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

CARMA

Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*
http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1},1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

Part II (as time permits) and Conclusions

toc

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1},1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1}, 1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1}, 1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!

Part II (as time permits) and Conclusions

Part II *Hypergeometric evaluations of the densities of short random walks*

http://carma.newcastle.edu.au/jon/densities-as.pdf

Conclusions

1. We still lack a complete accounting of $\mu_n(1+x+y)$ and are trying to resolve "the crisis of the 6th root in QFT."
2. Our log-sine and MZV algorithms uncovered many, many errors in the literature — old and new.
3. We are also filling gaps such as:
   - Euler sum values like $\zeta(\overline{2n+1}, 1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$.
4. Automated simplification, validation and correction tools are more and more important.
5. As are projects like the DDMF (INRIA's dynamic dictionary).
6. Thank you!