Digitally-assisted Discovery and Proof

ICMI Study 19
Proof and Proving in Mathematics Education
(National Taiwan Normal University, May 10-15, 2009)

Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

Laureate Professor University of Newcastle, NSW

"intuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.

Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning."

George Polya

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

Revised 08/06/09

International Commission on Mathematical Instruction

New ICMI Website

"*Mathematical proofs like diamonds should be hard and clear, and will be touched with nothing but strict reasoning.*" - John Locke

"*Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it.*" - Sir Alec Cairncross, 1996

ABSTRACT

Jonathan M. Borwein

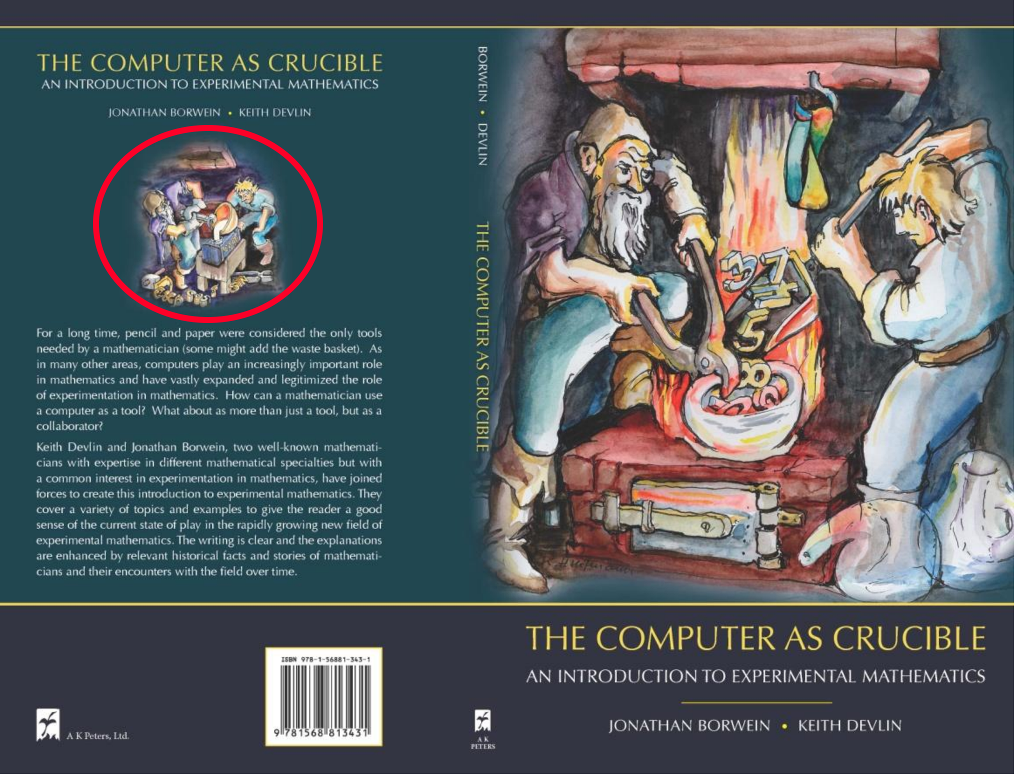
Dalhousie and

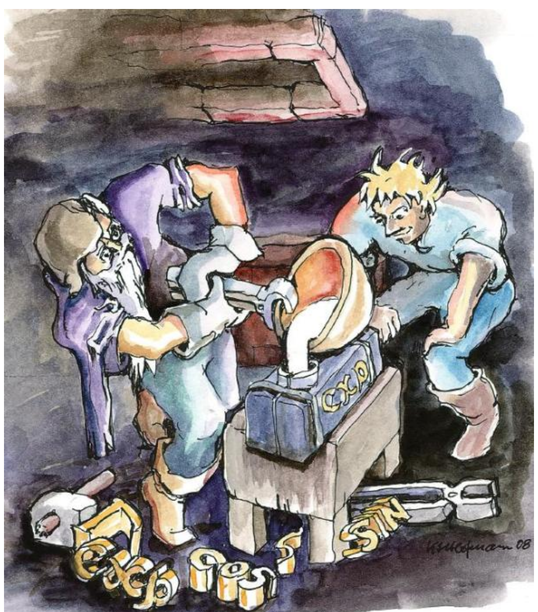
Director Newcastle Centre for

Computer Assisted Research Mathematics and its Applications (CARMA)

I will argue that the mathematical community (appropriately defined) is facing a great challenge to re-evaluate the role of proof in light of the power of current computer systems, of modern mathematical computing packages and of the growing capacity to data-mine on the internet. With great challenges come great opportunities. I intend to illustrate the current challenges and opportunities for the learning and doing of mathematics.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." – Jacques Hadamard





Jonathan Borwein
Keith Devlin
with illustrations by Karl H. Hofmann

Contents

OUTLINE

- Working Definitions of:
  - Discovery
  - Proof
  - Digital-Assistance

- Five (Tertiary) Core Examples:
  - **Number Theory**: What is that number?
  - **Calculus**: Why Pi is really not 22/7.
  - **Algebra**: Making abstract algebra concrete.
  - **Physics**: A more advanced foray into mathematical physics.
  - **Geometry**: dynamics I can visualize but have no proof of.

- Making Some Tacit Conclusions Explicit

- Additional Examples (as time permits)
  - Integer Relation Algorithms
  - Wilf-Zeilberger Summation

It won't!

WHAT is a DISCOVERY?

"discovering a truth has three components. First, there is the independence requirement, which is just that one comes to believe the proposition concerned by one's own lights, without reading it or being told. Secondly, there is the requirement that one comes to believe it in a reliable way. Finally, there is the requirement that one's coming to believe it involves no violation of one's epistemic state. …

In short , discovering a truth is coming to believe it in an independent, reliable, and rational way."

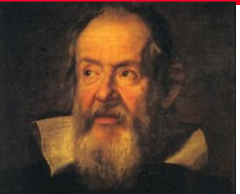
Marcus Giaquinto, Visual Thinking in Mathematics. An Epistemological Study, p. 50, OUP 2007

"All truths are easy to understand once they are discovered; the point is to discover them." – Galileo Galilei

Galileo was not alone in this view

"I will send you the proofs of the theorems in this book. Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding proofs of these same theorems.

For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge."



Archimedes to Eratosthenes in introduction to *The Method* in

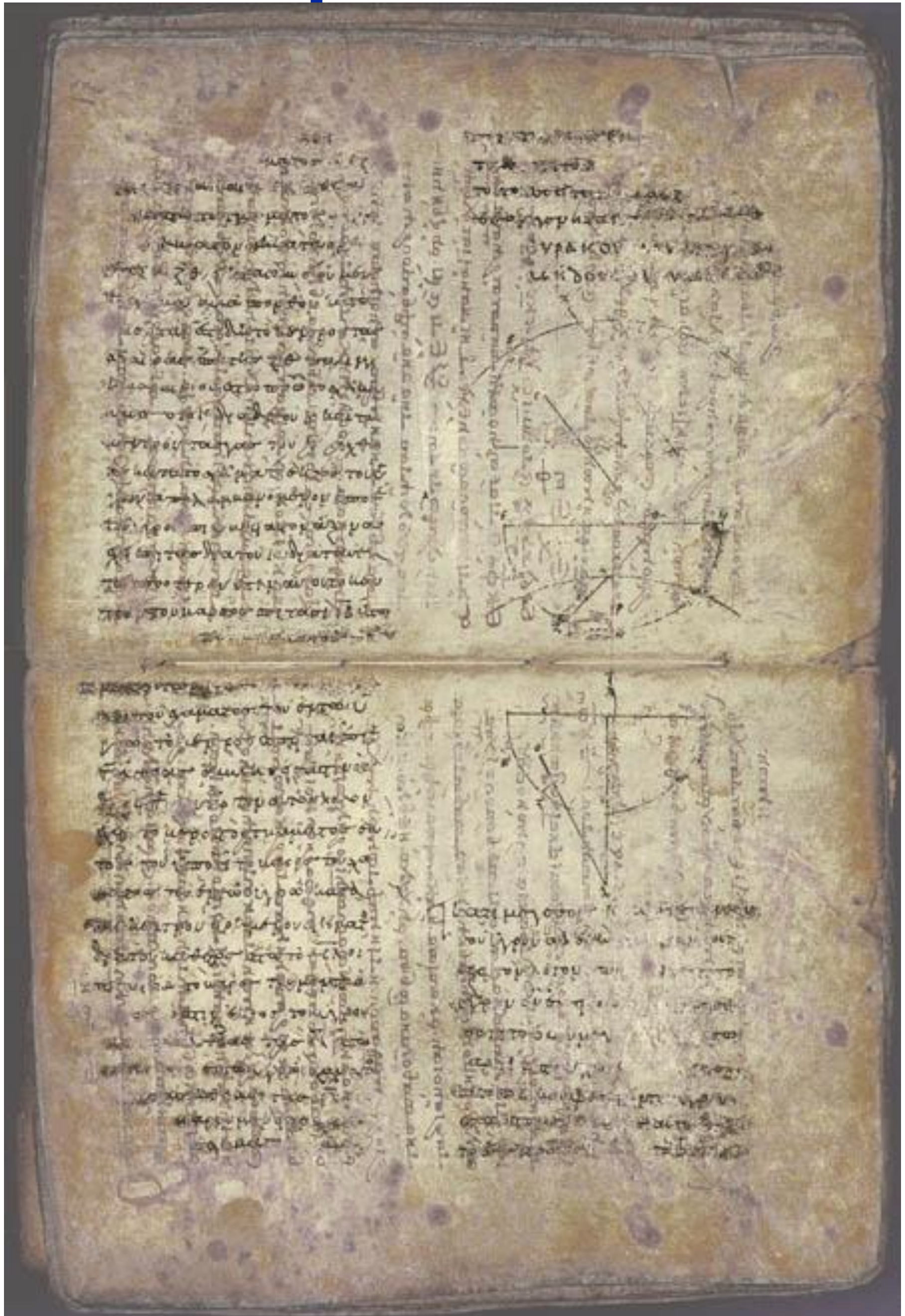
Mario Livio, *Is God a Mathematician?* Simon and Schuster, 2009

The Archimedes Palimpsest

- **1906** 10th-century palimpsest was discovered in Constantinople (Codex C). **1998** bought at auction for $2 million1**998-2008** “reconstructed”
- contained works of Archimedes that, sometime before April 14th **1229**, were partially erased, cut up, and overwritten by religious text
- after **1929** painted over with gold icons and left in a wet bucket in a garden. It included bits of 7 texts such as *On Floating Bodies* and of the *Method of Mechanical Theorems*, thought lost
- Archimedes used knowledge of levers and centres of gravity to envision ways of balancing geometric figures against one another which allowed him to compare their areas or volumes. He then used rigorous geometric argument to prove *Method* discoveries:

  "... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge." (*The Method*)
- Used Moore-Penrose inverses to reconstruct text and extract forgeries. See 2006 Google lecture at

http://video.google.com/videoplay?docid=8211813884612792878



WHAT is a PROOF?

"**PROOF,** *n.* a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the *conclusion* , is the statement of which the truth is thereby established. A *direct proof* proceeds linearly from premises to conclusion; an *indirect proof* (also called reductio ad absurdum) assumes the falsehood of the desired conclusion and shows that to be impossible. See also induction, deduction, valid. "

Borowski & JB, Collins Dictionary of Mathematics

INDUCTION , n. 3. (Logic) *a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them. Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.*

"No. I have been teaching it all my life, and I do not want to have my ideas upset."
- Isaac Todhunter (1820 - 1884) recording Maxwell's response when asked whether he would like to see an experimental demonstration of conical refraction.

Decide for yourself

WHAT is DIGITAL ASSISTANCE?

- Use of Modern Mathematical Computer Packages
  - Symbolic, Numeric, Geometric, Graphical, …
- Use of More Specialist Packages or General Purpose Languages
  - Fortran, C++, CPLEX, GAP, PARI, MAGMA, …
- Use of Web Applications
  - Sloane's Encyclopedia, Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java, Weeks' Topological Games, …
- Use of Web Databases
  - Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, DLMF, MacTutor, Amazon, …, Wolfram Alpha (??)
- All entail **data-mining** ["exploratory experimentation" and "widening technology" as in pharmacology, astrophysics, biotech… (Franklin)]
  - Clearly the boundaries are blurred and getting blurrier
  - Judgments of a given source's quality vary and are context dependent

"Knowing things is very 20th century. You just need to be able to find things."

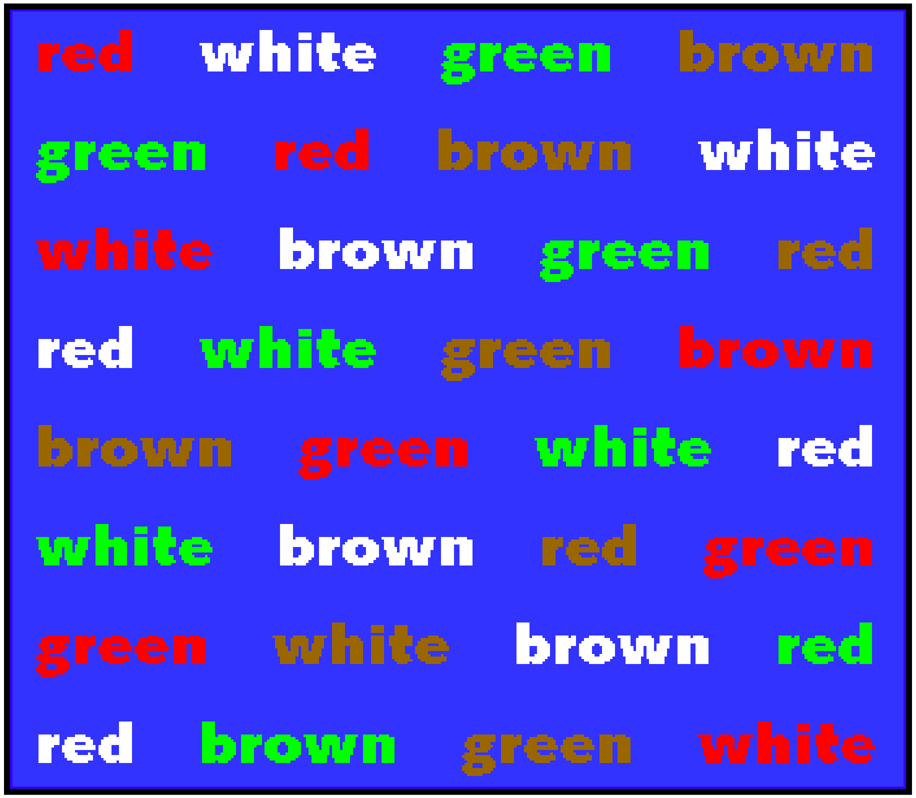
- Danny Hillis

- on how Google has already **changed how we think** in Achenblog, July 1 2008

- changing cognitive styles

Changing User Experience and Expectations

What is attention? (**Stroop** test, 1935)



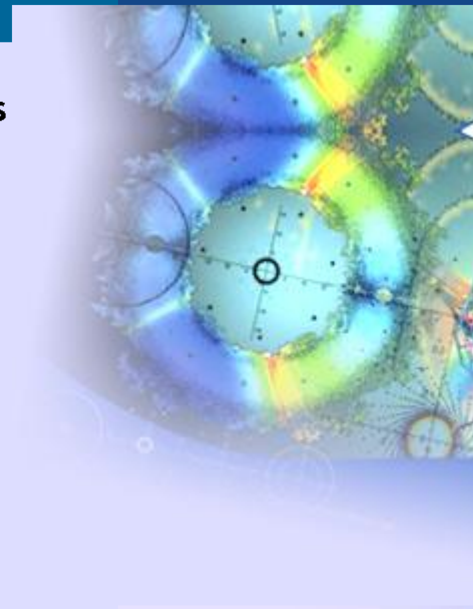
1. Say the **color** represented by the **word**.
2. Say the **color** represented by the **font** color.

High (young) multitaskers perform #2 very easily. They are great at suppressing information.

http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

Acknowledgements: Cliff Nass, CHIME lab, Stanford (interference and twitter?)

Jon Borwein's Mathematics Portal



The following is a list of useful math tools. The distinction between categories is somewhat arbitrary.

Utilities (General)

1. The On-Line Encyclopedia of Integer Sequences
2. ISC2.0: The Inverse Symbolic Calculator
3. 3D Function Grapher
4. Julia and Mandelbrot Set Explorer
5. The KnotPlot Site

Utilities (Special)

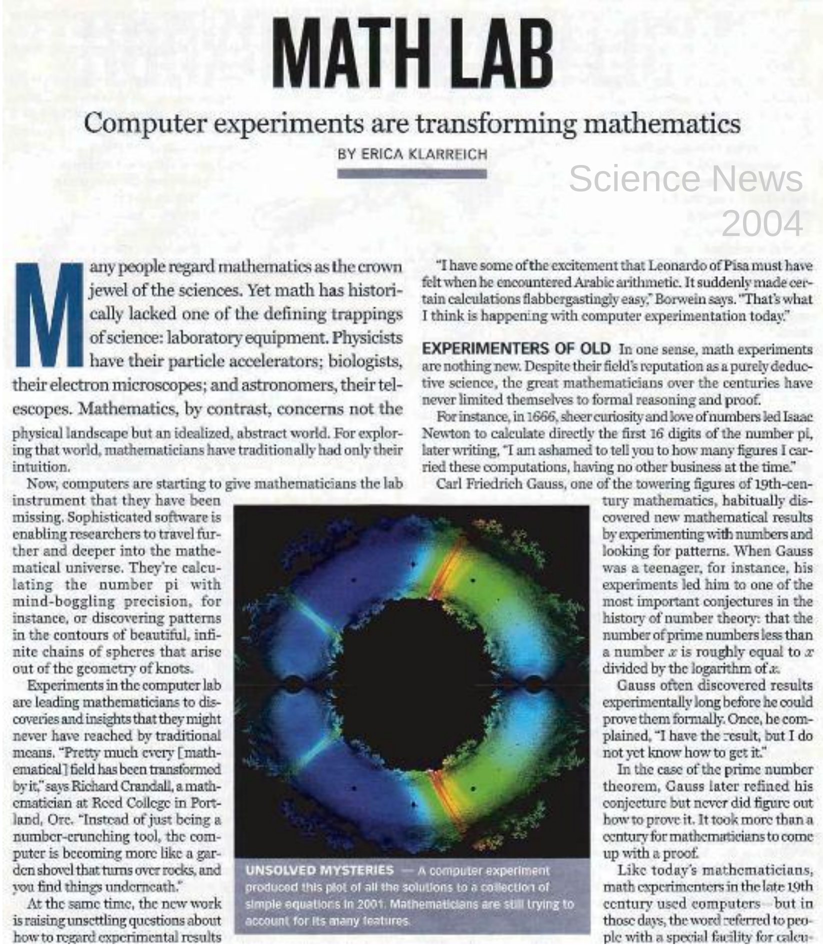
6. EZ Face : Evaluation of Euler Sums and Multiple Zeta Values
7. GraPHedron: Automated and Computer Assisted Conjectures in Graph Theory
8. Embree-Trefethen-Wright Pseudospectra and Eigenproblems
9. Symbolic and Numeric Convex Analysis Tools

Reference

10. NIST Digital Library of Mathematical Functions(X)
11. Experimental Mathematics Website
12. Numbers, Constants, and Computation
13. Numbers: the Competition
14. The Prime Pages

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Science News 2004

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

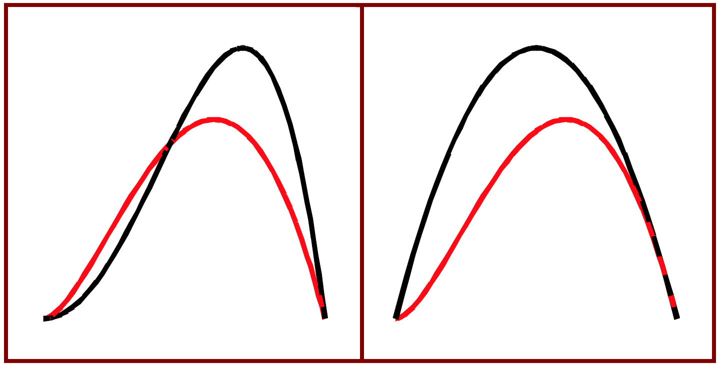
For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-



Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

Example 1. What is that number? (1995 -- 2008)

In **I995** or so Andrew Granville emailed me the number

$$\alpha := 1.433127426722312\ldots$$

and challenged me to identify it (our inverse calculator was new in those days).

Changing representations, I asked for its continued fraction? It was

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots] \qquad (1)$$

I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_0(2)}{I_1(2)}$$

where I_0 and I_1 are Bessel functions of the first kind. (Actually I knew that all arithmetic continued fractions arise in such fashion).

In 2009 there are at least three other strategies:

- Given (1), type "arithmetic progression", "continued fraction" into Google
- Type 1,4,3,3,1,2,7,4,2 into Sloane's Encyclopaedia of Integer Sequences

I illustrate the results on the next two slides:

"arithmetic progression", "continued fraction"

In Google on October 15 2008 the first three hits were

[*Continued Fraction* Constant -- from Wolfram MathWorld]

- 3 visits - 14/09/07Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. **...**
*mathworld.wolfram.com/**ContinuedFraction**Constant.html - 31k*

[HAKMEM -- *CONTINUED FRACTIONS* -- DRAFT, NOT YET PROOFED]

The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is I (2/D) A/D [A+D, A+2D, A+3D, . **...**
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

[On simple *continued fractions* with partial quotients in *arithmetic* ...]

0. This means that the sequence of partial quotients of the *continued fractions* under. investigation consists of finitely many *arithmetic progressions* (with **...**
www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the MathWorld entry includes

$$[A+D, A+2D, A+3D, \ldots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$.

Example 1: In the Integer Sequence Data Base

AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,4,3,3,1,2,7,4,2

Search Hints

Search: **1,4,3,3,1,2,7,4,2**

Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A060997 Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ... +20 5

1, 4, 3, 3, 1, 2, 7, 4, 2, 6, 7, 2, 2, 3, 1, 1, 7, 5, 8, 3, 1, 7, 1, 8, 3, 4, 5, 5, 7, 7, 5, 9, 9, 1, 8, 2, 0, 4, 3, 1, 5, 1, 2, 7, 6, 7, 9, 0, 5, 9, 8, 0, 5, 2, 3, 4, 3, 4, 4, 2, 8, 6, 3, 6, 3, 9, 4, 3, 0, 9, 1, 8, 3, 2, 5, 4, 1, 7, 2, 9, 0, 0, 1, 3, 6, 5, 0, 3, 7, 2, 6, 4, 3, 5, 7, 8, 6, 1, 1, 4, 6, 5, 9, 5, 0 (list; cons; graph; listen)

OFFSET 1,2

COMMENT The value of this continued fraction is the ratio of two Bessel functions: BesselI(0,2)/BesselI(1,2) = A070910/A096789. Or, equivalently, to the ratio of the sums: sum_{n=0..inf} 1/(n!n!) and sum_{n=0..inf} n/(n!n!). - Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003

FORMULA 1/A052119.

EXAMPLE C=1.433127426722311758317183455775 ...

MATHEMATICA RealDigits[FromContinuedFraction[Range[44]], 10, 110] [[1]]
(* Or *) RealDigits[BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]]
(* Or *) RealDigits[Sum[1/(n!n!), {n, 0, Infinity}] / Sum[n/(n!n!), {n, 0, Infinity}], 10, 110] [[1]]

CROSSREFS Cf. A052119, A001053.
Adjacent sequences: A060994 A060995 A060996 this_sequence A060998 A060999 A061000
Sequence in context: A016699 A060373 A090280 this_sequence A129624 A019975 A073871

KEYWORD cons,easy,nonn

AUTHOR Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001

The **Inverse Calculator** returns

Best guess:

BesI(0,2)/BesI(1,2)

- We show the ISC on another number next
- Most functionality of ISC is built into "identify" in Maple

"The price of metaphor is eternal vigilance." - Arturo Rosenblueth & Norbert Wiener quoted by R. C. Leowontin, *Science* p.1264, Feb 16, 2001 [Human Genome Issue].

The Inverse Symbolic Calculator (ISC) uses a combination of lookup tables and integer relation algorithms in order to associate with a user-defined, truncated decimal expansion (represented as a floating point expression) a closed form representation for the real number.

Drive NSERC CRSNG Maplesoft

The ISC in Action

isc+ inverse symbolic calculator

Standard lookup results for **12.587886229548403854**

exp(1)+Pi^2

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

3.146264370 Try it!

19.99909998 Try it!

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

The ISC presently accepts either floating point expressions or correct Maple syntax as input. However, for Maple syntax requiring too long for evaluation, a timeout has been implemented.

Visit

Jon Borwein's Webpage

David Bailey's Webpage

Math Resources Portal

- ISC+ runs on Glooscap
- Less lookup & more algorithms than 1995

Example 2. Pi and 22/7 (Year dot through 2008)

The following integral was made popular in a 1971 Eureka article

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x = \frac{22}{7} - \pi$$

• Set on a 1960 Sydney honours final it perhaps originated in 1941 with the author of the 1971 article [Dalzeil did not reference himself!]

Why trust the evaluation? Well Maple and Mathematica both 'do it'

• A better answer is to ask Maple for

$$\int_0^t \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x$$

• It will return

$$\int_0^t \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4\,t - 4\,\arctan(t)$$

and now differentiation and the Fundamental theorem of calculus proves the result.

• Not a conventional proof but a totally rigorous one. (An 'instrumental use' of the computer)

Example 3. Multivariate Zeta Values

In April 1993, Enrico Au-Yeung, then an undergraduate at the University of Waterloo, brought to my attention the result

17/360=0.047**222**..

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 k^{-2} = 4.59987\ldots \approx \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360}$$

I was very skeptical, but Parseval's identity computations affirmed this to high precision. This is a effectively a special case of the following class:

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1>n_2>\cdots>n_k>0} \prod_{j=1}^{k} n_j^{-|s_j|}\sigma_j^{-n_j},$$

where s_j are integers and σ_j= signum s_j. These can be rapidly computed as implemented at **www.cecm.sfu.ca/projects/ezface+**.

In the past 20 years they have become of more and more interest in number theory, combinatorics, knot theory and mathematical physics.

A marvellous example is Zagier's conjecture (found experimentally and now proven).

$$\zeta(\overbrace{3,1,3,1,\cdots,3,1}^{n}) = \frac{2\pi^{4n}}{(4n+2)!}$$

Example 3. Related Matrices (1993-2008)

In the course of studying such multiple zeta values we needed to obtain the closed form partial fraction decomposition for

$$\frac{1}{x^s(1-x)^t} = \sum_{j\geq 0} \frac{a_j^{s,t}}{x^j} + \sum_{j\geq 0} \frac{b_j^{s,t}}{(1-x)^j}$$

$$a_j^{s,t} = \binom{s+t-j-1}{s-j}$$

This was known to Euler but is easily discovered in Maple. We needed also to show that **M=A+B-C** was invertible where the n by n matrices A, B, C respectively had entries

$$(-1)^{k+1}\binom{2n-j}{2n-k}, \quad (-1)^{k+1}\binom{2n-j}{k-1}, \quad (-1)^{k+1}\binom{j-1}{k-1}$$

Thus, A and C are triangular and B is full.

After messing with many cases I thought to ask for M's minimal polynomial

> linalg[minpoly](M(12),t); $-2+t+t^2$

> linalg[minpoly](B(20),t); $-1+t^3$

> linalg[minpoly](A(20),t); $-1+t^2$

> linalg[minpoly](C(20),t); $-1+t^2$

$$M(6) = \begin{bmatrix} 1 & -22 & 110 & -330 & 660 & -924 \\ 0 & -10 & 55 & -165 & 330 & -462 \\ 0 & -7 & 36 & -93 & 162 & -210 \\ 0 & -5 & 25 & -56 & 78 & -84 \\ 0 & -3 & 15 & -31 & 35 & -28 \\ 0 & -1 & 5 & -10 & 10 & -6 \end{bmatrix}$$

Example 3. The Matrices Conquered

Once this was discovered proving that for all n >2

$$A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2$$

is a nice combinatorial exercise (by hand or computer). Clearly then

$$B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I$$

and the formula

$$M^{-1} = \frac{M + I}{2}$$

is again a fun exercise in formal algebra; as is confirming that we have discovered an amusing representation of the symmetric group S_3.

- characteristic and minimal polynomials --- which were rather abstract for me as a student --- now become members of a rapidly growing box of symbolic tools, as do many matrix decompositions, etc …
- a typical matrix has a full degree minimal polynomial

"Why should I refuse a good dinner simply because I don't understand the digestive processes involved?" - Oliver Heaviside (1850-1925)

Example 4. Numerical Integration (2006-2008)

The following integrals arise independently in mathematical physics in **Quantum Field Theory** and in **Ising Theory**:

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

We first showed that this can be transformed to a 1-D integral:

$$C_n = \frac{2^n}{n!}\int_0^\infty t K_0^n(t)\ dt$$

where K_0 is a **modified Bessel function**. We then computed 400-digit numerical values, from which we found these results (now proven):

$$\begin{aligned} C_3 &= \mathrm{L}_{-3}(2) = \sum_{n\geq 0}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right) \\ C_4 &= 14\zeta(3) \\ \lim_{n\to\infty} C_n &= 2e^{-2\gamma} \end{aligned}$$

The limit discovery showed the Bessel function representation to be fundamental

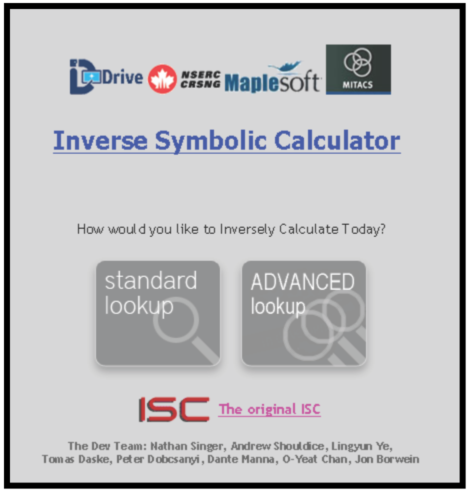
Example 4: Identifying the Limit Using the Inverse Symbolic Calculator (2.0)

We discovered the limit result as follow. We first calculated:

$$C_{1024} = 0.63047350337438679612204019271087890435458\ldots$$

We then used the Inverse Symbolic Calculator, the online numerical constant recognition facility, available at:

http://ddrive.cs.dal.ca/~isc/portal



Output: **Mixed constants, 2 with elementary transforms.**

630473503374386 = sr(2)^2/exp(gamma)^2

In other words

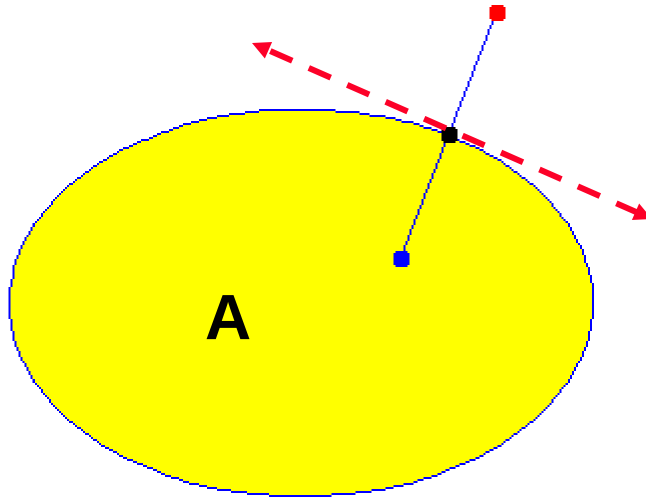
$$C_{1024} \approx 2e^{-2\gamma}$$

References. Bailey, Borwein and Crandall, "Integrals of the Ising Class," *J. Phys. A.*, **39** (2006)

Bailey, Borwein, Broadhurst and Glasser, "Elliptic integral representation of Bessel moments," *J. Phys. A,* **41** (2008) [IoP Select]

Example 5. Phase Reconstruction

Projectors and Reflectors: $P_A(x)$ is the metric projection or nearest point and $R_A(x)$ reflects in the tangent: x is red



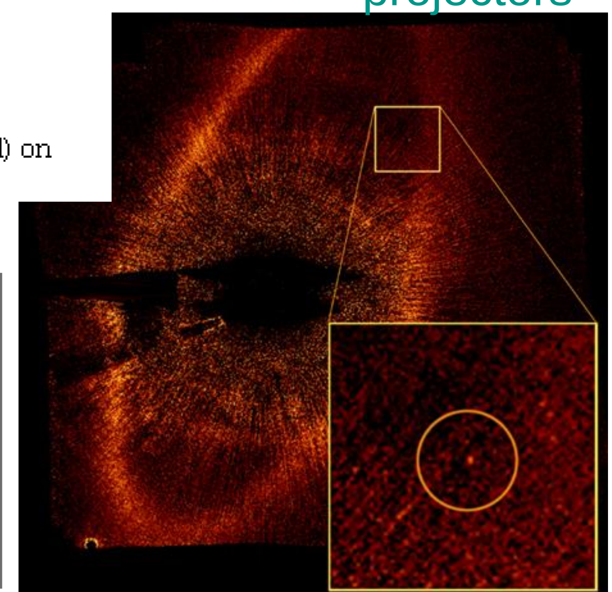
projection (black) and reflection (blue) of point (red) on boundary (blue) of ellipse (yellow)

Veit Elser, Ph.D.

2007 Solving Sudoku with reflectors

> "All physicists and a good many quite respectable mathematicians are contemptuous about proof."
> - G. H. Hardy (1877-1947)

2008 Finding exoplanet Fomalhaut in Piscis with projectors



Example 5. Why does it work?

In a wide variety of problems (protein folding, 3SAT, Sudoku) B is non-convex but “divide and concur” works better than theory can explain. It is:

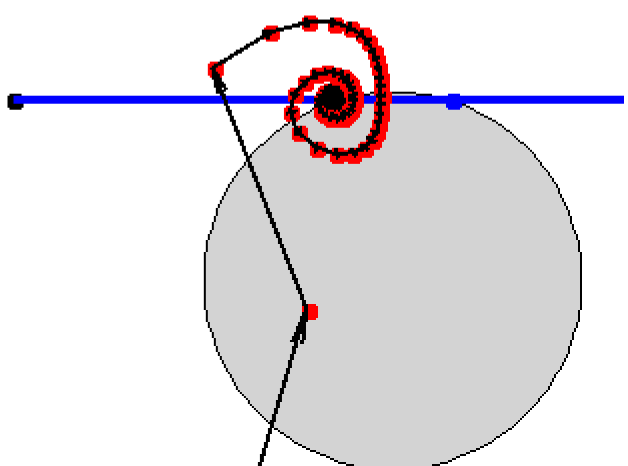
$$R_A(x) := 2\,P_A(x) - x \text{ and } x \to \frac{x + R_A(R_B(x))}{2}$$

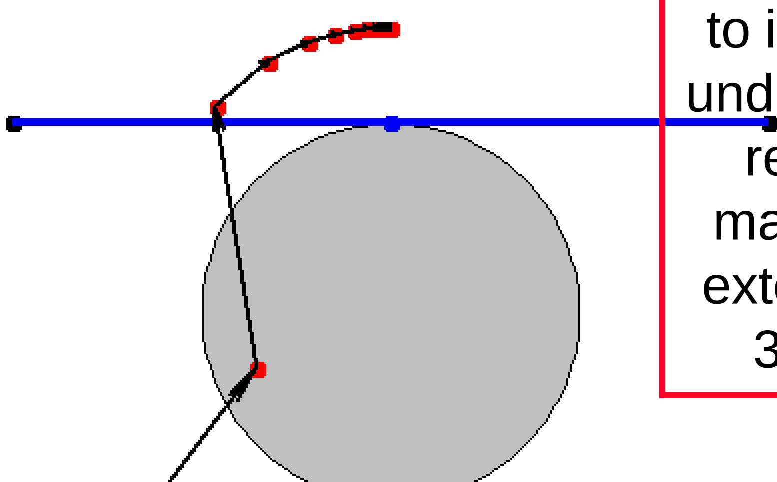
Consider the simplest case of a line A of height α and the unit circle B. With $z_n := (x_n, y_n)$ the iteration becomes

$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n, \qquad (\theta_n := \arg z_n)$$

For h=0 I can prove convergence to one of the two points in A ∩ B iff we do not start on the vertical axis (where we have chaos). For h>1 (infeasible) it is easy to see the iterates go to infinity (vertically). For h=1 we converge to an infeasible point. For h in (0,1) the pictures are lovely but proofs escape me. Two representative pictures follow:

An ideal problem to introduce early under-graduates to research, with many accessible extensions in 2 or 3 dimensions



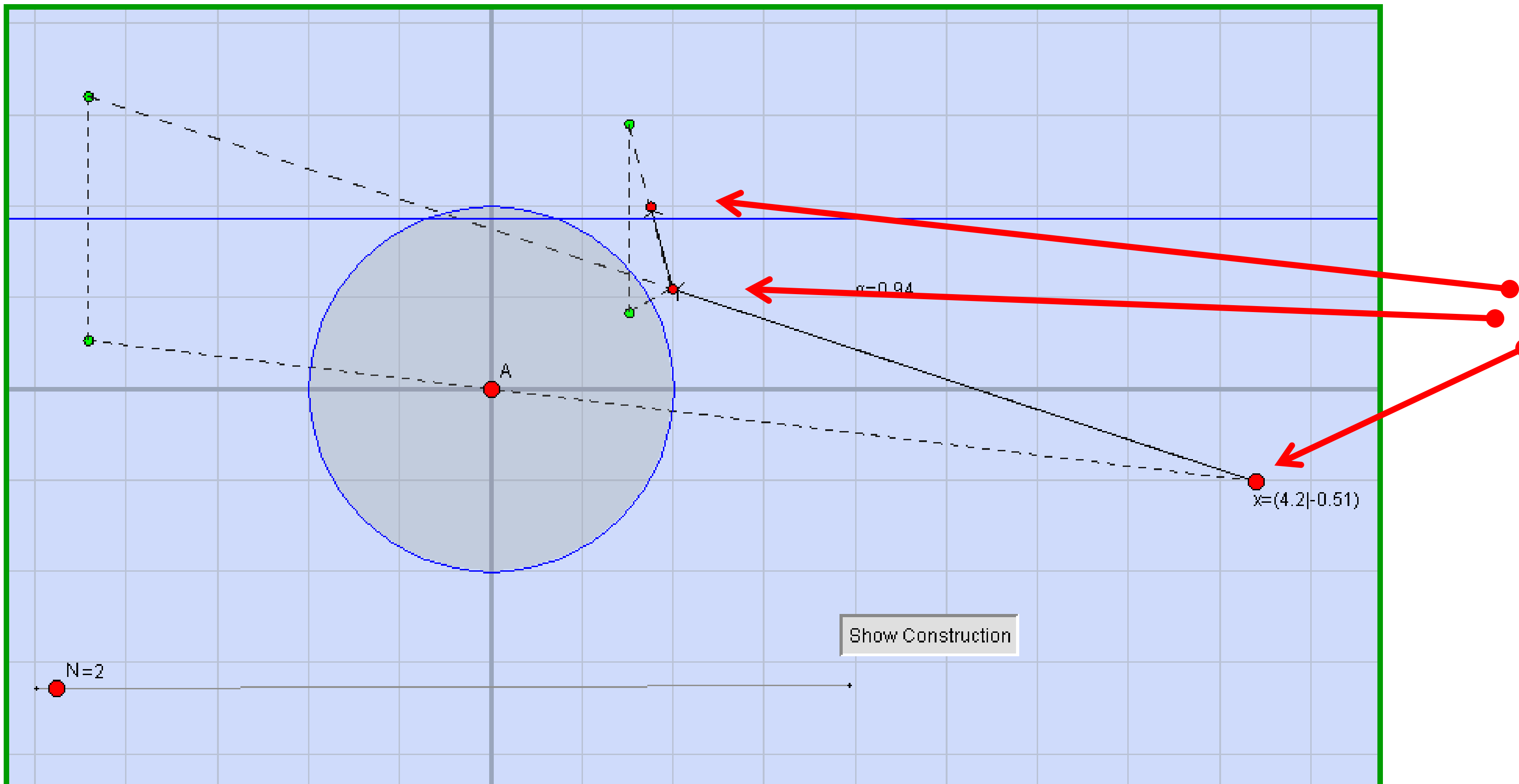


Interactive Phase Recovery in Cinderella

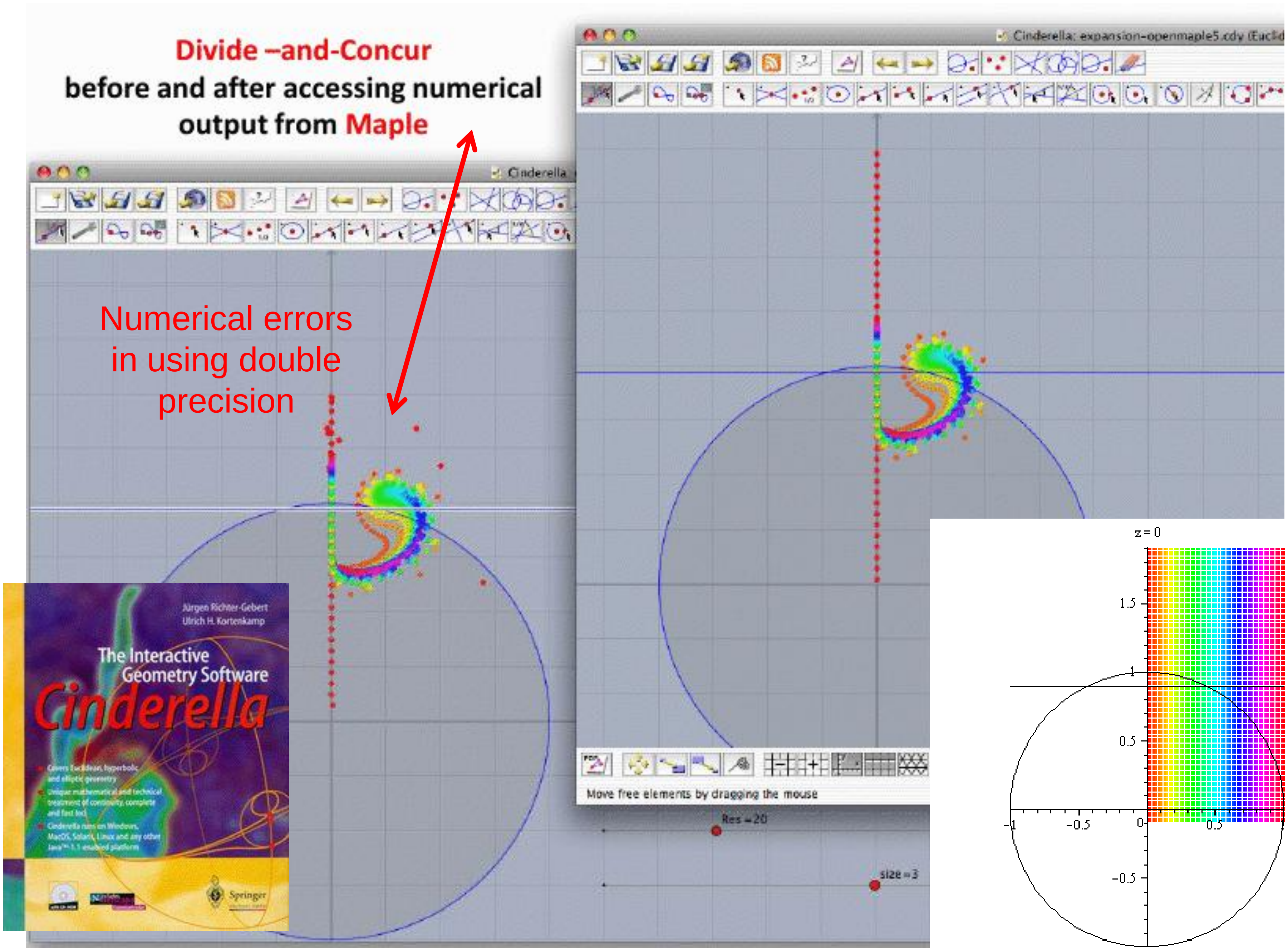
Recall the simplest case of a line A of height h and the unit circle B. With $z_n := (x_n, y_n)$ the iteration becomes

$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n, \qquad (\theta_n := \arg z_n)$$

The pictures are lovely but proofs escape me. A *Cinderella* picture of two steps from (4.2,-0.51) follows:



CAS+IGP: the Grief is in the GUI



"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."

Conclusions

- We like students of **2010** live in an information-rich, judgement-poor world
- The explosion of information is not going to diminish
  - nor is the desire (need?) to collaborate remotely
- So we have to learn and teach judgement (not obsession with plagiarism)
  - that means mastering the sorts of tools I have illustrated
- We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (few future mathematicians)
  - properly balanced, discovery and proof can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion
- **Impediments** to the assimilation of the tools I have illustrated are myriad
  - as I am only too aware from recent experiences
- These impediments include our own inertia and
  - organizational and technical bottlenecks (IT - not so much dollars)
  - under-prepared or mis-prepared colleagues
  - the dearth of good modern syllabus material and research tools
  - the lack of a compelling business model (societal goods)

"The plural of 'anecdote' is not 'evidence'."
- Alan L. Leshner (*Science's* publisher)

A Sad Story (UK)

1. **Teaching Maths In 1970** A logger sells a lorry load of timber for £1000. His cost of production is 4/5 of the selling price. What is his profit?
2. **Teaching Maths In 1980** A logger sells a lorry load of timber for £1000. His cost of production is 4/5 of the selling price, or £800. What is his profit?
3. **Teaching Maths In 1990** A logger sells a lorry load of timber for £1000. His cost of production is £800. Did he make a profit?
4. **Teaching Maths In 2000** A logger sells a lorry load of timber for £1000. His cost of production is £800 and his profit is £200. Underline the number 200.
5. **Teaching Maths In 2008** A logger cuts down a beautiful forest because he is a totally selfish and inconsiderate bastard and cares nothing for the habitat of animals or the preservation of our woodlands. He does this so he can make a profit of £200. What do you think of this way of making a living?

   Topic for class participation after answering the question: How did the birds and squirrels feel as the logger cut down their homes? (There are no wrong answers. If you are upset about the plight of the animals in question counselling will be available.)

A Sidebar: New Ramanujan-Like Identities

Guillera has recently found Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}.$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods.

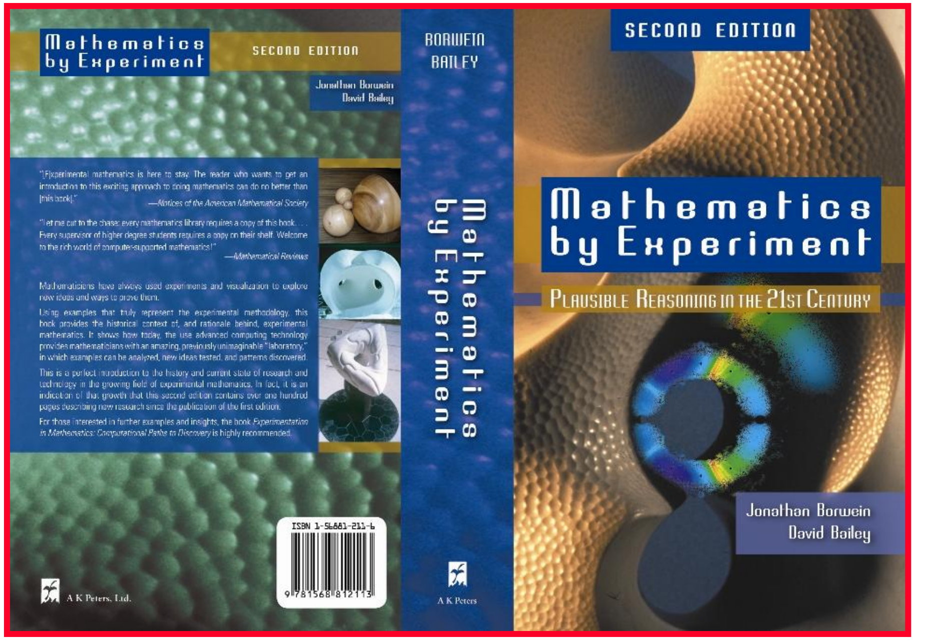
It is true but has no proof.

As far as we can tell there are no higher-order analogues!

Further Conclusions

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof (say via WZ) when one "knows" the answer is right.

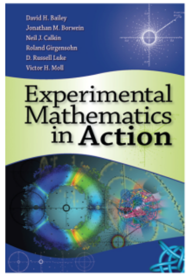


Full details of all the examples are in *Mathematics by Experiment* or its companion volume *Experimentation in Mathematics* written with Roland Girgensohn. A "Reader's Digest" version of these is available at **http://www.experimentalmath.info** along with much other material.

"Anyone who is not shocked by quantum theory has not understood a single word." - Niels Bohr

Experiencing Experimental Mathematics

Experimental Mathematics in Action
David H. Bailey, Jonathan M. Borwein, Neil J. Calkin, Roland Girgensohn, D. Russell Luke, Victor H. Moll



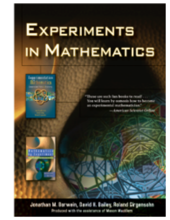
"David H. Bailey et al. have done a fantastic job to provide very comprehensive and fruitful examples and demonstrations on how experimental mathematics acts in a very broad area of both pure and applied mathematical research, in both academic and industry. Anyone who is interested in experimental mathematics should, without any doubt, read this book!"

—*Gazette* of the Australian Mathematical Society

978-1-56881-271-7; Hardcover; $49.00

Experiments in Mathematics (CD)
Jonathan M. Borwein, David H. Bailey, Roland Girgensohn

In the short time since the first edition of *Mathematics by Experiment: Plausible Reasoning in the 21st Century* and *Experimentation in Mathematics: Computational Paths to Discovery,* there has been a noticeable upsurge in interest in using computers to do real mathematics. The authors have updated and enhanced the book files and are now making them available in PDF format on a CD-ROM. This CD provides several "smart" features, including hyperlinks for all numbered equations, all Internet URLs, bibliographic references, and an augmented search facility assists one with locating a particular mathematical formula or expression.



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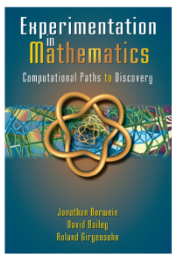
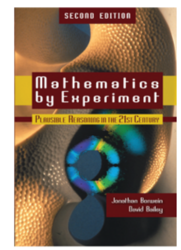
Experimentation in Mathematics Computational Paths to Discovery
Jonathan M. Borwein, David H. Bailey, Roland Girgensohn



Mathematics by Experiment — Second Edition
Plausible Reasoning in the 21st Century
Jonathan M. Borwein, David H. Bailey

"These are such fun books to read! Actually, calling them books does not do them justice. They have the liveliness and feel of great Web sites, with their bite-size fascinating factoids and their many human- and math-interest stories and other gems. But do not be fooled by the lighthearted, immensely entertaining style. You are going to learn more math (experimental or otherwise) than you ever did from any two single volumes. Not only that, you will learn by osmosis how to become an experimental mathematician."

—*American Scientist Online*



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Edited by J. M. Borwein, E. M. Rocha, J. F. Rodrigues

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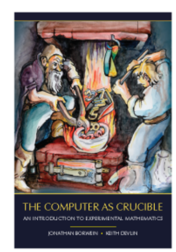
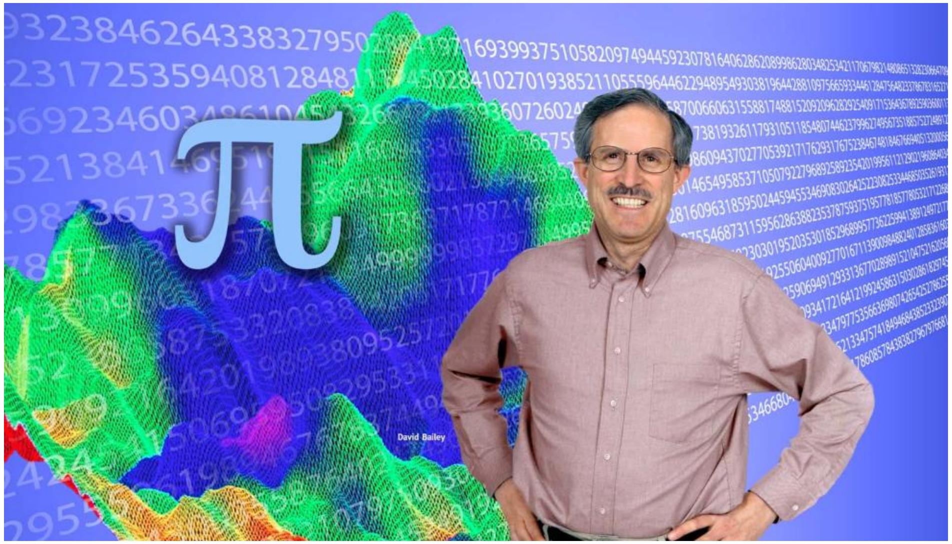
Keith Devlin and Jonathan Borwein cover a variety of topics and examples to give the reader a good sense of the current state of play in the rapidly growing new field of experimental mathematics. The writing is clear and the explanations are enhanced by relevant historical facts and stories of mathematicians and their encounters with the field over time.



978-1-56881-343-1; Paperback; $29.95

ADDITIONAL EXAMPLES of PSLQ and WILF-ZEILBERGER in ACTION

JM Borwein and DH Bailey



"Anyone who is not shocked by quantum theory has not understood a single word." - Niels Bohr

The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

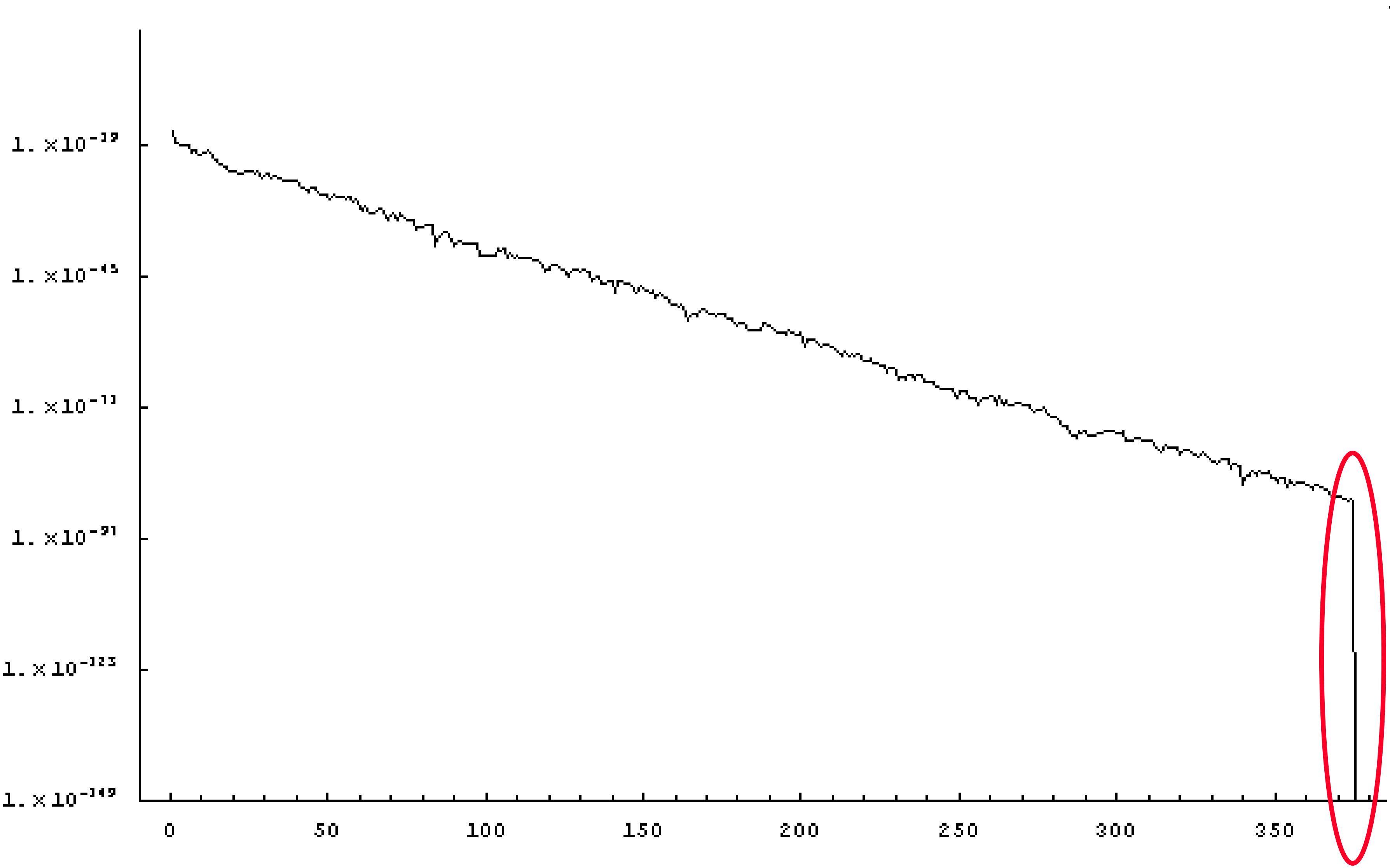
- At the present time, the PSLQ algorithm of mathematician-sculptor *Helaman Ferguson* is the best-known integer relation algorithm.
- PSLQ was named one of ten “algorithms of the century” by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least $d \times n$ digits, where d is the size (in digits) of the largest of the integers a_k.

Peter Borwein
in front of
Helaman Ferguson's
work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

Decrease of $\min_j |A_j x|$ in PSLQ: self-diagnosing



Peter Borwein's Observation

In 1996, Peter Borwein of SFU in Vancouver observed that the following well-known formula for $\log_e 2$

$$\log 2 \;=\; \sum_{n=1}^{\infty} \frac{1}{n 2^n} \;=\; 0.693147180559945309942\ldots$$

leads to a simple scheme for computing binary digits at an arbitrary starting position (here {} denotes fractional part):

$$\begin{aligned}\left\{2^d \log 2\right\} &= \left\{\sum_{n=1}^{d} \frac{2^{d-n}}{n}\right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \\ &= \left\{\sum_{n=1}^{d} \frac{2^{d-n} \bmod n}{n}\right\} + \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n}\end{aligned}$$

Fast Exponentiation Mod n

The exponentiation (2^{d-n} mod n) in this formula can be evaluated very rapidly by means of the binary algorithm for exponentiation, performed modulo n:

Example:

$3^{17} = (((3^2)^2)^2)^2 \times 3 = 129140163$

In a similar way, we can evaluate:

$3^{17} \bmod 10 = ((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \times 3 \bmod 10$

$3^2 \bmod 10 = 9$

$9^2 \bmod 10 = 1$

$1^2 \bmod 10 = 1$

$1^2 \bmod 10 = 1$

$1 \times 3 = 3$ Thus $3^{17} \bmod 10 = 3$.

Note: we never have to deal with integers larger than 81.

Is There a BBP-Type Formula for Pi?

The “trick” for computing digits beginning at an arbitrary position in the binary expansion of log(2) works for any constant that can be written with a formula of the form

$$\alpha = \sum_{n=1}^{\infty} \frac{p(n)}{2^n q(n)}$$

where p and q are polynomial functions with integer coefficients, and q has no zeroes at positive integer values.

- In 1995, no formula of this type was known for π.

Note however that if α and β have such a formula, then so does $\gamma = r\,\alpha + s\,\beta$, where r and s are integers. This suggests using PSLQ to find a formula for π.

The BBP Formula for Pi

In 1996, Simon Plouffe, using DHB's PSLQ program, discovered this formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

Indeed, this formula permits one to directly calculate binary or hexadecimal (base-16) digits of π beginning at an arbitrary starting position n, without needing to calculate any of the first n-1 digits.

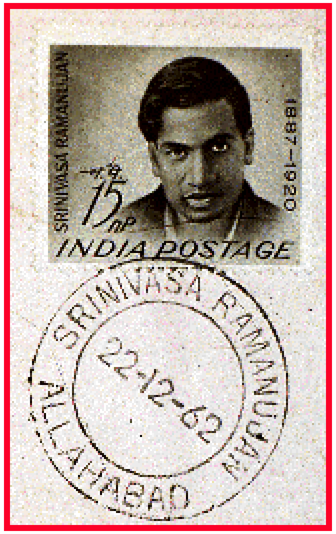
Proof of the BBP Formula

$$\int_0^{1/\sqrt{2}} \frac{x^{j-1}\,dx}{1-x^8} = \int_0^{1/\sqrt{2}} \sum_{k=0}^{\infty} x^{8k+j-1}\,dx = \frac{1}{2^{j/2}} \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}$$

Thus

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) \\
&\quad = \int_0^{1/\sqrt{2}} \frac{(4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5)\,dx}{1-x^8} \\
&\quad = \int_0^1 \frac{16(4-2y^3-y^4-y^5)\,dy}{16-y^8} \\
&\quad = \int_0^1 \frac{16(y-1)\,dy}{(y^2-2)(y^2-2y+2)} \\
&\quad = \int_0^1 \frac{4y\,dy}{y^2-2)} - \int_0^1 \frac{(4y-8)\,dy}{y^2-2y+2} \\
&\quad = \pi
\end{aligned}$$

Calculations Using the BBP Algorithm



Position	Hex Digits of Pi Starting at Position	
10^6	26C65E52CB4593	
10^7	17AF5863EFED8D	
10^8	ECB840E21926EC	
10^9	85895585A0428B	
10^{10}	921C73C6838FB2	
10^{11}	9C381872D27596	
1.25×10^{12}	07E45733CC790B	[1]
2.5×10^{14}	E6216B069CB6C1	[2]

[1] Fabrice Bellard, France, 1999

[2] Colin Percival, Canada, 2000

Some Other Similar New Identities

$$\pi\sqrt{3} = \frac{9}{32}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{16}{6k+1}-\frac{8}{6k+2}-\frac{2}{6k+4}-\frac{1}{6k+5}\right)$$

$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{144}{(6k+1)^2}-\frac{216}{(6k+2)^2}-\frac{72}{(6k+3)^2}-\frac{54}{(6k+4)^2}+\frac{9}{(6k+5)^2}\right)$$

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left(\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}-\frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}-\frac{72}{(12k+6)^2}-\frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right)$$

$$6\sqrt{3}\arctan\left(\frac{\sqrt{3}}{7}\right) = \sum_{k=0}^{\infty}\frac{1}{27^k}\left(\frac{3}{3k+1}+\frac{1}{3k+2}\right)$$

$$\frac{25}{2}\log\left(\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right) = \sum_{k=0}^{\infty}\frac{1}{5^{5k}}\left(\frac{5}{5k+2}+\frac{1}{5k+3}\right)$$

$$\sum_{n=0}^{\infty}\frac{1}{(-27)^n}\left(\frac{6}{6n+1}-\frac{2}{6n+3}+\frac{2/3}{6n+5}\right) = \sqrt{3}\,\pi$$

Stan Wagon
May 2009

Is There a Base-10 Formula for Pi?

Note that there is both a base-2 and a base-3 BBP-type formula for π^2. Base-2 and base-3 formulas are also known for a handful of other constants.

Question: Is there any base-n BBP-type formula for π, where n is NOT a power of 2?

Answer: No. This is ruled out in a 2004 paper by Jon Borwein, David Borwein and Will Galway.

This does not rule out some completely different scheme for finding individual non-binary digits of π.

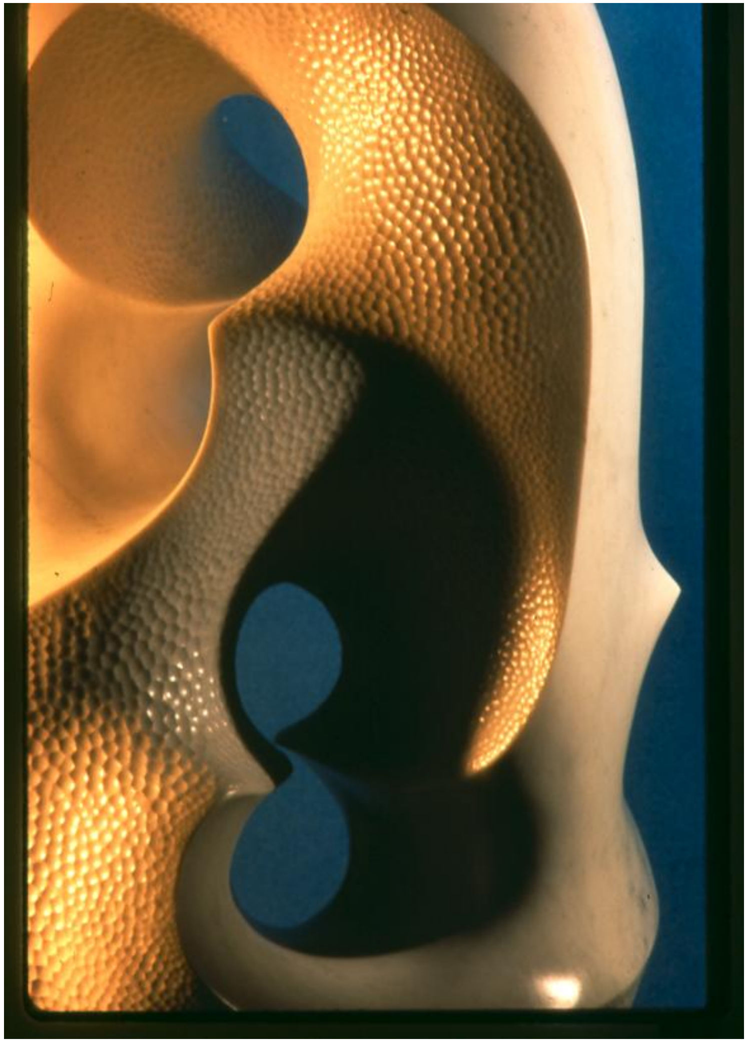
PSLQ and Sculpture

The complement of the figure-eight knot, when viewed in hyperbolic space, has finite volume

2.029883212819307250042…

David Broadhurst found, using PSLQ, that this constant is given by the formula:

$$V = \frac{\sqrt{3}}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{27^n}\left(\frac{18}{(6n+1)^2}-\frac{18}{(6n+2)^2}-\frac{24}{(6n+3)^2}-\frac{6}{(6n+4)^2}+\frac{2}{(6n+5)^2}\right)$$



Apery-Like Summations

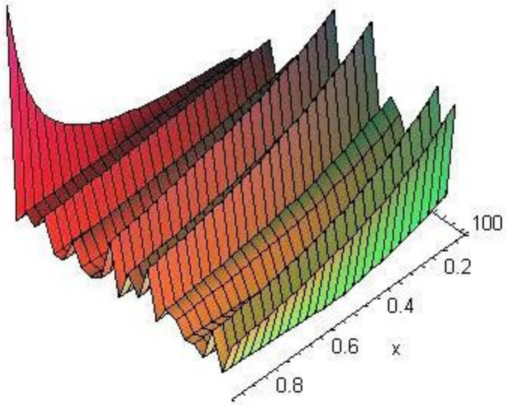
The following formulas for ζ(n) have been known for many decades:

$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}},$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}},$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

The RH in Maple



These results have led many to speculate that

$$Q_5 := \zeta(5)/\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}$$

might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits.

Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)^3} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^6}{(2n+1)!} \times \frac{\left(5(n+1)^2 + 6(a-1)(n+1) + 2(a-1)^2\right)}{\prod_{k=0}^{n} (a+k)^4}.$$

Note: *Maple* establishes this identity as

$$-1/2\,\Psi(2,a) = -1/2\,\Psi(2,a) - \zeta(3) + 5/4\,{}_4F_3([1,1,1,1],[3/2,2,2],-1/4)$$

Hence

$$\zeta(4) = -\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2m}{m} m^4} + \frac{10}{3}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\sum_{k=1}^{m}\frac{1}{k}}{\binom{2m}{m} m^3}.$$

The case a=0 above is Apery’s formula for $\zeta(3)$!

Example Usage of Wilf-Zeilberger

Two recent experimentally-discovered identities are

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}}\left(120n^2+34n+3\right) = \frac{32}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}^5}{2^{20n}}\left(820n^2+180n+13\right) = \frac{128}{\pi^2}$$

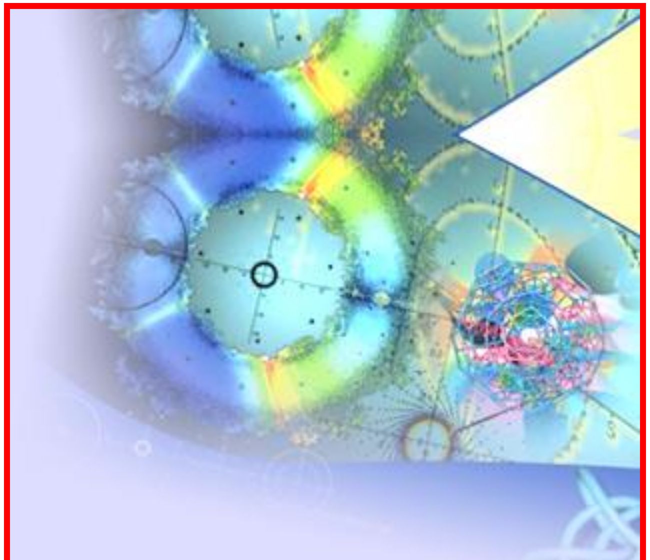
Guillera ***cunningly*** started by defining

$$G(n,k) = \frac{(-1)^k}{2^{16n}2^{4k}}\left(120n^2+84nk+34n+10k+3\right)\frac{\binom{2n}{n}^4\binom{2k}{k}^3\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

He then used the **EKHAD** software package to obtain the companion

$$F(n,k) = \frac{(-1)^k 512}{2^{16n}2^{4k}}\frac{n^3}{4n-2k-1}\frac{\binom{2n}{n}^4\binom{2k}{k}^3\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

Example Usage of W-Z, II



http://ddrive.cs.dal.ca/~isc/portal

When we define

$$H(n,k) \;=\; F(n+1,n+k)+G(n,n+k)$$

Zeilberger's theorem gives the identity

$$\sum_{n=0}^{\infty} G(n,0) \;=\; \sum_{n=0}^{\infty} H(n,0)$$

which when written out is

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4 \binom{4n}{2n}}{2^{16n}} \left(120n^2+34n+3\right) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n+7}} \frac{(n+1)^3}{2n+3} \frac{\binom{2n+2}{n+1}^4 \binom{2n}{n}^3 \binom{2n+4}{n+2}}{\binom{2n+2}{n}\binom{2n+1}{n+1}^2}$$

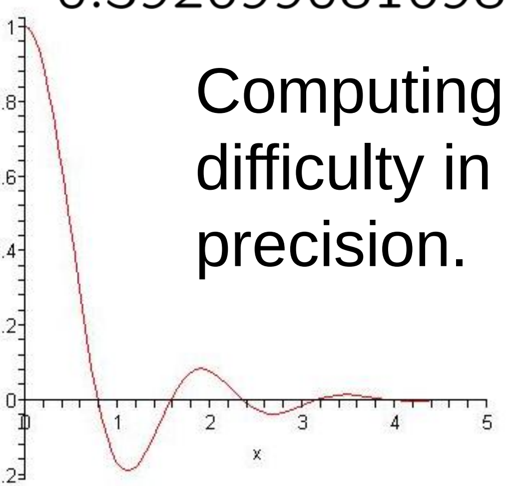
$$+\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}} \left(204n^2+44n+3\right) \binom{2n}{n}^5 \;=\; \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2+180n+13\right)$$

A limit argument completes the proof of Guillera’s identities.

A Cautionary Example

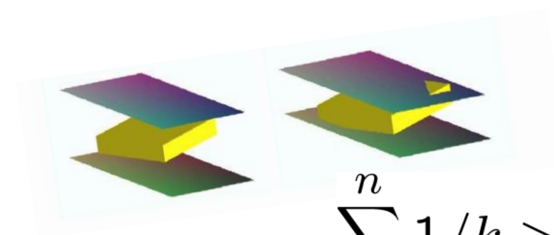
These **constants agree to 42 decimal digits** accuracy,
but are **NOT** equal:

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos(x/n)\, dx \quad =$$
$$0.392699081698724154807830422909937860524646\mathbf{543418723}\ldots$$
$$\frac{\pi}{8} \quad =$$
$$0.392699081698724154807830422909937860524646\mathbf{17492189}\ldots$$



Computing this integral is (or was) nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP) …



$$\sum_k^n 1/k > 2$$