

Mathematics by Experiment:

Plausible Reasoning in the 21st Century



Dorette Pronk and Jon Borwein



Canada Research Chair in IT
Dalhousie University

Halifax, Nova Scotia, Canada

Experimental Mathematics Circles

March Third, 2005

If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.

(Kurt Gödel, 1951)

www.cs.dal.ca/agatemath



NEXT YEAR IN D-DRIVE



www.cs.dal.ca/ddrive

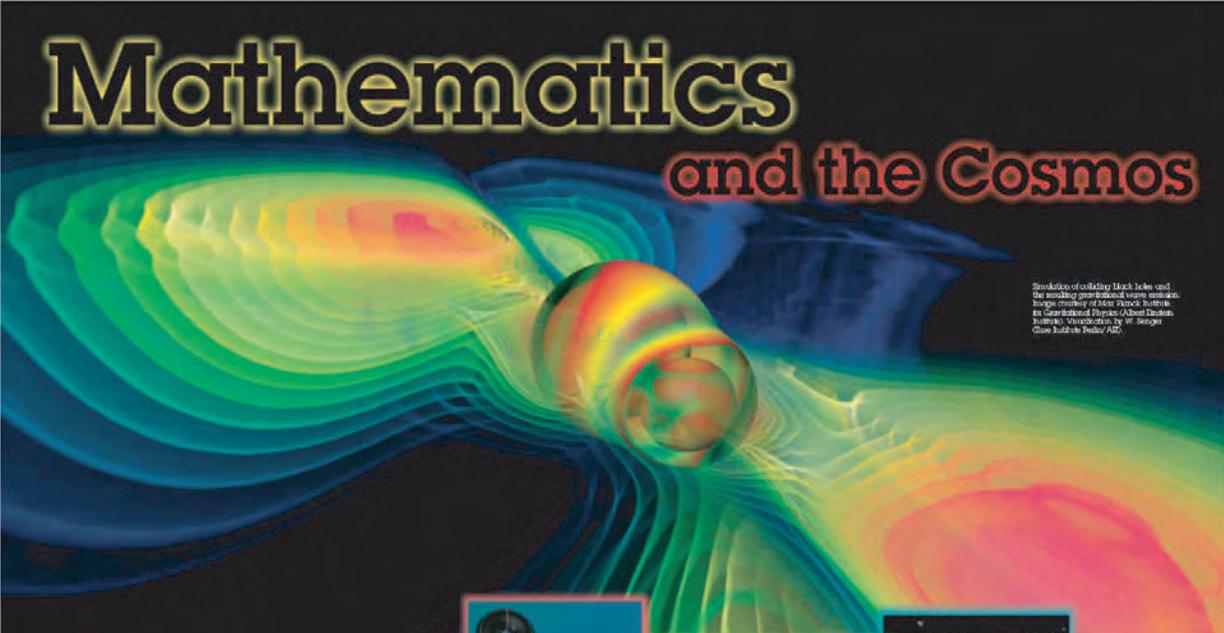
The Dalhousie Distributed Research Institute and Virtual Environment is opening this month.



www.cs.math.ca/ddrive

MATH AWARENESS MONTH is APRIL

- Interactive graphics will become integral part of math. **Features** gravitational boosting, gravity waves, Lagrange points ...



Mathematics and the Cosmos

Mathematics is at the core of our attempts to understand the universe at every level. Riemannian geometry and topology provide models of the universe, numerical simulations analyze large-scale interactions, celestial mechanics is used at the level of planetary systems, and a wide variety of mathematical tools go into actual space exploration.

Simulation of colliding black holes and the resulting gravitational wave emission. Image courtesy of Max Planck Institute for Gravitational Physics (Albert Einstein Institute). Visualization by M. Siegel, Gero Jutzsche-Petris / ARI.

A model of a two-dimensional torus (donut-shaped object) showing its complex geometry. Courtesy of Prof. Christian Voigt.

Artistic conception of the International Space Station. Courtesy of Dr. Shinya Ue, NASA, Air Pollution Laboratory, Osaka. The artist is Ch. Kozaki.

Artistic rendering of the Cassini spacecraft approaching Saturn. Courtesy of NASA/JPL, Caltech.

ISSO gravitational wave detector. Photo courtesy of LIGO laboratory.

APRIL 2005

Mathematics Awareness Month

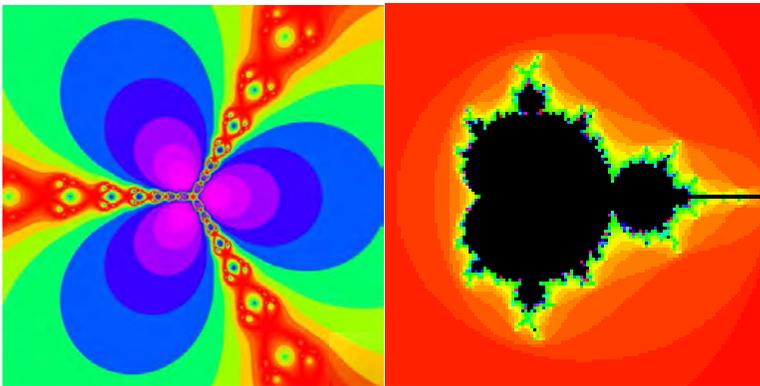
Sponsored by the Joint Policy Board for Mathematics

American Mathematical Society • American Statistical Association • Mathematical Association of America • Society for Industrial and Applied Mathematics

FOUR FORMS of EXPERIMENTS

- 1. **Kantian** example: generating “the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid’s axiom of parallels (or something equivalent to it) with alternative forms.”
- 2. The **Baconian** experiment is a contrived as opposed to a natural happening, it “is the consequence of ‘trying things out’ or even of merely messing about.”
- 3. **Aristotelian** demonstrations: “apply electrodes to a frog’s sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog’s dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble.”

- 4. The most important is **Galilean**: “a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction.”
- It is also the only one of the four forms which will make Experimental Mathematics a serious enterprise.
- From Peter Medawar’s *Advice to a Young Scientist*, Harper (1979).



**Julia
and
Mandelbrot
sets**

HOW NOT TO EXPERIMENT



Pooh Math

‘Guess and Check’ while **Aiming Too High**

DEFINITIONS

mathematics, n. a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their interrelationships, applications, generalizations and abstractions.

BOOKS

et al.

HISTORY OF SCIENCE

Solid Tools for Visualizing Science

Julie K. Brown

The general public has long been fascinated with three-dimensional models, for what they represent as well as for their sheer beauty and ingenious construction. However, until recently such objects have received little notice from historians of science and medicine. Now, with increasing attention being paid to the visual and material culture of science, scholars finally have begun to examine three-dimensional models for their part in "knowledge production." *Models: The Third Dimension of Science*, a collection of fascinating essays that grew out of a 1998 symposium in London hosted by the Wellcome Institute for the History of Medicine and the Science Museum, stands as the most important contribution to this topic to date.

In bringing together the work of 17 international historians of science who have previously written on their respective topics, the volume offers an opportunity to view both the continuity and the diversity in the uses of three-dimensional

three-dimensional models: as research tools ("epistemic objects") in the making of knowledge, as mediators of information for communication, and as visual objects for public display. Herbert Mehrtens points out that physical models were research tools for some German mathematicians such as Felix Klein but only during a short window of time, in the 1890s. However, their use, Mehrtens notes, as teaching tools, indicators of professional status, and aesthetic forms continued well into

the 20th century. In molecular biology the story is more intricate and layered, as Soraya de Chadarevian makes clear. Three-dimensional modeling was already an integral tool and an established research practice for crystallographers when, in the 1950s, molecular biologists adapted it for their studies of proteins. For publication purposes, however, these models had to be transposed back into two dimensions by specialized illustrators.

The life of these three-dimensional research models was further extended as they ap-



Mathematical model, 1886. The four plaster models (here, Clebsch's diagonal surface) reused to accompany (2) served as objects of beauty rather than research tools.

well as the growing rift between scientific authority and the demands of popular entertainment in the early 20th century.

The processes of crafting and constructing three-dimensional models provide a focus for several authors. Nick Hopwood discusses the wax embryological models created by Adolf and Friedrich Ziegler, drawing on his extensive documentation of their studio (1). Besides offering excellent descriptions of the

Science November 19, 2004

MODES OF REASONING

induction, n. *any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.*

and

deduction, n. *a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.*
b. a conclusion reached by this process.

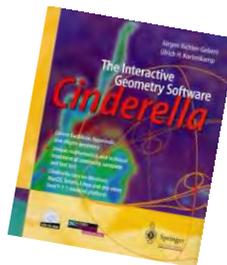
MATHEMATICS and COMPUTATION

The emergence of powerful mathematical computing environments like *Maple* and *Matlab*, the growing availability of correspondingly powerful (*multi-processor*) computers and the pervasive presence of the internet allow for research mathematicians, students and teachers, to proceed heuristically and ‘quasi-inductively’.

We may increasingly use symbolic and numeric computation, **geometry packages**:

Geometer’s SketchPad, Cabri and **Cinderella**

www.cinderella.de



Also visualization tools, simulation and data mining.

An aesthetic appreciation of mathematics may be given to a much broader audience.

THE TALK IS ORGANIZED SO IT ENDS WHEN IT ENDS

Many of the benefits of computation are accessible through low-end ‘electronic blackboard’ versions of experimental mathematics. This permits livelier classes, more realistic examples, and more collaborative learning.

The unique features of mathematics make this more problematic and challenging.

- For example, there is still no truly satisfactory way of displaying mathematical notation on the web;
- and we care more about the reliability of our literature than does any other science.

The traditional central role of proof in mathematics is arguably under siege.

QUESTIONS

- ★ What constitutes secure mathematical knowledge?
- ★ When is computation convincing? Are humans less fallible?
 - What tools are available? What methodologies?
 - What about the 'law of the small numbers'?
 - Who cares for certainty? What is the role of proof?
- ★ How is mathematics actually done? How should it be?

SOME SELF PROMOTION

- Experimental Mathematics is being discussed widely

MATH A Digital Slice of Pi

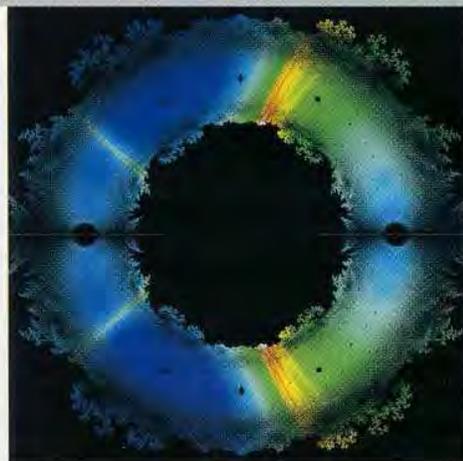
THE NEW WAY TO DO PURE MATH: EXPERIMENTALLY BY W. WAYT GIBBS

“**O**ne of the greatest ironies of the information technology revolution is that while the computer was conceived and born in the field of pure mathematics, through the genius of giants such as John von Neumann and Alan Turing, until recently this marvelous technology had only a minor impact within the field that gave it birth.” So begins *Experimentation in Mathematics*, a book by Jonathan M. Borwein and David H. Bailey due out in September that documents how all that has begun to change. Computers, once looked on by mathematical researchers with disdain as mere calculators, have gained enough power to enable an entirely new way to make fundamental discoveries: by running experiments and observing what happens.

The first clear evidence of this shift emerged in 1996. Bailey, who is chief technologist at the National Energy Research Sci-

entific Computing Center in Berkeley, Calif., and several colleagues developed a computer program that could uncover integer relations among long chains of real numbers. It was a problem that had long vexed mathematicians. Euclid discovered the first integer relation scheme—a way to work out the greatest common divisor of any two integers—around 300 B.C. But it wasn’t until 1977 that Helaman Ferguson and Rodney W. Forcade at last found a method to detect relations among an arbitrarily large set of numbers. Building on that work, in 1995 Bailey’s group turned its computers loose on some of the fundamental constants of math, such as $\log 2$ and π .

To the researchers’ great surprise, after months of calculations the machines came up with novel formulas for these and other nat-



COMPUTER RENDERINGS of mathematical constructs can reveal hidden structure. The bands of color that appear in this plot of all solutions to a certain class of polynomials (specifically, those of the form $\pm 1 \pm x \pm x^2 \pm x^3 \pm \dots \pm x^n = 0$, up to $n = 18$) have yet to be explained by conventional analysis.

MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

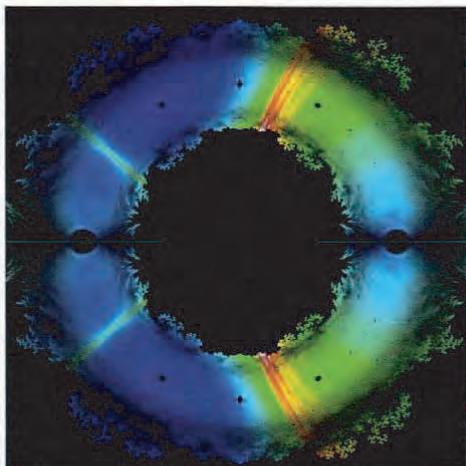
For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x .

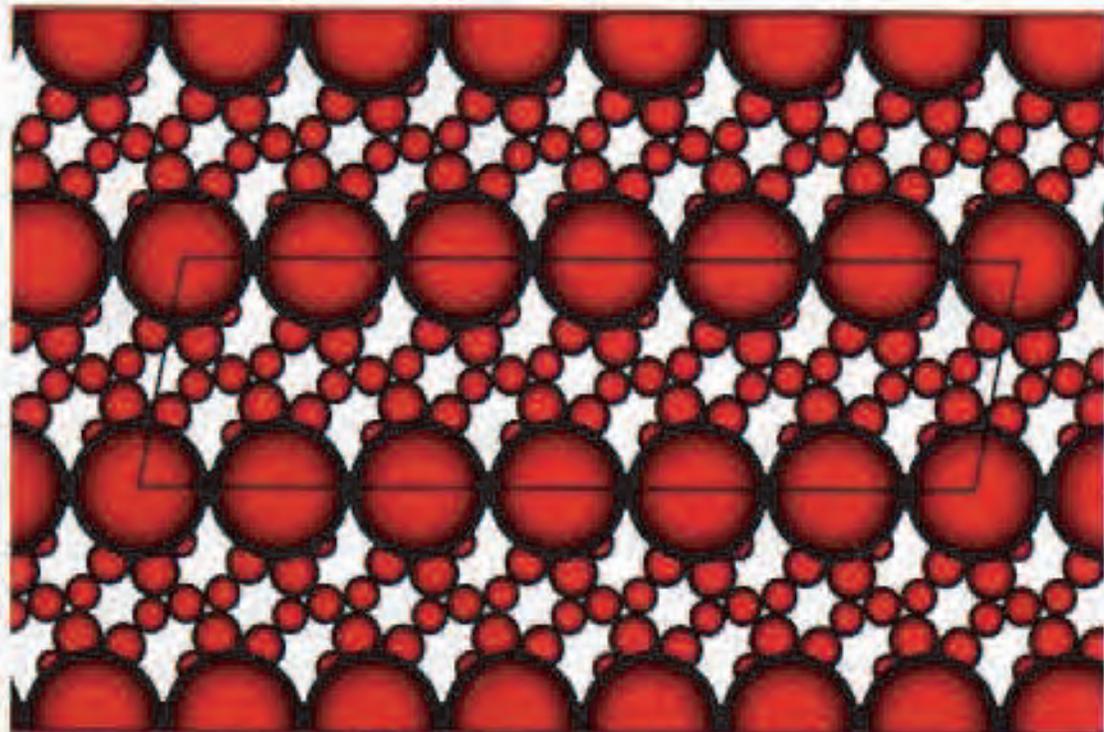
Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-



UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.



STRAIGHT CIRCLES — When mathematicians Colin Adams and Eric Schoenfeld created this image while playing with the computer program Snappea last year, they were stunned to see perfectly straight chains of spheres. The observation led them to an unexpected discovery about knots.

Bailey and Jonathan Borwein advance the controversial thesis that mathematics should move toward a more empirical approach. In it, formal proof would not be the only acceptable way to establish mathematical knowledge.

Mathematicians, Bailey and Borwein argue, should be free to work more like other scientists do, developing hypotheses through experimentation and then testing them in further experiments. Formal proof is still the ideal, they say, but it is not the only path to mathematical truth.

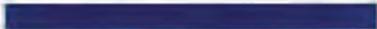
“When I started school, I thought mathematics was about proofs, but now I think it’s about having secure mathematical knowledge,” Borwein says. “We claim that’s not the same thing.”

Bailey and Borwein point out that mathematical proofs can run to hundreds of pages and require such specialized knowledge that only a few people are capable of reading and judging them.

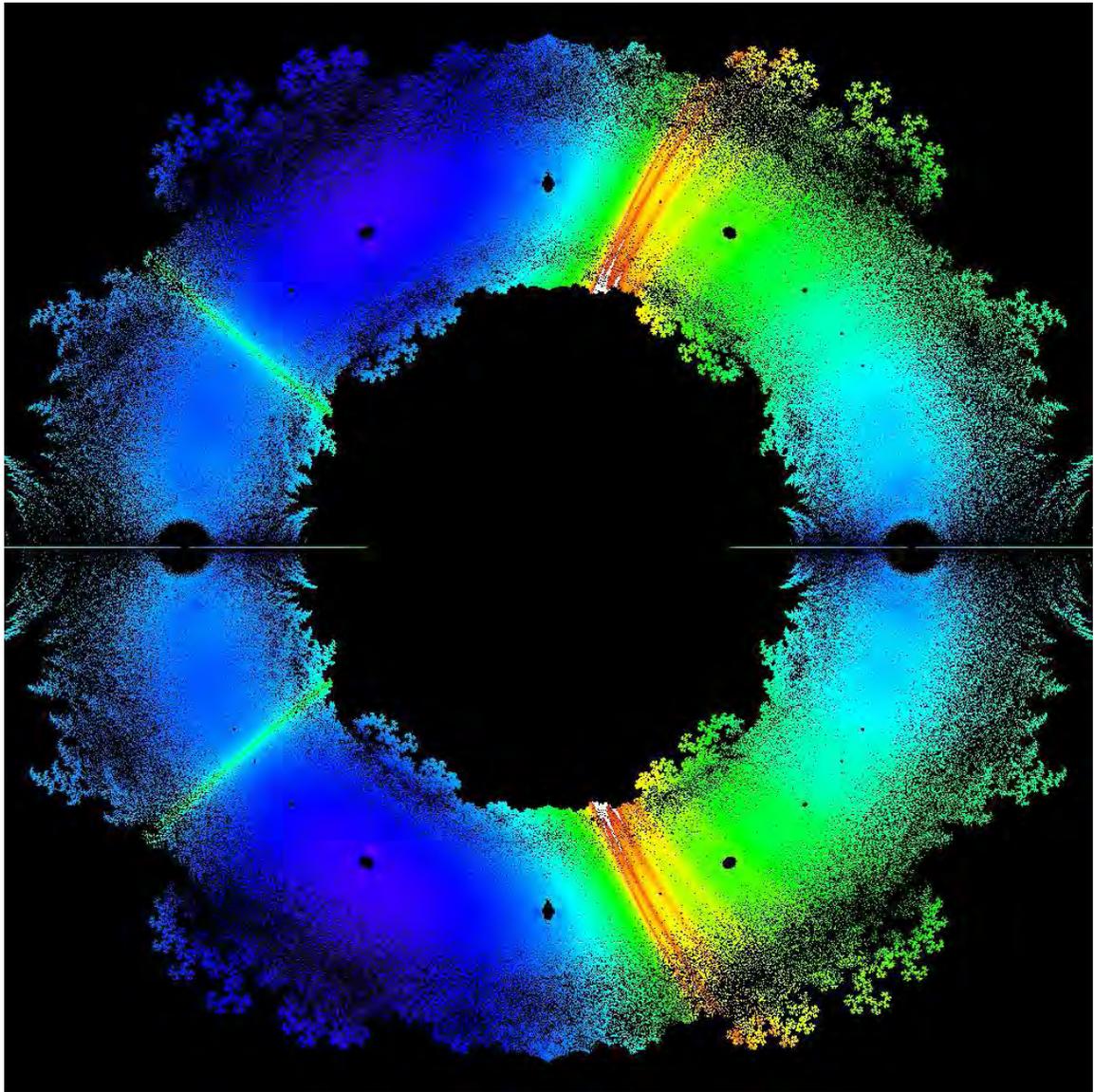
“We feel that in many cases, computations constitute very strong evidence, evidence that is at least as compelling as some of the more complex formal proofs in the literature,” Bailey and Borwein say in *Mathematics by Experiment*.

“One thing that’s happening is you can discover many more things than you can explain.”

— JONATHAN BORWEIN
DALHOUSIE UNIVERSITY



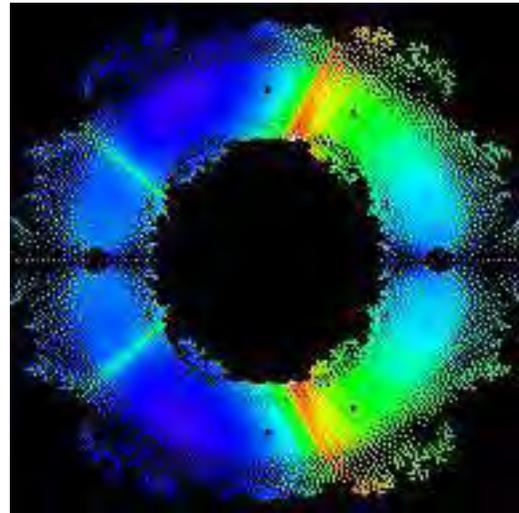
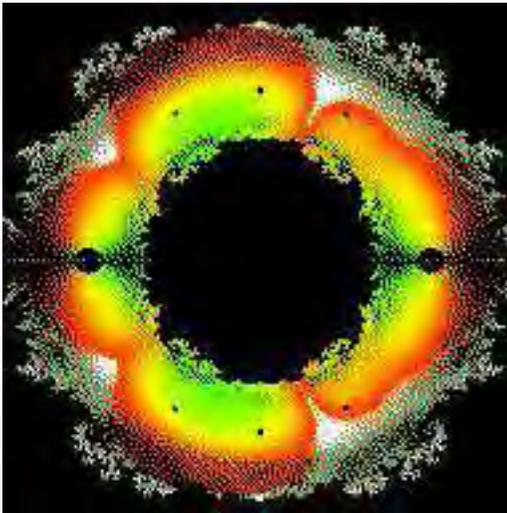
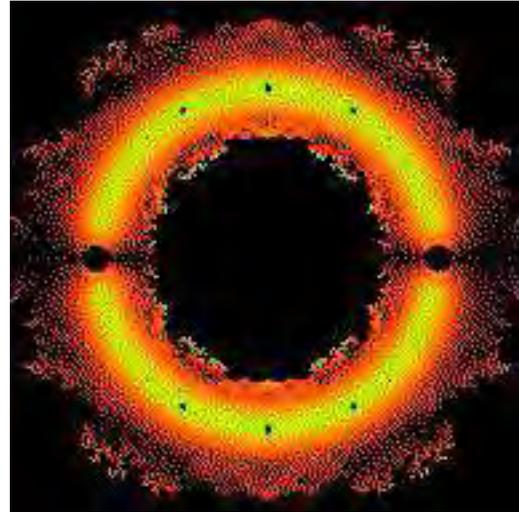
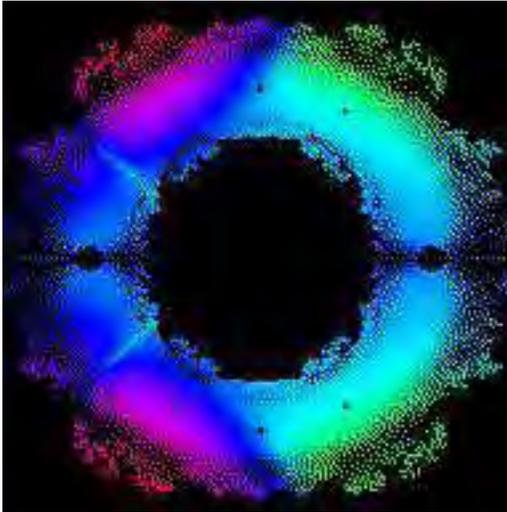
ZEROES of 0 – 1 POLYNOMIALS



Data mining in polynomials

- The striations are unexplained!

WHAT YOU DRAW is WHAT YOU SEE



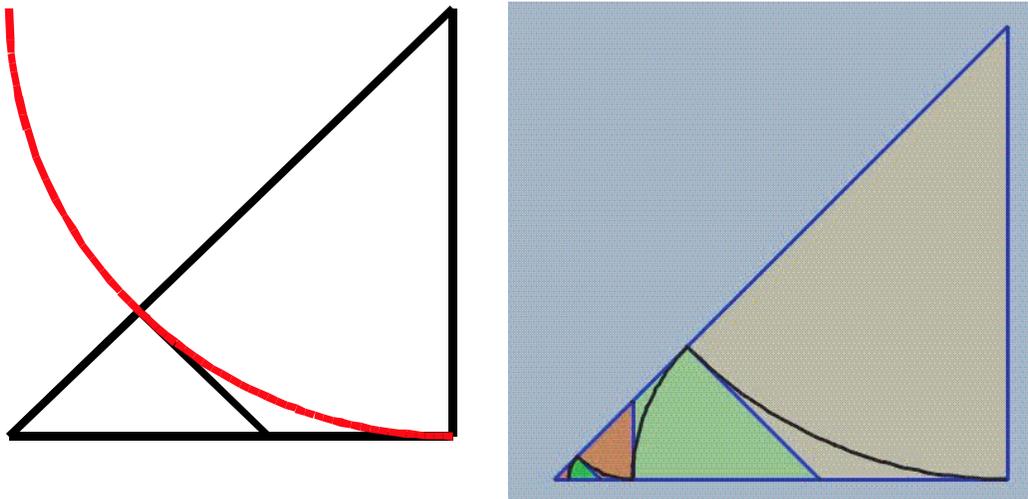
The price of metaphor is eternal vigilance

(Arturo Rosenblueth & Norbert Wiener)

A NEW PROOF $\sqrt{2}$ is IRRATIONAL

One can find new insights in the oldest areas:

- Here is Tom Apostol's lovely new graphical proof* of the irrationality of $\sqrt{2}$. I like very much that this was published in the present millennium.



Root two is irrational (static and self-similar pictures)

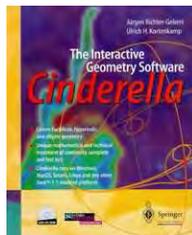
**MAA Monthly*, November 2000, 241–242.

PROOF. To say $\sqrt{2}$ is rational is to draw a right-angled isosceles triangle with integer sides. Consider the *smallest* right-angled isosceles triangle with integer sides —that is with shortest hypotenuse.

Circumscribe a circle of radius the vertical side and construct the tangent on the hypotenuse, as in the picture.

Repeating the process once more produces an even smaller such triangle in the same orientation as the initial one.

The *smaller* right-angled isosceles triangle again has integer sides **QED**



. . . AND more **CINDERELLA**

Rationality $\sqrt{2}$ also makes things rational:

$$\left(\sqrt{2}\sqrt{2}\right)^{\sqrt{2}} = \sqrt{2}(\sqrt{2}\cdot\sqrt{2}) = \sqrt{2}^2 = 2.$$

Hence by *the principle of the excluded middle*:

$$\text{Either } \sqrt{2}\sqrt{2} \in \mathbb{Q} \text{ or } \sqrt{2}\sqrt{2} \notin \mathbb{Q}.$$

In *either* case we can deduce that there are irrational numbers α and β with α^β rational.

But how do we know which ones? Compare the assertion that

$$\alpha := \sqrt{2} \text{ and } \beta := 2 \ln_2(3) \text{ yield } \alpha^\beta = 3$$

as *Mathematica* confirms.

- Again, verification is easier than discovery. Similarly multiplication is easier than factorization, as in secure encryption schemes for e-commerce.

There are eight possible (ir)rational triples:

$$\alpha^\beta = \gamma.$$

Can you find them?

DICTIONARIES ARE LIKE TIMEPIECES

- ▶ Samuel Johnson observed of watches that “the best do not run true, and the worst are better than none.” The same is true of tables and databases. Michael Berry “would give up Shakespeare in favor of Prudnikov, Brychkov and Marichev.”
- That excellent compendium contains

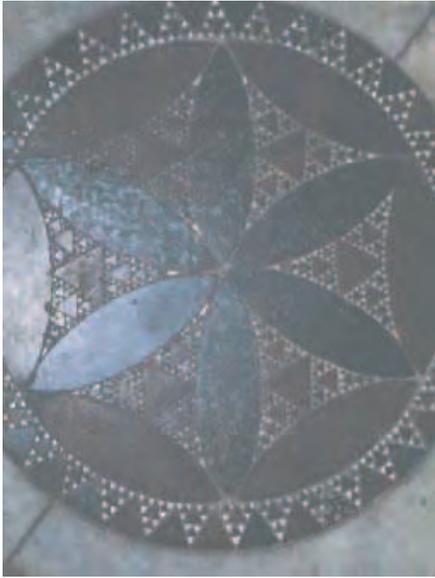
$$(1) \quad \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{k^2 (k^2 - kl + l^2)} = \frac{\pi^{\alpha} \sqrt{3}}{30},$$

where the “ α ” is *probably* “4” [volume 1, entry 9, page 750].

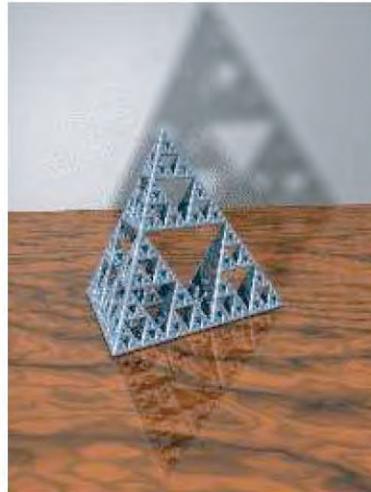
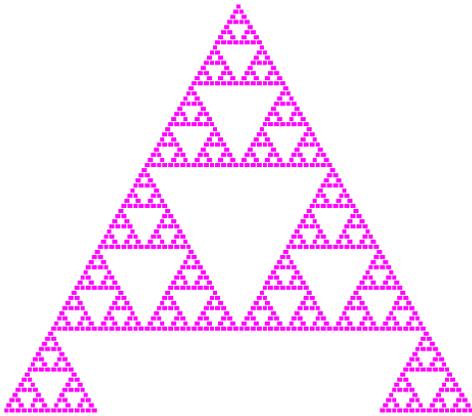
- ★ Integer relation methods suggest that **no reasonable value of α works.**

What is intended in (1)?

SELF-SIMILARITY

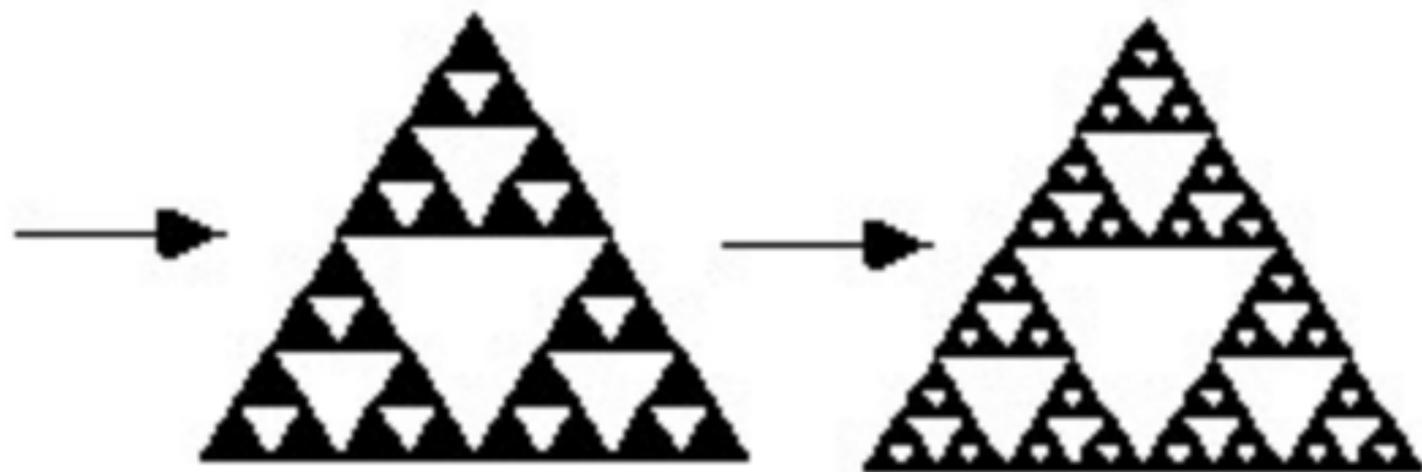


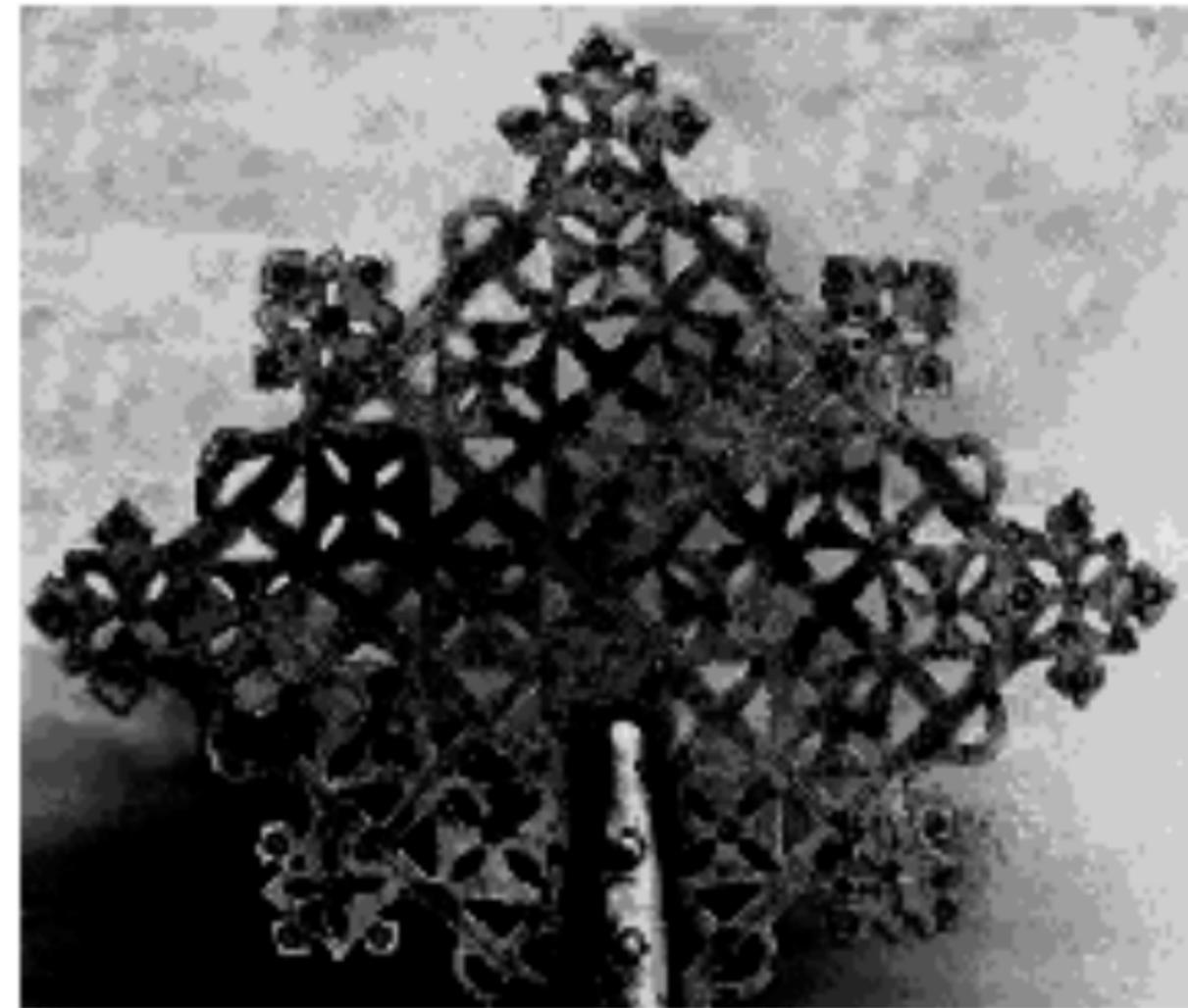
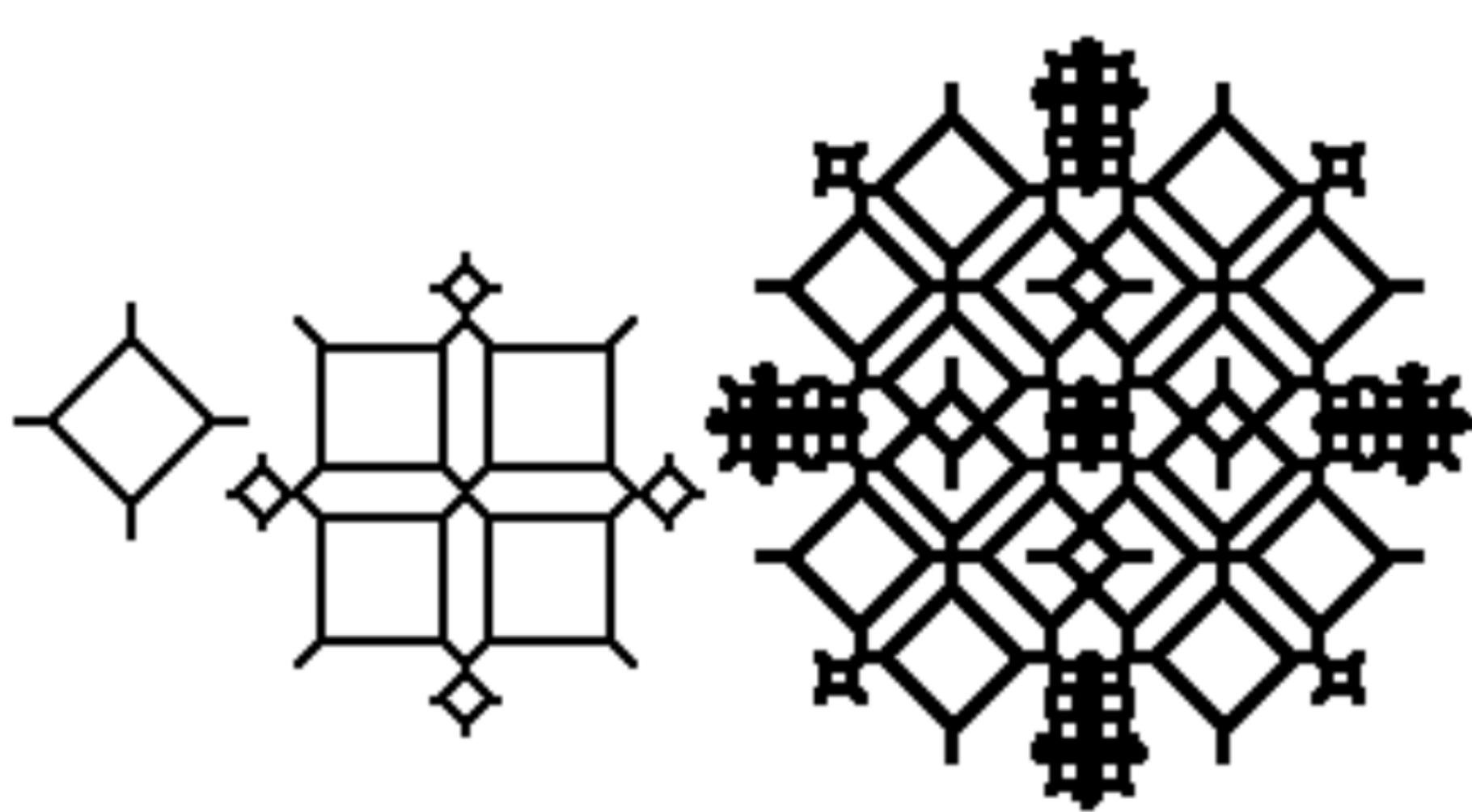
The **Sierpinski Gasket** or **Triangle** in a 13th Century Anagni Church



Pascal's Triangle modulo two

1, 11, 121, 1331, 14641, 15101051...



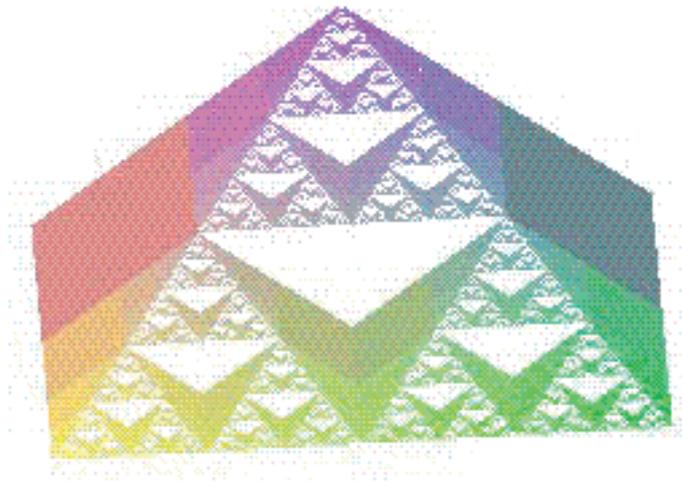




FRACTAL CARDS

Not all impressive discoveries require a computer.

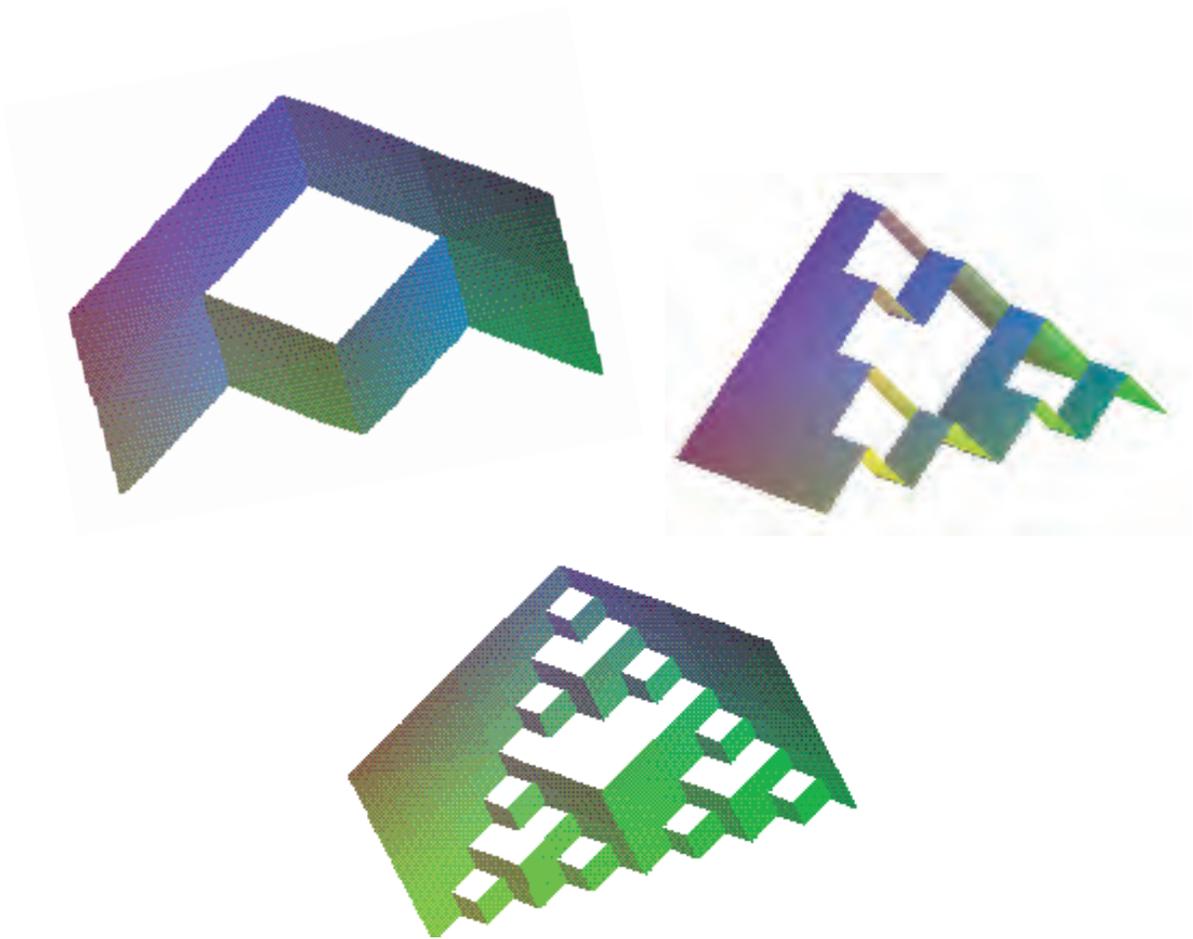
Elaine Simmt and Brent Davis describe lovely constructions made by repeated regular paper folding and cutting—but no removal of paper—that result in beautiful fractal, self-similar, “pop-up” cards*.



The 7th iterates of a Sierpinski triangle

*Fractal Cards: A Space for Exploration in Geometry and Discrete Maths, *Math Teacher*, **91** (1998), 102-8

- Nonetheless, we show various iterates of a pop-up Sierpinski triangle built in software, on turning those paper cutting and folding rules into an algorithm. This should let you start folding.

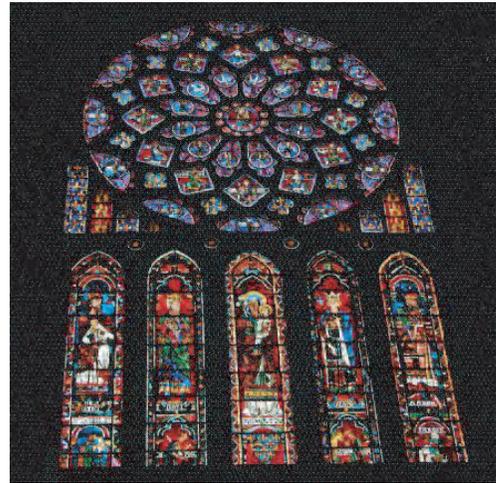
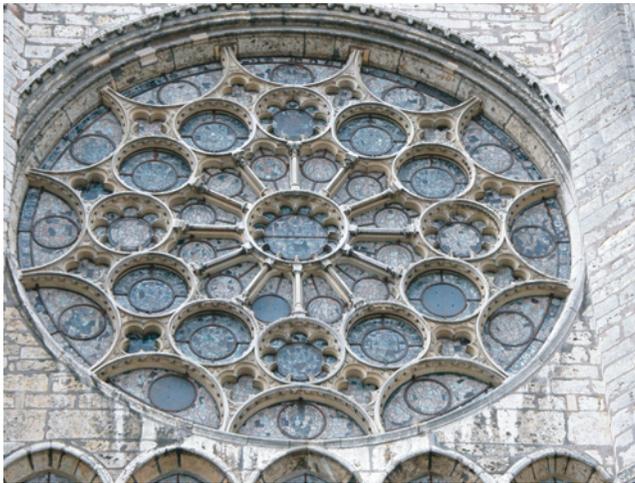


The 1st, 2nd and 3rd iterates

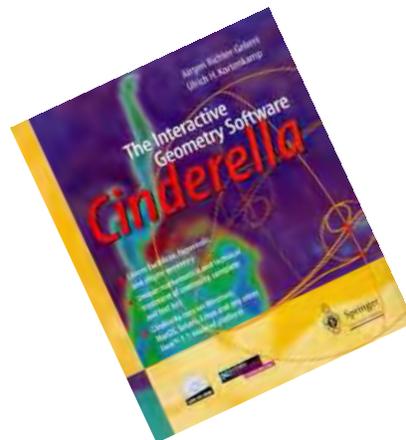




- Note the similarity to the **Pascal triangle**.
- And art can be an additional source of mathematical inspiration and stimulation



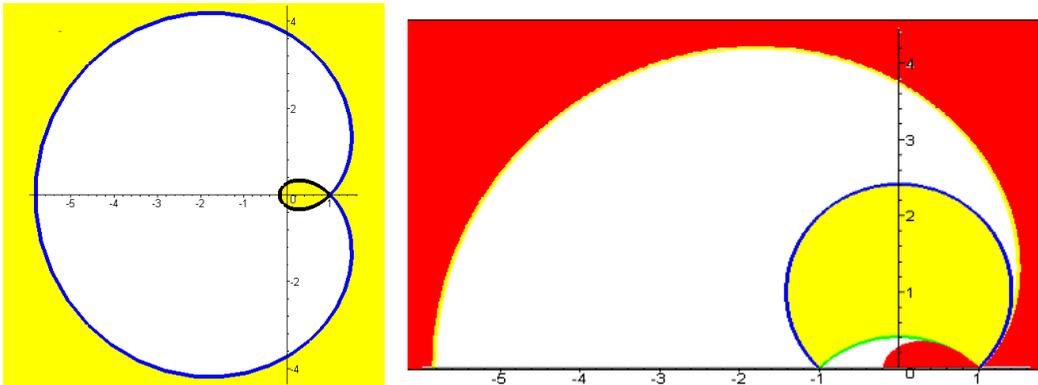
Self similarity at Chartres



and in

POLYA and HEURISTICS

*“[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.” (George Polya)**



In **Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving, 1968.*

Polya on Picture-writing

$$\begin{aligned} & (\square + \textcircled{1} + \textcircled{1}\textcircled{1} + \textcircled{1}\textcircled{1}\textcircled{1} + \dots) \cdot \\ & (\square + \textcircled{5} + \textcircled{5}\textcircled{5} + \textcircled{5}\textcircled{5}\textcircled{5} + \dots) \cdot \\ & (\square + \textcircled{10} + \textcircled{10}\textcircled{10} + \textcircled{10}\textcircled{10}\textcircled{10} + \dots) \cdot \\ & (\square + \textcircled{25} + \textcircled{25}\textcircled{25} + \textcircled{25}\textcircled{25}\textcircled{25} + \dots) \cdot \\ & (\square + \textcircled{50} + \textcircled{50}\textcircled{50} + \textcircled{50}\textcircled{50}\textcircled{50} + \dots) \cdot \\ & = \dots + \square \cdot \textcircled{5}\textcircled{5}\textcircled{5} \cdot \textcircled{10} \cdot \textcircled{25} \cdot \textcircled{50} + \dots \end{aligned}$$

Polya's illustration of the change solution

Polya, in his 1956 *American Mathematical Monthly* article provided three provoking examples of converting pictorial representations of problems into generating function solutions. We discuss the first one.

1. *In how many ways can you make change for a dollar?*

This leads to the (US currency) generating function $\sum_{k \geq 0} P_k x^k =$

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

which one can easily expand using a *Mathematica* command,

```
Series[  
1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)*(1-x^50))  
,{x,0,100}]
```

to obtain $P_{100} = 292$ (**243** for Canadian currency, which lacks a 50 cent piece but has a dollar coin).

- Polya's diagram is shown in the Figure.*

*Illustration courtesy the Mathematical Association of America

- To see why, we use geometric series and consider the so called *ordinary generating function*

$$\frac{1}{1 - x^{10}} = 1 + x^{10} + x^{20} + x^{30} + \dots$$

for dimes and

$$\frac{1}{1 - x^{25}} = 1 + x^{25} + x^{50} + x^{75} + \dots$$

for quarters etc.

- We multiply these two together and compare coefficients

$$\begin{aligned} \frac{1}{1 - x^{10}} \frac{1}{1 - x^{25}} &= 1 + x^{10} + x^{20} + x^{25} \\ &+ x^{30} + x^{35} + x^{40} + x^{45} \\ &+ 2x^{50} + x^{55} + 2x^{60} + \dots \end{aligned}$$

We argue that the *coefficient* of x^{60} on the right is precisely the number of ways of making 60 cents out of identical dimes and quarters.

- This is easy to check with a handful of change or a calculator, The general question with more denominations is handled similarly.
- I leave it open whether it is easier to decode the generating function from the picture or vice versa. In any event, symbolic and graphic experiment provide abundant and mutual reinforcement and assistance in concept formation.

“In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting.” (George Polya)

While by ‘beginner’ George Polya intended young school students, I suggest this is equally true of anyone engaging for the first time with an unfamiliar topic in mathematics.

ENIAC: Integrator and Calculator

SIZE/WEIGHT: ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.



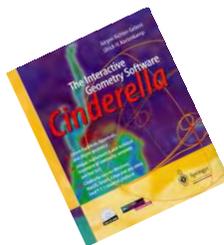
SPEED/MEMORY: A 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000 — 1,000 times faster than any earlier machine. The first stored-memory computer, ENIAC could store 200 digits.

ARCHITECTURE: Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line.

The accumulators were connected to each other manually.

- The 1949 computation of π to 2,037 places took 70 hours.
- It would have taken roughly **100,000** ENIACs to store the Smithsonian's picture!

... AND now we have



MOORE'S LAW

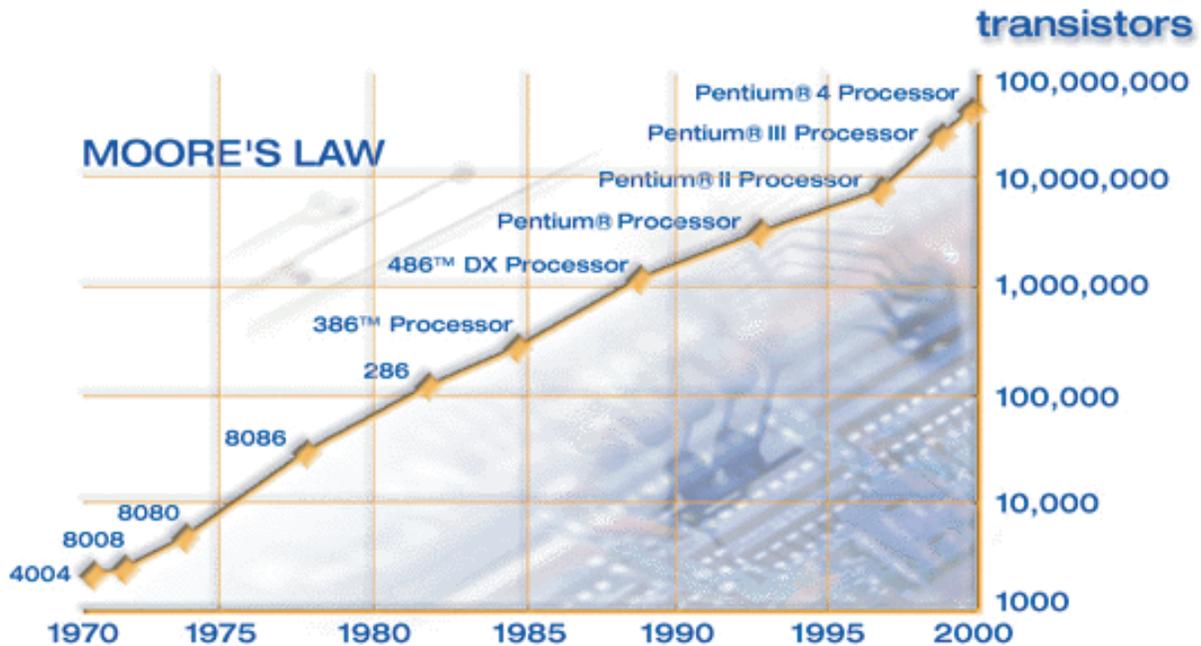
The complexity for minimum component costs has increased at a rate of roughly a factor of two per year. ... Certainly over the short term this rate can be expected to continue, if not to increase. Over the longer term, the rate of increase is a bit more uncertain, although there is no reason to believe it will not remain nearly constant for at least 10 years.

(Gordon Moore, Intel co-founder, 1965)

► “Moore’s Law” asserts that semiconductor technology approximately doubles in capacity and performance roughly every 18 to 24 months (not quite every year as Moore predicted).

This trend has continued unabated for 40 years, and, according to Moore and others, there is still no end in sight—at least another ten years is assured.

► An astounding record of sustained exponential progress, unique in history of technology.



What's more, mathematical computing tools are now being implemented on parallel computer platforms, which will provide even greater power to the research mathematician.

► Amassing huge amounts of processing power will not solve all mathematical problems, even those amenable to computational analysis.

VISUAL DYNAMICS

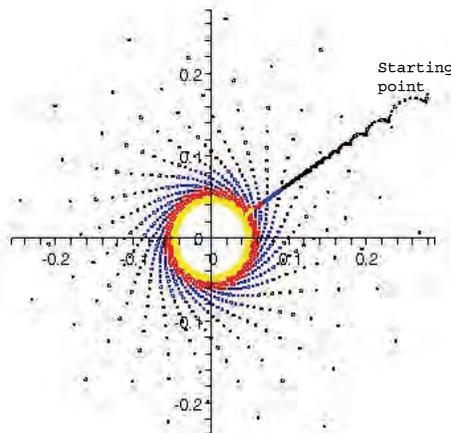
- In recent continued fraction work, we needed to study the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \leftrightarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively.

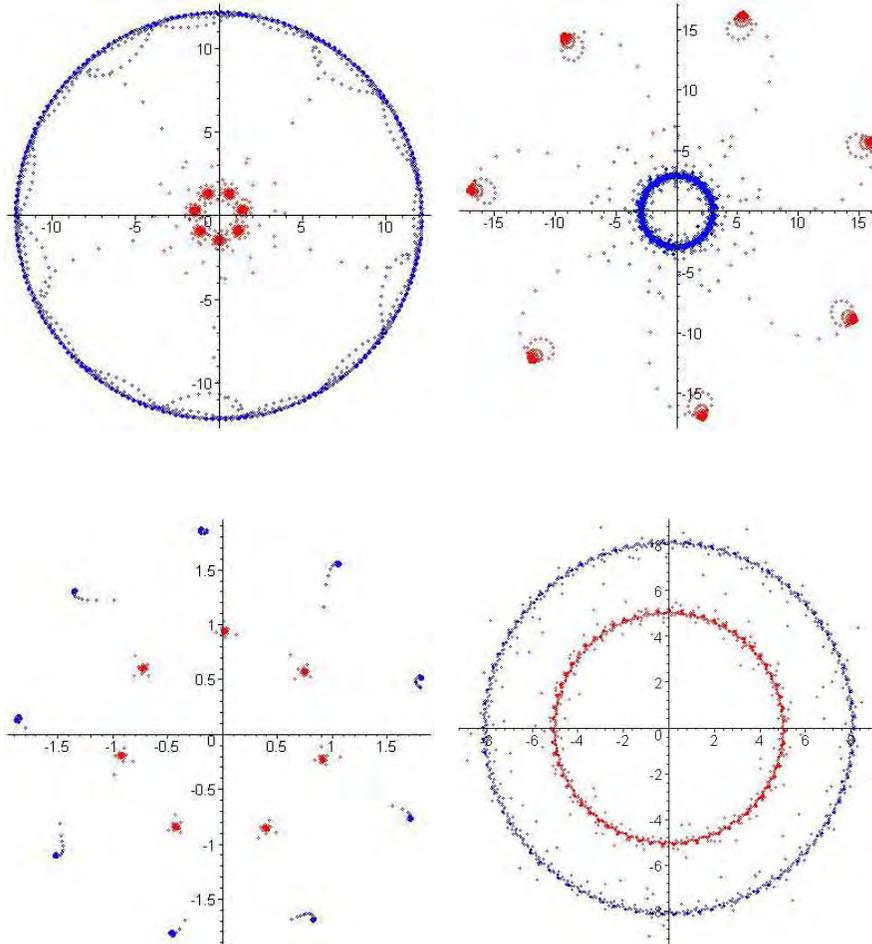
✓ Think of this as a **black box**.

- ▷ Numerically all one sees is $t_n \rightarrow 0$ slowly.
- ▷ Pictorially we *learn* significantly more*:



*... “Then felt I like a watcher of the skies, when a new planet swims into his ken.” (*Chapman’s Homer*)

- Scaling by \sqrt{n} , and coloring odd and even iterates, fine structure appears.



The **attractors** for various $|a| = |b| = 1$.

GAUSS and HADAMARD

Carl Friedrich Gauss, who drew (carefully) and computed a great deal, once noted, *I have the result, but I do not yet know how to get it.**

PAUCA SED MATVRA



Pauca sed Matura

The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.

- ◇ J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.

*Likewise the quote!

De lemniscata, elegantissime omnes expectationes
 superantia acquisivimus et quidem
 per methodos quae campum profectus
 nemini nobis aperuit. Gott. Jul.

Solutio problematis ballistici Gott. Oct.

Comentarum theoriam perfectiorem reddidi Gott. 1/2

Novus in analysisi campus se nobis aperuit,
 scilicet investigatio functionum etc.

Formas superiores considerare coepimus
 Mr. Oct. 1798

Formulas novas exactas pro paralleli
 curvis ————— Mr. Apr. 9.

Terminum medium arithmetico-geometricum
 inter 1 et $\sqrt{2}$ esse = $\frac{\pi}{10}$ usque
 ad figuram viderimus comprobavimus, quare
 demonstrata proventus novus campus in analysisi
 certo aperietur Mr. Maio.

In principiis Geometriae egregios progressus
 fecimus Mr. Sept. 1798

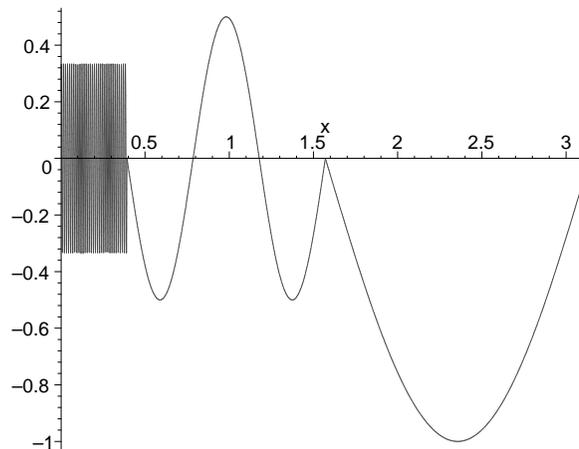
Circa terminos medios arithmetico-geometricos
 multae novae deteximus. Mr. Novemb.

Novus in analysisi campus se nobis aperuit

An excited young Gauss writes: "A new field of analysis has appeared to us, evidently in the study of functions etc." (October 1798)

A BRIEF HISTORY OF RIGOUR

- Greeks: trisection, circle squaring, cube doubling and $\sqrt{2}$.
- Newton and Leibniz: fluxions and infinitesimals.
- Cauchy and Fourier: limits and continuity.
- Frege and Russell, Gödel and Turing.



Fourier series need not converge

THE PHILOSOPHIES OF RIGOUR

- Everyman: **Platonism**—stuff exists (1936)
- Hilbert: **Formalism**—math is invented; formal symbolic games without meaning
- Brouwer: **Intuitionism**—many variants; (embodied cognition)
- Bishop: **Constructivism**—tell me how big; (social constructivism)

† Last two deny the *excluded middle*: $A \vee \tilde{A}$

HALES and KEPLER

- Kepler's conjecture: **the densest way to stack spheres is in a pyramid** is the oldest problem in discrete geometry.
- The most interesting recent example of computer assisted proof. Published in *Annals of Math* with an "only 99% checked" disclaimer.
- This has triggered very varied reactions. (In *Math, Computers Don't Lie. Or Do They?* *NYT* 6/4/04)
- Famous earlier examples: **The Four Color Theorem** and **The non existence of a projective plane of order 10.**
- The three raise and answer quite distinct questions—both real and specious.

Does the proof stack up?

Think peer review takes too long? One mathematician has waited four years to have his paper refereed, only to hear that the exhausted reviewers can't be certain whether his proof is correct. George Szpiro investigates.

P. TURNEYS, SHERBELL/R. BICKEL/CORBIS; H. SITTON/GETTY



Grocers the world over know the most efficient way to stack spheres — but a mathematical proof for the method has brought reviewers to their knees.

Just under five years ago, Thomas Hales made a startling claim. In an e-mail he sent to dozens of mathematicians, Hales declared that he had used a series of computers to prove an idea that has evaded certain confirmation for 400 years. The subject of his message was Kepler's conjecture, proposed by the German astronomer Johannes Kepler, which states that the densest arrangement of spheres is one in which they are stacked in a pyramid — much the same way as grocers arrange oranges.

Soon after Hales made his announcement, reports of the breakthrough appeared on the front pages of newspapers around the world. But today, Hales's proof remains in limbo. It has been submitted to the prestigious *Annals of Mathematics*, but is yet to appear in print. Those charged with checking it say that they believe the proof is correct, but are so exhausted with the verification process that they cannot definitively rule out any errors. So when Hales's manuscript finally does appear in the *Annals*, probably during the next year, it will carry an unusual editorial note — a statement that parts of the paper have proved impossible to check.

At the heart of this bizarre tale is the use of computers in mathematics, an issue that has split the field. Sometimes described as a 'brute force' approach, computer-aided

proofs often involve calculating thousands of possible outcomes to a problem in order to produce the final solution. Many mathematicians dislike this method, arguing that it is inelegant. Others criticize it for not offering any insight into the problem under consideration. In 1977, for example, a computer-aided proof was published for the four-colour theorem, which states that no more than four colours are needed to fill in a map so that any two adjacent regions have different colours^{1,2}. No errors have been found in the proof, but some mathematicians continue to seek a solution using conventional methods.

Pile-driver

Hales, who started his proof at the University of Michigan in Ann Arbor before moving to the University of Pittsburgh, Pennsylvania, began by reducing the infinite number of possible stacking arrangements to 5,000 contenders. He then used computers to calculate the density of each arrangement. Doing so was more difficult than it sounds. The proof involved checking a series of mathematical inequalities using specially written computer code. In all, more than 100,000 inequalities were verified over a ten-year period.

Robert MacPherson, a mathematician at the Institute for Advanced Study in Princeton, New Jersey, and an editor of the *Annals*,

was intrigued when he heard about the proof. He wanted to ask Hales and his graduate student Sam Ferguson, who had assisted with the proof, to submit their finding for publication, but he was also uneasy about the computer-based nature of the work.

The *Annals* had, however, already accepted a shorter computer-aided proof — the paper, on a problem in topology, was published this March³. After sounding out his colleagues on the journal's editorial board, MacPherson asked Hales to submit his paper. Unusually, MacPherson assigned a dozen mathematicians to referee the proof — most journals tend to employ between one and three. The effort was led by Gábor Fejes Tóth of the Alfréd Rényi Institute of Mathematics in Budapest, Hungary, whose father, the mathematician László Fejes Tóth, had predicted in 1965 that computers would one day make a proof of Kepler's conjecture possible.

It was not enough for the referees to rerun Hales's code — they had to check whether the programs did the job that they were supposed to do. Inspecting all of the code and its inputs and outputs, which together take up three gigabytes of memory space, would have been impossible. So the referees limited themselves to consistency checks, a reconstruction of the thought processes behind each step of the proof, and then a

S. TERRY/USPI

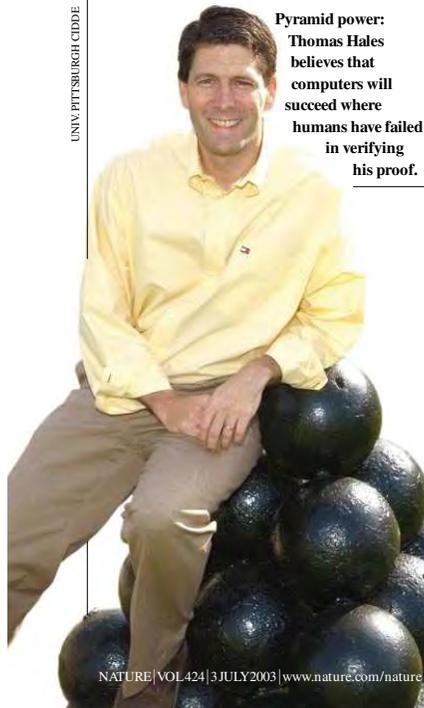
study of all of the assumptions and logic used to design the code. A series of seminars, which ran for full academic years, was organized to aid the effort.

But success remained elusive. Last July, Fejes Tóth reported that he and the other referees were 99% certain that the proof is sound. They found no errors or omissions, but felt that without checking every line of the code, they could not be absolutely certain that the proof is correct.

For a mathematical proof, this was not enough. After all, most mathematicians believe in the conjecture already — the proof is supposed to turn that belief into certainty. The history of Kepler's conjecture also gives reason for caution. In 1993, Wu-Yi Hsiang, then at the University of California, Berkeley, published a 100-page proof of the conjecture in the *International Journal of Mathematics*⁴. But shortly after publication, errors were found in parts of the proof. Although Hsiang stands by his paper, most mathematicians do not believe it is valid.

After the referees' reports had been considered, Hales says that he received the following letter from MacPherson: "The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy ... One can speculate whether their process would have converged to a definitive answer had they had a more clear manuscript from the beginning, but this does not matter now."

UNIV. PITTSBURGH CIDDE



Pyramid power: Thomas Hales believes that computers will succeed where humans have failed in verifying his proof.

NATURE | VOL 424 | 3 JULY 2003 | www.nature.com/nature



Star player: Johannes Kepler's conjecture has kept mathematicians guessing for 400 years.

The last sentence lets some irritation shine through. The proof that Hales delivered was by no means a polished piece. The 250-page manuscript consisted of five separate papers, each a sort of lab report that Hales and Ferguson filled out whenever the computer finished part of the proof. This unusual format made for difficult reading. To make matters worse, the notation and definitions also varied slightly between the papers.

Rough but ready

MacPherson had asked the authors to edit their manuscript. But Hales and Ferguson did not want to spend another year reworking their paper. "Tom could spend the rest of his career simplifying the proof," Ferguson said when they completed their paper. "That doesn't seem like an appropriate use of his time." Hales turned to other challenges, using traditional methods to solve the 2,000-year-old honeycomb conjecture, which states that of all conceivable tiles of equal area that can be used to cover a floor without leaving any gaps, hexagonal tiles have the shortest perimeter⁵. Ferguson left academia to take a job with the US Department of Defense.

Faced with exhausted referees, the editorial board of the *Annals* decided to publish the paper — but with a cautionary note. The paper will appear with an introduction by the editors stating that proofs of this type, which involve the use of computers to check a large number of mathematical statements, may be impossible to review in full. The matter might have ended there, but for Hales, having a note attached to his proof was not satisfactory.

This January, he launched the Flyspeck project, also known as the Formal Proof of Kepler. Rather than rely on human referees, Hales intends to use computers to verify

every step of his proof. The effort will require the collaboration of a core group of about ten volunteers, who will need to be qualified mathematicians and willing to donate the computer time on their machines. The team will write programs to deconstruct each step of the proof, line by line, into a set of axioms that are known to be correct. If every part of the code can be broken down into these axioms, the proof will finally be verified.

Those involved see the project as doing more than just validating Hales's proof. Sean McLaughlin, a graduate student at New York University, who studied under Hales and has used computer methods to solve other mathematical problems, has already volunteered. "It seems that checking computer-assisted proofs is almost impossible for humans," he says. "With luck, we will be able to show that problems of this size can be subjected to rigorous verification without the need for a referee process."

But not everyone shares McLaughlin's enthusiasm. Pierre Deligne, an algebraic geometer at the Institute for Advanced Study, is one of the many mathematicians who do not approve of computer-aided proofs. "I believe in a proof if I understand it," he says. For those who side with Deligne, using computers to remove human reviewers from the refereeing process is another step in the wrong direction.

Despite his reservations about the proof, MacPherson does not believe that mathematicians should cut themselves off from computers. Others go further. Freek Wiedijk, of the Catholic University of Nijmegen in the Netherlands, is a pioneer of the use of computers to verify proofs. He thinks that the process could become standard practice in mathematics. "People will look back at the turn of the twentieth century and say 'that is when it happened,'" Wiedijk says.

Whether or not computer-checking takes off, it is likely to be several years before Flyspeck produces a result. Hales and McLaughlin are the only confirmed participants, although others have expressed an interest. Hales estimates that the whole process, from crafting the code to running it, is likely to take 20 person-years of work. Only then will Kepler's conjecture become Kepler's theorem, and we will know for sure whether we have been stacking oranges correctly all these years. ■

George Szpiro writes for the Swiss newspapers *NZZ* and *NZZ am Sonntag* from Jerusalem, Israel. His book *Kepler's Conjecture* (Wiley, New York) was published in February.

1. Appel, K. & Haken, W. *Illinois J. Math.* **21**, 429–490 (1977).
2. Appel, K., Haken, W. & Koch, J. *Illinois J. Math.* **21**, 491–567 (1977).
3. Gabai, D., Meyerhoff, G. R. & Thurston, N. *Ann. Math.* **157**, 335–431 (2003).
4. Hsiang, W.-Y. *Int. J. Math.* **4**, 739–831 (1993).
5. Hales, T. C. *Discrete Comput. Geom.* **25**, 1–22 (2001).

Flyspeck

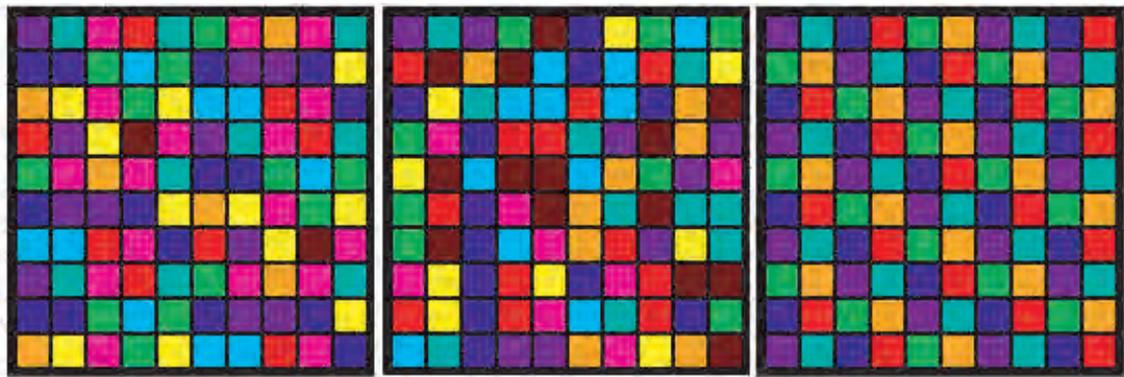
► www.math.pitt.edu/~thales/flyspeck/index.html

A PI INTEGRAL

A. Why $\pi \neq \frac{22}{7}$:

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

$$\left[\int_0^t \cdot = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4 \arctan(t) . \right]$$



Archimedes: $223/71 < \pi < 22/7$

The Colour Calculator

TWO INFINITE PRODUCTS

A. A rational evaluation:

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}.$$

...

B. And a transcendental one:

$$\prod_{n=2}^{\infty} \frac{n^2 - 1}{n^2 + 1} = \frac{\pi}{\sinh(\pi)}.$$

- The *Inverse Symbolic Calculator* can identify this product, as can **Maple**.
- f, Σ, Π are now largely algorithmic not **black arts**.

COINCIDENCE OR FRAUD

- Coincidences do occur

The approximations

$$\pi \approx \frac{3}{\sqrt{163}} \log(640320)$$

and

$$\pi \approx \sqrt{2} \frac{9801}{4412}$$

occur for deep number theoretic reasons—the first good to 15 places, the second to eight. By contrast

$$e^\pi - \pi = \mathbf{19.999099979}189475768\dots$$

most probably for no good reason.

- ✓ This seemed more bizarre on an eight digit calculator.

Likewise, as spotted by Pierre Lanchon recently,

$$e = \overline{\mathbf{10.10110111111000010}}101000101100\dots$$

while

$$\pi = 11.0010\overline{\mathbf{010000111110110101}}01000\dots$$

have 19 bits agreeing in base two—with one read right to left.

- More extended coincidences are almost always contrived ...

HIGH PRECISION FRAUD

$$\sum_{n=1}^{\infty} \frac{[n \tanh(\pi)]}{10^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to **268 places**; while

$$\sum_{n=1}^{\infty} \frac{[n \tanh(\frac{\pi}{2})]}{10^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to just **12 places**.

- Both are actually transcendental numbers.

Correspondingly the *simple continued fractions* for $\tanh(\pi)$ and $\tanh(\frac{\pi}{2})$ are respectively

[0, 1, **267**, 4, 14, 1, 2, 1, 2, 2, 1, 2, 3, 8, 3, 1]

and

[0, 1, **11**, 14, 4, 1, 1, 1, 3, 1, 295, 4, 4, 1, 5, 17, 7]

- Bill Gosper describes how continued fractions let you “see” what a number is. “[I]t’s completely astounding ... **it looks like you are cheating God somehow.**”

SEEING PATTERNS in PARTITIONS

The number of *additive partitions* of n , $p(n)$, is *generated* by

$$(2) \quad 1 + \sum_{n \geq 1} p(n)q^n = \frac{1}{\prod_{n \geq 1} (1 - q^n)}.$$

Thus, $p(5) = 7$ since

$$\begin{aligned} 5 &= 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1. \end{aligned}$$

Developing (2) is an introduction to enumeration via *generating functions* as discussed in Polya's change example.

Additive partitions are harder to handle than multiplicative factorizations, but they may be introduced in the elementary school curriculum with questions like: *How many 'trains' of a given length can be built with Cuisenaire rods?*

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable deep congruences like

$$p(\mathbf{5n+4}) \equiv 0 \pmod{\mathbf{5}}, \quad p(\mathbf{7n+5}) \equiv 0 \pmod{\mathbf{7}}$$

$$p(\mathbf{11n+6}) \equiv 0 \pmod{\mathbf{11}},$$

from relatively limited data like $P(q) =$

$$\begin{aligned} & 1 + q + 2q^2 + 3q^3 + \underline{5}q^{\mathbf{4}} + \overline{7}q^{\mathbf{5}} + 11q^6 + 15q^7 \\ & + 22q^8 + \underline{30}q^{\mathbf{9}} + 42q^{10} + 56q^{11} + \overline{77}q^{\mathbf{12}} \\ & + 101q^{13} + \underline{135}q^{\mathbf{14}} + 176q^{15} + 231q^{16} \\ & + 297q^{17} + 385q^{18} + \overline{490}q^{\mathbf{19}} \\ & + 627q^{20} + 792q^{21} + 1002q^{22} \\ & + \dots + p(200)q^{200} + \dots \end{aligned}$$

(3)

- Cases $5n + 4$ and $7n + 5$ are flagged in (3).
- Of course, it is easier to (heuristically) confirm than find these fine examples of **Mathematics: the science of patterns***

*Keith Devlin's 1997 book.

IS HARD or EASY BETTER?

A modern computationally driven question is *How hard is $p(n)$ to compute?*

- In **1900**, it took the father of combinatorics, Major Percy MacMahon (1854–1929), months to compute $p(200)$ using recursions developed from (2).
- By **2000**, *Maple* would produce $p(200)$ in seconds if one simply demands the 200'th term of the Taylor series. A few years earlier it required being careful to compute the series for $\prod_{n \geq 1} (1 - q^n)$ first and then the series for the *reciprocal* of that series!
- This baroque event is occasioned by *Euler's pentagonal number theorem*

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n = -\infty}^{\infty} (-1)^n q^{(3n+1)n/2}.$$

- The reason is that, if one takes the series for (2), the software has to deal with **200** terms on the bottom. But the series for $\prod_{n \geq 1} (1 - q^n)$, has only to handle the **23** non-zero terms in series in the pentagonal number theorem.
- If introspection fails, we can find the *pentagonal numbers* occurring above in Sloane and Plouffe's on-line '**Encyclopedia of Integer Sequences**': www.research.att.com/personal/njas/sequences/eisonline.html.
- This ex post facto algorithmic analysis can be used to facilitate independent student discovery of the pentagonal number theorem, and like results.

- The difficulty of estimating the size of $p(n)$ analytically—so as to avoid enormous or unattainable computational effort—led to some marvellous mathematical advances*.
- The corresponding ease of computation may now act as a retardant to insight.
- New mathematics is discovered only when prevailing tools run totally out of steam.
- This raises a caveat against mindless computing:

Will a student or researcher discover structure when it is easy to compute without needing to think about it? Today, she may thoughtlessly compute $p(500)$ which a generation ago took much, much pain and insight.

*By researchers including Hardy and Ramanujan, and Rademacher

The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen.

David Berlinski, "Ground Zero: A Review of *The Pleasures of Counting*, by T. W. Koerner," 1997.

- As all sciences rely more on 'dry experiments', via computer simulation, the boundary between physics (e.g., *string theory*) and mathematics (e.g., by *experiment*) is delightfully blurred.
- An early exciting example is provided by **gravitational boosting**.

GRAVITATIONAL BOOSTING

“The Voyager Neptune Planetary Guide” (JPL Publication 89–24) has an excellent description of Michael Minovitch’ computational and unexpected discovery of *gravitational boosting* (also known as slingshot magic) at the Jet Propulsion Laboratory in 1961.

The article starts by quoting Arthur C. Clarke

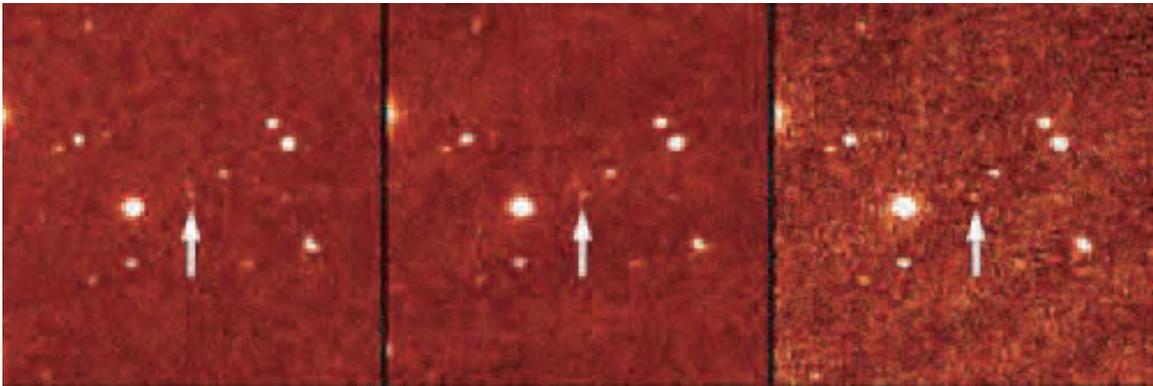
“Any sufficiently advanced technology is indistinguishable from magic.”



Sedna And Friends in 2004

Until he showed *Hohmann transfer ellipses* were not least energy paths to the outer planets:

“most planetary mission designers considered the gravity field of a target planet to be somewhat of a nuisance, to be cancelled out, usually by onboard Rocket thrust.”



- Without a boost from the orbits of [Saturn](#), [Jupiter and Uranus](#), the Earth-to-Neptune Voyager mission (achieved in 1989 in around a decade) would have taken over 30 years!
- We would still be waiting; longer to see Sedna confirmed (8 billion miles away—3 times further than Pluto).

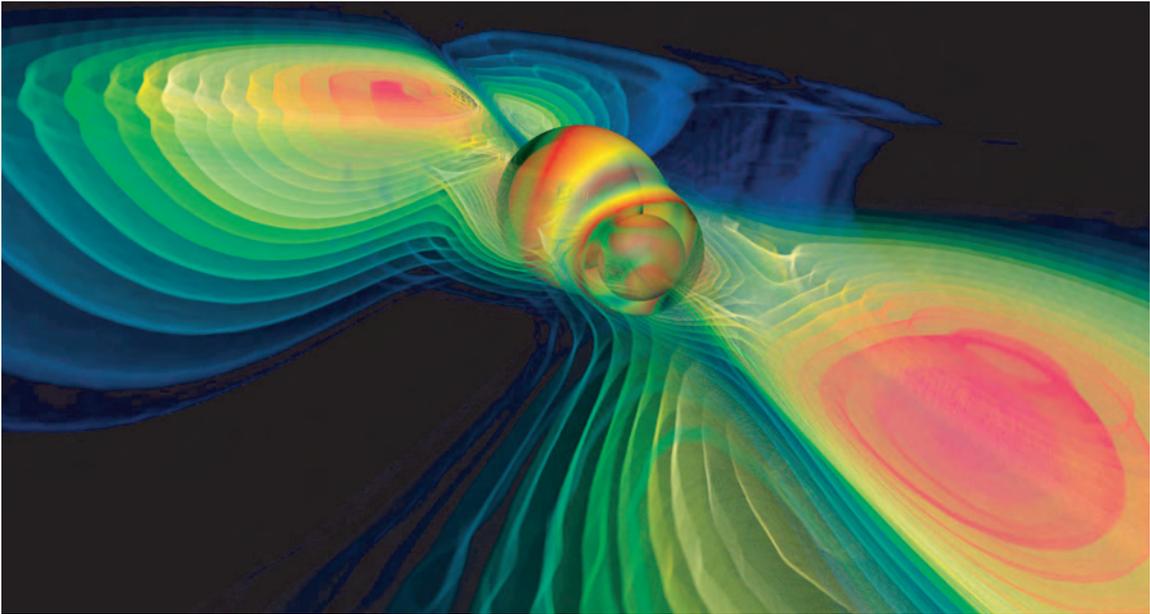
LIGO: MATH and the COSMOS

Einstein's theory of general relativity describes how massive bodies curve space and time; it realizes gravity as movement of masses along shortest paths in curved space-time.

- A subtle mathematical inference is that relatively accelerating bodies will produce ripples on the curved space-time surface, propagating at the speed of light: **gravitational waves**.

These extraordinarily weak cosmic signals hold the key to a new era of astronomy *if only* we can build detectors and untangle the mathematics to interpret them. The signal to noise ratio is tiny!

LIGO, the [Laser Interferometer Gravitational-Wave Observatory](#), is such a developing US gravitational wave detector.



One of the first 3D simulations of the gravitational waves arising when two black holes collide

- Only recently has the computational power existed to realise such simulations, on computers such as at WestGrid.

COME and VISIT in D-DRIVE



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WHERE MATHEMATICS GOES LIVE

- We are following in a great tradition of **Mathematical Models** as illustrated in the **APPENDIX** below . . .

19th C. MATHEMATICAL MODELS



Felix Klein's heritage

Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ...

I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane.

Augustus de Morgan (1806–71).

- de Morgan, first President of the London Mathematical Society, was equally influential as an educator and a researcher.
- There is evidence that young children see more naturally in three than two dimensions.

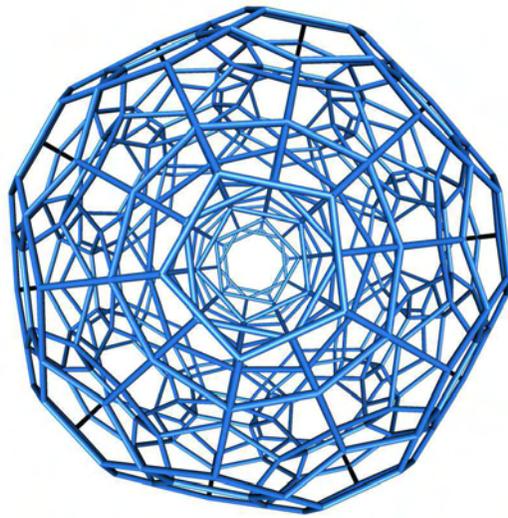
(See discussion at www.colab.sfu.ca/ICIAM03/)



Coxeter's octahedral kaleidoscope (circa 1925)

Modern science is often driven by fads and fashion, and mathematics is no exception. Coxeter's style, I would say, is singularly unfashionable. He is guided, I think, almost completely by a profound sense of what is beautiful.

(Robert Moody)



A four dimensional polytope with 120 dodecahedral faces

- In a **1997** paper, Coxeter showed his friend Escher, knowing no math, had achieved “mathematical perfection” in etching *Circle Limit III*. “Escher did it by instinct,” Coxeter wrote, “I did it by trigonometry.”
- Fields medalist David Mumford recently noted that Donald Coxeter (**1907-2003**) placed great value on working out details of complicated explicit examples.

In my book, Coxeter has been one of the most important 20th century mathematicians —not because he started a new perspective, but because he deepened and extended so beautifully an older esthetic. The classical goal of geometry is the exploration and enumeration of geometric configurations of all kinds, their symmetries and the constructions relating them to each other.

The goal is not especially to prove theorems but to discover these perfect objects and, in doing this, theorems are only a tool that imperfect humans need to reassure themselves that they have seen them correctly.

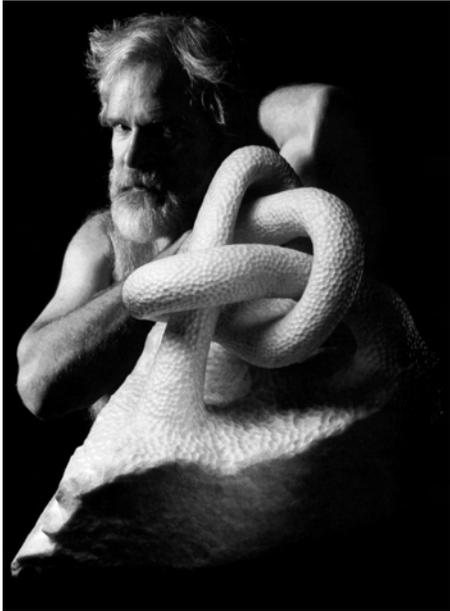
(David Mumford, 2003)

20th C. MATHEMATICAL MODELS



Fergusson's "Eight-Fold Way" sculpture

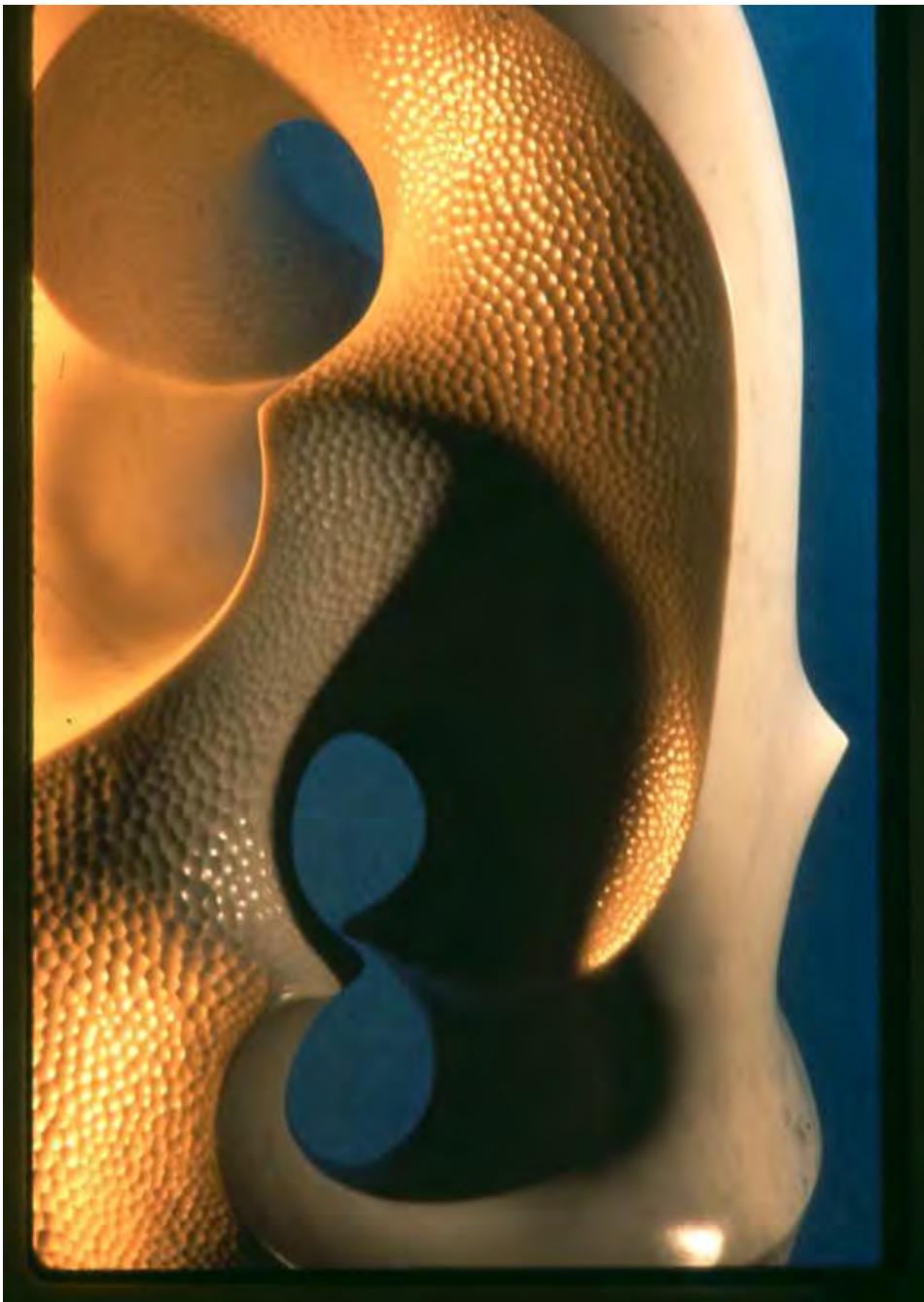
The Fergusons won the 2002 Communications Award, of the Joint Policy Board of Mathematics. The citation runs:



They have dazzled the mathematical community and a far wider public with exquisite sculptures embodying mathematical ideas, along with artful and accessible essays and lectures elucidating the mathematical concepts.

It has been known for some time that the hyperbolic volume V of the **figure-eight knot complement** is

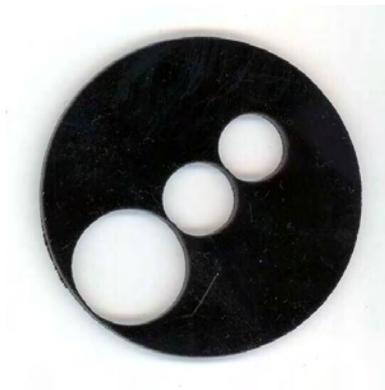
$$\begin{aligned} V &= 2\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{n \binom{2n}{n}} \sum_{k=n}^{2n-1} \frac{1}{k} \\ &= 2.029883212819307250042405108549 \dots \end{aligned}$$



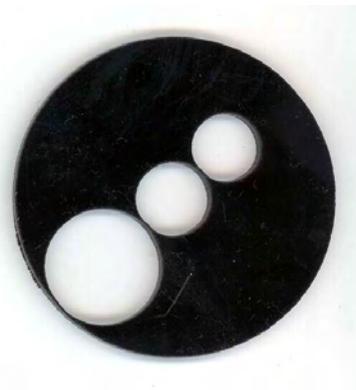
Ferguson's "Figure-Eight Knot Complement" sculpture

In 1998, British physicist David Broadhurst conjectured $V/\sqrt{3}$ is a *rational linear combination* of

$$(4) \quad C_j = \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n (6n + j)^2}.$$



Ferguson's
subtractive
image
of the
BBP Pi
formula



Indeed, as Broadhurst found, *using Ferguson's PSLQ*:

$$V = \frac{\sqrt{3}}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n} \times \left\{ \frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2} \right\}.$$

- Entering the following code in the *Mathematician's Toolkit*, at www.expmath.info:

```
v = 2 * sqrt[3] * sum[1/(n * binomial[2*n,n])
    * sum[1/k,{k, n,2*n-1}], {n, 1, infinity}]
```

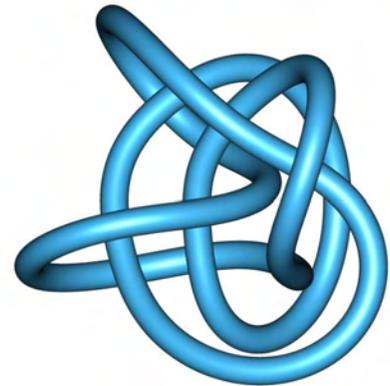
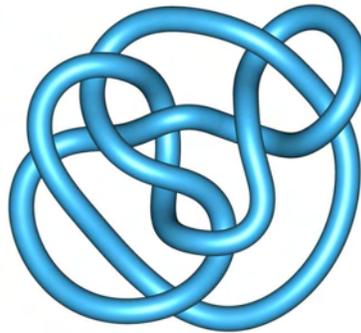
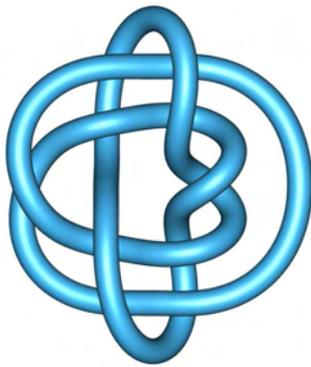
```
pslq[v/sqrt[3],
    table[sum[(-1)^n/(27^n*(6*n+j)^2),
    {n, 0, infinity}], {j, 1, 6}]]
```

recovers the solution vector

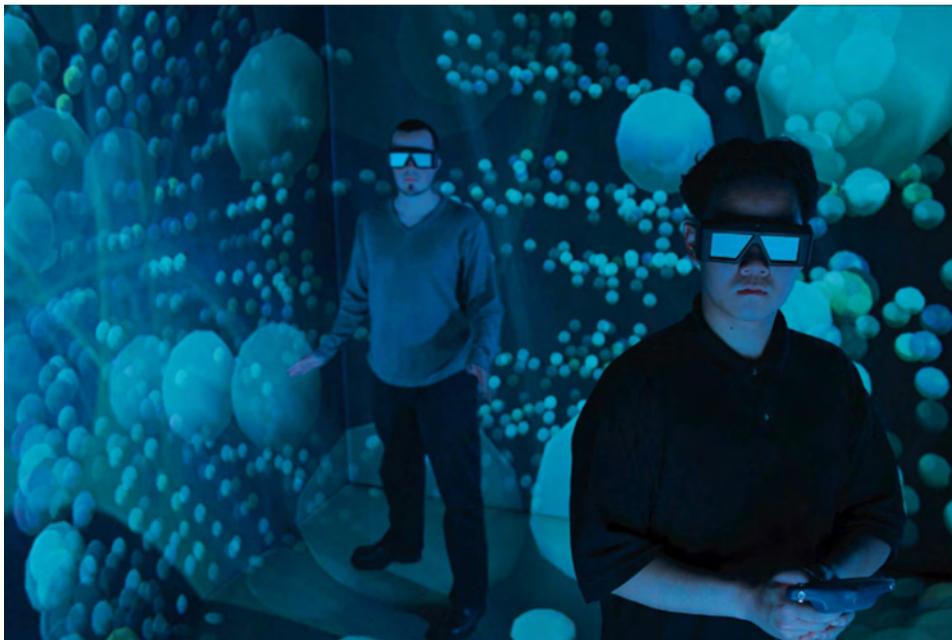
(9, -18, 18, 24, 6, -2, 0).

- The *first proof* that this formula holds is given in our new book.
- The formula is inscribed on each cast of the sculpture—marrying both sides of Helaman!

21st C. MATHEMATICAL MODELS



Knots 10₁₆₁ (L) and 10₁₆₂ (C) agree (R)*.



In NewMIC's Cave or Plato's?

***KnotPlot**: from Little (1899) to Perko (1974) and on.