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CONVEX RELATIONS IN ANALYSIS AND OPTIMIZATION

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In this paper our aim is to show some of the ways in which the use of convex relations simplifies, unifies and strengthens the study of convex constrained optimization problems with vector objectives. First we sketch the topological and analytic properties of convex relations which are of particular use in optimization and many of which are of considerable independent analytic interest. Subsequently we apply these considerations to the study of constrained optimization problems.

1. ANALYSIS

1.1 Motivation

Let X be a *(real, separated, topological) vector space* and consider the *ordinary convex program*

$$\text{(P)}\quad p = \inf_{x\in C}\left\{f(x)\ \middle|\ \begin{matrix} g_i(x) \le 0 & (i=1,\dots,m) \\ h_j(x) = 0 & (j=1,\dots,k)\end{matrix}\right\} \tag{1.1}$$

where $f, g_i, h_j : X \to \bar{R}$ and the f, g_i are *convex* while the h_j are *affine*, on the convex set C. Set

$$H_g(x) = \begin{cases} g(x) + R^m_+ , & x \in C \\ \emptyset , & x \notin C \end{cases} , \tag{1.2}$$

$$H_h(x) = \{h(x)\} , \tag{1.3}$$

where $g = (g_1,\dots,g_m)$, $h = (h_1,\dots,h_k)$ and R^m_+ is the non-negative orthant in R^m. Now set

$$H(x) = (H_g(x),\ H_h(x)) . \tag{1.4}$$

Then H is a *(multifunction, set-valued map) relation* between X and $Y = R^{m+k}$ and it is clear that we may rewrite (P) in the multivalued equation form

$$\text{(P)}\quad p = \inf\,\{f(x)\ |\ 0 \in H(x)\} . \tag{1.5}$$

This will be the form in which we will study (P) in the sequel. The reasons for this are various: it unifies the study of equality and

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inequality constraints via (1.4); it often simplifies proofs and strengthens results; also it is the author's contention that many results about constraint structure are intrinsically results about H and *not* about g and h.

Let us observe that H will turn out to be a convex relation as defined below exactly when g is convex and h is affine on a convex set C.

1.2 Some Basic Relationships

Before turning to convex relations let us summarize the central properties of relations with which we will be concerned. The reader is referred to Berge (1959), Dolecki (1981), Jameson (1972), Huard (1979), Robinson (1972; 1976a), and Rockafellar (1970; 1976) for more details.

Throughout the paper all spaces are supposed to be real separated topological vector spaces. If one wishes to study purely algebraic notions one may use the *finest locally convex* or *core* topology (see Oettli (1979) and Robertson and Robertson (1964)). Let $H:X \to 2^Y$ be a relation between X and Y. We will reserve lower case letters for relations and upper case letters for functions. Thus we will not usually distinguish between h and {h} and also we will consider $H:X \to Y$ although this is an abuse of notation. The *domain* of H, D(H), is defined by

$$D(H) = \{x \in X \mid H(x) \neq \emptyset\} . \tag{1.6}$$

Given any set C in X we define

$$H(C) = \cup\{H(x) \mid x \in C\} . \tag{1.7}$$

Then $H(X) = R(H)$ is the *range* of H. The *inverse* relation $H^{-1}:Y \to X$ is defined by

$$H^{-1}(y) = \{x \in X \mid y \in H(x)\} . \tag{1.8}$$

The *graph* of H is given by

$$\text{GrH} = \{(x,y) \mid y \in H(x)\} . \tag{1.9}$$

Thus H and H^{-1} essentially share their graph since (x,y) lies in GrH exactly if (y,x) lies in GrH^{-1}. Then one may identify relations between X and Y and sets in XxY via (1.9).

If $H:X \to Y$ and $K:Y \to Z$ the composite relation is defined by

$$(HK)(x) = H(K(x)) , \tag{1.10}$$

using (1.7). It is easily verified that

$$(HK)^{-1} = K^{-1}H^{-1} . \tag{1.11}$$

Now $H:X \to Y$ is said to be *lower semi-continuous (LSC) at* (x_0,y_0) if given any neighbourhood V of zero in Y one may find a neighbourhood U of zero in X with

$$H(x) \cap (V + y_0) \neq \emptyset \qquad x \in U + x_0 . \tag{1.12}$$

It is implicit in (1.12) that $y_0 \in \text{cl } H(x_0)$. If H is LSC at (x_0,y_0) for all y_0 in $H(x_0)$ we say that H is *LSC at* x_0.

Similarly H is *open at* (x_0,y_0) if given any neighbourhood U of zero in X one can find a neighbourhood V of zero in Y with

$$V + y_0 \subset H(x_0 + U). \tag{1.13}$$

If this holds for all x_0 in $H^{-1}(y_0)$ we say H is *open at* y_0.

Since (1.12) is equivalent to $U + x_0 \subset H^{-1}(V + y_0)$ we see that the following proposition holds.

PROPOSITION 1:

(a) H *is open at* (x_0,y_0) *if and only if* H^{-1} *is* LSC *at* (y_0,x_0).

(b) H *is open at* y_0 *if and only if* H^{-1} *is* LSC *at* y_0.

(c) *If* H *is open at* (x_0,y_0) *and* K *is open at* (y_0,z_0) *then* HK *is open at* (x_0,z_0).

(d) *If* H *is* LSC *at* (x_0,y_0) *and* K *is* LSC *at* (y_0,z_0) *then* HK *is* LSC *at* (x_0,z_0).

PROOF: (a) and (b) are clear from the previous discussion. Now (c) follows easily from (1.10) and (1.13) whence (d) follows from (1.11) (a) and (c). □

1.3 Convex Relations

A relation $H:X \to Y$ is *convex* if its graph is convex. Equivalently, one has

$$t\,H(x_1) + (1-t)\,H(x_2) \subset H(tx_1 + (1-t)x_2) \tag{1.14}$$

whenever $0 \le t \le 1$ and x_1, x_2 lie in X. It is immediate from the previous section that the inverse and composition of convex relations are convex and that the domain and range of a convex relation are convex. Moreover there is a one-to-one identification of convex relations between X and Y and convex sets in XxY.

Example 1:

(a) Let $g:C \subset X \to Y$ be a single valued function and $P \subset Y$ a convex cone (or just a convex set). Then

$$H_g(x) = \begin{Bmatrix} g(x) + P & x \in C \\ \emptyset & x \notin C \end{Bmatrix} \tag{1.15}$$

is a convex relation exactly when g is P-convex on C, (Borwein (1977a)). In particular this is true for H as in (1.2) or (1.4). The empty set here plays the role of $(+\infty)$ in convex analysis, the whole space the role of $(-\infty)$. Note that $\mathrm{Gr}H_g$ is exactly the P-*epigraph* of g on C.

(b) Let $A:X \to Y$ be a linear operator and let C and D be convex sets

$$H_A(x) = \begin{Bmatrix} A(x) + D & x \in C \\ \emptyset & x \notin C \end{Bmatrix}. \tag{1.16}$$

Then H_A is a convex relation. This is a particularly useful special case of (1.15).

(c) With H_g as in (1.15)

$$L_g(y) = H_g^{-1}(y) = \{x \in X \mid y \in g(x) + P, \quad x \in C\} \tag{1.17}$$
$$= g^{-1}(y-P) \cap C$$

is the "level set" relation and is convex exactly when H_g is. Thus in this framework L_g and H_g have entirely symmetric roles. For example, if g is continuous at x_0 in int C, H_g is LSC at x_0 and hence L_g is open at x_0.

(d) Let $H_1:X \to Y_1$, $H_2:X \to Y_2$ be convex. Then so is H given by

$$H(x) = (H_1,H_2)(x) = (H_1(x),H_2(x))\ . \tag{1.18}$$

The inverse of H is given by

$$H^{-1}(y_1,y_2) = H_1^{-1}(y_1) \cap H_2^{-1}(y_2)\ . \tag{1.19}$$

(e) Let H_1, $H_2:X \to Y$ be convex. So also is the *sum* $H_1 + H_2$ given by

$$(H_1 + H_2)(x) = H_1(x) + H_2(x) \tag{1.20}$$

and the *convolution* $H_1 \,\square\, H_2$ given by

$$H_1 \,\square\, H_2(x) = \{y_1 + y_2 \mid y_1 \in H(x_1),\ y_2 \in H(x_2),\ x_1 + x_2 = x\}. \tag{1.21}$$

Note that convolution corresponds to adding graphs and sum to adding images. As for convex function one may define partial convolution. In the real valued case we have

$$f \,\square\, g = \inf H_f \,\square\, H_g. \tag{1.22}$$

However, (1.21) makes sense in many situations in which (1.22) is not well defined.

(f) A convex relation is a *convex process* if its graph is a cone. For any convex relation H we may define the *convex process* P *generated by* H by

$$\mathrm{Gr}\ P = \mathrm{cone}\ (\mathrm{Gr}\ H) \tag{1.23}$$

or equivalently

$$P(x) = \bigcup_{\lambda>0} \{\lambda\ H(\tfrac{x}{\lambda})\}. \tag{1.24}$$

The reader is referred to Berge (1959), Rockafellar (1967; 1970), Robinson (1972) and Makarov and Rubinov (1977) for information on convex processes and to Borwein (1977a; 1979), Robinson (1972; 1976a) and Jameson (1970; 1972) for more information on convex relations.

Any other pieces of convex terminology are consistent with that of Ekeland and Temam (1976) or Robertson and Robertson (1964).

1.4 Convexity and Continuity

In a fashion analogous to that of convex functions the continuity properties of convex relations are considerably simpler than those of arbitrary relations.

PROPOSITION 2:

(a) *Suppose* $H:X \to Y$ *is convex and open* (LSC) *at* (x_0,y_0). *Then* H *is open* (LSC) *at* (x_1,y_1) *whenever* $y_1 \in H(x_1)$ *and* $y_1 \in \text{core } R(H)$ $(x_1 \in \text{core } D(H))$.

(b) *In particular if* H *is open* (LSC) *at* (x_0,y_0) *then* H *is open* (LSC) *at* $y_0(x_0)$.

(c) *If* $H_1(x) \subset H_2(x) \quad \forall\, x \in X$ (1.25)

where H_1 *is open* (LSC) *at* (x_0,y_0) *and* H_2 *is convex then* H_2 *is open* (LSC) *at* $y_0(x_0)$.

PROOF: We prove only the unbracketed assertions. The parenthetic assertions follow on reversing the roles of X and Y and using H^{-1}, as in Proposition 1.

(a) Since y_1 is in core $R(H)$ one may find $y_2 \in H(x_2)$ and $0 < \varepsilon < 1$ with $y_1 = \varepsilon y_0 + (1-\varepsilon)y_2$. Set $\bar{x}_1 = \varepsilon x_0 + (1-\varepsilon)x_2$. Then suppose that a neighbourhood of zero in X,W, is given and that there exists a neighbourhood V with

$$y_0 + V \subset H(x_0 + U) \tag{1.26}$$

for a balanced U with $U + U \subset W$. Then using convexity and (1.26)

$$\begin{aligned} y_1 + \varepsilon V = \varepsilon(y_0 + V) + (1-\varepsilon)y_2 &\subset \varepsilon H(x_0 + U) + (1-\varepsilon)H(x_2) \\ &\subset H(\bar{x}_1 + \varepsilon U). \end{aligned} \tag{1.27}$$

Now pick $0 < \delta < 1$ with $\delta(\bar{x}_1 - x_1) \in \varepsilon U$. Then

$$\begin{aligned} y_1 + \delta\varepsilon V = (1-\delta)y_1 + \delta(y_1 + \varepsilon V) &\subset (1-\delta)\, H(x_1) + \delta H(\bar{x}_1 + \varepsilon U) \\ &\subset H(x_1 + \delta(\bar{x}_1 - x_1) + \delta\varepsilon U) \end{aligned} \tag{1.28}$$

and thus

$$y_1 + \delta\varepsilon V \subset H(x_1 + \varepsilon W) \tag{1.29}$$

and H is open at (x_1,y_1). Note that one actually now gets

$$y_1 + \lambda\delta V \subset H(x_1 + \lambda W) \tag{1.30}$$

for $0 < \lambda \leq \varepsilon$. In Dolecki's terms (1981) H is *open at linear rate*. If X is locally convex the previous argument may be a little simplified.

(b) is now immediate since y_0 lies in core $R(H)$.

(c) Since H_1 is open at (x_0,y_0) it follows that H_2 is open at (x_0,y_0) and (b) now applies. □

In a normed or semi-normed setting the constants in (1.30) can be quantified as in Robinson (1976). Proceeding as above one can show that a convex relation is locally uniformly open or LSC at any point at which it is open or LSC: i.e. for (x_1,y_1) near (x_0,y_0) in Gr H, (Dolecki (1981)). In particular one derives the following.

PROPOSITION 3: Let $H:Y \to Y$ *be a convex relation between normed spaces. Let* H *be LSC at* x_0. *For all* $\eta > 0$ *sufficiently large there exist* ε, $k > 0$ *such that*

$$D(H(x) \cap \eta B,\ H(x') \cap \eta B) \leq k\, ||x-x'|| \tag{1.31}$$

whenever $||x-x_0||$, $||x'-x_0|| \leq \varepsilon$.

Here B is the unit ball in Y and D is the Hausdorff metric. A homogeneous form of the above is given by Robinson (1972) and in Tuy and Du'ong (1978). One can derive the corresponding Lipschitzness result for convex functions from (1.31). It is more convenient to establish the next lemma.

Recall that a cone S in Y is said to be *normal* if there is a base at zero of neighbourhoods V with

$$(V - S) \cap (S - V) \subset V. \tag{1.32}$$

Most commonly occurring cones are normal but by no means all. The reader is referred to Peressini (1967), Schaefer (1971), Jameson (1970), or Borwein (1980a) for more details.

LEMMA 4: Let $f:X \to Y$ *be S-convex. Let* $H_f(x) = f(x) + S$.

(a) Then H_f *is* LSC *at* x_0, *whenever* f *is continuous at* x_0.

(b) Conversely, if S *is normal and* H_f *is* LSC *at* x_0, f *is continuous at* x_0.

PROOF:

(a) is immediate.

(b) Let us suppose $x_0 = 0$, $f(x_0) = 0$. Since H_f is LSC at x_0 one can find for each neighbourhood V of zero a neighbourhood U of zero with

$$(f(x) + S) \cap V \neq \emptyset \qquad x \in U . \tag{1.33}$$

Since f is S-convex

$$f(x) + f(-x) \subset 2f(0) + S = S . \tag{1.34}$$

Hence for x in U

$$f(-x) \subset S - f(x) \subset (S + S - V) = (S - V)$$

using (1.33) and (1.34). Thus if x lies in $-u$

$$f(x) \subset (S - V) \cap (V - S) \subset V \tag{1.35}$$

on using (1.33) again and the normality of S. Thus f is actually continuous at x_0. □

It follows as in Borwein (1980a) and in Robinson (1976) that, in the normed case, f is actually locally Lipschitz throughout int(domf). When S is not normal it is usual that H_f is LSC while f is not continuous. In these cases one is much better advised to study H_f than f itself. A case of interest which is not normal occurs when Y = D[0,1] is the space of continuously differentiable functions on the unit interval with

$$||f|| = |f(0)| + \sup_{0 \le t \le 1} |f'(t)| \tag{1.36}$$

and S is the cone of non-negative functions. Then S is not normal since it takes no account of derivative behavior. This is a prototype for Sobelov space behavior. Note also that in the core topology any pointed cone is weakly normal.

For the final result in this section we need one more topological definition. A relation $H:X \to Y$ is *strongly open* at (x_0,y_0) if

$$y_0 \in \text{int } H(x_0) . \tag{1.37}$$

PROPOSITION 5: Let $H_1:X \to Y_1$, $H_2:X \to Y_2$ *be convex. Let* $y_1 \in H_1(x_1)$, $y_2 \in H_2(x_1)$. *Suppose that*

(a) H_1 *is* LSC *at* x_1 *and strongly open at* (x_1,y_1),

(b) H_2 *is open at* y_2.

Then

$$H(x) = (H_1(x),H_2(x))$$

is open at x_1.

A proof of this result is given in Borwein (1980b) and a special case may be found in Robinson (1976a). A dual result may be given for H as in (1.19) involving semi-continuity.

Example 2: If H_g is given by (1.15) H_g is strongly open at $(x_1,0)$ exactly when x_1 lies in C and

$$g(x_1) \in -\operatorname{int} P \quad (\textit{Slater's condition}). \tag{1.38}$$

Notice that while we know from Proposition 2 that H_g is then open at zero it is not generally strongly open at every point in $H_g^{-1}(0)$.

Proposition 5 allows us to replace various constraints by one product constraint while maintaining openness. For example, if in (1.2), (1.3), (1.4) g is continuous and satisfies Slater's condition at some point in the interior of C while h is open (in its range) then H given by (1.4) is open at zero. If one relativizes these conditions as one may in R^n one discovers that the standard constraint qualification given in Rockafellar (1970) is equivalent to the relative openness of H at 0.

In general, even for open continuous linear operators T_1 and T_2 between Banach spaces the product (T_1, T_2) need not have closed range and so cannot be open in its range.

Example 3: Let X be a reflexive Banach space. Let $K \subset X$ be a closed convex cone and K^+ its positive *dual* cone. Let $T:X \to X^*$ be a continuous linear operator and suppose T is *coercive* on K

$$(T(x),x) \geq c\,||x||^2 \qquad (\forall x \in K) \tag{1.39}$$

for some $c > 0$. Let

$$H(x) = \begin{cases} T(x)-K^+ & x \in K \\ \emptyset & x \notin K \end{cases}. \tag{1.40}$$

Then the theorem of Lions and Stampacchia given in Ekeland and Temam (1976) may be used to show that H is surjective. It will follow from Theorem 8 that H is open although in general T will not be surjective and K will have no interior. This has applications in variational inequality theory. See Borwein (1980a), Cryer and Dempster (1980), and Ekeland and Temam (1976).

1.5 CS-closed Relations and the Open Mapping Theorem

We now introduce CS-closed sets and prove a general form of the Open-Mapping-Closed Graph theorem for convex relations. We then give two easy applications.

DEFINITION 1: A set C *in a linear space* X *is* CS-closed *(convergent series-closed) if whenever* $\{\lambda_n\}$ *is a non-negative sequence with sum 1 and* $\{c_n\}$ *lies in* C *one has*

$$\sum_{n=1}^{\infty} \lambda_n c_n \in C \tag{1.41}$$

whenever $\sum^{\infty} \lambda_n c_n$ *exists in* X. *If* $\sum^{\infty} \lambda_n c_n$ *always exists,* C *is said to be* CS-compact.

A relation is CS-closed if its graph is. This definition is due to Jameson (1970; 1972). A similar notion of *ideal convexity* can be found in Holmes (1975).

It is obvious that CS-closed sets are convex. It is less obvious but true that if (1.41) holds for one infinitely positive sequence of λ_n it holds for all. We now gather up some of the signal properties of CS-closed sets.

PROPOSITION 6:

(a) *Finite dimensional, closed and open sets are* CS-*closed.*

(b) *The intersection of* CS-*closed sets is* CS-*closed; if in addition one of the sets is* CS-*compact so is the intersection.*

(c) *A convex* G_δ *in a Fréchet space is* CS-*closed.*

(d) *A bounded complete* CS-*closed subset of a normed space is* CS-*compact.*

(e) *If* H *is a* CS-*closed relation and* C *is* CS-*compact then* H(C) *is* CS-*closed.*

All of the above are easy and can be found in Jameson (1972) except for (c) which is much harder and is proven in Fremlin and Talagrand (1979).

Example 4: Let $H(x) = Tx + C_2$ where T is continuous and C_2 is CS-closed. Then part (e) shows $T(C_1) + C_2$ is CS-closed whenever C_1 is CS-compact.

In particular $C_1 + C_2$ is CS-closed.

The central topological property of CS-closed sets is that they are *semi-closed* (have the same interior as their closure) at least in a metrizable setting.

THEOREM 7 (Jameson 1972): Let Y *be a metrizable topological vector space. Let* $C \subset Y$ *be* CS-*closed. Then*

$$\text{int } C = \text{int cl}C. \tag{1.42}$$

This fails more generally. Let Y be B[0,1] the bounded functions in the product topology. Let C be the subspace of functions of countable support. Then C is sequentially closed and so CS-closed. However, clC = Y and (1.42) fails. The easiest way of showing a set is not CS-closed in a metrizable setting is showing (1.42) fails. Consider the convex hull of a countable dense subset of the unit ball in a separable Banach space.

We can now state simultaneously the Closed Graph and Open Mapping theorems.

THEOREM 8:

(a) Let $H:X \to Y$ *be a* CS-*closed relation between complete metrizable spaces. Then* H *is* LSC *throughout core* D(H) *and open throughout core* R(H).

(b) Let H *be a closed convex relation. Suppose* X *is barreled and* Y *is complete, metrizable and locally convex. Then* H *is* LSC *throughout core* D(H).

(c) Let H *be a closed convex relation. Suppose* X *is complete, metrizable and locally convex and* Y *is barreled. Then* H *is open throughout core* R(H).

PROOF: As before it is only necessary to establish the open version. Thus even in the linear case the standard proofs are simplified. Before

proceeding let us indicate various special cases. Parts (b) and (c) may be found in Ursescu (1975), albeit with slightly different terminology. The Banach space case of (b) is in Robinson (1976), while Jameson (1972) gives the Banach space case of (a) and other related results. When H is single valued these results reduce to the classical ones (Robertson & Robertson (1964), Rudin (1973)). When H is a convex process the condition that $0 \in$ core R(H) is equivalent to R(H) = Y. In this case one can weaken the requirement in (a) to having $R(H) \cap - R(H)$ second category.

1.6 A Proof of the Metrizable Case

We may suppose that $0 \in$ core R(H) and $0 \in H(0)$, by translation. We wish to show H is open at (0,0). Let us pick closed balanced neighbourhood bases $\{U_n\}$ at zero in X and $\{V_n\}$ at zero in Y such that

$$U_n + U_n \subset U_{n-1} \; ; \qquad V_n \subset V_{n-1} \,. \tag{1.43}$$

Since $0 \in$ core R(H) and each U_n is absorbing it follows easily from the convexity of H that

$$D_n = \overline{-H(U_n) \cap H(U_n)} \qquad (n=1,2,\ldots) \tag{1.44}$$

is absorbing. The Baire category theorem shows that some multiple of D_n has interior and by homothety D_n itself does. Let x_n be in int D_n. Then we may suppose that

$$\pm\, x_{n+1} + 2V_{n+1} \subset \overline{H(U_{n+1})}, \qquad (n=1,2,\ldots) \tag{1.45}$$

on selecting a subsequence from V_n if need be. This does not disturb (1.43). By convexity and (1.43)

$$2V_n \subset \overline{H(\tfrac{1}{2}U_{n+1} + \tfrac{1}{2}U_{n+1})} \subset \overline{H(U_n)} \subset H(U_n) + V_{n+1} \tag{1.46}$$

for each $n \geq 1$. Repeated use of (1.46) produces

$$V_1 \subset \tfrac{1}{2}H(U_1) + \tfrac{1}{4}H(U_2) + \ldots \frac{1}{2^{n-1}} H(U_{n-1}) + \frac{1}{2^{n-1}} V_n \,. \tag{1.47}$$

Let v_1 lie in V_1. Then there exist $y_i \in H(u_i)$, $u_i \in U_i$ such that

$$v_1 - \sum_{i=1}^{n-1} \frac{y_i}{2^i} \in V_n \ . \tag{1.48}$$

Since Y is metrizable one has

$$\sum_{i=1}^{\infty} \frac{y_i}{2^i} = v_1 \ . \tag{1.49}$$

Also

$$\sum_{i=m}^{k} \frac{u_i}{2^i} \in \sum_{i=m}^{k} \frac{1}{2^i} U_i \subset \frac{1}{2^m} U_{m-1} \tag{1.50}$$

on using (1.43) repeatedly. This shows that

$$\sum_{i=1}^{\infty} \frac{u_i}{2^i} = u_0 \tag{1.51}$$

exists since $\sum_{i=1}^{k} \frac{u_i}{2^i}$ is a Cauchy sequence and X is complete.

Moreover, $u_0 \in \frac{1}{2}U_0$ on using (1.50) with $m = 1$. Thus as $y_i \in H(u_i)$ and H is CS-closed (1.49) and (1.51) show that

$$v_1 \in H(u_0) \subset H(U_0) \ , \tag{1.52}$$

and as v_1 is arbitrary V_1 lies in $H(U_0)$. The identical considerations show that

$$V_{n+1} \subset H(U_n) \ , \tag{1.53}$$

for each $n \geq 0$ and so H is open at (0,0).

1.7 The Barreled Case

If actually H is closed and Y is barreled one proceeds in much the same fashion. Now we assume each U_n is convex and D_n turns out to be a barrel and so has zero in its interior. Then (1.46) will still hold except that V_n will not actually form a base at zero. Then one may fix n and pick an arbitrary neighbourhood V to replace V_n in (1.47) and (1.48). Now (1.48), (1.51) and the convexity of H in conjunction with $0 \in H(0)$ imply that

$$(u_0,v_1) = \sum_{i=1}^{n-1} \frac{1}{2^i}(u_i,y_i) + \left(\sum_{i=n}^{\infty} \frac{u_i}{2^i} \ , \ v_1 - \sum_{i=1}^{n-1} \frac{y_i}{2^i}\right) \tag{1.54}$$

and so

$$(u_0,v_1) \in \mathrm{Gr}(H) + (U_{n-1},V) \ . \tag{1.55}$$

Then (1.55) shows that (u_0,v_1) lies in $\overline{\mathrm{Gr}(H)}$ since U_{n-1} is in a neighbourhood base for zero in X and V is arbitrary. Since Gr(H) is closed the openness of H at (0,0) follows as before. □

Let us observe that while a CS-closed linear operator in a metrizable space is closed a CS-closed convex relation is generally not as is shown by Proposition 6 (a),(b),(c). Thus the generality of Theorem 8 is not spurious. Indeed it conveniently allows us to deal with constraint structures involving intersections of open, closed or finite dimensional sets and many others which may occur in infinite dimensional programming. Notice also that in the metrizable case neither X nor Y need be locally convex. This is useful in applications to partially ordered topological vector spaces. Let us derive some easy consequences.

COROLLARY 9: Let $f: X \to Y \cup \{\infty\}$ *be* S-*convex with* S *a normal cone. Suppose that either*

(a) f *has a* CS-*closed epigraph while* X *and* Y *are complete metrizable linear spaces,*

or

(b) f *has a closed epigraph while* X *is barreled and* Y *is a complete metrizable locally convex space.*

Then f *is continuous throughout the core of its domain.*

PROOF: Since the epigraph of f is just the graph of the associated relation with $H_f(x) = f(x) + S$, it follows from Theorem 8 that H_f is LSC throughout core $D(H_f)$ = core(domf). Lemma 4 finishes the result. □

Again in the normed case f is actually locally Lipschitz at core points of its domain. When Y is the real line and $S = R_+$ (b) is a well-known result. In the Banach space setting (b) is due to Robinson (1976a). Part (a) is new.

As a sample of how the open mapping theorem is used in partially ordered vector space theory we give the following:

COROLLARY 10: Let Y *be a complete metrizable linear space and suppose that* C_1, C_2 *are* CS-*closed sets in* Y *with*

$$0 \in \text{core } (C_1 + C_2) \ . \tag{1.56}$$

Then

$$0 \in \text{int } (C_1 \cap U) + (C_2 \cap U) \tag{1.57}$$

for any neighbourhood U *of zero.*

PROOF: Let $X = Y \times Y$ in the product topology, let $C = C_1 \times C_2$ and set

$$H(x) = \begin{cases} x_1 + x_2 & x_1 \in C_1, x_2 \in C_2 \\ \emptyset & \text{else} \end{cases} \ . \tag{1.58}$$

Then H is CS-closed and (1.56) shows $0 \in \text{core } R(H)$. Since (1.57) merely says H is open at 0 we are done. □

When C_1 and C_2 are cones a similar result is given in Jameson (1977) and can be used to derive and extend normal -B-cone duality Jameson (1970) and Schaefer (1971).

2. SOME FURTHER ANALYTIC RESULTS

2.1 The Principle of Uniform Boundedness

We suppose for simplicity in this development that X and Y are Banach spaces.

THEOREM 11: Let $H_i : X \to Y$ $(i \in I)$ *be a family of convex relations each* LSC *at* x_0. *Suppose that*

$$\sup_I \inf ||H_i(x)|| = m(x) < \infty \tag{2.1}$$

for each x *in some set* A *with* x_0 *in core* A. *Then if for* $c > 0$

$$y_{i_0} \in H_i(x_0) \ , \ ||y_{i_0}|| \le c \tag{2.2}$$

the relations H_i *are locally equi-Lipschitz at* (x_0, y_{i_0}).

PROOF: Let us set

$$K_i(x) = (H_i(x+x_0) - y_{i_0}) \cap B. \tag{2.3}$$

Then $0 \in K_i(0)$ and each K_i is still convex, is LSC at 0 and satisfies (2.1) for some set A with 0 in core A. Let

$$U = \bigcap_I \overline{K_i^{-1}(\bar{B})} \tag{2.4}$$

where B is the unit open ball in Y. Since (2.1) holds and K_i is convex it is easy to show that U is absorbing and being convex, closed in a Banach space is a neighbourhood of zero.

Since each $K_i^{-1}(B)$ actually has interior, being LSC at 0, we have

$$\text{int } U \subset \text{int } \overline{K_i^{-1}(\bar{B})} \subset K_i^{-1}(\bar{B}). \tag{2.5}$$

Now we may assume U is the closed unit ball in X and pick x_1 in int $\frac{1}{2}$ U and any y_{i_1} in $K_i(x_1)$. Thus y_{i_1} is in B and

$$x_1 + \frac{U}{2} \subset \text{int } U \subset K_i^{-1}(\bar{B}) \subset K_i^{-1}(y_{i_1} + 2\bar{B}) \ . \tag{2.6}$$

Since K_i^{-1} is convex and $x_1 \in K_i^{-1}(y_{i_1})$ we must have

$$x_1 + \lambda U \subset K_i^{-1}(y_{i_1} + 4\lambda\bar{B}) \tag{2.7}$$

whenever $0 < \lambda < \frac{1}{2}$. This shows that for i in I and any x in $x_1 + \lambda U$

$$H_i(x+x_0) \cap (B + y_{i_0}) \cap (y_{i_1} + y_{1_0} + 4\lambda\bar{B}) \ne \emptyset. \tag{2.8}$$

Since $y_{i_1} + y_{i_0}$ is an arbitrary point of $H_i(x_1+x_0) \cap (B + y_{i_0})$ one has

$$H_i(x_1+x_0) \cap (B + y_{i_0}) \subset H_i(x+x_0) \cap (B + y_{i_0}) + 4||x-x_1||\bar{B} \ . \tag{2.9}$$

Since x and x_1 play symmetric roles we see that actually

$$D(H_i(x_1) \cap (\bar{B} + y_{i_0}) \ , \ H_i(x) \cap (\bar{B} + y_{i_0})) \le 4||x-x_1|| \tag{2.10}$$

whenever $||x-x_0||$, $||x_1-x_0|| < \frac{1}{2}$. The choice of the unit ball in (2.10) is merely one of convenience. □

A slightly more extended argument allows one to replace (2.10) by (1.31) for each i. There is a more cumbersome dual form of this result involving open mappings.

It is convenient to have conditions which ensure that each H_i is LSC at x_0 and that (2.1) holds. One such condition is as follows. We say H_i is *image semicontinuous at* (x_0,i_0) if for each sufficiently small neighbourhood U of zero in X and each $\varepsilon > 0$ one has

$$H_{i_0}(x) \subset H_i(x) + \varepsilon B \qquad x \in U + x_0 , \tag{2.11}$$

for i sufficiently near i_0. Of course this supposes that I is topologized. This notion is due to Dolecki and Rolewicz (1979).

PROPOSITION 12: Suppose that H_i *is image semi-continuous at* (x_0,i_0) *and that each* H_i *has a* CS*-closed graph. Then for* i *near* i_0 *(2.1) holds. If* x_0 *is a core point of* $D(H_{i_0})$ *then actually all the* H_i *are* LSC *at* x_0 *for* i *near* i_0.

PROOF: Fix any U and ε for which (2.11) holds. Then for x in $U + x_0$ it is clear that (2.1) holds if i is near i_0. Moreover if x_0 is a core point of $D(H_{i_0})$, Theorem 8(a) shows that H_{i_0} is LSC at x_0. Let $x_0 + U$ be an open set inside $D(H_{i_0})$. Then (2.11) shows that for U sufficiently small and i near i_0 actually $x_0 + U$ lies in $D(H_i)$. Since each H_i has a CS-closed graph it is actually LSC at x_0. □

Example 5: Let $I = \{1,2,\ldots\}$ and let $g_i:X \to Y$ be S_i-convex on some open set C with

$$\lim_{i\to\infty} g_i(x) = g_0(x), \qquad (\forall x \in C) . \tag{2.12}$$

Suppose that for $i = 0,1,2,\ldots,g_i + S_i$ is LSC. It is immediate that (2.1) holds on C and that the result of Theorem 11 obtains. Thus $g_i(\cdot) + S_i$ satisfy (2.10) with $y_{i_0} = g_i(x_0)$. If in addition the S_i are *equi-normal* we may obtain from (2.10) that the g_i are equi-Lipschitz locally at x_0. This is Kosmol's central theorem (1977). He then applies this to the study of perturbed optimization problems of the form

$$h(i) = \inf \{f_i(x) \mid g_i(x) \in -S_i, x \in C\} . \tag{2.13}$$

We may use Theorem 11 and Proposition 12 to produce similar results without assuming normality of the S_i or even such a specific constraint form.

Finally, observe that if each $g_i + S_i$ is CS-closed $(i=1,2,\ldots,)$ and g_0 is continuous and if the convergence in (2.12) is uniform near x_0, then Proposition 12 applies. Note that Theorem 11 does include the classical Banach-Steinhaus result (Rudin (1973)).

2.2 The Dieudonné Closure Condition

Let us begin by recalling that the *recession cone* of a convex set C in X is defined by

$$\text{rec } C = \{x \in X \mid C + \lambda x \subset C, \quad \lambda \geq 0\} . \tag{2.14}$$

We refer to Holmes (1975) for details of the recession cone's properties in infinite dimensions. The central one is that a locally compact closed convex set is compact exactly when rec C = 0.

THEOREM 13: Let H *be a closed convex relation between topological vector spaces* X *and* Y. *Suppose that* C *is a closed convex set in* X *and* $0 \in R(H)$. *If*

(i) $D(H) \cap C$ *is relatively locally compact,* (2.15)

(ii) $\text{rec } C \cap \text{rec } H^{-1}(0)$ *is linear* (2.16)

then H(C) *is closed.*

PROOF: Let P be the projection of X×Y on Y given by P (x,y) = y. Now

$$H(C) = P\,(C\times Y \cap \text{Gr}H) \tag{2.17}$$

and since P is open and linear it follows much as in Holmes (1975) that H(C) is closed exactly when

$$P^{-1}(0) - (C\times Y \cap \mathrm{Gr}H)$$

is closed. Let $A = X\times 0 = P^{-1}(0)$ and $B = C\times Y \cap \mathrm{Gr}H$. Then A and B are closed convex sets and calculation shows

$$\begin{aligned} \text{rec } A \cap \text{rec } B &= (X\times 0) \cap (\text{rec } C\times Y) \cap (\text{rec Gr}H) \\ &= (\text{rec } C \cap \text{rec } H^{-1}(0), 0) . \end{aligned} \tag{2.18}$$

Thus all common recession directions for A and B are *lineality directions*. Dieudonné's theorem in Holmes (1975) would apply if either A or B were locally compact. This fails in the present case. However, the following extension of that theorem is valid and completes the proof of the present result.

LEMMA 14: Let A *and* B *be closed convex subsets of a product space* $X\times Y$ *such that*

(i) rec $A \cap$ rec B *is linear*, (2.19)

(ii) $P_X A$ *and* $P_Y B$ *are locally compact (relatively).* (2.20)

Then $A - B$ *is closed.*

PROOF: The proof is a slight complication of that of Dieudonné's theorem which is the case that $X = 0$ above. □

Note that Lemma 14 covers a variety of cases in which Dieudonné's theorem is inapplicable.

Gwinner (1977) gives a similar result for upper semi-continuous relations and indicates applications to duality theory, splines and best approximation. It should be clear that Theorem 12 has applications whenever one needs closure of an image set as is frequently the case in convex analysis. One can also derive in a straight forward fashion conditions for the composition of closed relations to be closed since

$$R(H_1, H_2) = \mathrm{Gr}(H_1H_2^{-1}) . \tag{2.21}$$

Application of those considerations yields the following condition for $H_1H_2^{-1}$ to be closed.

(i) $D(H_1) \cap D(H_2)$ is relatively locally compact, (2.22)

(ii) rec $H_1^{-1}(0) \cap$ rec $H_2^{-1}(0)$ is linear. (2.23)

In finite dimensions (2.22) is, of course, always satisfied.

Often one has in mind the case where $H_1(x) = f(x) + R_+$ and $H_2(x)$ is an arbitrary convex relation. Then if the *value function* h is defined by

$$h(u) = \inf\{f(x) \mid u \in H_2(x)\}, \tag{2.24}$$

$$\mathrm{Epi}h = \overline{\mathrm{Gr}(H_1H_2^{-1})}\,; \qquad (\text{i.e. } h(u) + R_+ = \mathrm{cl}(H_1H_2^{-1})(u)) \tag{2.25}$$

and (2.22) and (2.23) give conditions for h to be (single-valued) lower semi-continuous and to have the infimum in (2.24) attained. This has of course, the usual implications for duality theory. See Gwinner (1977) and Levine and Pomerol (1979) for further details and in particular discussion of the relationships between the closure of the epigraph of the dual value function and the existence of Lagrange multipliers for (P) in (1.5).

One may apply (2.22) and (2.23) to semi-infinite programming pairs such as in Charnes et al (1963) and derive conditions ruling out a *duality gap* (Duffin and Karlovitz (1965)) either from the primal or the dual.

Note that in (2.16) or (2.23) 0 may be replaced by any point y_0 in R(H) and the hypothesis that $0 \in R(H)$ abandoned. Indeed it is shown in Borwein (1977a) that when H is closed rec $H^{-1}(y_0)$ is independent of y_0. Thus one could define a recession relation in analogy to the recession function of convex analysis.

2.3 The Inverse Function Theorem for Approximately Convex Relations

Let us, in keeping with the terminology in Michel (1974) and Pomerol (1979), call a relation Θ *approximately convex* at x_0 if Θ satisfies

$$\Theta(x) = f(x) + H(x) \ , \tag{2.26}$$

where f is a single-valued continuously Fréchet differentiable mapping and H is a LSC (on D(H)) convex relation, in a neighbourhood of x_0. In this case there is an associated "derivative"

$$\nabla\Theta(x_0)(h) = \nabla f(x_0)(h) + f(x_0) + H(x_0+h) \tag{2.27}$$

where $\nabla f(x_0)$ is the derivative of f at x_0. These approximately convex relations admit much simpler convex approximates than is usual for more general relations. Indeed the heart of the next theorem can be established, using considerably more terminology, in the general framework of Dolecki (1981) and Dolecki and Rolewicz (1979). The more direct relationships indicated below seem to be limited to approximately convex relations.

Example 6:

(i) If $H(x) = h(x) + P$ where h is P convex one is in the setting of Michel (1974) and Pomerol (1979).

(ii) If $H(x) = \begin{cases} K & x \in C \\ \emptyset & x \notin C \end{cases}$ and K is a convex cone and C a convex set one is in Robinson's (1976a) and Du'ong and Tuy's (1978) setting, while if K is only a convex set one is in a setting that various authors have considered.

(iii) If f is zero one is again considering arbitrary convex relations.

THEOREM 15: Suppose Θ *is a closed approximately convex relation at* x_0 *and that* $\nabla\Theta(x_0)$ *is open at* $y_0 \in \Theta(x_0)$. *Then there exist* $K > 0$, $\varepsilon > 0$ *such that*

$$d(y \mid \Theta(x)) \geq K\, d(x \mid \Theta^{-1}(y)) \tag{2.28}$$

whenever $||x-x_0|| \leq \varepsilon$ *and* $||y-y_0|| \leq \varepsilon$.

PROOF: The proof is a little delicate. We sketch its outline. Fix $y \in Y$. We may assume that $y_0 = 0, x_0 = 0$ on translating Θ. Define

$$P(h) = \nabla\Theta(x_0)^{-1}(\nabla f(x_0)h + f(x_0) + y - f(x_0 + h)) \tag{2.29}$$

and, for $\delta > 0$, set

$$Q(h) = P(h) \cap B_\delta(0) \ . \tag{2.30}$$

Since $\nabla\Theta(x_0)$ is open at 0, $\nabla\Theta(x_0)^{-1} \cap B_\delta$ is k-locally Lipschitz at 0 using (1.31) or (2.10) and, for sufficiently small $\varepsilon > 0$, Q is a multi-valued contraction mapping on $B_\varepsilon(0)$ since f is strictly differentiable. Also, one can show that for x in $D(\Theta)$

$$d(y \mid \Theta(x)) \to 0 \quad \text{as } x \to x_0 \ , \ y \to y_0 \tag{2.31}$$

since Θ is LSC on $D(\Theta)$ at x_0, and as $\nabla f(x_0)x + H(x) \subset \nabla\Theta(x_0)(x)$,

$$d(x \mid Q(x)) \leq k\, d(y \mid \Theta(x)) \ . \tag{2.32}$$

It follows from (2.30) and (2.32) that for δ sufficiently small and $||x-x_0||$, $||y-y_0|| < \varepsilon$ one has a fixed point x(y) with

$$x(y) \in Q(x(y)) \quad , \quad d(x(y), x) \leq K\, d(y \mid \Theta(x)) \quad , \tag{2.33}$$

for some $K > 0$ (independent of x and y) as promised by the multivalued contraction mapping theorem (Robinson (1976b or Tuy and Du'ong (1978)). Note that P, and so Q, does have closed convex images since the continuity of f and closure of Θ forces closure of H and hence of $\nabla\Theta(x_0)$. Now a fixed point x(y) of Q satisfies

$$y \in f(x_0 + x(y)) + H(x_0 + x(y)) = \Theta(x_0 + x(y)) \tag{2.34}$$

and in conjunction with (2.32) this shows that (since $x_0 = 0$)

$$d(x \mid \Theta^{-1}(y)) \leq K\, d(y \mid \Theta(x)) \tag{2.35}$$

as required. □

Let us recall that the *tangent cone* to a set E at x_0 is defined by

$$T(E,x_0) = \{x \mid t_n(x_n - x_0) \to x,\ x_n \to x_0\quad x_n \in E,\ t_n \geq 0\}. \quad (2.36)$$

Let us call Θ *regular* at (x_0, y_0) if (2.28) holds. Then this corresponds with the usage in Ioffe (1972) and we can derive the following extended Ljusternik theorem (Luenberger (1969)).

THEOREM 16: Suppose Θ *is approximately convex and regular at* $(x_0,0)$. *Then if* $0 \in \Theta(x_0)$

$$\nabla\Theta(x_0)^{-1}(0) \subset T(\Theta^{-1}(0),\ x_0) . \quad (2.37)$$

PROOF: Suppose $0 \in \nabla\Theta(x_0)(h)$. Since $0 \in \nabla\Theta(x_0)(0)$ and $\nabla\Theta(x_0)$ is convex one has

$$0 \in \nabla\Theta(x_0)(th) \quad (2.38)$$

for $0 \leq t \leq 1$. Also

$$\begin{aligned} d(0 \mid \Theta(x_0 + th)) &\leq \|f(x_0 + th) - f(x_0) - \nabla f(x_0)(th)\| \\ &\quad + d(o \mid \nabla\Theta(x_0)(th)) \\ &= o(t) \end{aligned}$$

since f is (strictly) differentiable at x_0. By the regularity of Θ at $(x_0,0)$ one has

$$d(x_0 + th \mid \Theta^{-1}(0)) = o(t) , \quad (2.39)$$

and it is easily verified that

$$h \in T(\Theta^{-1}(0),\ x_0) . \qquad \square$$

This type of approximation result can be extended as in Oettli (1980), Dolecki (1981) and Tuy and Du'ong (1978) to cases where $\nabla f(x_0)(\cdot)$ is a convex approximate to f in a sufficiently strong sense.

3. MATHEMATICAL PROGRAMMING

3.1 A General Theorem of the Alternative

Let $K:X \to Y$, $H:X \to Z$ be convex relations. Let Y* and Z* be the topological duals of Y and Z.

THEOREM 17: Suppose that H *is open at* 0, *that* $x_0 \in H^{-1}(0)$ *and that* K *is* LSC *at* x_0. *Suppose also that for some* x_1, $\operatorname{int} K(x_1) \neq \emptyset$. *Then of the following exactly one has a solution:*

$$(i)\quad 0 \in H(x) \quad , \quad 0 \in \operatorname{int} K(x) ; \quad (3.1)$$

$$(ii)\quad y' K(x) + z' H(x) \geq 0 , \quad (3.2)$$

for all x *in* X *and some* y′ *in* Y*, z′ *in* Z* *with* y′ *non-zero.*

PROOF: Let $G(x) = \operatorname{int} K(x)$. It is straightforward that G is a convex relation. Also $\operatorname{int} D(K) \subset D(G)$ since for $0 < t \leq 1$ and any x in X

$$t \operatorname{int} K(x_1) + (1 - t) K(x) \subset \operatorname{int} K(tx_1 + (1 - t)x). \quad (3.3)$$

Now given x_2 in int D(K) we may find x in D(K) and $0 < t < 1$ with $x_2 = tx_1 + (1 - t)x$. Then (3.3) shows that x_2 is in D(G). Let y_0 lie in $G(x_0)$. Since G is LSC at x_0 and strongly open at (x_0,y_0) Proposition 5 shows that

$$F = (G,\ H) \quad (3.4)$$

is open at the image point $(y_0,0)$. Thus R(F) is a convex set with non-empty interior and the failure of (3.1) says that (0,0) does not lie in R(F). The Mazur separation theorem guarantees the existence of y′ in Y* and z′ in Z*, not both zero, such that

$$y'(y) + z'(z) \geq 0 \quad \forall y \in G(x) \ , \ z \in H(x) . \quad (3.5)$$

Since x_0 lies in int D(G) and H is open at 0 a standard argument shows y′ is non-zero. It remains to show that G can be replaced by K in (3.5). Let y lie in K(x) and z in H(x) for some fixed x. Set

$$z_\varepsilon = (1-\varepsilon)z \ , \ y_\varepsilon = \varepsilon y_0 + (1-\varepsilon)y \ , \ x_\varepsilon = \varepsilon x_0 + (1-\varepsilon)x, \quad (3.6)$$

for $0 < \varepsilon < 1$. Then z_ε lies in $H(x_\varepsilon)$ since H is convex while y_ε lies in $G(x_\varepsilon)$ on applying (3.3) with x_0 replacing x_1. Thus (3.5) shows that

$$y'(y_\varepsilon) + z'(z_\varepsilon) \geq 0 . \quad (3.7)$$

If we let ε tend to zero we establish (3.3). The converse is easy. $\square$

We say that a relation M is S-*convex* if M + S is a convex relation, Borwein (1977a). An immediate application of Theorem 17 is with K = M + S when int S is non-empty. Then (3.1) and (3.2) become, when S is a cone,

$$\text{(i)}' \quad 0 \in M(x) + \operatorname{int} S \ , \quad 0 \in H(x) \ , \tag{3.8}$$

$$\text{(ii)}' \quad s^+M(x) + z'H(x) \geq 0 \tag{3.9}$$

for some non-zero s^+ in S^+ and z' in z^*. Here $S^+ = \{s^+ \in Y^* | s^+(s) \geq 0, s \in S\}$ is the *dual cone* to S.

This alternative was established in Borwein (1977a) and similar results may be found in Oettli (1980); and used to prove weak Lagrange multiplier theorems extending those in Oettli (1980) and in Craven and Mond (1973).

One may also treat Pareto alternatives in this framework, extending results in Borwein (1977b) and elsewhere.

3.2 The Lagrange Multiplier Theorem

We next establish a Lagrange multiplier theorem for vector convex programs and derive various consequences.

PROPOSITION 18 (Borwein (1980b): Let $K:X \to Y$ *be a convex relation. Suppose that* Y *is partially ordered by an order complete normal cone* S. *Suppose that* K *is* LSC *at* 0 *and*

$$K(0) \geq 0 \ . \tag{3.10}$$

Then there exists a continuous linear operator $T:X \to Y$ *such that*

$$T(x) \leq K(x) \qquad (\forall x \in X) \ . \tag{3.11}$$

This result is established in Borwein (1980b) from the Hahn-Banach theorem (Day (1973)) which holds exactly when S is order-complete (Elster and Nehse (1978)).

THEOREM 19: Let $H:X \to Z$ *be a convex relation. Let* $f:X \to Y \cup \{\infty\}$ *be* S-*convex with respect to an order complete normal cone* S. *Consider*

$$\text{(P)} \quad \mu = \inf_S \{f(x) \mid 0 \in H(x)\} \quad (\mu \in Y). \tag{3.12}$$

Suppose H *is open at zero and* f *is continuous at* x_1 *for some* x_1 *in* $H^{-1}(0)$. *Then there exists a continuous linear operator* $T:Z \to Y$ *such that*

$$\mu \leq f(x) + T\,H(x) \qquad (\forall x \in X). \tag{3.13}$$

Moreover the set of all such T *is equicontinuous. Also* h *given by*

$$h(u) = \inf_S \{f(x) \mid u \in H(x)\} \tag{3.14}$$

is well defined convex and continuous in some neighbourhood of 0.

PROOF: This is a specialization of the Lagrange multiplier result given by the author in Borwein (1980b). Define

$$K(u) = H_f\, H^{-1}(u) - \mu \tag{3.15}$$

where $H_f(x) = f(x) + S$. Then $K:Z \to Y$ is a convex relation and $K(0) \geq 0$ by (3.12). Since H is open at $(x_1, 0)$ H^{-1} is LSC at $(0, x_i)$ and also H_f is LSC at $(x_1, f(x_1))$. Then Proposition 1 shows that K is LSC at $(0, f(x_1) - \mu)$ and hence at 0. Proposition 18 now guarantees the existence of a continuous linear T with

$$K(u) \geq -T(u). \tag{3.16}$$

Equivalently,

$$H_f(x) + Tu \geq \mu \qquad (x,u) \in G(H) \tag{3.17}$$

or

$$f(x) + T\,H(x) \geq \mu \qquad x \in X. \tag{3.18}$$

Since K is LSC at 0 (3.16) shows that

$$h(u) = \inf K(u) \tag{3.19}$$

is well defined and finite (and so convex) on some neighbourhood U of zero. Moreover Propositions 1 and 2 show that H_h is LSC at zero and thus Lemma 4 shows h is continuous at zero since S is normal. Finally, the solutions T of (3.13) are exactly

$$-\partial h(0) = -\{T \in B[Z,Y] \mid T(u) \leq h(u) - h(0)\}. \tag{3.20}$$

This last set is the *subgradient* of h at zero. Again the proof of Lemma 4 shows that $\partial h(0)$ is equi-continuous. □

Define the *vector Lagrangian* L by

$$L(T) = \inf f(x) + T\,H(x). \tag{3.21}$$

Then, as in the scalar case, L is S-concave and one always has *weak duality*

$$\sup_T L(T) \leq h(0). \tag{3.22}$$

The considerations of the last proof show that the openness of H at zero in conjunction with some continuity assumption on f is a *constraint qualification* which ensures that *strong duality* holds:

$$\sup_T L(T) = h(0) \tag{3.23}$$

with attainment on the left hand side of (3.23). Indeed T is a solution in (3.23) exactly when

$$T \in -\,\partial h(0). \tag{3.24}$$

It seems much harder than it is in the scalar case to give semi-continuity conditions for (3.23) to hold rather than (3.24).

Equation (3.24) (Borwein, Penot and Thera (1980)) shows that multipliers may still be interpreted as marginal prices for the program (P). In many situations the multiplier is actually unique and is the Gateaux or Fréchet derivative of -h at zero (Borwein (1980a).

Let us also observe that if the infimum in (3.12) is attained by a feasible point x_0 one actually has the *complementary slackness* condition

$$\min T\,H(x_0) = 0, \tag{3.25}$$

which reduces to the standard slackness condition when H is given by (1.4) or (1.15).

Example 6: The Hahn-Banach theorem is a special case of Theorem 19 in the algebraic core topology. Let X be a vector space, M a subspace of X, p a sublinear operator and T a linear operator dominated by p on M. Then

$$0 = \inf_S \{p(x) - T(x) \mid 0 \in x - M\} \tag{3.26}$$

and application of Theorem 19 provides an appropriate multiplier. Note that the constraint is open because in the algebraic core topology a convex relation is, not surprisingly, open at any core point of its range.

Remark: Suppose in Theorem 19 that Z is replaced by Z_0 equal to span R(H). Then one may establish the existence of a Lagrange multiplier assuming only that H is open at zero in Z_0 or equivalently that H is *relatively open* at zero. The multiplier T will only be defined on Z_0. If S has non-empty interior (as is true in the scalar case) T will have a continuous extension to all of Z (Peressini (1967)). In any case T will have a linear extension to all of Z. Thus in the algebraic case it suffices that zero be a relative core point of R(H) as in Kutateladze (1979) or Oettli (1980).

Finally, some remark is in order about which cones are order complete. Considerable detail may be found in Jameson (1970), Peressini (1967) or Schaefer (1971). In short, however, the only finite dimensional cones which are order complete are those with linearly independent generators (Peressini (1967). In an infinite dimensional setting all dual Banach lattices are order complete as are many other spaces.

3.3 Vector Perturbational Duality

Now we examine the relationship between our Lagrange duality and the vector version of the Fenchel-Rockafellar bifunctional duality given in Ekeland and Temam (1976) or in Rockafellar (1970, 1974b). As we will see below nothing is lost in our framework and certain results may be strengthened. Moreover, it is the author's contention that the constraint structure of (3.12) is more intuitive and easier to work with than that of (3.27).

Let $f: X \times Z \to Y \cup \{\infty\}$ be an S-convex (bi)function.

Define

$$h(u) = \inf_S f(x,u),\ \mu = \inf f(x, 0) \tag{3.27}$$

where $h(u) = -\infty$ if the infimum fails to exist. We may rewrite (3.12) in this form by setting

$$f(x,u) = \begin{Bmatrix} f(x) & u \in H(x) \\ +\infty & u \notin H(x) \end{Bmatrix} . \tag{3.28}$$

Conversely if we set

$$u \in K(x, y) \qquad f(x, u) \leq y \tag{3.29}$$

we may write

$$h(u) = \inf_S \{y \mid u \in K(x, y)\}. \tag{3.30}$$

Then K is a convex relation since its graph is essentially the epigraph of f. The usual regularity condition on f is that $f(x_0, \cdot)$ should be continuous at zero for some x_0 (Ekeland and Temam (1976)). This is easily seen to be equivalent (when S is normal) to the openness of $K(x_0, \cdot)$ at zero (Propositions 1, 2 and Lemma 4). In turn this implies that K is open at zero, but in general the converse is false. Thus the openness of K is a weaker requirement to impose in (3.27) or (3.29). A related result is given in Robinson (1976a) and Rockafellar (1974b). Now (3.13) becomes

$$\begin{aligned} h(0) &= \inf_S \{y + Tu \mid f(x, u) \leq y\} \\ &= -\sup_S \{-Tu - f(x, u)\} = -f^*(0, -T), \end{aligned} \tag{3.31}$$

where f* is by the definition the *vector conjugate* of f defined in Zowe (1975a). Now (3.31) and (3.23) give the conjugate duality result obtained by Zowe and others under stronger regularity conditions. We will have more to say about regularity conditions below.

Let us rederive the ordinary Lagrange multiplier theorem in our framework.

COROLLARY 20: Consider the ordinary convex program given by (1.1). Suppose that g *is continuous at some point* x_0 *in* int C *with* $h(x_0) = 0$ *and* $g(x_0) < 0$. *Then there exist non-negative numbers* $\lambda_1, \ldots, \lambda_m$ *and real numbers* $\mu_1, \ldots, \mu_k$ *such that*

$$f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{k} \mu_j h_j(x) \geq p \qquad \forall x \in C. \tag{3.32}$$

PROOF: We have supposed implicitly that h is everywhere defined and that C lies in the domain of each function. Then the associated relation H of (1.4) is open considered as a mapping between X and $R^m x R(h)$ by Proposition 5. Indeed H_h is open since R(h) is a finite dimensional subspace and h is surjective while H_g is LSC at x_0 and strongly open. Then (3.13) becomes

$$f(x) + \sum_{i=1}^{m} \lambda_i(g_i(x) + R_+) + \mu\, h(x) \geq p \qquad \forall x \in C; \tag{3.33}$$

for some $\lambda_1, \ldots, \lambda_m$ and some $\mu \in R(h)^*$. Since we may suppose that μ is extended to R^k, $\mu = (\mu_1, \ldots, \mu_k)$ for some μ_j in R and now (3.33) is easily seen to show that $\lambda_i \geq 0$ and (3.32) holds. □

The analogous considerations apply to more general mixtures of inequality and equality constraints. If $X = R^n$ the continuity assumptions on g and C may be relativized.

Various other duality calculations are given in Borwein (1980b). One may also consider *augmented Lagrangians* (Rockafellar (1974a)) and *limiting Lagrangians* (Duffin (1973); Duffin and Jeroslow (1978)) profitably in this framework.

3.4 The Krein-Rutman Theorem

We prove a form of the Krein-Rutman theorem given in Peressini (1967) which will be applied in section 4 to produce Kuhn-Tucker conditions.

Let us define the S-*dual* cone to a set C in X by

$$C^S = \{T \in B[X,Y] \mid T(C) \subset S\}. \tag{3.34}$$

When $S = R_+$, $Y = R$, C^S is just C^+.

THEOREM 21 (Borwein (1980b)): Let A *be a linear mapping between* X *and* Z *and let* K_1 *and* K_2 *be convex cones in* X *and* Z *respectively. Set*

$$H_A(x) = \begin{cases} A(x) - K_2 & x \in K_1 \\ \emptyset & x \notin K_1 \end{cases} . \qquad (3.35)$$

Suppose that S *is an order complete normal cone in* Y. *Then*

$$(K_1 \cap A^{-1}(K_2))^S = K_1^S + K_2^S A , \qquad (3.36)$$

whenever H_A *is open at zero.*

PROOF: The right hand side of (3.36) always lies in the left hand side. The converse is established by considering the convex program

$$0 = \inf \{T_0(x) \mid 0 \in H_A(x)\} \qquad (3.37)$$

where T_0 is any member of $(K_1 \cap A^{-1}(K_2))^S = (H_A^{-1}(0))^S$. The Lagrange multiplier T then satisfies

$$T_0 + TA \in K_1^S \quad , \quad - T \in K_2^S \qquad (3.38)$$

and (3.36) follows. □

In particular H_A is open if either (a) $R(H_A) = A(K_1) - K_2$ is Z in a situation in which Theorem 8 holds or (b) in more general spaces if $A(K_1) \cap \text{int } K_2 \neq \emptyset$. The scalar version of (a) is obtained in Banach space by Kurcuysz (1976), while (b) is the classic Krein-Rutman result given by Peressini (1967).

A similar result can be proven in similar ways assuming that K_1 and K_2 are only convex sets. This extends a result in Ben Tal and Zowe (1980) and elsewhere.

3.5 A Gauvin-Tolle Type Result

We show that for convex programming openness constraints is the weakest constraint qualification with various pleasant properties. In the next section this will be seen to imply the same type of behavior for differentiable programming as in Gauvin and Tolle (1977).

THEOREM 22: Consider the scalar convex program

$$(P) \quad h(u) = \inf \{f(x) \mid u \in H(x)\} \qquad (3.39)$$

where $f:X \to R$ *is a continuous function and* $H:X \to Y$ *is a convex relation. Consider the following statements.*

(a) H *is open at zero and* h(o) *is finite.*

(b) The value function h *is continuous at zero.*

(c) The Lagrange multiplier set at zero is equicontinuous and non-empty.

Then the following relationships hold:

(i) (a) => (b) => (c).

(ii) If X *is locally convex, complete, metrizable and either* H *has a* CS-*closed graph and* Y *is complete, metrizable or* H *has a closed graph and* Y *is barreled then (a) and (b) are equivalent.*

(iii) If in addition R(H) *is supportable at boundary points, as occurs if* R(H) *is finite dimensional or if* H *is open at some point* y_0*, then (a), (b) and (c) are equivalent.*

PROOF:

(i) Follows from Theorem 19 and (3.20).

(ii) In either case the openness of H is equivalent to $0 \in \text{int } R(H)$ which since dom f = X and h(o) is finite is equivalent to $0 \in \text{int(dom h)}$.

(iii) By (ii), if H is not open at zero, zero lies in the boundary of R(H). Let $\Phi \in Y^*$ be a non-zero vector with

$$\Phi(R(H)) \geq 0. \qquad (3.40)$$

If λ is a Lagrange multiplier for (P) at zero one has

$$f(x) + \lambda\, H(x) \geq h(0). \qquad (3.41)$$

It follows from (3.40) and (3.41) that

$$f(x) + (\lambda + n\Phi)\, H(x) \geq h(0) , \qquad (3.42)$$

and as $\lambda + n\Phi$ lies in the Lagrange multiplier set for each $n \geq 0$ this set is unbounded. Thus (c) implies (a). □

If R(H) is finite dimensional one may add to the above equivalences the condition that h is finite in some neighbourhood of zero.

Suppose that H has a closed graph, that D(H) is locally compact, and that the ε-solution mapping S_ε given by

$$S_\varepsilon(u) = \{x \mid u \in H(x),\ \ f(x) \leq h(u) + \varepsilon\} \tag{3.43}$$

has $S_\varepsilon(o)$ compact. Then the continuity of h forces *upper semi-continuity* of S_ε at zero (in ε and u). Thus the upper semi-continuity of S_ε may be added as an equivalence in (iii).

If actually H is strongly open at $(0,x_0)$ $S_\varepsilon(u)$ can be shown to be lower semi-continuous (in ε and u) at zero.

4. VECTOR KUHN-TUCKER THEORY

4.1 Karush-Kuhn-Tucker Conditions in Guignard's Form

In this section we consider the non-linear program

$$(\mathrm{P})\qquad \min_S\{f(x) \mid g(x) \in B\ ,\ x \in C\}\ , \tag{4.1}$$

where $f:X \to Y$, $g:X \to Z$ are Fréchet or Hadamard differentiable at X_0 (as in Craven (1978)) and B and C are arbitrary sets. Let $A = g^{-1}(B) \cap C$ and suppose x_0 lies in A and $f(x_0)$ minimizes (vectorially) f over A.

For an arbitrary set E recall that the *tangent cone* to E at x_0 is defined by (2.35) and the *pseudo-tangent* cone to E at x_0 by

$$P(E,\ x_0) = \overline{\mathrm{co}}\ T(E,\ x_0)\ . \tag{4.2}$$

Let the *linearizing cone* K be defined by

$$K = \nabla g(x_0)^{-1}(P(B,\ g(x_0)))\ , \tag{4.3}$$

and let G be any closed convex cone in X such that

$$G \cap K \subset P(A,\ x_0)\ . \tag{4.4}$$

Define H_G by

$$H_G(x) = \begin{Bmatrix} \nabla g(x_0)(x) - P(B,g(x_0)), & x \in G \\ \emptyset & x \in G \end{Bmatrix}\ . \tag{4.5}$$

THEOREM 23:

(a) If H_G *is open at zero and* S *is normal and order complete then*

$$(G \cap K)^S = G^S + P(B,\ g(x_0))^S \nabla g(x_0)\ . \tag{4.6}$$

(b) If (4.6) *holds and* S *is closed then one obtains the necessary condition*

$$\nabla f(x_0) - T\ \nabla g(x_0) \in G^S \tag{4.7}$$

for some T *in* $P(B,\ g(x_0))^S$.

PROOF:

(a) This is the Krein-Rutman theorem for H_G. Thus in a complete metrizable setting it holds if $\nabla g(x_0)(G) - P(B,\ g(x_0)) = Z$.

(b) Now (4.4) shows that

$$P(A,\ x_0)^S \subset G^S + P(B,\ g(x_0))^S \nabla g(x_0)\ . \tag{4.8}$$

Thus, once we show that $\nabla f(x_0)$ lies in $P(A,\ x_0)^S$ we will obtain (4.7). Let h lie in $T(A,\ x_0)$. Then

$$h_n \to h,\ h_n = t_n(a_n - x_0), t_n > 0,\ a_n \to x,\ a_n \in A.$$

It follows that $x_0 + t_n^{-1}h_n$ is feasible and so

$$t_n[f(x_0 + t_n^{-1}h_n) - f(x_0)] \subset S. \tag{4.9}$$

The differentiability assumptions imply that

$$\nabla f(x_0)(h) = \lim t_n[f(x_0 + t_n^{-1}h_n) - f(x_0)] \tag{4.10}$$

and since S is closed

$$\nabla f(x_0)(T(A,\ x_0)) \subset S\ . \tag{4.11}$$

Since $\nabla f(x_0)$ is linear and continuous one may replace $T(A,\ x_0)$ by $P(A,\ x_0)$ in (4.11). □

For more details about the tangent cones and the scalar conditions in this form we refer to Guignard (1969) or Borwein (1978). Even in the scalar case Theorem 23(b) slightly extends the necessary results in these papers while (a) is entirely new.

Ideally one wishes to take $G = P(C, x_0)$ or even larger. The Kuhn-Tucker constraint qualification of Kuhn and Tucker (1951) ensures that this is possible.

Remark: Suppose we take f and g linear in (4.1) and take B a closed convex cone with C equal to X and $x_0 = 0$. Then one verifies that one may take $G = X$ in (4.4) and that (4.6) becomes

$$(g^{-1}(B))^S = B^S g . \tag{4.12}$$

Then (4.1) becomes

$$g(x) \in B \quad \text{implies} \quad f(x) \in S \tag{4.13}$$

and (4.7) is

$$f(x) = Tg(x) \quad , \quad T(B) \subset S . \tag{4.14}$$

Thus one sees that Theorem 23 includes the Farkas lemma in Craven (1978; 1980) and in Elster and Nehse (1978). It is shown in Nehse (1978) that, even assuming Slater's condition holds for g and B, the Farkas lemma only holds when S is order complete (Day (1973)). The same, therefore, is true of the Kuhn-Tucker conditions unless one places more restrictions such as surjectivity on the constraints (Craven (1980) and Ritter (1970)). This is also true of the Krein-Rutman Theorem.

In the case that $S = R_+$, $Y = R$ (4.6) is equivalent to the closure of $G^+ + P(B, g(x_0))^+\nabla g(x_0)$. This can be ensured either by making G and $P(B, g(x_0))$polyhedral or by applying the Dieudonné closure condition of Theorem 13.

Sufficiency conditions may be given for (4.7) to guarantee the minimality of a feasible x_0. These are essentially unchanged from those given in Borwein (1978) and in Guignard (1969) for the scalar case.

4.2 Karush-Kuhn-Tucker Conditions in Robinson's Form

We now consider a program in Banach space of the form

$$\text{(P)} \quad \min_S \{f(x) + m(x) \mid 0 \in g(x) + H(x)\} \tag{4.15}$$

where $\Theta(x) = g(x) + H(x)$ is approximately convex at x_0 as in Theorem 15 and $f(x) + m(x) + S$ is also approximately convex at x_0. We assume that $0 \in \theta(x_0)$, that the minimum occurs at x_0 and that H has a closed graph.

THEOREM 24: Suppose that S *is order-complete closed and normal and that*

$$\nabla\Theta(x_0)(h) = \nabla g(x_0)(h) + g(x_0) + H(x_0 + h) \tag{4.16}$$

is open at zero.

Then the following necessary condition for a minimum holds at x_0:

$$\begin{aligned} 0 \leq_S\ & \nabla f(x_0)(h) + m(x_0 + h) - m(x_0) + \\ & T(\nabla g(x_0)(h) + g(x_0) + H(x_0 + h)) \end{aligned} \tag{4.17}$$

for some continuous linear T.

PROOF: The extended Ljusternik theorem of Theorem 16 shows that if $0 \in \nabla\Theta(x_0)(h)$ then $h \in T(\Theta^{-1}(0), x)$. Much as in the last theorem we have $h_n \to h$, $h_n = t_n^{-1}(x_n - x_0)$; $0 \in \Theta(x_n)$ and $x_n \to x_0$, $t_n > 0$. Then

$$\frac{f(x_n) - f(x_0)}{t_n} + \frac{m(x_n) - m(x_0)}{t_n} \geq_S 0 . \tag{4.18}$$

Now as before

$$\frac{f(x_n) - f(x_0)}{t_n} \to \nabla f(x_0)(h) , \tag{4.19}$$

while, for large n,

$$\frac{m(x_0 + t_n h_n) - m(x_0)}{t_n} \leq_S m(x_0 + h_n) - m(x_0) \tag{4.20}$$

since m is S-convex. Since S is closed and m is continuous we may derive from (4.18), (4.19) and (4.20) that

$$\nabla f(x_0)(h) + m(x_0 + h) - m(x_0) \geq_S 0 . \tag{4.21}$$

Thus one has

$$0 = \min_S \{\nabla f(x_0)(h) + m(x_0 + h) - m(x_0) \mid 0 \in \nabla\Theta(x_0)(h)\}. \tag{4.22}$$

Now (4.22) is a convex program which has an open constraint and a continuous objective. Theorem 19 now produces the desired result. □

In addition, one has as a consequence of Theorem 22 that openness of $\nabla\Theta(x_0)$ at 0 is essentially the weakest constraint qualification yielding a bounded set of multipliers in (4.17) and a continuous value function for (P) of (15.8). This extends some of the results in Gauvin & Tolle (1977), Kurcyusz & Zowe (1979), Lempio & Zowe (1980) and elsewhere.

For $S = R^+$, $Y = R$ the result with H induced by a convex function is due to Pomerol (1979). For m = 0 and H a convex cone it is due to Robinson (1976). One may think of (4.16) as a generalized *Mangasarian-Fromovitz* constraint qualification (Mangasarian-Fromovitz (1967)).

One may also use Theorem 16 to derive second order conditions for (P) of (4.15) and analogous to those in Ben Tal and Zowe (1980) and extending those in Maurer and Zowe (1979), Lempio and Zowe (1980). The *second order variation* for θ looks something like

$$\nabla g(x_0)(d) + r\,\nabla g(x_0)(d_1) + \tfrac{1}{2}\nabla^2 g(x_0)(d_1, d_1) + M(x_0 + rd_1 + d)\ . \tag{4.23}$$

Here d and r are variable while x_0 and d_1 are fixed.

4.3 Constraint Regularization

We conclude with an example showing a new interpretation of the LSC hull of the value function h of (3.14). Suppose that in

$$\text{(P)}\quad h(u) = \inf_S \{f(x) \mid u \in H(x)\}, \tag{4.24}$$

H is not open at zero. Let U be an arbitrary neighbourhood of zero in Z and define

$$H_U(x) = H(x) - U\ . \tag{4.25}$$

Then H_U is strongly open at zero. Consider

$$(P_U)\quad P_U = \inf_S \{f(x) \mid 0 \in H_U(x)\}. \tag{4.26}$$

This has infimal value

$$P_U = \inf_{u\in U} h(u)\ . \tag{4.27}$$

Assume that P_U is finite for some U. Then Theorem 19 applies to (P_U) and we derive that

$$\inf_{u\in U} h(u) = \inf[f(x) + T_U(H_U(x))] \tag{4.28}$$

for some continuous linear T_U. Thus

$$\sup_U \inf_{u\in U} h(u) = \sup_U \inf[f(x) + T_U(H_U(x))]$$

$$\le \sup_T \inf_X [f(x) + T\,H(x)] \tag{4.29}$$

$$= \sup_T L(T)\ .$$

The left hand side of (4.29) is the *order lower semi-continuous hull* of h at zero, $\bar{h}(0)$, while the right hand side is easily seen to be equal to the *second conjugate* of h at zero, $h^{**}(0)$, as in Zowe (1975a). Thus we have shown that

$$\bar{h}(0) \le h^{**}(0) \le h(0) \tag{4.30}$$

whenever $\bar{h}(0)$ is finite. In restricted cases such as the scalar case equality actually holds in (4.30). More generally, however, linear continuous operators T may exist with

$$\bar{T}(0) \equiv -\infty,\ T(0) = T^{**}(0)\ , \tag{4.31}$$

as the example of the Hilbert matrix in Peressini (1967) shows. The positive result is as follows.

THEOREM 25: Suppose that $h:X \to Y$ *is* S-*convex and finite at zero. Suppose that* S *is an order-complete, normal, Daniell lattice ordering with interior. Then*

$$\bar{h}(0) = h^{**}(0)\ . \tag{4.32}$$

PROOF: By (4.30) $h^{**}(0)$ is finite. Also as above

$$\sup_T L(T) = h^{**}(0)\ . \tag{4.33}$$

For each finite subset F of B[X,Y] let

$$Y_F = \max_F L(T)\ . \tag{4.34}$$

Then $\{Y_F\}$ is an increasing net with supremum $h^{**}(0)$. A Daniell order is one in which the supremum is actually the topological limit (Borwein (1980a); Penot (1975-6)). Hence one can find F such that

$$Y_F \geq h^{**}(0) - s_0 \tag{4.35}$$

where s_0 is an interior point of S. Since each T in F is continuous one can find a neighbourhood U with

$$T(U) \leq s_0 \tag{4.36}$$

for each T in F. Then, for these T and that U,

$$\inf_{x,u} f(x) + T(H(x) - U) \geq \underline{L}(T) - s_0 \tag{4.37}$$

and so combining (4.35) and (4.37)

$$\sup_T \inf_{x,u} [f(x) + T(H(x) - U)] \geq h^{**}(0) - 2s_0 . \tag{4.38}$$

Finally (4.29) shows that

$$\overline{h}(0) \geq h^{**}(0) - 2s_0 . \tag{4.39}$$

Since S is linearly closed and s_0 is arbitrary in int S we have (4.32). □

Related results may be found in Borwein, Penot and Thera (1980).

Similar considerations allow us to extend the Transposition theorem of Theorem 17 to include what Holmes calls "Tuy's inconsistency condition" (Holmes (1975)).

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