Adventures with the OEIS

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Abstract

This paper is a somewhat expanded companion to a talk[1] with the same title presented in December 2015 at a 2015 workshop celebrating Tony Guttmann's seventieth birthday. My main intention is to further advertise the wonderful resource that the Online Encyclopedia of Integer Sequences (OEIS) has become.

1 Introduction

What began in 1964 with a small set of personal file cards has grown over half a century into the current wonderful online resource: the *Online Encyclopedia of Integer Sequences* (OEIS).

1.1 Introduction to Sloane's on-and-off line encyclopedia

I shall describe five encounters over nearly 30 years with Neil Sloane's (Online) Encyclopedia of Integer Sequences. Its brief chronology is as follows:

[1] Available at http://www.carma.newcastle.edu.au/jon/OEIStalk.pdf.

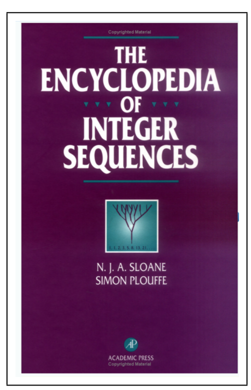

Figure 1: The second edition of 1995

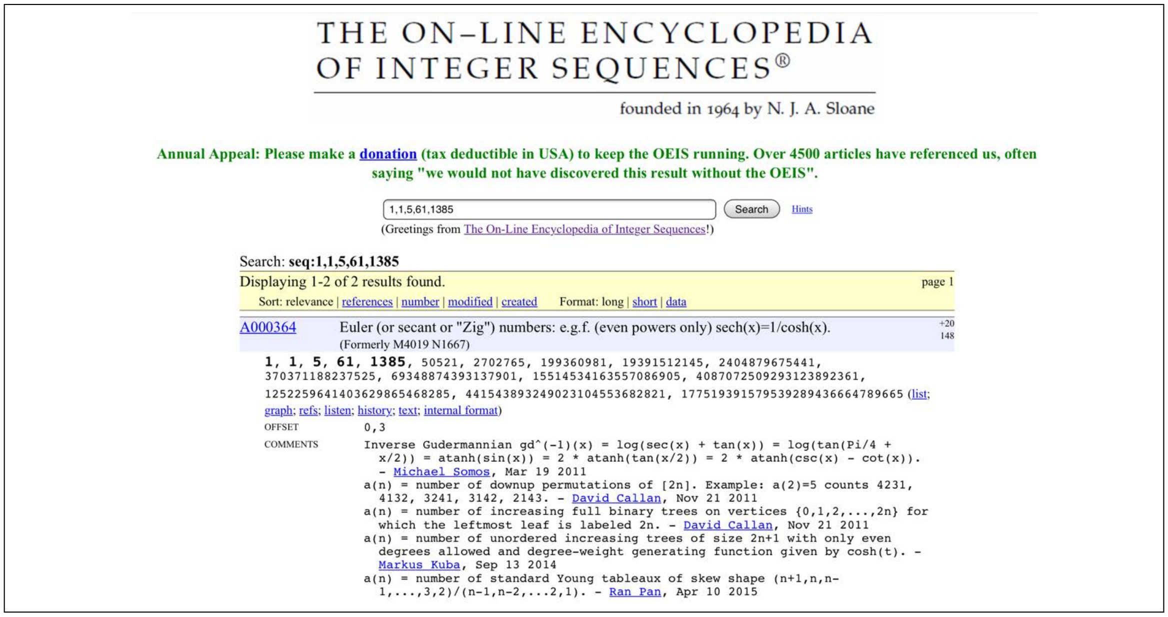

Figure 2: The OIES in action

- In **1973** a published book (Sloane) with 2,372 entries appeared. This was based on file cards kept since 1964.
- In **1995** a revised and expanded book (by Sloane & Simon Plouffe) with 5,488 entries appeared.
  - See the book review in *SIAM Review* by Rob Corless and me of the 1995 book at `https://carma.newcastle.edu.au/jon/sloane/sloane.htm`.
- Soon after the world wide web went public, between **1994–1996**, the OEIS went on line with approximately 16,000 entries.
- As of **Nov 15 21:28 EST 2015** OEIS had 263,957 entries
  - all sequences used in this paper/talk were accessed between Nov 15–22, 2015.

1.2 The OEIS in action

As illustrated in Figure 2 taken from `https://oeis.org/` the OEIS is easy to use, entering an integer sequence which it recognizes, one is rewarded with meanings, generating functions, computer code, links and references, and other delights.

1.3 OEIS has some little known features

The OEIS also now usefully recognises numbers: entering 1.4331274267223117583... yields the following answer.

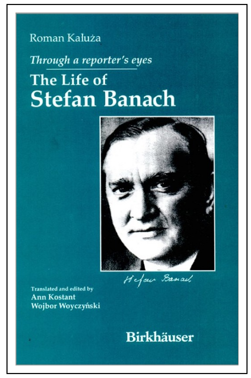

Figure 3: A fine biography of Banach

Answer 1.1 (A060997)**.** *Decimal representation of continued fraction* $1, 2, 3, 4, 5, 6, 7, \ldots$ (as a ratio of Bessel functions $I_0(2)/I_1(2)$).

The OEIS currently has excellent search facilities, by topic or author, and so on. For instance entering "Bell numbers" returned over 850 results while entering "Alladi" yielded 23 sequences. The third sequence listed on the page is:

Answer 1.2 (A000700)**.** *Expansion of product* $(1+x^{2k+1}), k = 0..\infty$*; number of partitions of* n *into distinct odd parts; number of self-conjugate partitions; number of symmetric Ferrers graphs with* n *nodes.*

The sequence begins

$$1, 1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 8, 8, 9, 11, 12, 12, 14, 16, 17, 18, 20, 23, 25, 26, 29, 33, 35$$

In the page we are told Krishna Alladi showed this is also the number of partitions of n into parts $\neq 2$ and differing by $\geqslant 6$ with strict inequality if a part is even.

Alladi's paper "A variation on a theme of Sylvester — a smoother road to Göllnitz's (Big) theorem", *Discrete Math.*, **196** (1999), 1–11, through a link to http://www.sciencedirect.com/science/article/pii/S0012365X98001939 is also provided.

The OEIS also has an email-based 'super-seeker' facility.

1.4 Stefan Banach (1892–1945) … the OEIS notices analogies

The MacTutor website, see www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html, quotes Banach as saying:

> A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.

In a profound way the OEIS helps us – greater or lesser mathematicians – find analogies between theories.

2 1988: James Gregory & Leonard Euler (1707–1783)

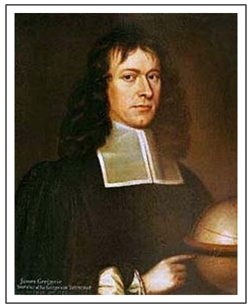

Figure 4: James Gregory (1638–1885)

Sequence 2.1 (A000364 (1/2))**.**

$$2, -2, 10, -122, 2770\ldots$$

Answer 2.2 (A011248)**.** *Twice* `A000364`[2] *Euler (or secant or "Zig") numbers: e.g.f. (even powers only)* $\operatorname{sech}(x) = 1/\cosh(x)$.

Story 2.3. In 1988 Roy North observed that Gregory's series for π,

$$\pi \quad = \quad 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \; = \; 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{1}$$

when truncated to 5,000,000 terms, gives a value differing strangely from the true value of π. Here is the truncated Gregory value and the true value of π:

3.141592**4**535897932384646433832795027**8**41971693993873058209749418**22**30781640...

3.14159265358979323846**2**64338327950288419716939937**51**0582097494459230781640...

Errors: 2 – 2 10 – 122 2770

The series value differs, as one might expect from a series truncated to $5,000,000$ terms, in the seventh decimal place—a "4" where there should be a "6." But the next 13 digits are correct!

Then, following another erroneous digit, the sequence is once again correct for an additional 12 digits. In fact, of the first 46 digits, only four differ from the corresponding decimal digits of π.

Further, the "error" digits appear to occur in positions that have a period of 14, as shown above.

We note that each integer is *even*; dividing by two, we obtain (1, −1, 5, −122, 1385). Sloane has told us we have the *Euler numbers* defined in terms of Taylor's series for $\sec x$:

$$\sec x \quad = \quad \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} x^{2k}}{(2k)!}. \tag{2}$$

[2] Two sequences are found which we flag via (1/2). It is interesting too see how many terms are needed to uniquely define well-known sequences. We indicate the same information in the next two examples.

Figure 5: Siméon Poisson (1781–1840)

Indeed, we see the *asymptotic expansion* base 10 on the screen:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \quad \approx \quad \sum_{m=0}^{\infty} \frac{(-1)^m\, E_{2m}}{N^{2m+1}} \tag{3}$$

This works in hex (!!) and $\log 2$ yields the *tangent numbers*.

In 1988 we only had recourse to the original printed book and had to decide to divide the sequence by two before finding it. Now this sort of preprocessing and other such transformations are typically done fro one by the OEIS. But it does not hurt to look for variants of ones sequence – such as considering the odd or square free parts – if the original is not found.

Nico Temme's 1995 Wiley book *Special Functions: An Introduction to the Classical Functions of Mathematical Physics* starts with this motivating example.

References 2.4. The key references are

1. J.M. Borwein, P.B. Borwein, and K. Dilcher, "Euler numbers, asymptotic expansions and pi," *MAA Monthly*, **96** (1989), 681–687.

2. See also *Mathematics by Experiment* [1, §2.10] and "I prefer Pi," *MAA Monthly*, March 2015.

3 1999: Siméon Poisson & ET Bell (1883–1960)

Sequence 3.1 (A000110 (1/10))**.**

$$1, 1, 2, 5, 15, 52, 203, 877, 4140 \ldots$$

Answer 3.2. *Bell or exponential numbers: number of ways to partition a set of n labeled elements.*

Story 3.3 (MAA Unsolved Problem)**.** For $t > 0$, let

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t)\, \frac{t^k}{k!}$$

be the n-th moment of a *Poisson distribution* with parameter t. Let $c_n(t) = m_n(t)/n!\,$. Show

(a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex for all $t > 0$.

(b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.

(c*) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geqslant 1$.

Proof. (b) As

$$m_{n+1}(t) = t \sum_{k=0}^{\infty} (k+1)^n \exp(-t) \frac{t^k}{k!},$$

on applying the binomial theorem to $(k+1)^n$, we see that

$$m_{n+1}(t) = t \sum_{k=0}^{n} \binom{n}{k} m_k(t), \qquad m_0(t) = 1.$$

In particular for $t = 1$, we obtain the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$$

These we have learned are the *Bell numbers*.

The OEIS `A001861` also tells us that for $t = 2$, we have *generalized Bell numbers*, and gives us the exponential generating functions. [The Bell numbers – as with many other discoveries – were known earlier to Ramanujan.]

Now an explicit computation shows that

$$t \, \frac{1+t}{2} = c_0(t)\, c_2(t) \leqslant c_1(t)^2 = t^2$$

exactly if $t \geqslant 1$. Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n \geqslant 0} c_n u^n = \exp\left(t(e^u - 1)\right). \tag{4}$$

(c*) (The * indicates this was unsolved.) We appeal to a then recent theorem due to Canfield. A search in 2001 on *MathSciNet* for "Bell numbers" since 1995 turned up 18 items. Canfield showed up as paper #10. Later, *Google* found the paper immediately!

Theorem 3.4 (Canfield)**.** *If a sequence* $1, b_1, b_2, \cdots$ *is non-negative and log-concave, then so is* $1, c_1, c_2, \cdots$ *determined by the generating function equation*

$$\sum_{n \geqslant 0} c_n u^n = \exp\left(\sum_{j \geqslant 1} b_j \frac{u^j}{j}\right).$$

Our desired application has $b_j \equiv 1$ for $j \geqslant 1$. Can the theorem be adapted to deal with eventually log concave sequences? □

References 3.5. The key references are

1. *Experimentation in Mathematics* [2, §1.11].

2. E.A. Bender and R.E. Canfield, "Log-concavity and related properties of the cycle index polynomials," *J. Combin. Theory Ser. A* **74** (1996), 57–70.

3. Solution to "Unsolved Problem 10738." posed by Radu Theodorescu in the 1999 *American Mathematical Monthly*.

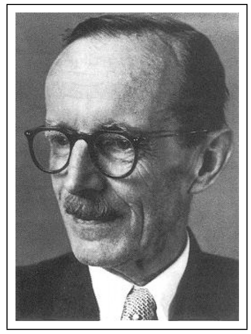

Figure 6: Erwin Madelung (1881–1972)

4 2000: Erwin Madelung & Richard Crandall (1947–2012)

Sequence 4.1 (A055745 (1/3)).

$$1, 2, 6, 10, 22, 30, 42, 58, 70, 78, 102, 130190, 210, 330, 462\ldots$$

Answer 4.2. *Squarefree numbers not of form $ab + bc + ca$ for $1 \leqslant a \leqslant b \leqslant c$ (probably the list is complete).*

A034168 *Disjoint discriminants (one form per genus) of type 2 (doubled).*

Story 4.3. A lovely 1986 formula for $\theta_4^3(q)$ due to Andrews is

$$\theta_4^3(q) = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^n} - 2\sum_{n=1,|j|<n}^{\infty} (-1)^j q^{n^2-j^2} \frac{1-q^n}{1+q^n}. \tag{5}$$

From (5) Crandall obtained

$$\sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s} = -4\sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(nm+mp+pn)^s} - 6\alpha^2(s). \tag{6}$$

Here $\alpha(s) = \left(1 - 2^{1-s}\right)\zeta(s)$ is the alternating zeta function.

Crandall used Andrew's formula (6) to find representations for *Madelung's constant*, $M_3(1)$, where

$$M_3(2s) := \sum_{n,m,p>0}^{\infty} \frac{(-1)^{n+m+p}}{(n^2+m^2+p^2)^s}.$$

The nicest integral consequence of (6) is

$$M_3(1) = -\frac{1}{\pi}\int_0^1\int_0^{2\pi} \frac{1 + 3\,r^{\sin(2\theta)-1}}{\left(1 + r^{\sin(2\theta)-1}\right)\left(1 + r^{\cos^2\theta}\right)\left(1 + r^{\sin^2\theta}\right)}\,\mathrm{d}\theta\,\mathrm{d}r.$$

A beautiful evaluation due to Tyagi also follows:

$$M_3(1) \quad = \quad -\frac{1}{8} - \frac{\log 2}{4\pi} - \frac{4\pi}{3} + \frac{1}{2\sqrt{2}} + \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\pi^{3/2}\sqrt{2}} \tag{7}$$

$$- 2\sum\nolimits'_{m,n,p} \frac{(-1)^{m+n+p}(m^2+n^2+p^2)^{-1/2}}{\exp\left[8\pi\sqrt{m^2+n^2+p^2}\right]-1}, \tag{8}$$

Here the 'closed form' part (7) – absent the rapidly convergent series (8) – is already correct to ten places of the total: $-1.74756459463318219063621\ldots$. No fully closed form for $M_3(1)$ is known.

Although not needed for his work, the ever curious Crandall then asked me what natural numbers were not of the form

$$ab + bc + ca.$$

It was bed-time in Vancouver so I asked my ex-PDF Roland Girgensohn in Munich. When I woke up, Roland had used MATLAB to send all 18 solutions up to $50,000$. Also $4, 18$ are the only non-square free solutions.

I recognised the square-free numbers as exactly the *singular values* of type II (Dickson), discussed in [3, §9.2]. One more 19-th solution $s > 10^{11}$ might exist but only without GRH.

4.0.1 Ignorance can be bliss

Luckily, we only looked at the OEIS *after* the paper was written. In this unusual case, the entry was based only on a comment supplied by two correspondents. Had we seen it originally, we should have told Crandall and left the subject alone. As it is, two other independent proofs appeared around the time of our paper.

4.1 The Newcastle connection

> . . . Born decided to investigate the simple ionic crystal-rock salt (sodium chloride) – using a ring model. He asked Lande to collaborate with him in calculating the forces between the lattice points that would determine the structure and stability of the crystal. Try as they might, the mathematical expression that Born and Lande derived contained a summation of terms that would not converge. Sitting across from Born and watching his frustration, Madelung offered a solution. His interest in the problem stemmed from his own research in Goettingen on lattice energies that, six years earlier, had been a catalyst for Born and von Karman's article on specific heat.
>
> The new mathematical method he provided for convergence allowed Born and Lande to calculate the electrostatic energy between neighboring atoms (a value now known as the Madelung constant). Their result for lattice constants of ionic solids made up of light metal halides (such as sodium and potassium chloride), and the compressibility of these crystals agreed with experimental results.

Actually, soon after, Born and Lande discovered they had forgotten to divide by two in the compressibility analysis. This ultimately led to the abandonment of the Bohr-Sommerfeld planar model of the atom.

Max Born was singer-and -actress Olivia Newton-John's maternal grandfather. Newton John's father Brinley (1914–1992) was the first Provost of the University of Newcastle. He was a fluent

Figure 7: Cyril Domb (1920–2012)

German speaker who interrogated Hess after his mad flight to Scotland in 1941. So Olivia has a fine academic background.

References 4.4. The key references are

1. Jonathan Borwein and Kwok-Kwong Stephen Choi, "On the representations of $xy+yz+zx$," *Experimental Mathematics*, **9** (2000), 153–158.

2. J. Borwein, L. Glasser, R. McPhedran, J. Wan, and J. Zucker, *Lattice Sums: Then and Now.* Encyclopedia of Mathematics and its Applications, **150**, Cambridge University Press, 2013.

5 2015: Cyril Domb & Karl Pearson (1857–1936)

Sequence 5.1 (A002895 & A253095)**.**

$$1, 4, 28, 256, 2716, 31504, 387136, 4951552\ldots$$

and

$$1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276\ldots$$

Answer 5.2. *Respectively:*

(a) Domb numbers: number of $2n$*-step polygons on diamond lattice.*

(b) Moments of 4-step random walk in 2 and 4 dimensions.

Story 5.3. We developed the following expression for the even moments. It is only entirely integer for $d = 2$ and $d = 4$.

In two dimensions it counts *abelian squares*. What does it count in four space?

Theorem 5.4 (Multinomial sum for the moments)**.** *The even moments of an* n*-step random walk in dimension* $d = 2\nu + 2$ *are given by*

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \ldots, k_n} \binom{k+n\nu}{k_1+\nu, \ldots, k_n+\nu}.$$

Story 5.5 (Generating function for three steps in four dimensions)**.** For $d = 4$, so $\nu = 1$, the moments are sequence `A103370`. The OEIS also records a hypergeometric form of the generating function, as the linear combination of a hypergeometric function and its derivative, added by Mark van Hoeij. On using linear transformations of hypergeometric functions, we have more simply that

$$\sum_{k=0}^{\infty} W_3(1;2k)x^k = \frac{1}{2x^2} - \frac{1}{x} - \frac{(1-x)^2}{2x^2(1+3x)}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 2\end{matrix}\,\middle|\,\frac{27x(1-x)^2}{(1+3x)^3}\right),$$

which we are able to generalise (the planar o.g.f has the same "form") – note the Laurent polynomial.

Theorem 5.6 (Generating function for even moments with three steps)**.** *For integers $\nu \geqslant 0$ and $|x| < 1/9$, we have*

$$\sum_{k=0}^{\infty} W_3(\nu;2k)x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}}\frac{(1-1/x)^{2\nu}}{1+3x}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1+\nu\end{matrix}\,\middle|\,\frac{27x(1-x)^2}{(1+3x)^3}\right) - q_\nu\left(\frac{1}{x}\right), \tag{9}$$

where $q_\nu(x)$ is a polynomial (that is, $q_\nu(1/x)$ is the principal part of the hypergeometric term on the right-hand side). In particular

$$\sum_{k=0}^{\infty} W_3(0;2k)x^k = \frac{1}{1+3x}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\,\middle|\,\frac{27x(1-x)^2}{(1+3x)^3}\right).$$

References 5.7. The key references are

1. J.M. Borwein, A. Straub and C. Vignat, "Densities of short uniform random walks in higher dimensions," *JMAA*, to appear 2016. See
   http://www.carma.newcastle.edu.au/jon/dwalks.pdf.

2. J. Borwein, A. Straub, J. Wan and W. Zudilin, with an Appendix by Don Zagier, "Densities of short uniform random walks," *Canadian. J. Math.* **64** (5), (2012), 961–990.

We finish with another recent example that again illustrates Richard Crandall's nimble mind.

6 2015: Poisson & Crandall

Sequence 6.1 (A218147)**.** $2, 2, 4, 4, 12, 8, 18, 8, 30, 16, 36, 24, 32, 32, 64, 36, 90, 32, 96, 60, 132, 64, 100, 72\ldots$

Notice that this is the first non-monotonic positive sequence we have studied.

Answer 6.2. *We are told it is the:*

(a) Conjectured degree of polynomial satisfied by

$$m(n) := \exp(8\pi\phi_2(1/n, 1/n).$$

(as defined in (10) *below)*

Figure 8: Richard Crandall (1947-2012)

(b) `A079458`: $4\,m(n)$ *is the number of Gaussian integers in a reduced system modulo* n.

Story 6.3. The lattice sums in question are defined by

$$\phi_2(x,y) := \frac{1}{\pi^2}\sum_{m,n \text{ odd}} \frac{\cos(m\pi x)\cos(n\pi y)}{m^2+n^2}. \tag{10}$$

Crandall conjectured while developing a deblurring algorithm illustrated in Figure 9 — and I then proved — that when x, y are rational

$$\phi_2(x,y) = \frac{1}{\pi}\log A, \tag{11}$$

where A is *algebraic*. Again, this number-theoretic discovery plays no role in the performance of the algorithm. Both computation and proof exploited the Jacobian theta-function representation [3, §2.7]:

$$\phi_2(x,y) = \frac{1}{2\pi}\log\left|\frac{\theta_2(z,q)\theta_4(z,q)}{\theta_1(z,q)\theta_3(z,q)}\right|, \tag{12}$$

where $q = e^{-\pi}$ and $z = \frac{\pi}{2}(y+ix)$

In Table 1 we display the recovered polynomial for $s = 35$. Note how much structure the picture reveals and how far from 'random' it is.

Story 6.4. *Remarkably, in* **2012** Jason Kimberley (University of Newcastle) observed that the degree $m(s)$ of the minimal polynomial for $x = y = 1/s$ appears to be as follows. Set $m(2) = 1/2$. For primes p congruent to 1 mod 4, set $m(p) = \mathrm{int}^2(p/2)$, where int denotes greatest integer, and for p congruent to 3 mod 4, set $m(p) = \mathrm{int}\,(p/2)(\mathrm{int}\,(p/2)+1)$. Then with prime factorisation $s = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$,

$$m(s) \stackrel{?}{=} 4^{r-1}\prod_{i=1}^{r} p_i^{2(e_i-1)}\,m(p_i). \tag{13}$$

- **2015** (13) holds for all tested cases where s now ranges up to 50 – save $s = 41, 43, 47, 49$, which are still too costly to test.
- Kimberly has recently conjectured a closed form for the polynomials, see Conjecture 6.5.

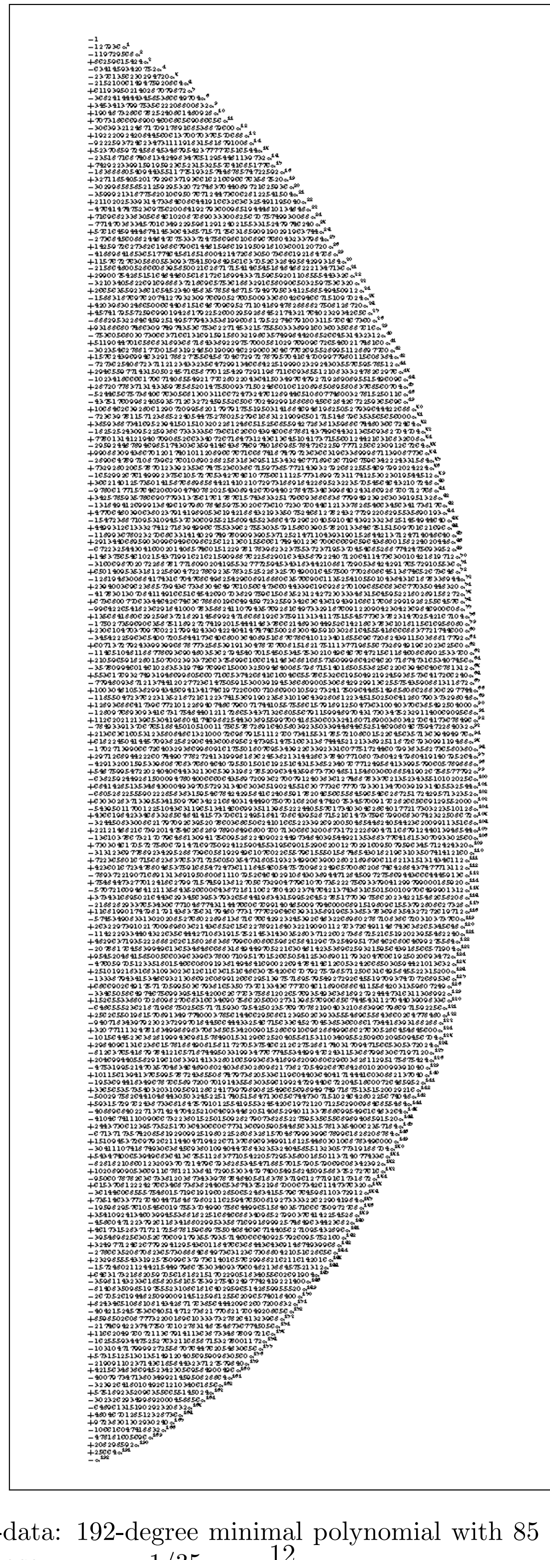

Table 1: Visualizing big-data: 192-degree minimal polynomial with 85 digit coefficients found by multipair PSLQ for the case $x = y = 1/35$.

Figure 9: Crandall's deblurring with a Poisson filter

Searching for 387221579866, from the polynomial for $s = 11$, P_{11}, we learn that Gordan Savin and David Quarfoot (2010) defined a sequence of polynomials $\psi_s(x, y)$ with $\psi_0 = \psi_1 = 1$, while $\psi_2 = 2y, \psi_3 = 3x^4 + 6x^2 + 1$, $\psi_4 = 2y(2x^6 + 10x^4 - 10x^2 - 2)$ and

$$\psi_{2n+1} = \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3 \qquad (n \geqslant 2) \tag{14}$$

$$2y\psi_{2n} = \psi_n \left(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2\right) \qquad (n \geqslant 3). \tag{15}$$

This led Kimberly to the following:

Conjecture 6.5 (Kimberley)**.** *We conjecture:*

(a) *For each integer $s \geqslant 1$, $P_s(-x^2)$ is a prime factor of $\psi_s(x)$. In fact, it is the unique prime factor of degree $2 \times A218147(s)$.*

(b) *The algebraic quantity is the largest real root of P_s.*

(c) *(Divisibility) For integer $m, n > 1$ when $m \mid n$ then $\psi_m \mid \psi_n$.*

(d) *(Irreducibility) For primes of form $4n + 3$, $\psi_s(x)$ is irreducible over $Q(i)$.*

- Conjecture (a) is *confirmed* for $s = 52$ and (b) has been checked up to $s = 40$. Parts (c) and (d) have been confirmed for $n \leqslant 120$.
- The full discovery remains to be disentangled.

References 6.6. The key references are

1. D.H. Bailey, J.M. Borwein, R.E. Crandall and I.J. Zucker, "Lattice sums arising from the Poisson equation." *Journal of Physics A*, **46** (2013) #115201 (31pp).
2. D.H. Bailey and J.M. Borwein, "Discovery of large Poisson polynomials using the MPFUN-MPFR arbitrary precision software." Submitted *ARITH23*, October 2015.
3. G. Savin and D. Quarfoot, "On attaching coordinates of Gaussian prime torsion points of $y^2 = x^3 + x$ to $Q(i)$," March 2010. www.math.utah.edu/~savin/EllipticCurvesPaper.pdf.

7 Conclusion

When I started showing the OEIS in talks twenty years ago, only a few hands would go up when asked who had heard of it. Now often half the audience will claim some familiarity. So there has been much progress but there is still work to be done to further advertize the OEIS.

- The OEIS is an amazing *instrumental* resource. I recommend everyone read Sloane's 2015 interview in *Quanta*

  - `https://www.quantamagazine.org/20150806-neil-sloane-oeis-interview/`

  It is now a fifty year old model both for *curation* and for *moderation* of a web resource.

  - Since Neil Sloane retired from ATT, the OEIS has moved to an edited and wiki-based resousce run by the OEIS foundation.

- As with all tools, the OEIS can help (very often) as as in the examples of Section 2 and Section 3, and it can hinder (much less often) as in the Example of Section 4.

- If a useful sequence occurs in your work, please contribute to the OEID as we did with the examples of Section 4 and Section 6.

  - Many of the underlying issues of technology and mathematics are discussed in [4] and more fully in: J. Monaghan, L. Troché and JMB, *Tools and Mathematics*, Springer (Mathematical Education), 2015.

We finish with another quotation.

> Algebra is generous; she often gives more than is asked of her. (Jean d´Alembert, 1717–1783).

As generous as algebra is, the OEIS usually has something more to add.

References

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