NON-SMOOTH ANALYSIS, OPTIMISATION THEORY AND BANACH SPACE THEORY

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Abstract. The questions listed here do not necessarily represent the most significant problems from the areas of Non-smooth Analysis, Optimisation theory and Banach space theory, but rather, they represent a selection of problems that are of interest to the authors.

1. Weak Asplund spaces

Let X be a Banach space. We say that a function $\varphi : X \to \mathbb{R}$ is *Gâteaux differentiable at* $x \in X$ if there exists a continuous linear functional $x^* \in X^*$ such that

$$x^*(y) = \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \quad \text{for all } y \in X.$$

In this case, the linear functional x^* is called the *Gâteaux derivative of* φ *at* $x \in X$. If the limit above is approached uniformly with respect to all $y \in B_X$-the closed unit ball in X, then φ is said to be *Fréchet differentiable at* $x \in X$ and x^* is called the *Fréchet derivative of* φ *at* x.

A Banach space X is called a *weak Asplund space* [*Gâteaux differentiability space*] if each continuous convex function defined on it is Gâteaux differentiable at the points of a *residual subset* (i.e., a subset that contains the intersection of countably many dense open subsets of X) [dense subset] of its domain.

Since 1933, when S. Mazur [55] showed that every separable Banach space is weak Asplund, there has been continued interest in the study of weak Asplund spaces. For an introduction to this area see, [61] and [32]. Also see the seminal paper [1] by E. Asplund.

The main problem in this area is given next.

Question 1.1. *Provide a geometrical characterisation for the class of weak Asplund spaces.* **1001 ?**

Note that there is a geometrical dual characterisation for the class of Gâteaux differentiability spaces, see [67, §6]. However, it has recently been shown that there are Gâteaux differentiability spaces that are not weak Asplund [58]. Hence

The first named author was supported by NSERC and by the Canada Research Chair Program. The second named author was supported by a Marsden Fund research grant, UOA0422, administered by the Royal Society of New Zealand.

the dual characterisation for Gâteaux differentiability spaces cannot serve as a dual characterisation for the class of weak Asplund spaces.

The description of the next two related problems requires some additional definitions.

Let $A \subseteq (0,1)$ and let $K_A := [(0,1] \times \{0\}] \cup [(\{0\} \cup A) \times \{1\}]$. If we equip this set with the order topology generated by the lexicographical (dictionary) ordering (i.e., $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$) then with this topology K_A is a compact Hausdorff space [46]. In the special case of $A = (0,1)$, K_A reduces to the well-known "double arrow" space.

? 1002 **Question 1.2.** *Is* $(C(K_A), \|\cdot\|_\infty)$ *weak Asplund whenever* A *is perfectly meagre?*

Recall that a subset $A \subseteq \mathbb{R}$ is called *perfectly meagre* if for every perfect subset P of $\mathbb{R}$ the intersection $A \cap P$ is meagre (i.e., first category) in P. An affirmative answer to this question will provide an example (in ZFC) of a weak Asplund space whose dual space is not weak* fragmentable, see [58] for more information on this problem. For example, it is shown in [58] that if A is perfectly meagre then $(C(K_A), \|\cdot\|_\infty)$ is *almost weak Asplund* i.e., every continuous convex function defined on $(C(K_A), \|\cdot\|_\infty)$ is Gâteaux differentiable at the points of an everywhere second category subset of $(C(K_A), \|\cdot\|_\infty)$. Moreover, it is also shown in [58] that if $(C(K_A), \|\cdot\|_\infty)$ is weak Asplund then A is obliged to be perfectly meagre.

Our last question on this topic is the following well-known problem.

? 1003 **Question 1.3.** *Is* $(C(K_{(0,1)}), \|\cdot\|_\infty)$ *a Gâteaux differentiability space?*

The significance of this problem emanates from the fact that $(C(K_{(0,1)}), \|\cdot\|_\infty)$ is not a weak Asplund space as the norm $\|\cdot\|_\infty$ is only Gâteaux differentiable at the points of a first category subset of $(C(K_{(0,1)}), \|\cdot\|_\infty)$, [32]. Hence a positive solution to this problem will provide another, perhaps more natural, example of a Gâteaux differentiability space that is not weak Asplund.

2. Bishop-Phelps Problem

For a Banach space $(X, \|\cdot\|)$, with closed unit ball B_X, the *Bishop-Phelps set* is the set of all linear functionals in the dual X^* that attain their maximum value over B_X; that is, the set $\{x^* \in X^* : x^*(x) = \|x\| \text{ for some } x \in B_X\}$. The Bishop-Phelps Theorem [4] says that the Bishop-Phelps set is always dense in X^*.

? 1004 **Question 2.1.** *Suppose that* $(X, \|\cdot\|)$ *is a Banach space. If the Bishop-Phelps set is a residual subset of* X^* *(i.e., contains, as a subset, the intersection of countably many dense open subsets of* X^**) is the dual norm necessarily Fréchet differentiable on a dense subset of* X^**?*

The answer to this problem is known to be positive in the following cases:

(i) if X^* is weak Asplund, [36, Corollary 1.6(i)];
(ii) if X admits an equivalent weakly mid-point locally uniformly rotund norm and the weak topology on X is σ-fragmented by the norm, [59, Theorem 3.3 and Theorem 4.4];

(iii) if the weak topology on X is Lindelöf, [49].

The assumptions in (ii) can be slightly weakened, see [37, Theorem 2]. It is also known that each equivalent dual norm on X^* is Fréchet differentiable on a dense subset on X^* whenever the Bishop-Phelps set of each equivalent norm on X is residual in X^*, [57, Theorem 4.4]. Note that in this case X has the Radon-Nikodým property.

For an historical introduction to this problem and its relationship to local uniformly rotund renorming theory see, [48].

Next, we give an important special case of the previous question.

Question 2.2. *If the Bishop-Phelps set of an equivalent norm $\|\cdot\|$ defined on $(\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)$ is residual, is the corresponding closed unit ball dentable?* **1005 ?**

Recall that a nonempty bounded subset A of a normed linear space X is *dentable* if for every $\varepsilon > 0$ there exists a $x^* \in X^* \setminus \{0\}$ and a $\delta > 0$ such that

$$\|\cdot\| - \operatorname{diam}\{a \in A : x^*(a) > \sup_{x \in A} x^*(x) - \delta\} < \varepsilon.$$

It is well-known that if the dual norm has a point of Fréchet differentiability then B_X is dentable [75].

3. The Complex Bishop-Phelps Property

For S a subset of a (real or complex) Banach space X, we may recast the notion of *support functional* as follows: a nonzero functional $\varphi \in X^*$ is a support functional for S and a point $x \in S$ is a *support point* of S if $|\varphi(x)| = \sup_{y \in S} |\varphi(y)|$.

Let us say a set is *supportless* if there is no such φ.

As Phelps observed in [66] while the Bishop-Phelps construction resolved Klee's question [51] of the existence of support points in real Banach space, it remained open in the complex case. Lomonosov, in [52], gives the first example of a closed convex bounded convex set in a complex Banach space with no support functionals.

Question 3.1. *Characterise (necessarily complex) Banach spaces which admit supportless sets.* **1006 ?**

It is known that they must fail to have the Radon-Nikodým property [52, 53].

A Banach space X has the *attainable approximation property (AAP)* if the set of support functionals for any closed bounded convex subset $W \subseteq X$ is norm dense in X^*. In [53] Lomonosov shows that if a uniform dual algebra $\mathcal{R}$ of operators on a Hilbert space has the (AAP) then $\mathcal{R}$ is self-adjoint.

Question 3.2. *Characterise complex Banach spaces with the AAP. In particular do they include $L^1[0,1]$?* **1007 ?**

4. Biorthogonal Sequences and Support Points

Uncountable biorthogonal systems provide the easiest way to produce sets with prescribed support properties.

4.1. **Constructible Convex Sets and Biorthogonal Families.** A closed convex set is *constructible* [10] if it is expressible as the countable intersection of closed half-spaces. Clearly every closed convex subset of a separable space is constructible.

More generally:

Theorem 4.1. [10] *Let X be a Banach space, then the following are equivalent.*

(i) *There is an uncountable family $\{x_\alpha\} \subseteq X$ such that*

$$x_\alpha \notin \overline{\mathrm{conv}}(\{x_\beta : \beta \neq \alpha\})$$

for all α.

(ii) *There is a closed convex subset in X that is not constructible.*

(iii) *There is an equivalent norm on X whose unit ball is not constructible.*

(iv) *There is a bounded uncountable system $\{x_\alpha, \phi_\alpha\} \subseteq X \times X^*$ such that $\phi_\alpha(x_\alpha) = 1$ and $|\phi_\alpha(x_\beta)| \leq a$ for some $a < 1$ and all $\alpha \neq \beta$.*

Example 4.1. [10] *The sequence space c_0 considered as a subspace of ℓ_∞ is not constructible. Consequently, no bounded set with nonempty interior relative to c_0 is constructible as a subset of ℓ_∞. In particular the unit ball of c_0 is not constructible when viewed as a subset of ℓ_∞.*

In particular, if X admits an uncountable biorthogonal system then it admits an non-constructible convex set. Under additional set-theoretic axioms, there are nonseparable Banach spaces in which all closed convex sets are constructible. These are known to include: (i) the $C(K)$ space of Kunen constructed under the Continuum Hypothesis (CH) [64], and (ii) the space of Shelah constructed under the diamond principle [73]. In consequence, these non-separable spaces of Kunen and Shelah have the property that for each equivalent norm, the dual unit ball is weak*-separable, [10].

? 1008 **Question 4.1.** *Can one construct an example of a nonseparable space whose dual ball is weak* separable for each equivalent norm using only Zermelo-Fraenkel set theory along with the Axiom of Choice?*

In contrast, it is shown in [10] that there are general conditions under which nonseparable spaces are known to have uncountable biorthogonal systems. Suppose X is a nonseparable Banach space such that

(i) X is a dual space, or
(ii) $X = C(K)$, for K compact Hausdorff, and one assumes Martin's axiom along with the negation of the Continuum Hypothesis.

Then X admits an uncountable biorthogonal system. Part(ii) is a deep recent result of S. Todorcevic, see for example, [41, p. 5].

? 1009 **Question 4.2.** *When, axiomatically, does a continuous function space always admit an uncountable biorthogonal system?*

4.2. **Support Sets.** In a related light, consider the question:

? 1010 **Question 4.3.** *Does every nonseparable $C(K)$ contain a closed convex set entirely composed of support points (the tangent cone is never linear)?*

In [9] it is shown that this is equivalent to $C(K)$ admitting an uncountable *semi-biorthogonal system*, i.e., a system $\{x_\alpha, f_\alpha\}_{1\leq\alpha<\omega_1} \subseteq X \times X^*$ such that $f_\alpha(x_\beta) = 0$ if $\beta < \alpha$, $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \geq 0$ if $\beta > \alpha$. Moreover, [9] observes that Kunen's space is an example where this happens without there being an uncountable biorthogonal system assuming the Continuum Hypotheses. Thus, the answer is 'yes' except perhaps when Martin's Axiom fails (along with CH).

4.3. **Supportless Sets.** For a set C in a normed space X, $x \in C$ is a *weakly supported point* of C if there is a linear functional f such that the restriction of f to C is continuous and nonzero. Fonf [35], extending work of Klee [50] (see also Borwein-Tingley [8]) proves the following result which is in striking contrast to the Bishop-Phelps theorem in Banach spaces: *Every incomplete separable normed space X contains a closed bounded convex set C such that the closed linear span of C is all of X and C contains no weakly supported points.*

Let us call such a closed bounded convex set *supportless*. It is known that there are *Fréchet spaces* (complete metrizable locally convex spaces) which admit supportless sets. In [65] Peck shows that *for any sequence of nonreflexive Banach spaces $\{X_i\}$, in the product space $E = \prod_{i=1}^{\infty} X_i$, there is a closed bounded convex set that has no E^*-support points.* Peck also provides some positive results.

Question 4.4. *Characterise when a Fréchet space contains a closed convex supportless convex set?* 1011 ?

5. Best Approximation

Even in Hilbert spaces and reflexive Banach spaces some surprising questions remain open.

Question 5.1. *Is there a non-convex subset A of a Hilbert space H with the property that every point in $H \setminus A$ has a unique nearest point?* 1012 ?

Such a set is called a *Chebyshev set* and must be closed and bounded. For a good up-to-date general discussion of best approximation in Hilbert space we refer to [27]. Asplund [2] shows that if non-convex Chebysev sets exist then among them are so called *Asplund caverns*—complements of open convex bodies. In finite dimensions, the Motzkin-Klee theorem establishes that all Chebyshev sets are convex. Four distinct proofs are given in [7, §9.2] which highlight the various obstacles in infinite dimensions.

Question 5.2. *Is there a closed nonempty subset A of a reflexive Banach space X with the property that no point outside A admits a best approximation in A? Is this possible in an equivalent renorm of a Hilbert space?* 1013 ?

The *Lau-Konjagin Theorem* (see [5]) states that in a reflexive space, *for every closed set A there is a dense (or generic) set in $X \setminus A$ which admits best approximations if and only if the norm has the Kadec-Klee property.* Thus, any counter example must have a non-Kadec-Klee norm and must be unbounded—via the Radon-Nikodym property. In [5], a class of reflexive non-Kadec Klee norms is exhibited for which some nearest points always exist.

By contrast, in every non-reflexive space, James Theorem [34] provides a closed hyperplane H with no best approximation: equivalently $H + B_X$ is open. More exactingly, two closed bounded convex sets with nonempty interior are called *companion bodies* and *anti-proximinal* if their sum is open. Such research initiates with Edelstein and Thompson [31].

? 1014 **Question 5.3.** *Characterise Banach spaces (over $\mathbb{R}$) that admit companion bodies.*

Such spaces include c_0 [31, 22, 6] and again do not include any space with the Radon-Nikodym property [5].

6. Metrizability of compact convex sets

One facet of the study of compact convex subsets of locally convex spaces is the determination of their metrizability in terms of topological properties of their extreme points. For example, a compact convex subset K of a Hausdorff locally convex space X is metrizable if, and only if, the extreme points of K (denoted $\mathrm{Ext}(K)$) are *Polish* (i.e., homeomorphic to a complete separable metric space), [23].

Since 1970 there have been many papers on this topic (e.g. [23, 24, 45, 54, 69] to name but a few).

? 1015 **Question 6.1.** *Let K be a nonempty compact convex subset of a Hausdorff locally convex space (over $\mathbb{R}$). Is K metrizable if, and only if, $A(K)$ - the continuous real-valued affine mappings defined on K, is separable with respect to the topology of pointwise convergence on* $\mathrm{Ext}(K)$*?*

The answer to this problem is known to be positive in the following cases:

(i) if $\overline{\mathrm{Ext}(K)}$ is Lindelöf, [60];

(ii) if $\overline{\mathrm{Ext}(K)} \setminus \mathrm{Ext}(K)$ is countable, [60].

Question 6.1 may be thought of as a generalisation of the fact that a compact Hausdorff space K is metrizable if, and only if, $C_p(K)$ is separable. Here $C_p(K)$ denotes the space of continuous real-valued functions defined on K endowed with the topology of pointwise convergence on K.

7. The Boundary Problem

Let $(X, \|\cdot\|)$ be a Banach space. A subset B of the dual unit ball B_{X^*} is called a *boundary* if for any $x \in X$, there is $x^* \in B$ such that $x^*(x) = \|x\|$. A simple example of boundary is provided by the set $\mathrm{Ext}(B_{X^*})$ of extreme points of B_{X^*}. This notion came into light after James' characterisation of weak compactness [44], and has been studied in several papers (e.g. [74, 70, 76, 38, 39, 19, 17, 16, 40, 18]). In spite of significant efforts, the following question is still open (see [38, Question V.2] and [30, Problem I.2]):

Question 7.1. *Let A be a norm bounded and $\tau_p(B)$ compact subset of X. Is A weakly compact?* **1016 ?**

The answer to the boundary problem is known to be positive in the following cases:

(i) if A is convex, [74];
(ii) if $B = \mathrm{Ext}(B_{X^*})$, [12];
(iii) if X does not contain an isomorphic copy of $l_1(\Gamma)$ with $|\Gamma| = \mathfrak{c}$, [17, 18];
(iv) if $X = C(K)$ equipped with their natural norm $\|\cdot\|_\infty$, where K is an arbitrary compact space, [16].

Case (i) can be also obtained from James' characterisation of weak compactness, see [39]. The original proof for (ii) given in [12] uses, among other things, deep results established in [11]. Case (iii) is reduced to case (i): if $l_1(\Gamma) \not\subset X$, $|\Gamma| = \mathfrak{c}$, and $C \subset B_{X^*}$ is 1-norming (i.e., $\|x\| = \sup\{|x^*(x)| : x^* \in C\}$), it is proved in [17, 18] that for any norm bounded and $\tau_p(C)$-compact subset A of X, the closed convex hull $\overline{\mathrm{co}}^{\tau_p(C)}(A)$ is again $\tau_p(C)$-compact; the class of Banach spaces fulfilling the requirements in (iii) is a wide class of Banach spaces that includes: weakly compactly generated Banach spaces or more generally weakly Lindelöf Banach spaces and spaces with dual unit ball without a copy of $\beta\mathbb{N}$. The techniques used in case (iv) are somewhat different, and naturally extend the classical ideas of Grothendieck, [42], that led to the fact that norm bounded $\tau_p(K)$-compact subsets of spaces $C(K)$ are weakly compact. It should be noted that it is easy to prove that for any set Γ, the boundary problem has also positive answer for the space $\ell^1(\Gamma)$ endowed with its canonical norm, see [16, 18].

We observe that the solution in full generality to the boundary problem without the concourse of James' theorem of weak compactness would imply an alternative proof of the following version of James' theorem itself: a Banach space X is reflexive if, and only if, each element $x^* \in X^*$ attains its maximum in B_X.

Finally, we point out that in the papers [71, 79], it has been claimed that the boundary problem was solved in full generality. Unfortunately, to the best of our knowledge both proofs appear to be flawed.

8. Separate and Joint Continuity

If X, Y and Z are topological spaces and $f : X \times Y \to Z$ is a function then we say that f is *jointly continuous at* $(x_0, y_0) \in X \times Y$ if for each neighbourhood W of $f(x_0, y_0)$ there exists a product of open sets $U \times V \subseteq X \times Y$ containing (x_0, y_0) such that $f(U \times V) \subseteq W$ and we say that f is *separately continuous on* $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$ are both continuous on Y and X respectively.

Since the paper [3] of Baire first appeared there has been continued interest in the question of when a separately continuous function defined on a product of "nice" spaces admit a point (or many points) of joint continuity and over the years there have been many contributions to this area (e.g. [15, 20, 21, 25, 26, 49, 63, 56, 68, 72, 77] etc.). Most of these results can be classified into one of two types. (I) The existence problem, i.e., if $f : X \times Y \to \mathbb{R}$ is separately continuous find conditions on either X or Y (or both) such that f has at least one point of joint continuity. (II) The fibre problem, i.e., if $f : X \times Y \to \mathbb{R}$ is separately continuous find conditions on either X or Y (or both) such that there exists a nonempty subset R of X such that f is jointly continuous at the points of $R \times Y$.

The main existence problem is, [78]:

? 1017 **Question 8.1.** *Let X be a Baire space and let Y be a compact Hausdorff space. If $f : X \times Y \to \mathbb{R}$ is separately continuous does f have at least one point of joint continuity?*

We will say that a topological space X has the *Namioka Property* of has property $\mathcal{N}$ if for every compact Hausdorff space Y and every separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense G_δ-subset G of X such that f is jointly continuous at each point of $G \times Y$. Similarly, we will say that a compact Hausdorff space Y has the *co-Namioka Property* or has property $\mathcal{N}^*$ if for every Baire space X and every separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense G_δ-subset G of X such that f is jointly continuous at each point of $G \times Y$.

The main fibre problems are:

? 1018 **Question 8.2.** *Characterise the class of Namioka spaces.*

There are many partial results.

(i) Every Namioka space is Baire, [72];
(ii) Every separable Baire space and every Baire p-space is a Namioka space, [72];
(iii) Not every Baire space is a Namioka space [78];
(iv) Every Lindelöf weakly α-favourable space is a Namioka space, [49]
(v) Every space expressible as a product of hereditarily Baire metric spaces is a Namioka space, [20].

? 1019 **Question 8.3.** *Characterise the class of co-Namioka spaces.*

There are many partial results.

(i) $\beta\mathbb{N}$ is not a co-Namioka space, [28];
(ii) Every Valdivia compact is a co-Namioka space, [13, 29];
(iii) The co-Namioka spaces are stable under products, [15];
(iv) All scattered compacts K with $K^{(\omega_1)} = \emptyset$ are co-Namioka, where $K^{(\alpha)}$ denotes the α^{th} derived set of K, [28];
(v) There exists a non co-Namioka compact space K such that $K^{(\omega_1)}$ is a singleton, [43].

A partial characterisation, in terms of a topological game on $C_p(K)$, is given in [47] for the class of compact spaces K such that: *for every weakly α-favourable space X and every separately continuous mapping $f : X \times K \to \mathbb{R}$ there exists a dense G_δ subset G of X such that f is jointly continuous at each point of $G \times K$.*

For an introduction to this topic see, [56, 68]. Also see the seminal paper [62] by I. Namioka, as well as, the paper [63].

9. Acknowledgements

The material on the "Boundary problem" was provided by B. Cascales.

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