Tools for visualizing real numbers.
Part I: Planar number walks

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May 13, 2012

Abstract

Motivated by the desire to visualize large mathematical data sets, especially in number theory, we offer various tools for representing floating point numbers as planar (or three dimensional) walks and for quantitatively measuring their "randomness".

1   Introduction

In the recent paper [4], by accessing the results of several extremely large recent computations [44, 45], the authors tested positively the normality of a prefix of roughly four trillion hexadecimal digits of π. This result was used by a Poisson process model of normality of π: in this model, it is extraordinarily unlikely that π is not asymptotically normal base 16, given the normality of its initial segment. During that work, the authors of [4] like many others looked for visual methods of representing their large mathematical data sets. Their chosen tool was representation as walks in the plane. In this work, based in part on sources such as [21, 22, 20, 18, 14] we make a more rigorous and quantitative study of such walks on numbers.

The organization of the paper is as follows. In Section 2 we describe and exhibit uniform walks on various numbers, both rational and irrational, artificial and 'natural'. In

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the next two sections, we look at quantifying two of the best-known features of random walks: the expected distance travelled after N steps (Section 3) and the number of sites visited (Section 4). In Section 5 we discuss measuring the fractal (actually box) dimension of our walks. In Section 6 we describe two classes for which normality and non-normality results are known and one for which we have only surmise. In Section 7 we show some further examples and leave some open questions. Finally, in Appendix 8 we collect the numbers we have examined.

2 Walking on numbers

One of our tasks is to compare pseudo-random walks to deterministic walks of the same length. For example, in Figure 1 we draw a uniform pseudo random-walk with a million base-4 steps.
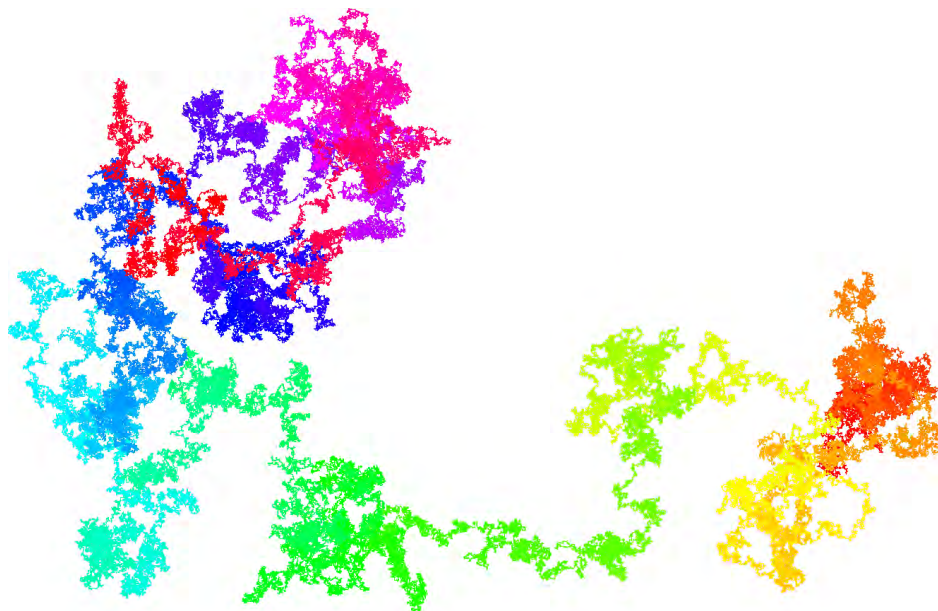

Figure 1: A uniform pseudo-random walk.

Below one will see very similar and widely dissimilar walks on the numbers we study.

2.1 A first few steps

At first glance the following two rational numbers $Q1$ and $Q2$ look quite different:

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Q1=
1049012271677499437486619280565448601617567358491560876166848380843
1443584472528755516292470277595555704537156793130587832477297720217
7081818796590637365767487981422801328592027861019258140957135748704
71229026746515131280595419539975042020613803738223389597133391954
/
1612226962694290912940490066273549214229880755725468512353395718465
1913530173488143140175045399694454793530120643833272670970079330526
2920303509209736004509554561365966493250783914647728401623856513742
95294530896122681527488756156580761624107880751845999421938774835

Q2=
7278984857066687413042833612434773655776009792025799724606605320967
1510416153622193809833373062647935595578496622633151106310912260966
7568778977976821682512653537303069288477901523227013159658247897670
30435402490295493942131091063934014849602813952
/
1118707184315428172047608747409173378543817936412916114431306628996
5259377090978187244251666337745459152093558288671765654061273733231
7877736113382974861639142628415265543797274479692427652260844707187
53215525487295285372502631868599749526213465215
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    Their decimal expansion gives us some additional information: they both agree in the first 240 digits (in base 4, their first 400 digits are the same). In Figure 2 a plot of the walk on the digits of $Q1$ and $Q2$ in base 4 is shown. In order to create it, we have transformed the rational numbers $Q1$ and $Q2$ into their base 4 decimal expansion, obtaining two (periodic) sequences whose elements are either 0, 1, 2 or 3. At each step a new point is drawn adjacent to the previous one: 0, 1, 2 and 3 draw the new point to the east, north, west and south, respectively.

2.2 Random and deterministic walks

The idea is the same as when a random walk is plotted, with the difference being that the direction followed at each step is encoded in the digits of the number instead of being chosen randomly. The color is shifted up the spectrum (red-orange-yellow-green-cyan-blue-purple-red) following an HSV[1] model with S and V equal to one, which permits to visualize the path followed by the walk. Following this same method, in each of the pictures in Figures 2 through 8, a digit string for a given number in a particular base is used to determine the angle of unit steps (multiples of 120 degrees for base three, 90 degrees for base four, etc). In Figure 3 the origin has been marked. Since this information is not that important for our purposes and it can be approximately deduced by the color in most of the cases (and

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[1]HSV (hue, saturation, and value) is a cylindrical-coordinate representation that permits to get an easy rainbow-like range of colors.

the pictures look less esthetic, let us face it), we have not marked it in the rest. Figure 6 is colored in a different way: by the number of returns to each point.

For the reader's convenience, we have included an appendix with the definitions of the mathematical constants addressed in this paper, see Section 8. We also exhibit there a few digits in various bases for each number.

The rational numbers $Q1$ and $Q2$ represent the two possibilities when one computes a walk on a rational number: either the walk is bounded like in Figure 2(a) (for any walk with more than 440 steps one obtains the same plot), or it is unbounded but repeating some pattern after a finite number of digits like in Figure 2(b).

(a) A 440 step walk on $Q1$ base 4.
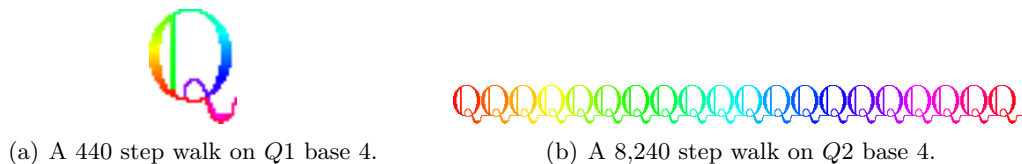(b) A 8,240 step walk on $Q2$ base 4.

Figure 2: Walks on the rational numbers $Q1$ and $Q2$.

Of course, not all rational numbers are that easily identified by plotting their walk. It is possible to create a rational number whose period is of any desired length. For example, the following rational numbers from [37],

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{1000000000061},$$

have a periodic part in base 10 with length[2] 1,750,000,000 and 1,000,000,000,060, respectively. A walk on the first million digits of both numbers is plotted in Figure 4.

2.3 Numbers as walks

In the following, given some positive integer base b, we will say that a real number x is b-normal if every m-long string of base-b digits appears in the base-b expansion of x with precisely the expected limiting frequency $1/b^m$. It follows, from basic measure theory, as shown by Borel, that almost all real numbers are b-normal for any specific base b and even for all bases simultaneously [15].

But proving normality for specific constants of interest in mathematics has proven remarkably difficult. It is useful to know that while small in measure, the absolutely abnormal numbers are residual in the sense of topological category [1] (moreover, the Hausdorff-Besicovitch dimension of the set of real numbers having no asymptotic frequencies is equal to 1). Likewise the Liouville numbers are measure zero but second category [17, p. 352].

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[2]The numerators and denominators of $Q3$ and $Q4$ are relatively prime, and the denominators are not congruent to 2 or 5, thus the period is simply the discrete logarithm of the denominator.
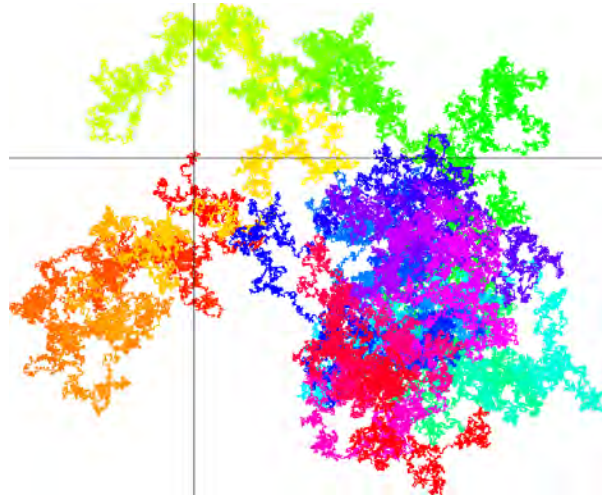
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Figure 3: A million step base-4 walk on e.

(a) $Q3 = \frac{3624360069}{7000000001}$

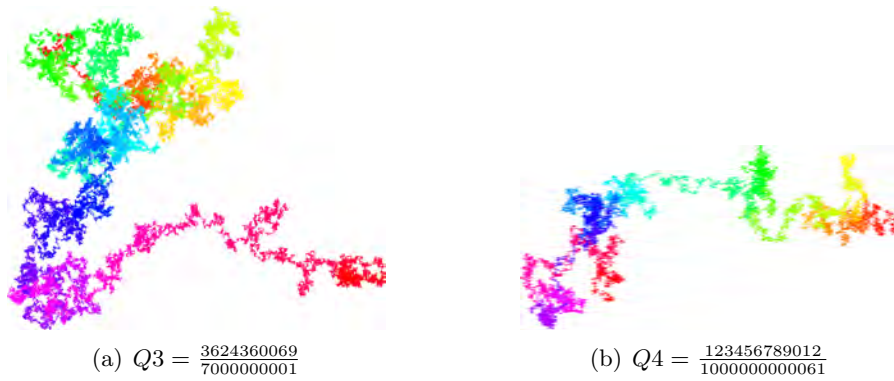(b) $Q4 = \frac{123456789012}{1000000000061}$

Figure 4: Walks on the first million base 10-digits of the rationals $Q3$ and $Q4$ from [37].

For example, a reasonable conjecture is that *every* irrational algebraic number (i.e., the irrational roots of polynomial equations with integer coefficients) is *b*-normal for *every* integer base *b*. Yet there is no proof of this conjecture, not for *any* specific algebraic number or for *any* specific integer base. The tenor of current knowledge is illustrated by [43, 14, 33, 36, 38, 37, 42].

In Figure 5 we show a walk on the first 10 billion base 4 digits of π. This may be viewed in more detail[3] online at http://gigapan.org/gigapans/99214/. In Figure 6 a 100 million base 4 walk on π is shown where the color is determined by the number of returns to the point. It is unknown whether π is normal or not. In [4] the authors have empirically tested the normality of roughly four trillion of its hexadecimal digits by using a Poisson process model, concluding that it is "extraordinarily unlikely" that π is not 16-normal.
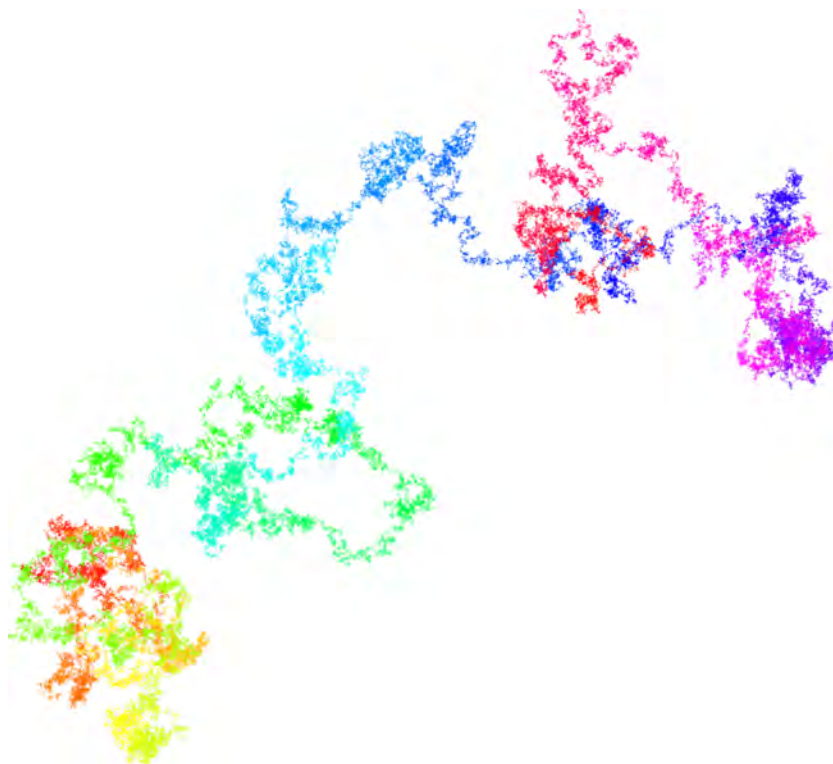

Figure 5: A walk on the first 10 billion base-4 digits of π (normal?).

In what follows, we propose various methods to analyze real numbers and visualize them as walks. Other methods widely used to visualize numbers include the matrix rep-

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[3]The full-size picture has a resolution of $149,818 \times 136,312$ pixels, that is, 20.42 gigapixels.
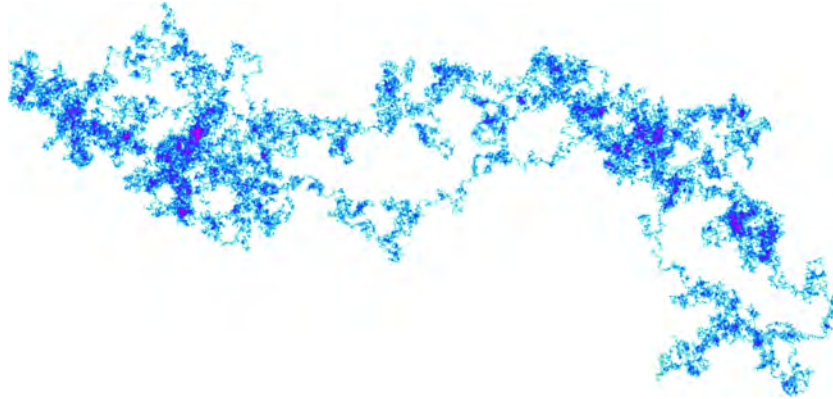
Figure 6: A walk on the 100 million base-4 digits of π, colored by number of returns (normal?). Color follows an HSV model (green-cyan-blue-purple-red) depending on the number of returns to each point (where the maximum is colored in pink/red).

Figure 7: A walk on the first 100,000 bits of the primes $(CE(2))$ base two (normal).

(a) A million step walk on $\alpha_{2,3}$ base 3 (normal?).

(b) A 100,000 step walk on $\alpha_{2,3}$ base 6 (abnormal).

(c) A million step walk on $\alpha_{2,3}$ base 2 (normal).

(d) A 100,000 step walk on Champernowne's number C_4 base 4 (normal).

(e) A million step walk on $EB(2)$ base 4 (normal?).

(f) A million step walk on $CE(10)$ base 4 (normal?).

Figure 8: Walks on prefixes of various numbers in different bases.

resentations shown in Figure 9, where each pixel is colored depending on the value of the digit, following a left-to-right up-to-down direction (in base 4 the colors used for 0, 1, 2 and 3 are red, green, cyan and purple, respectively). This method has been mainly used to visually test "randomness." In some cases, it clearly shows the features of some numbers, as for small periodic rationals, see Figure 9(c). It also shows the nonnormality of the number $\alpha_{2,3}$, see Figure 9(e) (where the horizontal red bands correspond to the strings of zeroes), and it captures some of the special peculiarities of the Champernowne's number C_4 (normal) in Figure 9(d). Nevertheless, it does not reveal the apparently non-random behavior of numbers like the Erdös-Borwein constant: compare Figure 9(f) with Figure 8(e).

As we will see in what follows, the study of real numbers as walks will permit us to compare them with random walks, obtaining in this manner a new way to empirically test "randomness" in their digits.

3   Expected distance to the origin

Let $b \in \{3, 4, \ldots\}$ be a fixed base[4], and let $X_1, X_2, X_3, \ldots$ be a sequence of independent bivariate discrete random variables whose common probability distribution is given by

$$P\left(X = \left(\begin{array}{c} \cos\left(\frac{2\pi}{b}k\right) \\ \sin\left(\frac{2\pi}{b}k\right) \end{array}\right)\right) = \frac{1}{b} \quad \text{for } k = 1, \ldots, b.$$

Then the random variable $S^N := \sum_{m=1}^{N} X_m$ represents a base-b random walk in the plane of N steps.

The following result on the asymptotic expectation of the distance to the origin of a base-b random walk is probably known, but being unable to find any reference in the literature, we provide a proof.

Theorem 3.1. *The expected distance to the origin of a base-b random walk of N steps is asymptotically equal to $\sqrt{\pi N}/2$.*

Proof. By the multivariate central limit theorem, the random variable $1/\sqrt{N}\sum_{m=1}^{N}(X_m - \mu)$ is asymptotically bivariate normal with mean 0_2 and covariance matrix M, where μ is the bi-dimensional mean vector of X and M is its 2×2 covariance matrix. By applying Lagrange's trigonometric identities, one gets

$$\mu = \left(\begin{array}{c} \frac{1}{b}\sum_{k=1}^{b}\cos\left(\frac{2\pi}{b}k\right) \\ \frac{1}{b}\sum_{k=1}^{b}\sin\left(\frac{2\pi}{b}k\right) \end{array}\right) = \frac{1}{b}\left(\begin{array}{c} -\frac{1}{2} + \frac{\sin\left((b+1/2)\frac{2\pi}{b}\right)}{2\sin(\pi/b)} \\ \frac{1}{2}\cot(\pi/b) - \frac{\cos\left((b+1/2)\frac{2\pi}{b}\right)}{2\sin(\pi/b)} \end{array}\right) = 0_2.$$

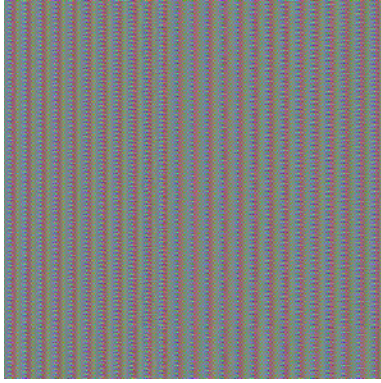[4]We treat the case $b = 2$ as a base-4 walk. We could consider it as a base-2 walk on a line, but the pictures would be much less interesting.
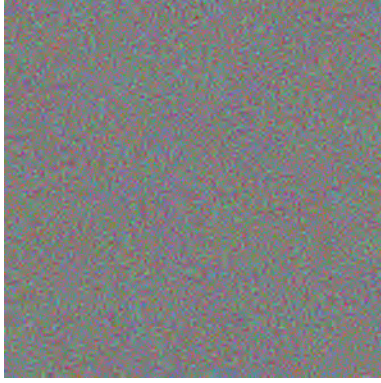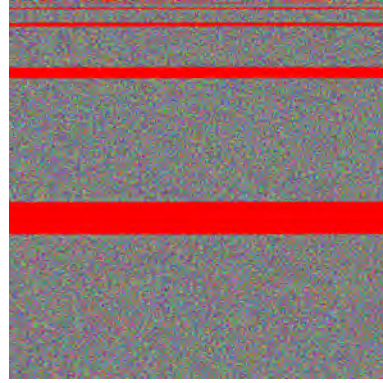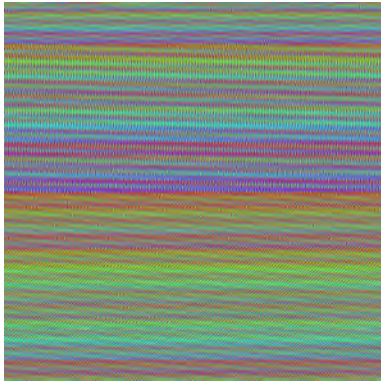
(a) π base 4
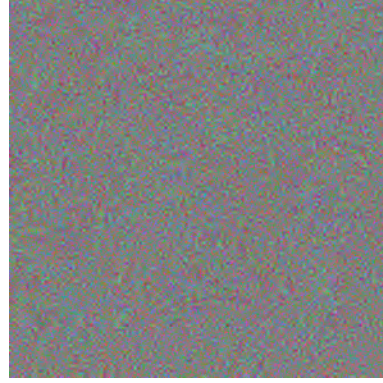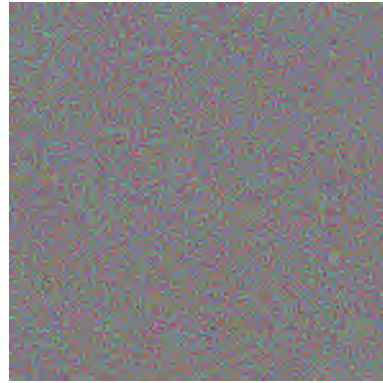

(b) (Pseudo)random number base 4

(c) The rational number $Q1$ base 4

(d) $\alpha_{2,3}$ base 6

(e) Champernowne's number C_4 base 4

(f) $EB(2)$ base 4

Figure 9: Horizontal color representation of a million digits of various numbers.

Number	Base	Steps	Average normalized distance to the origin	Normal
Mean of 10,000 random walks	4	1,000,000	1.00315	Yes
$\alpha_{2,3}$	3	1,000,000	0.89275	?
$\alpha_{2,3}$	4	1,000,000	0.25901	Yes
$\alpha_{2,3}$	5	1,000,000	0.88104	?
$\alpha_{2,3}$	6	1,000,000	108.02218	No
$\alpha_{4,3}$	3	1,000,000	1.07223	?
$\alpha_{4,3}$	4	1,000,000	0.24268	Yes
$\alpha_{4,3}$	6	1,000,000	94.54563	No
$\alpha_{4,3}$	12	1,000,000	371.24694	No
$\alpha_{3,5}$	3	1,000,000	0.32511	Yes
$\alpha_{3,5}$	5	1,000,000	0.85258	?
$\alpha_{3,5}$	15	1,000,000	370.93128	No
π	4	1,000,000	0.84366	?
π	6	1,000,000	0.96458	?
π	10	1,000,000	0.82167	?
π	10	10,000,000	0.56856	?
π	10	100,000,000	0.94725	?
π	10	1,000,000,000	0.59824	?
e	4	1,000,000	0.59583	?
$\sqrt{2}$	4	1,000,000	0.72260	?
$\log 2$	4	1,000,000	1.21113	?
Champernowne C_{10}	10	1,000,000	59.91143	Yes
$EB(2)$	4	1,000,000	6.95831	?
$CE(10)$	4	1,000,000	0.94964	?
Rational number Q_1	4	1,000,000	0.04105	No
Rational number Q_2	4	1,000,000	58.40222	No
Euler constant γ	10	1,000,000	1.17216	?
Fibonacci $\mathcal{F}$	10	1,000,000	1.24820	
$\zeta(2) = \frac{\pi^2}{6}$	4	1,000,000	1.57571	?
$\zeta(3)$	4	1,000,000	1.04085	?
Catalan's constant G	4	1,000,000	0.53489	?
Thue-Morse $\mathcal{TM}_2$	4	1,000,000	531.92344	?
Paper-folding $\mathcal{P}$	4	1,000,000	0.01336	?

Table 1: Average of the normalized distance to the origin (i.e. multiplied by $2/\sqrt{\pi N}$, where N is the number of steps) of the walk of various constants in different bases.

Thus,

$$M = \frac{1}{b} \left[\begin{array}{cc} \sum_{k=1}^{b} \cos^2 \left(\frac{2\pi}{b} k \right) & \sum_{k=1}^{b} \cos \left(\frac{2\pi}{b} k \right) \sin \left(\frac{2\pi}{b} k \right) \\ \sum_{k=1}^{b} \cos \left(\frac{2\pi}{b} k \right) \sin \left(\frac{2\pi}{b} k \right) & \sum_{k=1}^{b} \sin^2 \left(\frac{2\pi}{b} k \right) \end{array} \right].$$

Since

$$\sum_{k=1}^{b} \cos^2 \left(\frac{2\pi}{b} k \right) = \sum_{k=1}^{b} \frac{1 + \cos \left(\frac{4\pi}{b} k \right)}{2} = \frac{b}{2},$$

$$\sum_{k=1}^{b} \sin^2 \left(\frac{2\pi}{b} k \right) = \sum_{k=1}^{b} \frac{1 - \cos \left(\frac{4\pi}{b} k \right)}{2} = \frac{b}{2},$$

$$\sum_{k=1}^{b} \cos \left(\frac{2\pi}{b} k \right) \sin \left(\frac{2\pi}{b} k \right) = \sum_{k=1}^{b} \frac{\sin \left(\frac{4\pi}{b} k \right)}{2} = 0,$$

one has

$$M = \left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right].$$

Hence, $1/\sqrt{N} S^N$ is asymptotically bivariate normal with mean 0_2 and covariance matrix M. Since its components $(1/\sqrt{N} S_1^N, 1/\sqrt{N} S_2^N)^T$ are uncorrelated, then they are independent random variables, whose distribution is (univariate) normal with mean 0 and variance $1/2$. Therefore, the random variable

$$\sqrt{\left(\frac{\sqrt{2}}{\sqrt{N}} S_1^N \right)^2 + \left(\frac{\sqrt{2}}{\sqrt{N}} S_2^N \right)^2}$$

converges in distribution to a χ random variable with two degrees of freedom. Then, for N sufficiently large,

$$E \left(\sqrt{(S_1^N)^2 + (S_2^N)^2} \right) = \frac{\sqrt{N}}{\sqrt{2}} E \left(\sqrt{\left(\frac{\sqrt{2}}{\sqrt{N}} S_1^N \right)^2 + \left(\frac{\sqrt{2}}{\sqrt{N}} S_2^N \right)^2} \right)$$

$$\approx \frac{\sqrt{N}}{\sqrt{2}} \frac{\Gamma(3/2)}{\Gamma(1)} = \frac{\sqrt{\pi N}}{2},$$

where $E(\cdot)$ stands for the expectation of a random variable. Therefore, the expected distance to the origin of the random walk is asymptotically equal to $\sqrt{\pi N}/2$. $\qquad \square$

As a consequence of this result, for any random walk of N steps in any given base, the expectation of the distance to the origin multiplied by $2/\sqrt{\pi N}$ (which we will call

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normalized distance to the origin) must approach asymptotically to 1 as N goes to infinity. Therefore, for a "sufficiently" big random walk, one would expect that the arithmetic mean of the normalized distances (which will be denominated as the *average normalized distance* to the origin) should be close to 1.

We have created a sample of 10,000 (pseudo)random walks base-4 of one million points each in Python[5], and we have computed their average normalized distance to the origin. The arithmetic mean of these numbers for the mentioned sample of pseudorandom walks is 1.0031, while its variance is 0.1351, so the asymptotic result fits quite well. In Table 1 we show the average normalized distance to the origin of various numbers.

4 Number of points visited during an N-step base-4 walk

The number of distinct points visited during a walk of a given constant (on a lattice) can be also used as an indicator of how "random" the digits of that constant are. It is well known that the expectation of an N-step random walk on a two-dimensional lattice is asymptotically equal to $\pi N / \log(N)$, see e.g. [34, page 338] or [13, page 27]. This result was first proven by Dvoretzky and Erdös in [32, Theorem 1]. The main practical problem with this asymptotic result is that its convergence is rather slow, specifically it has order of $O\left(N \log \log N / (\log N)^2\right)$. In [30, 31], Downham and Fotopoulos show the following bounds on the expectation of the number of distinct points,

$$\left] \frac{\pi(N + 0.84)}{1.16\pi - 1 - \log 2 + \log(N + 2)}, \frac{\pi(N + 1)}{1.066\pi - 1 - \log 2 + \log(N + 1)} \right[, \tag{1}$$

which provide a tighter estimate on the expectation than the asymptotic limit $\pi N / \log(N)$. For example, for $N = 10^6$, these bounds are]199,256.1, 203,059.5[while $\pi N / \log(N) = 227,396$, which overestimates the expectation.

In Table 2 we have calculated the number of distinct points visited by the base-4 walks on several constants. One can see that the values for different step walks on π fit quite well the expectation. On the other hand, numbers that are known to be normal like $\alpha_{2,3}$ or the base-4 Champernowne number substantially differ from the expectation of a random walk. These constants, despite being normal, do not have a "random" appearance when one draws the associated walk, see Figure 8(d).

At first visit, the walk on the constant $\alpha_{2,3}$ might seem random, see Figure 8(c). A closer look, shown in Figure 11, reveals a more complex structure: the walk appears to be somehow self-repeating. This helps explain why the number of sites visited by the base-4 walk on $\alpha_{2,3}$ or $\alpha_{4,3}$ is smaller than the expectation for a random walk. A detailed discussion of the Stoneham constants and their walks is given in Section 6.2 where the precise structure of Figure 11 is conjectured.

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[5]Python uses the Mersenne Twister as the core generator and produces 53-bit precision floats, with a period of $2^{19937} - 1 \approx 10^{6002}$. Compare the length of this period to the comoving distance from Earth to the edge of the observable universe in any direction, which has a magnitude of $4.6 \cdot 10^{29}$ millimeters.
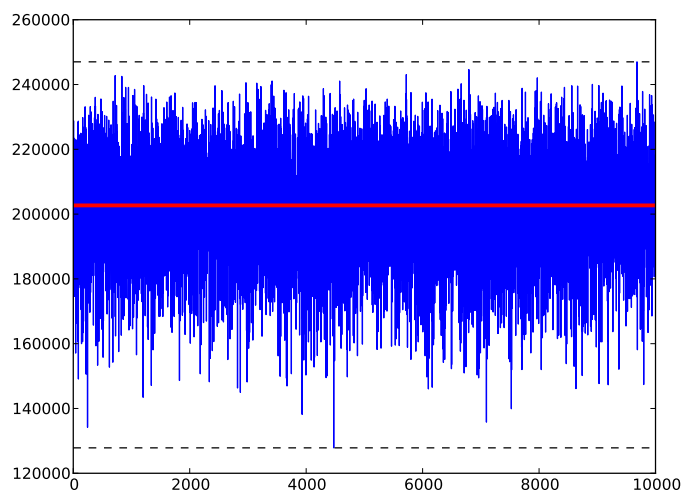
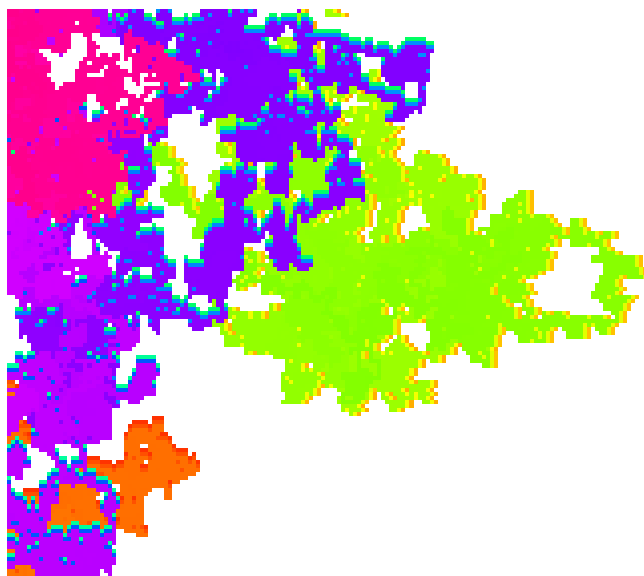Figure 10: Number of points visited by 10^4 base-4 (pseudo)random million steps walks.

Figure 11: Zooming in on the base-4 walk on $\alpha_{2,3}$ of Figure 8(c) and Conjecture 6.5.

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Number	Steps	Sites visited	Bounds on the expectation of sites visited by a random walk	
			Lower bound	Upper bound
Mean of 10,000 random walks	1,000,000	202,684	199,256	203,060
$\alpha_{2,3}$	1,000,000	95,817	199,256	203,060
$\alpha_{4,3}$	1,000,000	68,613	199,256	203,060
$\alpha_{3,2}$	1,000,000	195,585	199,256	203,060
π	1,000,000	204,148	199,256	203,060
π	10,000,000	1,933,903	1,738,645	1,767,533
π	100,000,000	16,109,429	15,421,296	15,648,132
π	1,000,000,000	138,107,050	138,552,612	140,380,926
e	1,000,000	176,350	199,256	203,060
$\sqrt{2}$	1,000,000	200,733	199,256	203,060
$\log 2$	1,000,000	214,508	199,256	203,060
Champernowne C_4	1,000,000	548,746	199,256	203,060
$EB(2)$	1,000,000	279,585	199,256	203,060
CE(10)	1,000,000	190,239	199,256	203,060
Rational number Q_1	1,000,000	378	199,256	203,060
Rational number Q_2	1,000,000	939,322	199,256	203,060
Euler constant γ	1,000,000	208,957	199,256	203,060
$\zeta(2)$	1,000,000	188,808	199,256	203,060
$\zeta(3)$	1,000,000	221,598	199,256	203,060
Catalan's constant G	1,000,000	195,853	199,256	203,060
$\mathcal{TM}_2$	1,000,000	1,000,000	199,256	203,060
Paper-folding $\mathcal{P}$	1,000,000	21	199,256	203,060

Table 2: Number of points visited in various N-step base-4 walks. The values of the two last columns are upper and lower bounds on the expectation of the number of distinct sites visited during an N-step random walk, see [30, Theorem 2] and [31].

5 Fractal and box-dimension

We now turn to our estimates of fractal dimension of number walks. One can appreciate in each of the pictures in Figures 1 through 8 that the walks on numbers exhibit a fractal-like structure. The fractal dimension has been used as an appropriate tool to measure the geometrical complexity of a set, characterizing its space-filling capacity (see e.g. [5] for a nice introduction about fractals). The *box-counting dimension*, also known as the *Minkowski-Bouligand dimension*, permits to estimate the fractal dimension of a given set. If we denote by $\#\text{box}_\epsilon(A)$ the number of boxes of side length $\epsilon > 0$ required to cover a compact set $A \subset \mathbb{R}^n$, the box-counting dimension is defined as

$$\text{d}_{\text{box}}(A) := \lim_{\epsilon \to 0} \frac{\log\left(\#\text{box}_\epsilon(A)\right)}{\log(1/\epsilon)}. \tag{2}$$

The box-counting dimension of a given image can be easily estimated by dividing the image into a non-overlapping regular grid and counting the number of filled boxes for different grid sizes. Then the box-counting dimension can be approximated by the slope of a linear regression model on $\log(1/\epsilon)$ and the logarithm of the number of nonempty boxes for different values of box-size ϵ, see Figure 12.

A random walk, being space-filling, has fractal dimension 2. For the mentioned sample of 10,000 pseudo-random walks of one million steps, the average of their box-counting dimension is 1.752, with a variance of 0.0011. The average of the box-counting dimension of these same walks with 500,000 steps is 1.738, with a variance of 0.0013. This method seems to be both efficient and stable for analyzing "randomness", see also Figure 13. The box-counting dimension of various constants is collected in Table 3.

6 Copeland-Erdös, Stoneham and Erdös-Borwein constants

As well as the classical numbers—such as e, π, γ—listed in the Appendix we also considered various other constructions. Most notably as described in the next three subsections.

6.1 Champernowne number and its concatenated relatives

The first mathematical constant proven to be 10-normal is the *Champernowne number*, which is defined as the concatenation of the decimal values of the positive integers, i.e., $C_{10} = 0.12345678910111213141516\ldots$. Champernowne proved that C_{10} is 10-normal in 1933 [23]. This was later extended to base-b normality (for base-b versions of the Champernowne constant) as in Theorem 6.1. In 1946, Copeland and Erdös established that the corresponding concatenation of primes $0.23571113171923\ldots$ and also the concatenation of composites $0.46891012141516\ldots$, among others, are also 10-normal [25]. In general they proved that concatenation leads to normality if the sequence grows slowly enough. We call such numbers *concatenation numbers*:
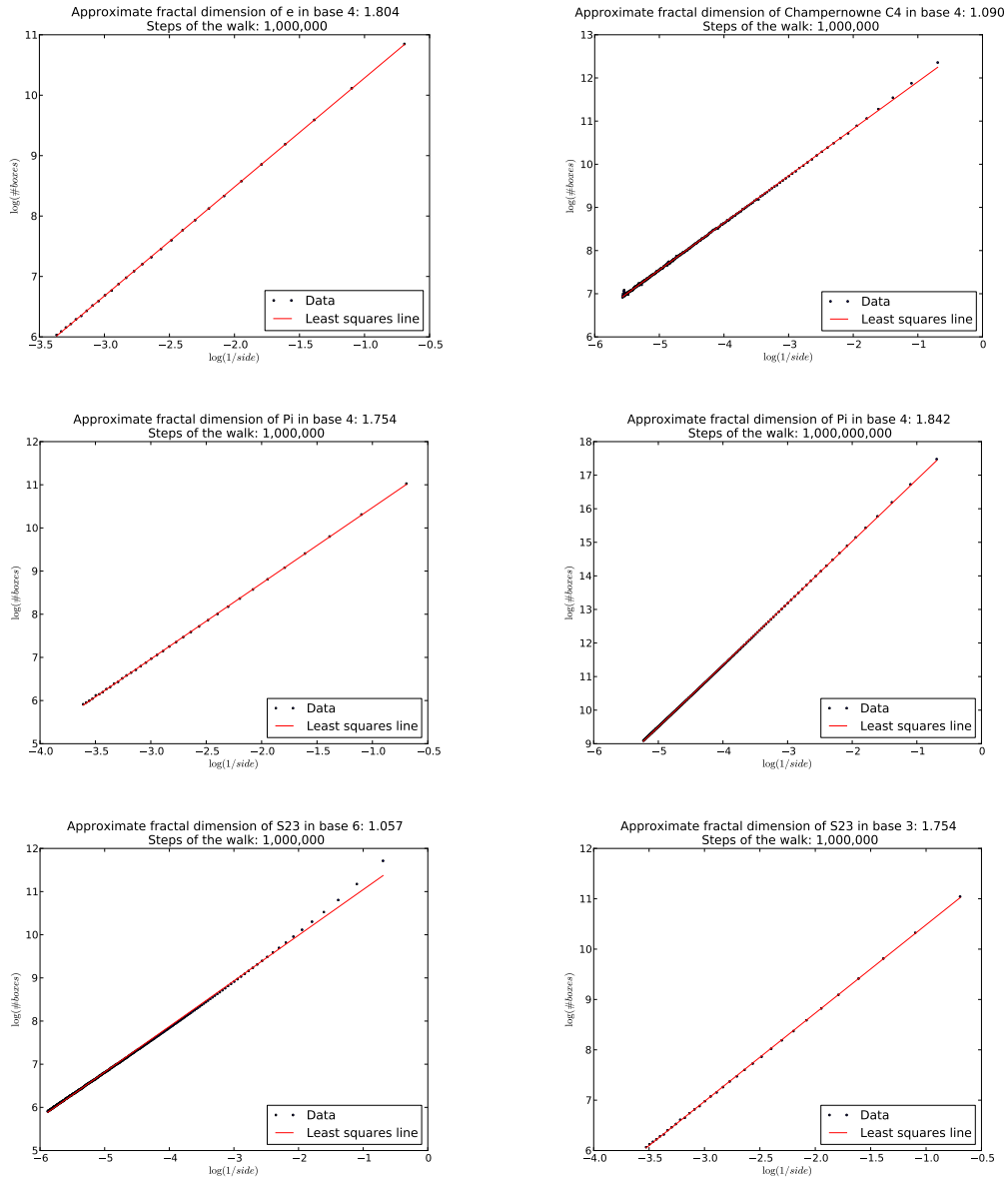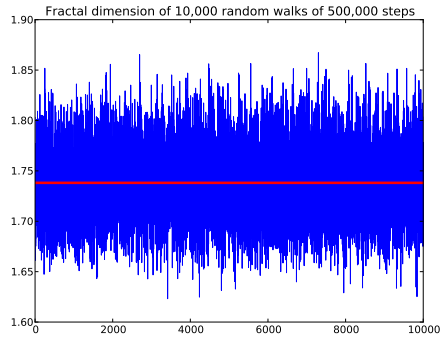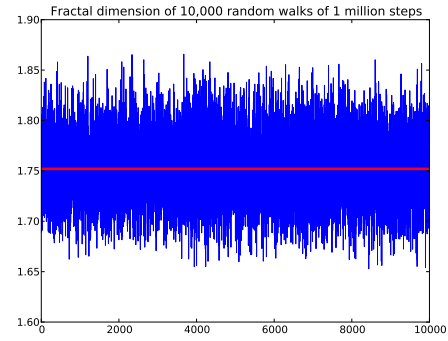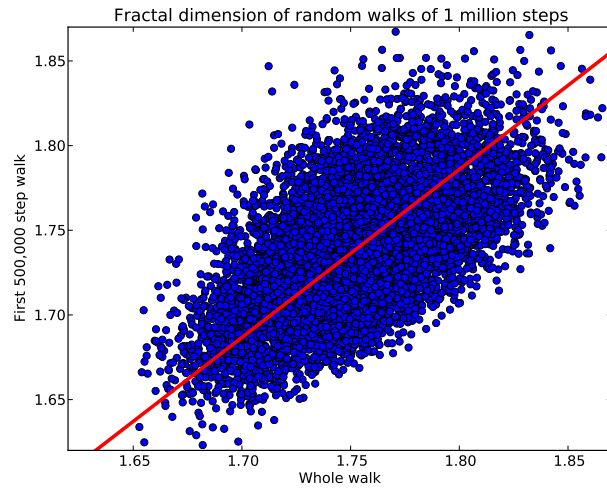
16

Figure 12: Approximate box-counting dimensions of prefixes of various walks.

(a)

(b)

(c)

Figure 13: Comparison of the approximate box dimension of 10,000 random walks.

Theorem 6.1 ([25]). *If $a_1, a_2, \cdots$ is an increasing sequence of integers such that for every $\theta < 1$ the number of a_i's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal*

$$0.a_1 a_2 a_3 \cdots$$

is normal with respect to the base b in which these integers are expressed.

This result clearly applies the Champernowne numbers (Figure 8(d)), to the primes of the form $ak + c$ with a and c relatively prime in any given base, and to the integers which are the sum of two squares (since every prime of the form $4k + 1$ is included). In further illustration, using the primes in binary lead to normality in base two of the number

$$0.101110111110111101100011001110111111011111110010110100110101110111\ldots,$$

as shown as a planar walk in Figure 7.

6.1.1 Strong normality

In [14] it is shown that C_{10} fails the following stronger test of normality which we now discuss. The test is is a simple one, in the spirit of Borel's test of normality, as opposed to other more statistical tests discussed in [14]. If the digits of a real number α are chosen at random in the base b, the *asymptotic frequency* $m_k(n)/n$ of each 1-string approaches $1/b$ with probability 1. However, the *discrepancy* $m_k(n) - n/b$ does not approach any limit, but fluctuates with an expected value equal to the standard deviation $\sqrt{(b-1)n}/b$. (Precisely $m_k(n) := m_k(n) = \#\{i\colon a_i = k, i \le n\}$ when α has fractional part $0.a_0\, a_1\, a_2 \cdots$ in base b.)

Kolmogorov's law of the iterated logarithm allows one make a precise statement about the discrepancy of a random number. Belshaw and P. Borwein [14] use this to define their criterion and then show that almost every number is absolutely strongly normal.

Definition 6.2 (Strong normality [14]). *For real α, and $m_k(n)$ as above, α is* simply strongly normal *in the base b if for each $0 \le k \le b - 1$ one has*

$$\limsup_{n \to \infty} \frac{m_k(n) - n/b}{\sqrt{2n \log \log n}} = -\liminf_{n \to \infty} \frac{m_k(n) - n/b}{\sqrt{2n \log \log n}} = \sqrt{1 - \frac{1}{b}}. \tag{3}$$

A number is strongly normal *in base b if it is simply strongly normal in each base b^j, $j = 1, 2, 3, \ldots$, and is* absolutely strongly normal *if it is strongly normal in every base.*

In paraphrase (absolutely) strongly normal numbers are those that distributionally oscillate as much as is possible.

Belshaw and Borwein show that strongly normal numbers are indeed normal. They also make the important observation that Champernowne's base-b number is not strongly normal in base b. Indeed, there are b^{k-1} digits of length k and they all start with a digit between 1 and $b-1$ while the following $k-1$ digits take values between 0 and $b-1$ equally.

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In consequence, there is a dearth of zeroes. This is easiest to analyse base 2. As illustrated below, the concatenated numbers start

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111$$

where for $k = 3$ there are 4 zeroes and 8, for $k = 4$ there are 12 zeroes and 20 ones, and for $k = 5$ there are 32 zeroes and 48 ones.

In general, let $n_k := 1 + (k-2)2^k$. Then one has $m_0(n_k) = 1 + (k-2)\,2^{k-1}$ and so

$$m_1(n_k) = n_k - m_0(n_k) = k\,2^{k-1}$$

and so $2m_1(n_k) \geq n_k$ for $k \geq 1$. Thence

$$\liminf_{n \to \infty} \frac{m_1(n) - n/2}{\sqrt{2n \log \log n}} \geq 0 \neq -\sqrt{1 - \frac{1}{2}}.$$

It seems likely that by appropriately shuffling the integers, one should be able to display a strongly normal variant. Relatedly Martin [38] has shown how to construct an explicit absolutely abnormal number.

Finally, while the $\log \log$ limiting-behaviour required by (3) appears hard to test numerically to any significant level, it appears reasonably easy computationally to check whether other sequences such as many of the concatenation sequences of Theorem 6.1 fail to be strongly normal for similar reasons. Heuristically, we would expect the number $CE(2)$ to fail to be strongly normal since each prime of length k both starts and ends with a one, while intermediate bits should show no skewing.

To Finalize **DAVID to test strong normality for** $CE(2)$ **and** $\alpha_{2,3}$

6.2 Stoneham numbers: a class containing provably normal and abnormal constants

Giving further motivation for these studies is the recent provision of rigorous proofs of normality for an uncountably infinite class of explicit real numbers, the *Stoneham numbers* defined by

$$\alpha_{b,c} := \sum_{m \geq 1} \frac{1}{c^m b^{c^m}} \tag{4}$$

for relatively prime integers b, c.

Relatedly, Bailey and Crandall showed that given a real number r in $[0, 1)$, and r_k denoting the k-th binary digit of r, that the real number

$$\alpha_{2,3}(r) : \quad = \quad \sum_{k=0}^{\infty} \frac{1}{3^k 2^{3^k + r_k}} \tag{5}$$

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is 2-normal. It can be seen that if $r \neq s$, then $\alpha_{2,3}(r) \neq \alpha_{2,3}(s)$, so that these constants are all distinct. Thus, this class of constants is uncountably infinite. A similar result applies if 2 and 3 in this formula are replaced by any pair of co-prime integers (b, c) greater than one [10] [15, pg. 141–173]. More recently, Bailey and Misieurwicz were able to establish this normality result by a much simpler argument, based on techniques of ergodic theory [11] [15, pg. 141–173].

Theorem 6.3 (Normality of Stoneham constants [3]). *For every coprime pair of integers (b, c) with $b \geq 2$ and $c \geq 2$, the constant $\alpha_{b,c} = \sum_{m \geq 1} 1/(c^m b^{c^m})$ is b-normal.*

So, for example, the constant $\alpha_{2,3} = \sum_{k \geq 0} 1/(3^k 2^{3^k}) = 0.0418836808315030\ldots$ is provably 2-normal. This special case was proven by Stoneham in 1973 [41]. A similar result applies for all $\alpha_{b,c}(r)$ as above.

Equally interesting is the following result:

Theorem 6.4 (Abnormality of Stoneham constants [3]). *Given coprime integers $b \geq 2$ and $c \geq 2$, and integers $p, q, r \geq 1$, with neither b nor c dividing r, let $B = b^p c^q r$. Assume that the condition $D = c^{q/p} r^{1/p} / b^{c-1} < 1$ is satisfied. Then the constant $\alpha_{b,c} = \sum_{k \geq 0} 1/(c^k b^{c^k})$ is B-nonnormal.*

In various of the Figures and Tables we explore the striking differences of behaviour—proven and unproven—for $\alpha_{b,c}$ as we vary the base. For instance, the abnormality of $\alpha_{2,3}$ base six was proved just before we started to draw walks. Contrast Figure 8(b) to Figure 8(c) and Figure 8(a). Now compare the values given in Table 1 and Table 2. Clearly, from this sort of visual and numeric data, the discovery of other cases of Theorem 6.4 is very easy.

As illustrated also in the 'zoom' of Figure 11, we can use the pictures to discover more subtle structure. We conjecture the following relations on the digits of $\alpha_{2,3}$ in base 4 (which explain the values in Tables 1 and 2).

Conjecture 6.5 (Base 4 structure of $\alpha_{2,3}$). *Denote by a_k the kth digit of $\alpha_{2,3}$ in its base 4 expansion; that is, $\alpha_{2,3} = \sum_{k=1}^{\infty} a_k/4^k$, with $a_k \in \{0, 1, 2, 3\}$ for or all k. Then, for all $n = 0, 1, 2, \ldots$ one has:*

$$(i) \sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n} e^{a_k \pi i/2} = \frac{(-1)^{n+1}-1}{2} + \frac{(-1)^n-1}{2} i = - \left\{ \begin{array}{ll} i, & n \text{ odd} \\ 1, & n \text{ even} \end{array} \right. ;$$

$$(ii) \ a_k = a_{k+3^n} = a_{k+2 \cdot 3^n} \text{ for all } k = \frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+1, \ldots, \frac{3}{2}(3^n+1)+3^n-1.$$

In Figure 14 we show the position of the walk after $\frac{3}{2}(3^n+1)$, $\frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2 \cdot 3^n$ steps for $n = 0, 1, \ldots, 11$, which, together with Figures 8(c) and 11, graphically explains Conjecture 6.5. Similar results seem to hold for other Stoneham constants in other bases. For instance, for $\alpha_{3,5}$ base 3 we conjecture the following.
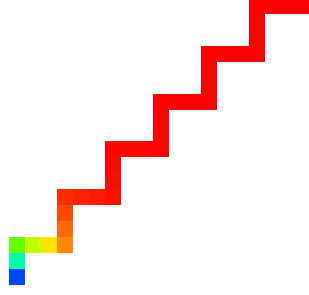
21

Figure 14: A pattern in the digits of $\alpha_{2,3}$ base 4. We show only positions of the walk after $\frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2\cdot 3^n$ steps for $n=0,1,\ldots,11$.

Conjecture 6.6 (Base 3 structure of $\alpha_{3,5}$). *Denote by a_k the kth digit of $\alpha_{3,5}$ in its base 3 expansion; that is, $\alpha_{3,5} = \sum_{k=1}^{\infty} a_k/3^k$, with $a_k \in \{0,1,2\}$ for all k. Then, for all $n=0,1,2,\ldots$ one has:*

(i) $$\sum_{k=2+5^{n+1}}^{2+5^{n+1}+4\cdot 5^n} e^{a_k \pi i/2} = \frac{(-1)^n\left(-1+\sqrt{3}i\right)}{2} = e^{(3n+2)\pi i/3};$$

(ii) $a_k = a_{k+4\cdot 5^n} = a_{k+8\cdot 5^n} = a_{k+12\cdot 5^n} = a_{k+16\cdot 5^n}$ for $k = 5^{n+1}+j, j = 2,\ldots,2+4\cdot 5^n$.

6.3 The Erdös-Borwein constants

The constructions of the previous two subsections exhaust most of what is known of concrete irrational examples. By contrast, we finish this section with a truly tantalizing case:

In a base $b \geq 2$, we define the *Erdös-(Peter) Borwein constant* $EB(b)$ by the *Lambert series* [17]:

$$EB(b) := \sum_{n\geq 1} \frac{1}{b^n-1} = \sum_{n\geq 1} \frac{\sigma(n)}{b^n} \tag{6}$$

where σ_k the sum of the k-th power of the divisors and $\sigma = \sigma_1$. It is known that the numbers $\sum 1/(q^n-r)$ are irrational for r rational and $q=1/b, b=2,3,\ldots$ [19]. Whence – as provably irrational numbers other than the standard examples are few and far between, it is interesting to consider their normality.

Crandall [26] has used the BBP-like structure [7, 15] made obvious in (6), and some non-trivial knowledge of the arithmetic properties of σ to establish results such as that the googol-th bit–that is, the bit in position 10^{100} to the right of the floating point—is a 1.

In [26] Crandall also computed the full first 2^{43} bits of $EB(2)$ (a Terabyte in about a day), and finds that there are **4359**105565638 zeroes and **4436**987456570 ones. There is

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corresponding variation in the second and third place in the single digit hex distributions. This certainly leaves some doubt as to its normality. See also Figure 8(e) but contrast it to Figure 9(f).

Our own more modest computations of $EB(10)$ base-ten again leave it far from clear that $EB(10)$ is 10-normal. Likewise, Crandall finds that in the first $10,000$ decimal positions after the quintillionth digit 10^{18}), the respective digit counts for digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are $104, 82, 87, 100, 73, 126, 87, 123, 114, 104$.

We should note that for computational purposes we used the identity

$$\sum_{n \geq 1} \frac{q^n}{1 - q^n} = \sum_{n \geq 1} q^{n^2} \frac{1 + q^n}{1 - q^n},$$

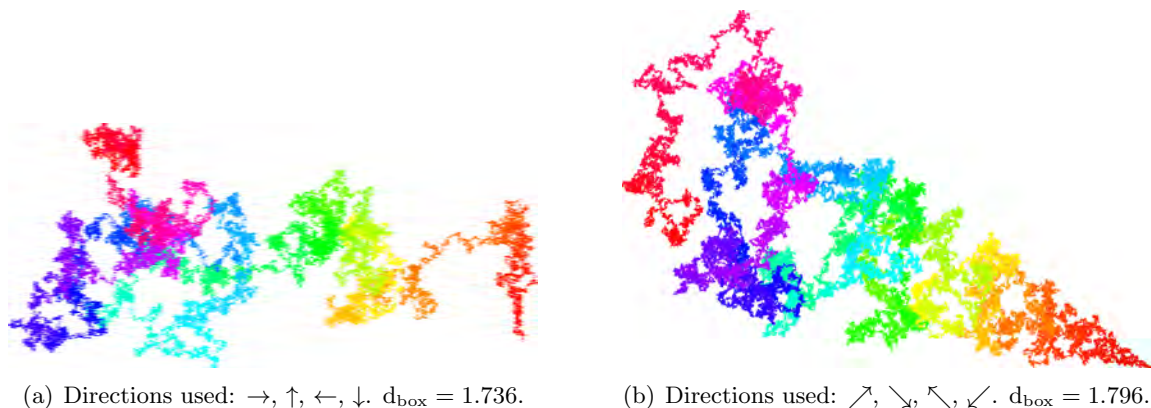for $|q| < 1$, due to Clausen, as did Crandall [26].

(a) Directions used: $\rightarrow, \uparrow, \leftarrow, \downarrow$. $d_{box} = 1.736$.　　(b) Directions used: $\nearrow, \searrow, \nwarrow, \swarrow$. $d_{box} = 1.796$.

Figure 15: Two different rules for plotting a base 2 walk on the first million values of $\lambda(n)$ (the Liouville number λ_2).

7 Other avenues and concluding remarks

Let us recall two further examples utilized in [14], that of the *Liouville function* which counts the parity of the number of primes factors of n, namely $\Omega(n)$, see Figure 15 and of the human genome taken from the *UCSC Genome Browser* at http://hgdownload. cse.ucsc.edu/goldenPath/hg19/chromosomes/, see Figure 17. Note the similarity of the genome walk to the those of concatenation sequences. We have explored a wide variety of walks on genomes but reserve the results for future paper.

We should emphasize that, to the best of our knowledge the normality and transcendence status of the numbers explored is unresolved other than in the cases indicated in

Number		Steps	Box dimension
Random average	base 4	500,000	1.738
Random average	base 4	1,000,000	1.752
e	base 4	1,000,000	1.804
C_4	base 4	1,000,000	1.090
π	base 4	1,000,000	1.754
π	base 4	1,000,000,000	1.842
$\alpha_{2,3}$	base 6	1,000,000	1.057
$\alpha_{2,3}$	base 3	1,000,000	1.754
$\mathcal{P}$	angle $\pi/3$	10,000,000	1.921
$\mathcal{P}$	angle $2\pi/3$	1,000,000	1.783
$\mathcal{TM}_2$	angle $\pi/3$	100,000	1.353
π	angle $\pi/3$	1,000,000	1.783

Table 3: Box-counting dimensions of various walks (Fig. 12) and turtle plots (Fig. 19) .

sections 6.1 and 6.2 and indicated in Appendix 8. While one of the clearly non-random numbers (say Stoneham or Copeland-Erdös) may pass muster on one or other measure of the walk, it is generally the case that it fails another. Thus, the Liouville number λ_2, see Figure 15, shows a much more structured drift than π or e but looks more like them than like Figure 17(a).

This situation gives us hope for more precise future analyses. We conclude by remarking on some unresolved issues and on our plans for future research.

7.1 Three dimensions

We have also explored three-dimensional graphics—using base-6 for directions—both in perspective as in Figure 16, and in a large passive (glasses-free) three-dimensional viewer outside the CARMA laboratory; but have not yet quantified these excursions.

7.2 Genome comparison

Genomes are made up of so called purine and pyrimidines nucleotides. In DNA, purine necleotide bases are adenine and guanine (A and G), while the pyrimidine bases are thymine and cytosine (T and C). Thymine is replaced by uracyl in RNA. The haploid human genome (i.e., 23 chromosomes) is estimated to hold about 3.2 billion base pairs and so to contain $20,000 - 25,000$ distinct genes. Thence, there are many ways of representing a stretch of a chromosome as a walk, say as a base-four uniform walk on the symbols (A,G,T,C) illustrated in Figure 17 (where A, G, T and C draw the new point to the south, north, west and east, respectively, and we have not plotted undecoded or unused portions), or as a three dimensional logarithmic walk inside a tetrahedron. We have also compared random
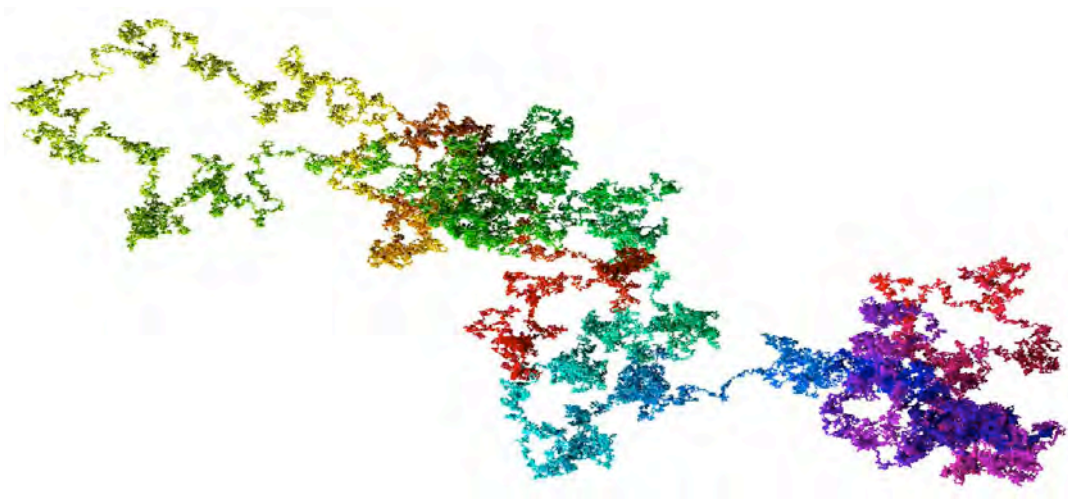
Figure 16: A 3D walk on the first million base 6 digits of π.

chaos games in a square with genomes and numbers plotted by the same rules, as in [16, pp. 194–205].

7.3 Automatic numbers

We have also explored numbers originating with finite state automata such as those of the *paper-folding* and the *Thue-Morse* sequences, $\mathcal{P}$ and $\mathcal{TM}_2$, see [2] and Section 8. Automatic numbers are never normal and are typically transcendental; by comparison the Liouville number λ_2 is not p-automatic for any prime p [24].

The walks on $\mathcal{P}$ and $\mathcal{TM}_2$ have a similar shape, see Figure 18, but while the Thue-Morse sequence generates very large pictures, the paper-folding sequence generates very small ones since it is highly self-replicating, see also the values in Tables 1 and 2.

A *turtle plot*[6] on these constants exhibits some of their striking features, see Figure 19. For instance, drawn with a rotating angle of $\pi/3$, $\mathcal{TM}_2$ converges to a Koch snowflake [39], see Figure 19(c). We show a corresponding turtle graphic of π in Figure 19(d). Corresponding features occur for the paper folding sequence as described in [27, 28, 29]and two variants are shown in Figures 19(a) and 19(b). While the walk specific metrics make little sense to measure, we do supply their fractal dimensions in Table 3 below. As might be expected turtle plots of the same length appear to exhibit higher box-dimensions. The corresponding dimensions for our two variants of Liouville walks are recorded below their pictures in Figure 15.

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[6]In base 2, each digit correspond to one the following orders: either "forward motion" of length one or "rotate the Logo turtle" a fixed angle.
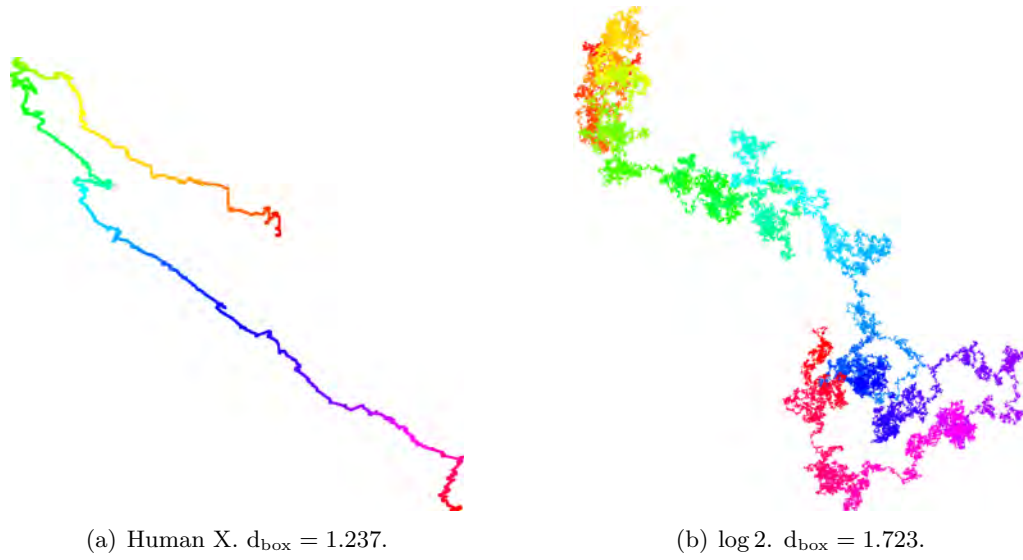
(a) Human X. $d_{box} = 1.237$.

(b) $\log 2$. $d_{box} = 1.723$.

Figure 17: Base four walks on 10^6 bases of the X-chromosome and 10^6 digits of $\log 2$.

(a) A thousand digits of the Thue-Morse sequence $\mathcal{TM}_2$ base 2.

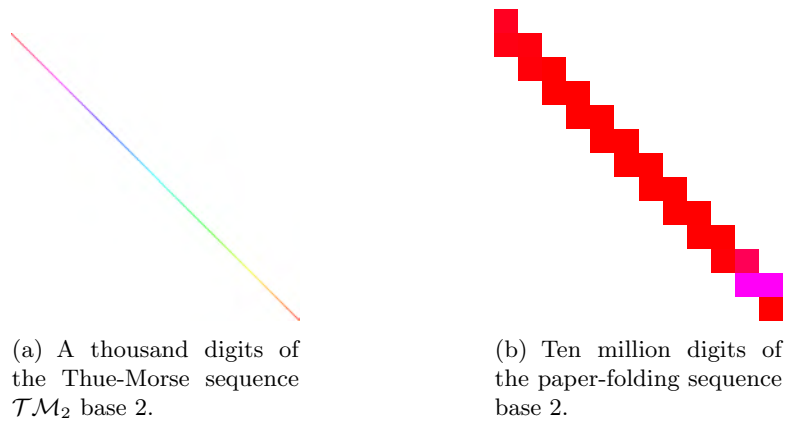(b) Ten million digits of the paper-folding sequence base 2.

Figure 18: Walks on two automatic and abnormal numbers.

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7.4 Continued fractions

Simple continued fractions often encode more information about a real number. Basic facts are that a continued fraction terminates or repeats if and only if the number is rational or a quadratic irrationality [15, 7]. By contrast, the simple continued fractions for π and e start as follows in the standard compact form:

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, \ldots]$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 1, \ldots],$$

from which the surprising regularity of e and apparent irregularity of π as continued fraction is apparent. The counterpart to Borel's theorem—that almost all numbers are normal—is that almost all numbers have 'normal' continued fractions $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ for which the *Gauss-Kuzmin distribution* holds [15]: for each $k = 1, 2, 3, \ldots$

$$\mathrm{Prob}\{a_n = k\} = -\log_2\left(1 - \frac{1}{(k+1)^2}\right) \tag{7}$$

so that roughly 41.5% of the terms are 1, 16.99% are 2, 9.31% are 3, etc.

In Figure 20 we show an histogram of the first 100 million terms[7] of the continued fraction of π. We have not yet found a satisfactory way to embed this in a walk on a continued fraction but in the Figure 21 we show base-4 walks on π and e where we use the remainder modulo four to build the walk (with 0 being right, 1 being up 2 being left and 3 being down). We also show turtle plots on π, e.

Andrew Mattingly has observed that:

Proposition 7.1. *With probability one, such a mod four random walk on the simple continued fraction coefficients of a real number is asymptotic to a line making a positive angle with the x-axis of:*

$$\arctan\left(\frac{1}{2}\frac{\log_2(\pi/2) - 1}{\log_2(\pi/2) - 2\log_2\left(\Gamma\left(3/4\right)\right)}\right) \approx 110.44°.$$

Proof. The result comes by summing the expected Gauss-Kuzmin probabilities of each step being taken as given by (7). □

This is illustrated in Figure 21(a) with a 90° anticlockwise rotation; thus making the case that one must have some a priori knowledge before designing tools.

It is also instructive to compare at digits and and continued fractions of numbers as horizontal matrix plots of the form already used in Figure9. In Figure 22 we show six pairs of million terms digits strings and their corresponding fractions. In some cases both look random, in others one or the other does.
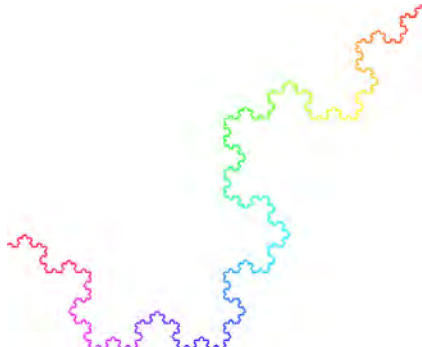
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[7]Computed by Neil Bickford and accessible at http://neilbickford.com/picf.htm.
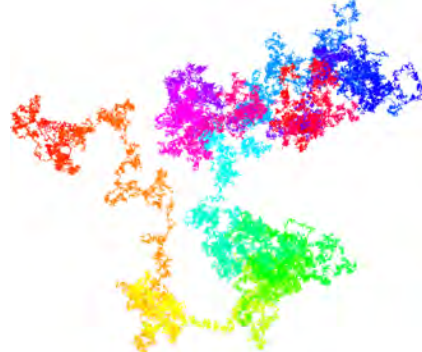
(a) Ten million digits of the paper-folding sequence with rotating angle $\pi/3$. $d_{box} = 1.921$.

(b) Dragon curve from one million digits of the paper-folding sequence with rotating angle $2\pi/3$. $d_{box} = 1.783$.
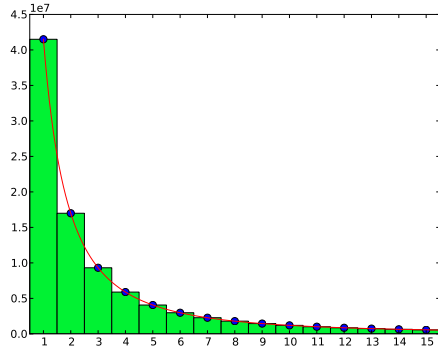

(c) Koch snowflake from 100,000 digits of the Thue-Morse sequence $\mathcal{TM}_2$ with rotating angle $\pi/3$. $d_{box} = 1.353$.
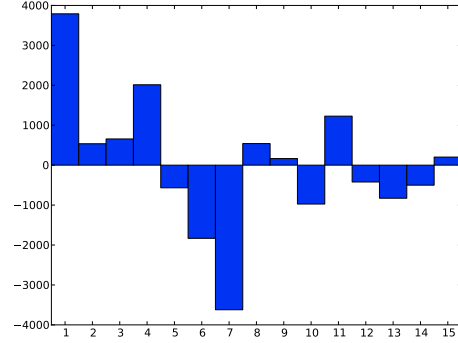

(d) One million digits of π with rotating angle $\pi/3$. $d_{box} = 1.760$.

Figure 19: Turtle plots on various constants with different rotating angles in base 2—where '0' gives forward motion and '1' rotation by a fixed angle.
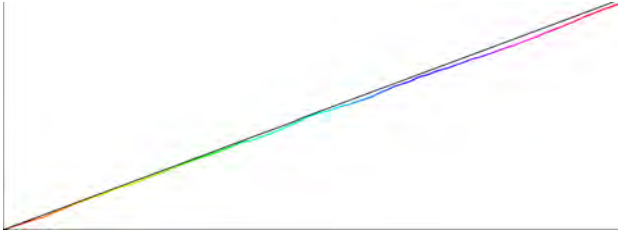
(a) Histogram of the terms in green, Gauss-Kuzmin function in red.

(b) Difference between the expected and computed values of the Gauss-Kuzmin function.

Figure 20: Expected values of the Gauss-Kuzmin distribution of (7) and the values of 100 million terms of the continued fraction of π.
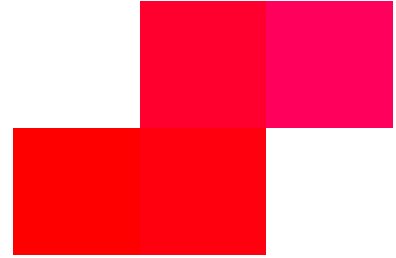

(a) A 100,000 step walk on the continued fraction of π modulo 4.

(b) A 100 step walk on the continued fraction of e modulo 4.

(c) A one million step turtle walk on the continued fraction of π modulo 2 with rotating angle $\pi/3$.

(d) A 100 step turtle walk on the continued fraction of e modulo 2 with rotating angle $\pi/3$.

Figure 21: Simple continued fraction based uniform walks on π and e.
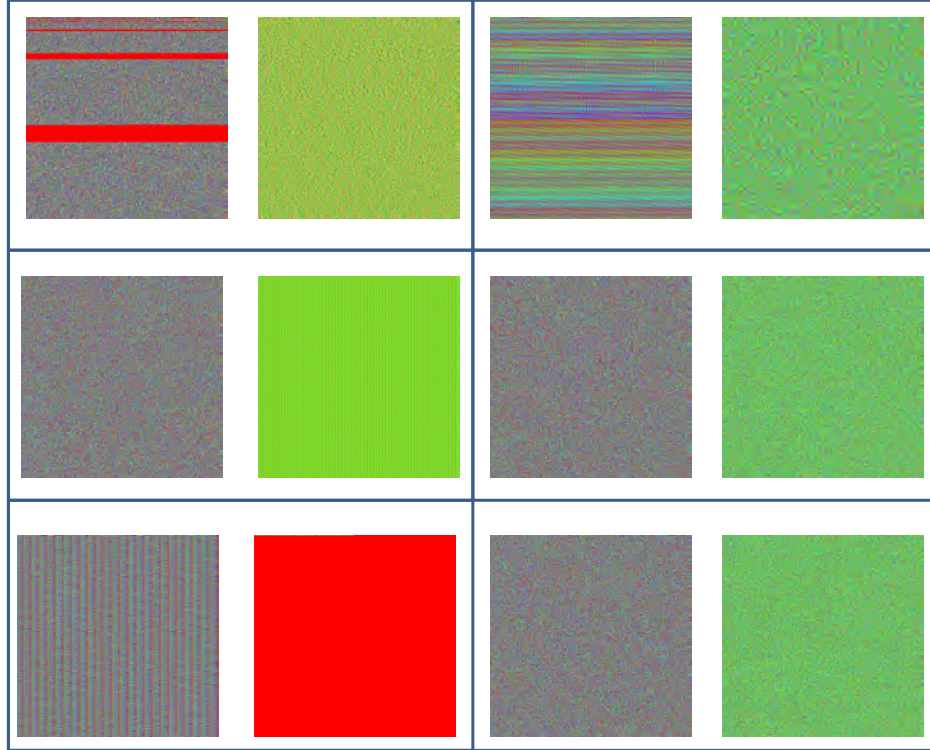
Figure 22: Million step comparisons of base-4 digits and fractions. Row 1: $\alpha_{2,3}$(base 6) and C_4. Row 2: e and π. Row 3: Q_1 and pseudorandoms; as listed from top left to bottom right.

In conclusion, we have only tapped the surface of what is becoming possible in a period in which data—five hundred million terms of the continued fraction or five trillion bits of π, full genomes and much more— can be downloaded from the internet, then rendered—and visually mined—with fair rapidity.

8 Appendix: Selected numerical constants

In what follows, we denote $x := 0._b a_1 a_2 a_3 a_4 \ldots$ the base-b expansion of the number x, that is, $x = \sum_{k=1}^{\infty} \frac{a_k}{b^k}$. Base 10 is denoted without a subindex.

Archimedes constant (transcendental):

$$\pi := 2 \int_{-1}^{1} \sqrt{1 - x^2} \, dx \tag{8}$$

$$= \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

$$= 3.1415926535\ldots$$

Catalan's constant (irrational?):

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.9159655941\ldots \tag{9}$$

Champernowne numbers (irrational and normal):

$$C_b := \sum_{k=1}^{\infty} \frac{\sum_{m=b^{k-1}}^{b^k-1} m b^{-k[m-(b^{k-1}-1)]}}{b \sum_{m=0}^{k-1} m(b-1)b^{m-1}} \tag{10}$$

$$C_{10} = 0.123456789101112\ldots$$

$$C_4 = 0._4 1231011121320212223\ldots$$

Copeland-Erdös constants (irrational and normal):

$$CE(b) := \sum_{k=1}^{\infty} p_k b^{-\left(k + \sum_{m=1}^{k} \lfloor \log_b p_m \rfloor\right)}, \text{ where } p_k \text{ is the } k^{\text{th}} \text{ prime number.} \tag{11}$$

$$CE(10) = 0.2357111317\ldots$$

$$CE(2) = 0._2 1011101111\ldots$$

Exponential constant (transcendental):

$$e := \sum_{k=0}^{\infty} \frac{1}{k!} = 2.7182818284\ldots \tag{12}$$

Erdös-Borwein constants (irrational):

$$EB(b) := \sum_{k=1}^{\infty} \frac{1}{b^k - 1}. \tag{13}$$

$$EB(2) = 1.6066951524\ldots = 0._4212311001\ldots$$

Euler-Mascheroni constant (irrational?):

$$\gamma := \lim_{m \to \infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \log m \right) = 0.5772156649\ldots \tag{14}$$

Fibonacci constant (irrational?):

$$\mathcal{F} := \sum_{k=1}^{\infty} F_k 10^{-\left(1 + k + \sum_{m=1}^{k} \lfloor \log_{10} F_m \rfloor\right)}, \quad \text{where } F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} \tag{15}$$

$$= 0.011235813213455\ldots$$

Liouville number (irrational, not p-automatic):

$$\lambda_2 := \sum_{k=1}^{\infty} \frac{1}{2^{(\lambda(k)+1)/2}} \text{ where } \lambda(k) := (-1)^{\Omega(k)} \tag{16}$$

where $\Omega(k)$ counts prime factors of k.

Logarithmic constant (transcendental):

$$\log 2 := \sum_{k=1}^{\infty} \frac{1}{k2^k} = 0.6931471806\ldots \tag{17}$$

Riemann zeta function (transcendental for n even, irrational for $n = 3$):

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \tag{18}$$

whence

$$\zeta(2) = \frac{\pi^2}{6} = 1.6449340668\ldots$$

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}\,B_{2n} \quad \text{(in terms of Bernoulli numbers)}$$

$$\zeta(3) = \text{Apéry's constant} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}} = 1.2020569031\ldots$$

Stoneham constants (normal or abnormal irrationals):

$$\alpha_{b,c} := \sum_{k=1}^{\infty}\frac{1}{b^{c^k}c^k}. \tag{19}$$

$\alpha_{2,3} = 0.0418836808\ldots = 0._40022232032\ldots = 0._60130140430003334\ldots$
$\alpha_{4,3} = 0.0052087571\ldots = 0._40001111111301\ldots = 0._60010430041343502130000\ldots$
$\alpha_{3,2} = 0.0586610287\ldots = 0._30011202021212121\ldots = 0._6020400520003054400002\ldots$
$\alpha_{3,5} = 0.0008230452\ldots = 0._300000012101210121\ldots = 0._{15}002ba00000061d2\ldots$

Thue-Morse constant (transcendental, 2-automatic hence abnormal):

$$\mathcal{TM}_2 := \sum_{k=1}^{\infty}\frac{1}{2^{t(n)}} \text{ where } t(0) = 0, \text{ while } t(2n) = t(n) \text{ and } t(2n+1) = 1 - t(n). \tag{20}$$
$$= 0.4124540336\ldots = 0._20110100110010110100101100110 1001\ldots$$

Paper-folding constant (transcendental, 2-automatic hence abnormal):

$$\mathcal{P} := \sum_{k=0}^{\infty}\frac{8^{2^k}}{2^{2^{k+2}}-1} = 0.8507361882\ldots = 0._21101100111001001\ldots \tag{21}$$

Acknowledgements

References

[1] S. Albeverioa, M. Pratsiovytyie and, G. Torbine, "Topological and fractal properties of real numbers which are not normal." *Bulletin des Sciences Mathḿatiques*, **129** (2005), 615-630.

[2] J.-P. Allouche and J. Shallit, *Automatic sequences. Theory, applications, generalizations.* Cambridge University Press, Cambridge, 2003.

[3] D.H. Bailey and J.M. Borwein, "Normal Numbers and Pseudorandom Generators." *Proceedings of the Workshop on Computational and Analytical Mathematics in Honour of Jonathan Borweins 60th Birthday*, Springer. In press 2012.

[4] D.H. Bailey, J.M. Borwein, C. S. Calude, M. J. Dinneen, M. Dumitrescu, and A. Yee, "An empirical approach to the normality of pi." *Experimental Mathematics.* In press 2012.

[5] M. Barnsley, *Fractals everywhere.* Academic Press, Inc., Boston, MA, 1988.

[6] D.H. Bailey, J.M. Borwein, Richard E. Crandall and Carl Pomerance, "On the Binary Expansions of Algebraic Numbers." *Journal of Number Theory Bordeaux*, **16** (2004), 487–518.

[7] D.H. Bailey, P.B. Borwein and S. Plouffe, "On the rapid computation of various polylogarithmic Constants." *Mathematics of Computation*, **66**, no. 218 (1997), 903–913.

[8] D.H. Bailey and D.J. Broadhurst, "Parallel integer relation detection: Techniques and applications." *Mathematics of Computation*, **70**, no. 236 (2000), 1719–1736.

[9] D.H. Bailey and R.E. Crandall, "On the random character of fundamental constant expansions." *Experimental Mathematics*, **10**, no. 2 (2001), 175–190.

[10] D.H. Bailey and R.E. Crandall, "Random generators and normal numbers." *Experimental Mathematics*, **11** (2002), no. 4, 527–546.

[11] D.H. Bailey and M. Misiurewicz, "A strong hot spot theorem." *Proceedings of the American Mathematical Society*, **134** (2006), no. 9, 2495–2501.

[12] L. Berggren, J.M. Borwein and P.B. Borwein, *Pi: a Source Book.* Springer-Verlag, Third Edition, 2004.

[13] M.N. Barber and B.W. Ninham, *Random and Restricted Walks: Theory and Applications.* Gordon and Breach, New York, 1970.

[14] A. Belshaw and P.B. Borwein, "Champernownes Number, Strong Normality, and the X Chromosome." *Proceedings of the Workshop on Computational and Analytical Mathematics in Honour of Jonathan Borweins 60th Birthday*, Springer. In press 2012.

[15] J.M. Borwein and D.H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century.* A K Peters, Natick, MA, second edition, 2008.

[16] J. Borwein, D. Bailey, N. Calkin, R. Girgensohn, R. Luke, and V. Moll, *Experimental Mathematics in Action.* A.K. Peters, 2007.

[17] J.M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity.* John Wiley, New York, 1987. Paperback 1998.

[18] J.M. Borwein, P. B. Borwein, R.M. Corless, L. Jörgenson and N. Sinclair, "What is organic mathematics?" *Organic mathematics* (Burnaby, BC, 1995), CMS Conf. Proc., **20**, Amer. Math. Soc., Providence, RI, 1997, 1–18.

[19] P.B. Borwein, "On the Irrationality of Certain Series." *Math. Proc. Cambridge Philos. Soc.* **112** (1992), 141–146.

[20] P.B. Borwein and L. Jörgenson, "Visible structures in number theory." *Amer. Math. Monthly* **108** (2001), no. 10, 897910.

[21] C.S. Calude, "Borel normality and algorithmic randomness," in G. Rozenberg, A. Salomaa (eds.). *Developments in Language Theory*, World Scientific, Singapore, 1994, 113–129.

[22] C.S. Calude, *Information and Randomness: An Algorithmic Perspective.* 2nd Edition, Revised and Extended, Springer-Verlag, Berlin, 2002.

[23] D.G. Champernowne, "The construction of decimals normal in the scale of ten." *Journal of the London Mathematical Society*, **8** (1933), 254–260.

[24] M. Coons, "(Non)automaticity of number theoretic functions." *J. Théor. Nombres Bordeaux*, **22** (2010), no. 2, 339–352.

[25] A.H. Copeland and P. Erdös, "Note on normal numbers." *Bulletin of the American Mathematical Society*, **52** (1946), 857–860.

[26] R.E. Crandall, "The googol-th bit of the ErdösBorwein constant." *Integers*, A23, 2012. `http://www.westga.edu/~integers/cgi-bin/get.cgi`.

[27] M. Dekking, M. Mendès France and A. van der Poorten, "Folds." *Math. Intelligencer* **4** (1982), no. 3, 130–138.

[28] M. Dekking, M. Mendès France and A. van der Poorten, "Folds II." *Math. Intelligencer* **4** (1982), no. 4, 173–181.

[29] M. Dekking, M. Mendès France and A. van der Poorten, "Folds III." *Math. Intelligencer* **4** (1982), no. 4, 190–195.

[30] D.Y. Downham and S.B. Fotopoulos, "The transient behaviour of the simple random walk in the plane." *J. Appl. Probab.*, **25** (1988), no. 1, 58–69.

[31] D.Y. Downham and S.B. Fotopoulos, "A note on the simple random walk in the plane." *Statist. Probab. Lett.*, **17** (1993), no. 3, 221–224.

[32] A. Dvoretzky and P. Erdös, "Some problems on random walk in space." *Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability*, (1951), 353–367.

[33] P. Hertling, "Simply Normal Numbers to Different Bases." *Journal of Universal Computer Science*, **8**, no. 2 (2002), 235–242.

[34] B.D. Hughes, *Random walks and random environments. Vol. 1. Random walks.* Oxford Science Publications, New York, 1995.

[35] H. Kaneko, "On normal numbers and powers of algebraic numbers." *Integers*, **10** (2010), 31–64.

[36] D. Khoshnevisan, "Normal numbers are normal." *Clay Mathematics Institute Annual Report* (2006), 15 & 27–31.

[37] G. Marsaglia, "On the Randomness of Pi and Other Decimal Expansions." Preprint, 2010.

[38] G. Martin, "Absolutely Abnormal Numbers," *Amer. Math. Monthly*, **108** (2001), no. 8, 746-754.

[39] J. Mah and J. Holdener, "When Thue-Morse Meets Koch." *Fractals*, **13** (2005), no. 3, 191–206.

[40] S.M. Ross, *Stochastic Processes.* John Wiley & Sons, New York, 1983.

[41] R. Stoneham, "On absolute (j, ε)-normality in the rational fractions with applications to normal numbers." *Acta Arithmetica*, **22** (1973), 277–286.

[42] M. Queffelec, "Old and new results on normality." *Lecture Notes – Monograph Series*, **48**, *Dynamics and Stochastics*, 2006, Institute of Mathematical Statistics, 225–236.

[43] W. Schmidt, "On normal numbers." *Pacific Journal of Mathematics*, **10** (1960), 661–672.

[44] A.J. Yee, "y-cruncher-multi-threaded pi program," http://www.numberworld.org/y-cruncher, 2010.

[45] A.J Yee and S. Kondo, "10 Trillion Digits of Pi: A Case Study of summing Hypergeometric Series to high precision on Multicore Systems." Preprint 2011. Available at http://hdl.handle.net/2142/28348.