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THE SPECTRUM OF AN ARRAY AND ITS APPLICATION TO THE STUDY OF THE TRANSLATION PROPERTIES OF A SIMPLE CLASS OF ARITHMETICAL FUNCTIONS

Part Two

On the Translation Properties of a Simple Class of Arithmetical Functions

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§1. Let ξ be a simple q-th root of unity, q being any positive integer greater than 1. Let $\bar{\xi}$ be the conjugate complex number, so that

$$\xi\bar{\xi}=1.$$

We then define the arithmetical function $\rho(n)$ by the functional equations

$$\begin{cases} \rho(0)=1; \\ \rho(qn+l)=\xi^{l}\rho(n) \text{ for } \begin{bmatrix} l=0,\ 1,\ 2,\ \cdots q-1 \\ n=0,\ 1,\ 2,\ \cdots\cdot \end{bmatrix}. \end{cases} \tag{1}$$

We thus have defined $\rho(n)$ unambiguously for every positive integer n. We may write

$$\rho(n)=\xi^{q(n)},$$

where $q(n)$ is the sum of the digits of n in the q-ary system of notation.

Our problem here is to give an asymptotic evaluation of

$$S_k(n)=\sum_{l=0}^{n-1}\rho(l)\bar{\rho}(l+k) \tag{2}$$

for arbitrary positive integral values of k and large values of n. Here $\bar{\rho}(l)$ denotes the complex number conjugate to $\rho(l)$, so that

$$\rho(l)\bar{\rho}(l)=1.$$

If $k=0$, we have the obvious formula

$$S_0(n)=n. \tag{3}$$

We shall use this as a basis on which to determine

$$S_1(n)=\sum_{l=0}^{n-1}\rho(l)\bar{\rho}(l+1).$$

We may deduce at once from our fundamental equation (1) the functional equations of $S_1(n)$: namely,

$$\begin{cases} S_1(0)=0; \\ S_1(qn+l)=\bar{\xi}S_1(n)+[(q-1)n+l]\bar{\xi}. \quad [l=0,1,\cdots,q-1]. \end{cases} \tag{4}$$

As is obvious, these equations determine $S_1(n)$ unambiguously.

We now see, however, that the series

$$S(n)=\bar{\xi}\left\{[n]-\left[\frac{n}{q}\right]\right\}+\bar{\xi}^2\left\{\left[\frac{n}{q}\right]-\left[\frac{n}{q^2}\right]\right\}+\bar{\xi}^3\left\{\left[\frac{n}{q^2}\right]-\left[\frac{n}{q^3}\right]\right\}+\cdots$$[1]

satisfies the same functional equations (4) as $S_1(n)$, and hence is identical with $S_1(n)$. We thus have

$$S_1(n)=\bar{\xi}\left\{[n]-\left[\frac{n}{q}\right]\right\}+\bar{\xi}^2\left\{\left[\frac{n}{q}\right]-\left[\frac{n}{q^2}\right]\right\}+\bar{\xi}^3\left\{\left[\frac{n}{q^2}\right]-\left[\frac{n}{q^3}\right]\right\}+\cdots. \tag{5}$$

Now let

$$q^r\leq n<q^{r+1}. \tag{6}$$

We see that

$$\begin{aligned} S_1(n)&=\bar{\xi}\left\{[n]-\left[\frac{n}{q}\right]\right\}+\bar{\xi}^2\left\{\left[\frac{n}{q}\right]-\left[\frac{n}{q^2}\right]\right\}+\cdots+\bar{\xi}^{r+1}\left\{\left[\frac{n}{q^r}\right]-\left[\frac{n}{q^{r+1}}\right]\right\} \\ &=n\bar{\xi}\left(1-\frac{1}{q}\right)\left(1+\frac{\bar{\xi}}{q}+\frac{\bar{\xi}^2}{q^2}+\cdots+\frac{\bar{\xi}^r}{q^r}\right)+O(r) \\ &=n\bar{\xi}\left(1-\frac{1}{q}\right)\frac{1}{1-\frac{\bar{\xi}}{q}}+O(1)+O(r), \end{aligned}$$

[1] (x) denotes the greatest integer not exceeding x.

or by (6),

$$S_1(n)=\frac{q-1}{q\xi-1}n+O(\log n). \tag{7}$$

Formulæ (3) and (7) are only special cases of the corresponding formula for arbitrary k. We obtain this in the following manner.

Since

$$S_k(qn+l)=S_k(qn)+O(1), \qquad [l=0,1,2,\cdots,q-1]$$

we need only consider $S_k(qn)$. For this we have the formula

$$S_{q\kappa+\lambda}(qn)=\bar{\xi}^{\lambda}\{(q-\lambda)S_\kappa(n)+\lambda S_{\kappa+1}(n)\}. \tag{8}$$

We define a sequence $\sigma(k)$ by the functional equations

$$\begin{cases}\sigma(0)=1; \\ \sigma(q\kappa+\lambda)=\bar{\xi}^{\lambda}\left(\frac{q-\lambda}{q}\sigma(\kappa)+\frac{\lambda}{q}\sigma(\kappa+1)\right)\end{cases} \tag{9}$$

Then it is always true that

$$S_k(n)=\sigma(k)n+O(\log n). \tag{10}$$

To begin with, we have proved this theorem for $k=0$ and $k=1$. Formula (8) shows, however, that we may prove (10) in general by a mathematical induction with respect to k.

$\sigma(k)$ is a very complicated arithmetical function. For small values of its argument ($\kappa=0, 1, \cdots, q-1$; $\lambda=0, 1, \cdots, q-1$), we have

$$\sigma(\lambda)=\frac{\bar{\xi}^{\lambda}(q-\lambda)+(\lambda-1)\bar{\xi}}{q-\bar{\xi}};$$

$$\sigma(\kappa q+\lambda)$$
$$=\bar{\xi}^{\kappa+\lambda}\frac{(q-\kappa)(q-\lambda)+[(\kappa-1)(q-\lambda)+(q-\kappa-1)\lambda]\bar{\xi}+\kappa\lambda\bar{\xi}^2}{q(q-\bar{\xi})}.$$

It is natural to extend our definition of $\sigma(k)$ to negative values of by the formula

$$\sigma(-k)=\overline{\sigma(k)}.$$

Formula (10) is then true for negative as well as for positive arguments.

§2. It is natural to investigate the functions

$$T_k(n) = \sum_{0}^{r-1} \sigma(l)\bar{\sigma}(l+k),$$

which arise from σ in the same fashion in which S arises from ρ. We shall confine ourselves to the case

$$q=2,\ \xi=\bar{\xi}=-1.$$

We have here the equations

$$\begin{aligned} \sigma(k) &= \bar{\sigma}(k); \\ \sigma(2k) &= \sigma(k); \\ \sigma(2k+1) &= -\frac{\sigma(k)+\sigma(k+1)}{2}. \end{aligned} \tag{11}$$

Hence we have the following formulæ:

$$T_{2k}(2n) = \sum_{m=0}^{n-1} (\sigma(2m)\sigma(2m+2k)+\sigma(2m+1)\sigma(2m+2k+1))$$

$$= \sum_{m=0}^{n-1} \left(\sigma(m)\sigma(m+k) + \frac{(\sigma(m)+\sigma(m+1))(\sigma(m+k)+\sigma(m+k+1))}{4}\right)$$

or

$$\left| T_{2k}(2n) - \frac{3}{2}T_k(n) - \frac{1}{4}T_{k-1}(n) - \frac{1}{4}T_{k+1}(n) \right| < \text{const.},$$

and further

$$T_{2k+1}(2n) = \sum_{m=0}^{n-1} (\sigma(2m)\sigma(2m+2k+1)+\sigma(2m+1)\sigma(2m+2k+2))$$

$$= -\sum_{m=0}^{n-1} \left(\sigma(m)\frac{\sigma(m+k)+\sigma(m+k+1)}{2} + \frac{\sigma(m)+\sigma(m+1)}{2}\sigma(m+k+1)\right)$$

or

$$\left| T_{2k+1}(2n) + T_k(n) + T_{k+1}(n) \right| < \text{const.}$$

The array

$$(\cdots \rho(n), \cdots \rho(1), \rho(0), \rho(1), \cdots \rho(n), \cdots)$$

is regular in the sense of the preceding paper of Mr. Wiener, since

$$\sigma(k) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \rho(l)\rho(l+k)$$

exists for every k. Hence, by a theorem contained in that paper, $\sigma(k)$ forms a spectral array, and

$$\tau(k) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \sigma(l)\sigma(l+k)$$

exists for every k. Since

$$\tau(k) = \lim_{n\to\infty} \frac{T_k(n)}{n}, \tag{12}$$

we may conclude from our equations for T_k that

$$\begin{cases} \tau(2k) = \dfrac{\tau(k-1)+6\tau(k)+\tau(k+1)}{8}\,; \\ \tau(2k+1) = -\dfrac{\tau(k)+\tau(k+1)}{2}\,. \end{cases} \tag{13}$$

It follows that if

$$\tau(0)=0, \qquad \tau(1)=0,$$

then for every k

$$\tau(k)=0.$$

We now put $k=0$ in (13), remembering that

$$\tau(-k)=\tau(k).$$

We obtain

$$8\tau(0)=2\tau(1)+6\tau(0);$$
$$2\tau(1)=-\big(\tau(0)+\tau(1)\big);$$

or

$$\tau(0)-\tau(1)=0;$$
$$\tau(0)+\tau(1)=0.$$

Hence $\tau(0)=\tau(1)=0$, and $\tau(k)$ is identically zero. In other words,

$$T_k(n)=o(n). \tag{14}$$

As $\sigma(2k)=\sigma(k)$, $\sigma(1)=-1/3$, we see that we cannot have

$$\lim_{k\to\infty} \sigma(k)=0.$$

Hence by a theorem in the preceding paper of Mr. Wiener, the spectral function of $\rho(k)$ cannot be the integral of its derivative. Since

$$\tau(0)=0,$$

another theorem from the same paper shows that the spectral function of $\rho(k)$ must be continuous. Hence the spectral function of $\rho(k)$, which is monotone, is by a theorem of Fréchet the sum of a (possibly null) function which is the integral of a summable function, and a function, certainly not null, which vanishes at 0 and has a derivative almost everywhere equal to 0. The previous paper also shows how we may construct from the array $\rho(k)$ a function with a spectrum of the same type.