On the fractional parts of the powers of a rational number.

By

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Let u and v be two coprime integers with $u > v > 1$, such that $\frac{u}{v} > 1$, suppose that

$$\rho_n = \left(\frac{u}{v}\right)^n - \left[\left(\frac{u}{v}\right)^n\right].$$

Then the following results, as special cases of more general theorems, are proved in this paper:

a:
$$\lim_{n \to \infty} v^n \rho_n = \infty.$$

b: When ε is a positive constant and

$$\rho_n \leqslant u^{-\varepsilon n}$$

for an infinite sequence of positive integers $n = n_1, n_2, n_3, \ldots$ with $n_{\nu+1} > n_\nu$, then

$$\limsup_{n \to \infty} \frac{n_{\nu+1}}{n_\nu} = \infty.$$

The proofs of a) and b) depend on generalizations of the Thue-Siegel theorem, due to Schneider or myself, and are very simple.

I.

1) Some years ago, I proved the following theorem [1]):

[1]) Math. Annalen 107 (1932), 691—730, in particular Satz 2, p. 722.

LEMMA 1: *Let $F(x,y)$ be an irreducible binary form of degree $n \geq 3$ with integer coefficients, x and y two coprime integers, $P_1, P_2, \ldots, P_t$ $(t \geq 1)$ a finite number of different prime numbers, and $Q(x,y) = P_1^{h_1} P_2^{h_2} \ldots P_t^{h_t}$ the greatest product of powers of these primes, which divides $F(x,y)$. Then*

$$Q(x,y) \leqslant c_0 \max(|x|, |y|)^{2\sqrt{n}},$$

where $c_0 > 0$ is a constant, which does not depend on x and y.

From this lemma, the following one is a trivial consequence:

LEMMA 2: *Let a, b, x be three non-vanishing integers, $n \geqslant 5$ a prime number, v an integer $\geqslant 2$, and $q(x) = v^\nu$ the highest power of v, which divides $ax^n - b$. Then*

$$q(x) \leqslant c_1 |x|^{2\sqrt{n}+1},$$

where $c_1 > 0$ is a constant, which does not depend on x.

Proof: Since n is an odd prime, the binary form $F(x,y) = ax^n - by^n$ either is irreducible, or of the form

$$F(x,y) = (\alpha x - \beta y)\ G(x,y),$$

where α, β are integers, and $G(x,y)$ is an irreducible binary form of degree $n-1$. Suppose that $P_1, P_2, \ldots, P_t$ are the different prime factors of v. Then apply Lemma 1 with $y = 1$ to $F(x,y)$ in the first case, and to $G(x,y)$ in the second case. Then we get

$$q(x) = O(|x|^{2\sqrt{n}})$$

in the first case, and

$$q(x) = O(|x| \cdot |x|^{2\sqrt{n-1}})$$

in the second case, since $\alpha x - \beta = O(|x|)$.

THEOREM 1: *Let a, b, u, v be four non-vanishing integers with $u > v > 1$. Then the equation*

$$\text{(1)}: \qquad a u^x - v^x y = b$$

has at most a finite number of solutions in integers $x \geqslant 0$ and y.

Proof: Let λ be the number

$$\lambda = \frac{\log v}{\log u};$$

thus $0 < \lambda < 1$. Take for n a prime number $\geqslant 5$, such that

$$1+2\sqrt{n} < \lambda n;$$

this condition is satisfied, for instance, when

$$n \geqslant \left(\frac{3}{\lambda}\right)^2.$$

Obviously, to every solution x,y of (1), there are two integers ξ and ν with

$$x = n\xi + \nu, \quad \xi \geqslant 0, \quad 0 \leqslant \nu \leqslant n-1, \quad au^\nu (u^\xi)^n - b = v^\nu y\, (v^\xi)^n.$$

Hence

$$a\, u^\nu X^n - b, \qquad \textit{where} \qquad X = u^\xi,$$

is divisible by a power of v, which, at least, is equal to

$$(v^\xi)^n = X^{\lambda n}.$$

But by Lemma 2, applied to each of the n polynomials

$$au^\nu X^n - b \qquad (\nu = 0, 1, \ldots, n-1),$$

this power of v must be

$$O\,(X^{2\sqrt{n}+1}),$$

and therefore X and x connot be arbitrarily large, i. e., (1) has at most a finite number of solutions, q. e. d.

THEOREM 2: *Under the conditions of theorem 1, the congruence*

$$a\,u^x \equiv d\,(mod\ v^x)$$

can hold only for a finite number of integers $x \geqslant 0$.

THEOREM 3: *Suppose that a, u, v are integers with* $a \neq 0$, $u > v > 1$, $v \nmid u$. Then

$$\lim_{n\to\infty} v^n \left\{ a\left(\frac{u}{v}\right)^n - \left[a\left(\frac{u}{v}\right)^n\right]\right\} = \infty.$$

These two theorems are trivial consequences of Theorem 1. In the case of Theorem 3, the additional condition $v \nmid u$ makes it impossible, that $au^x - v^x y = 0$ has an infinity of solutions.

II.

2) The following theorem can be proved:

LEMMA 3: *Let $\vartheta \neq 0$ be an algebraic number and* p_1/q_1, p_2/q_2, p_3/q_3,

... *an infinite sequence of simplified fractions with the following properties:*

a: $$1 \leqslant q_1 < q_2 < q_3 < \dots$$

b: For every n, p_n and q_n can be written as

$$p_n = P_1^{h_1} \dots P_s^{h_s} p_n^* , \qquad q_n = Q_1^{k_1} \dots Q_t^{k_t} q_n^* ,$$

where $P_1, \dots, P_s, Q_1, \dots, Q_t$ is a given finite system of different prime numbers, $h_1, \dots, h_s, k_1, \dots, k_t$ are integers $\geqslant 0$ and p_n^*, q_n^* are integers, such that as $n \to \infty$

$$p_n^* = O(p_n^\alpha), \quad q_n^* = O(q_n^\beta),$$

where α, β are given constants with $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$.

c: For every n

$$\left| \vartheta - \frac{p_n}{q_n} \right| \leqslant q_n^{-\gamma},$$

where γ is a constant with $\gamma > \alpha + \beta$.

Then

$$\limsup_{n \to \infty} \frac{\log q_{n+1}}{\log q_n} = \infty .$$

For $\alpha = \beta = 1$, $s = t = 0$, this theorem was proved by Th. Schneider [2], and by using his method, I proved it [3] for $\alpha = 0$, $\beta = 1$, $t = 0$, or for $\alpha = 1$, $\beta = 0$, $s = 0$, or for $\alpha = \beta = 0$. The same method, however, leads also to the general result of Lemma 3, as a study of the proof shows. (It is sufficient for this purpose, to use approximation polynomials of the form

$$R(z_1, z_2, \dots, z_n) = \sum R_{l_1 l_2 \dots l_k} z_1^{l_1} z_2^{l_2} \dots z_k^{l_k} ,$$

where the summation sign refers to all integers $l_1, l_2, \dots, l_k$ with

$$0 \leqslant l_1 \leqslant r_1 , 0 \leqslant l_2 \leqslant r_2 , \dots , 0 \leqslant l_k \leqslant r_k , \; \frac{k}{2}(1 - \varepsilon) \leqslant \sum_{\varkappa=1}^{k} \frac{l_\varkappa}{r_\varkappa} \leqslant \frac{k}{2}(1 + \varepsilon).$$

Compare Kapitel 1 of my paper, in particular § 6 and § 8).

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[2]) Journal reine u. angew. Math. 175 (1937), „Über die Approximation algebraischer Zahlen".

[3]) Proceedings Royal Academy Amsterdam, 39 (1937), 633—640, 729—737.

THEOREM 4: *Suppose that* $\vartheta \neq 0$ *is an algebraic number and that* u *and* v *are integers with* $u > v > 1$, $v \nmid u$, *that* ε *is a positive constant, and that* $n = n_1, n_2, n_3, \ldots$ *is an infinite increasing sequence of positive integers, for which*

$$(2): \qquad \vartheta \left(\frac{u}{v}\right)^n - \left[\vartheta \left(\frac{u}{v}\right)^n\right] \leqslant u^{-\varepsilon n}.$$

Then

$$\limsup_{\nu \to \infty} \frac{n_{\nu+1}}{n_\nu} = \infty .$$

Proof: If again

$$\lambda = \frac{\log v}{\log u},$$

then (2) obviously is equivalent to

$$0 \leqslant \vartheta - \frac{v^n \left[\vartheta \left(\frac{u}{v}\right)^n\right]}{u^n} \leqslant \left(\frac{v}{u}\right)^n u^{-\varepsilon n} = u^{-(1-\lambda+\varepsilon)n}.$$

Hence, Lemma 3 can be applied with

$$p = v^n \left[\vartheta \left(\frac{u}{v}\right)^n\right], \; p^* = \left[\vartheta \left(\frac{u}{v}\right)^n\right], \; q = u^n, \; q^* = 1,$$

so that

$$\alpha = 1 - \lambda, \; \beta = 0, \; \alpha + \beta < \gamma = 1 - \lambda + \varepsilon,$$

and the assertion follows at once.

Probably, (2) has only a finite number of solutions for n.

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