KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

A Theorem on inhomogeneous Diophantine Inequalities

BY

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Reprinted from: Proceedings Vol. XLI, N^{0}. 6, 1938

Mathematics. — *A Theorem on inhomogeneous Diophantine Inequalities.* By KURT MAHLER. [1]) (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 28, 1938.)

Two years ago (Math. Ann. **113**, 1936, 398—415), KHINTCHINE proved that if the system of homogeneous linear inequalities

$$0 < x < \gamma t^n, \quad |x\theta_i - y_i| < \frac{1}{t} \qquad (i = 1, 2, \ldots, n),$$

where $\gamma = \gamma(\theta_1, \ldots, \theta_n) > 0$ does not depend on t, has no integer solution for any $t > 1$, then the system of inhomogeneous linear inequalities

$$0 < x < \Gamma t^n, \quad |x\theta_i - y_i - \alpha_i| < \frac{1}{t} \quad (i = 1, 2, \ldots, n),$$

where $\Gamma = \Gamma(\gamma, \theta_1, \ldots, \theta_n) > 0$ does not depend on t and $\alpha_1, \ldots, \alpha_n$, has an integer solution for all $t > 1$. MORDELL generalized this result and at the same time gave a simpler proof (Journal London Math. Soc. **12**, 1937, 34—36 and 166—167). In this note, I shall prove a still more general theorem (Theorem 2), which contains the results of KHINTCHINE and MORDELL as special cases, while its proof remains nearly as simple as that of MORDELL.

1. Let $F(x_1, \ldots, x_n)$ be a real function of the n real variables $x_1, \ldots, x_n$ with the following properties:

(1): $F(0, \ldots, 0) = 0$, but $F(x_1, \ldots, x_n) > 0$ for $\sum_{k=1}^{n} x_k^2 > 0$.

(2): $F(tx_1, \ldots, tx_n) = |t| F(x_1, \ldots, x_n)$ for all real values of t.

(3): $F(x_1 + y_1, \ldots, x_n + y_n) \leq F(x_1, \ldots, x_n) + F(y_1, \ldots, y_n)$.

(4): The convex body defined by the inequality $F(x_1, \ldots, x_n) \leq 1$ has the volume I.

Then, by a theorem of MINKOWSKI (Geometrie der Zahlen, p. 218), there exist n^2 integers $X_k^{(l)}$ with determinant

$$d = |X_k^{(l)}|_{k, l = 1, 2, \ldots, n} \neq 0,$$

such that

$$(5): \qquad \prod_{l=1}^{n} F(X_1^{(l)}, \ldots, X_n^{(l)}) \leq \frac{2^n}{I}.$$

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[1]) I wish to express my thanks to Prof. MORDELL for his help with the manuscript.

MINKOWSKI also proved (G. d. Z., p. 189 and 192) that $|d| \leq n!$ and in particular $|d| = 1$ for $n = 2$; in general, however, $|d| = 1$ need not be true. But the following weaker theorem holds:

Theorem 1: *Under the conditions* (1)—(4), *there are* n^2 *integers* $x_{hk}^{(l)}$ *with determinant*

$$|x_k^{(l)}|_{k,l=1,2,\ldots,n} = 1,$$

such that

$$\text{(6):} \qquad \prod_{l=1}^{n} F(x_1^{(l)}, \ldots, x_n^{(l)}) \leq \frac{2^n n!}{I}.$$

Proof: Suppose that $X_k^{(l)}$ are MINKOWSKI's integers, and that, if

$$F(X_1^{(l)}, \ldots, X_n^{(l)}) = S_l \qquad (l = 1, 2, \ldots, n),$$

then without loss of generality

$$\text{(7):} \qquad S_1 \leq S_2 \leq \ldots \leq S_n.$$

By MINKOWSKI's method of "adaptation" of the lattice of all points $(x_1, \ldots, x_n)$ with respect to the n lattice points $(X_1^{(l)}, \ldots, X_n^{(l)})$ $(l = 1, \ldots, n)$ (G. d. Z., p. 173—176), n lattice points $(x_1^{(l)}, \ldots, x_n^{(l)})$ $(l = 1, \ldots, n)$ with determinant

$$|x_k^{(l)}|_{k,l=1,\ldots,n} = 1,$$

exist, such that in vector notation for $l = 1, \ldots, n$

$$(x_1^{(l)}, \ldots, x_n^{(l)}) = \sum_{k=1}^{l} \beta_k^{(l)} (X_1^{(k)}, \ldots, X_n^{(k)}),$$

where the $\beta_k^{(l)}$ are real numbers satisfying

$$|\beta_k^{(l)}| \leq 1 \qquad (k, l = 1, \ldots, n;\ k \leq l).$$

Hence by (2), (3) and (7)

$$F(x_1^{(l)}, \ldots, x_n^{(l)}) \leq S_1 + S_2 + \ldots + S_l \leq l S_l \leq l F(X_1^{(l)}, \ldots, X_n^{(l)}),$$

so that (6) follows at once from (5).

2. From the last result we shall obtain:

Theorem 2: *Suppose that the conditions* (1)—(4) *are satisfied, that* $\xi_1, \ldots, \xi_n$ *are real numbers, that* τ *is a positive number, and that there is no other integer solution of the inequality*

$$\text{(8):} \qquad F(X_1, \ldots, X_n) \leq \frac{2\tau}{\sqrt[n]{I}},$$

than the trivial one $X_1 = \ldots = X_n = 0$. *Then the inequality*

$$(9): \qquad F(x_1 + \xi_1, \ldots, x_n + \xi_n) \leq \frac{(n+1)! + 1}{\tau^{n-1} \sqrt[n]{I}}$$

has a solution in integers $x_1, \ldots x_n$.

Proof: Let $X_1, \ldots, X_n, Y$ be $n+1$ variables and consider the domain in $n+1$ dimensions defined by

$$(10): \qquad F(X_1 + \xi_1 Y, \ldots, X_n + \xi_n Y) \leq \frac{2\tau}{\sqrt[n]{I}}, \quad |Y| \leq \frac{1}{\tau^n}.$$

It is easy to see that this domain is a convex body of volume

$$I\left(\frac{2\tau}{\sqrt[n]{I}}\right)^n \cdot \frac{2}{\tau^n} = 2^{n+1}.$$

Hence there are $n+1$ integers $x_1^*, \ldots, x_n^*, y$, which are not all zero (G. d. Z., p. 76), such that

$$(11): \qquad F(x_1^* + \xi_1 y, \ldots, x_n^* + \xi_n y) \leq \frac{2\tau}{\sqrt[n]{I}}, \quad |y| \leq \frac{1}{\tau^n}.$$

Here $y \neq 0$, since otherwise there would be a non-trivial integer solution $x_1, \ldots, x_n$ of (8), against hypothesis.

By theorem 1, there is a system of n^2 integers $x_k^{(l)}$ of determinant 1 satisfying (6). Obviously the numbers $x_1^{(l)}, \ldots, x_n^{(l)}$ do not vanish simultaneously for any l; hence by assumption

$$F(x_1^{(l)}, \ldots, x_n^{(l)}) > \frac{2\tau}{\sqrt[n]{I}} \qquad (l = 1, \ldots, n)$$

and therefore by (6)

$$(12): \qquad F(x_1^{(l)}, \ldots, x_n^{(l)}) \leq \frac{2n!}{\tau^{n-1} \sqrt[n]{I}} \qquad (l = 1, \ldots, n).$$

Now consider the system of n linear congruences in $u_1, \ldots, u_n$:

$$x_k^* + \sum_{l=1}^{n} x_k^{(l)} u_l \equiv 0 \pmod{y} \qquad (k = 1, \ldots, n).$$

Since its determinant is 1, there is at least one integer solution $u_1^*, \ldots, u_n^*$, and then all integer solutions can be written as

$$u_l = u_l^* + y v_l \qquad (l = 1, \ldots, n),$$

where $v_1, \ldots, v_n$ are arbitrary integers. Hence we may assume that

$$|u_l| \leqq \frac{|y|}{2} \qquad (l = 1, \ldots, n).$$

With these values of the u's, put

$$x_k = \frac{1}{y}\left\{x_k^* + \sum_{l=1}^{n} x_k^{(l)} u_l\right\} \qquad (k = 1, \ldots, n),$$

so that $x_1, \ldots, x_n$ are integers and

$$x_k + \xi_k = \frac{1}{y}(x_k^* + y\,\xi_k) + \sum_{l=1}^{n} \frac{u_l}{y} x_k^{(l)},$$

and since $|y| \geqq 1$,

$$F(x_1 + \xi_1, \ldots, x_n + \xi_n) \leqq \frac{1}{|y|} F(x_1^* + \xi_1 y, \ldots, x_n^* + \xi_n y) + \sum_{l=1}^{n} \left|\frac{u_l}{y}\right| F(x_1^{(l)}, \ldots, x_n^{(l)})$$

$$\leqq \frac{2\tau}{\sqrt[n]{I}} + \frac{n}{2} \cdot \frac{2n!}{\tau^{n-1}\sqrt[n]{I}} \leqq \frac{2 + n.n!}{\tau^{n-1}\sqrt[n]{I}} \leqq \frac{(n+1)! + 1}{\tau^{n-1}\sqrt[n]{I}},$$

since by MINKOWSKI's theorem necessarily

$$\tau < 1,$$

q. e. d.

The example $F(x_1, \ldots, x_n) = max(\tau|x_1|, \tau|x_2|, \ldots, \tau|x_{n-1}|, \tau^{-(n-1)}|x_n|)$, $\xi_1 = \ldots = \xi_n = \frac{1}{2}$ shows, that the exponent $n-1$ of τ in theorem 2 cannot be improved.

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Eastern 1938.