

SOME ARITHMETICAL PROPERTIES OF THE CONVERGENTS OF A CONTINUED FRACTION

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In this note, we consider the greatest prime factor $G(B_n)$ of the denominator of the n -th convergent A_n/B_n of an infinite continued fraction

$$\zeta = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots,$$

where the a_1, a_2, \dots are positive integers.

We show in §1 that, for "almost all" ζ , $G(B_n)$ increases rapidly with n (Theorem 1). In §2, we prove that ζ is a Liouville number (*i.e.* $B_n < B_{n+1}^\epsilon$ for arbitrary $\epsilon > 0$ and an infinity of n) if $G(B_n)$ is bounded for all n (Theorem 2); and, in fact, there are Liouville numbers with bounded $G(B_n)$. If the denominators a_{n+1} are bounded or increase slowly, then we can prove sharper results (B and C); but we omit the proofs, since they are similar to that of Theorem 2.

Corresponding results hold for the numerators A_n of the convergents A_n/B_n of ζ .

Notation. In the following, ζ is a positive irrational number,

$$\zeta = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots$$

is its regular continued fraction, and

$$\frac{A_{-1}}{B_{-1}} = \frac{1}{0}, \quad \frac{A_0}{B_0} = \frac{a_0}{1}, \quad \frac{A_1}{B_1} = \frac{a_0 a_1 + 1}{a_1}, \quad \dots$$

is the sequence of its convergents. If

$$\{P\} = \{P_1, P_2, \dots, P_t\}$$

is an arbitrary finite set of different prime numbers, then $M(\{P\})$ denotes the set of all indices n for which all prime factors of B_n belong to $\{P\}$, and

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$N(x; \{P\})$ denotes the number of elements $n \leq x$ of this set. Finally, $G(k)$ denotes the greatest prime factor of $k \neq 0$.

I.

1. In this first paragraph, we prove that for "almost all" ξ the function $G(B_n)$ increases rapidly with n .

LEMMA 1. *Let S be the set of all positive integers k for which*

$$k \geq \xi, \quad G(k) \leq \exp\left(\frac{\log k}{20 \log \log k}\right);$$

then, for large $\xi > 0$,

$$\sum_{k \text{ in } S} \frac{1}{k} = O((\log \xi)^{-3}).$$

Proof. We divide the set of all positive integers k for which

$$(1) \quad k \leq x, \quad G(k) \leq \exp\left(\frac{\log k}{20 \log \log k}\right)$$

into three classes A , B , and C , such that A consists of those elements which are divisible by a square greater than or equal to $(\log x)^{10}$, and the remaining elements k belong to B or C , according as $k \geq \sqrt{x}$ or $k < \sqrt{x}$. Then A has at most

$$\sum_{r > (\log x)^5} \frac{x}{r^2} = O\left(\frac{x}{(\log x)^5}\right)$$

elements. Next, let k be an element of B , and let

$$k = P_1^{h_1} P_2^{h_2} \dots P_t^{h_t}$$

be its representation as a product of powers of different primes. Then, if an exponent $h \geq 2$, either P^{h-1} or P^h is a square factor of k , and therefore

$$P^{h-1} < (\log x)^{10}.$$

Since $\sqrt{x} \leq k \leq x$, for large x , we have

$$P^h \leq (\log x)^{10h/(h-1)} \leq (\log x)^{20} \leq \exp\left(\frac{\log x}{40 \log \log x}\right) \leq \exp\left(\frac{\log k}{20 \log \log k}\right).$$

Since this inequality holds also for $h = 1$, we have

$$k \leq \exp\left(\frac{t \log k}{20 \log \log k}\right),$$

and k is divisible by at least $20 \log \log k \geq 10 \log \log x$ different prime numbers, when x is sufficiently large. Therefore the number of divisors of k

$$d(k) \geq 2^{10 \log \log x} \geq (\log x)^5.$$

Now
$$\sum_{k \leq x} d(k) = O(x \log x),$$

so that B has at most

$$(\log x)^{-5} O(x \log x) = O\left(\frac{x}{(\log x)^4}\right)$$

elements. Since U has less than \sqrt{x} elements, there are therefore only

$$O\left(\frac{x}{(\log x)^5}\right) + O\left(\frac{x}{(\log x)^4}\right) + \sqrt{x} = O\left(\frac{x}{(\log x)^4}\right)$$

integers k satisfying (1).

Suppose now that

$$k_1, k_2, k_3, \dots \quad (1 \leq k_1 < k_2 < k_3 < \dots)$$

is the sequence of all positive integers k for which

$$G(k) \leq \exp\left(\frac{\log k}{20 \log \log k}\right).$$

Then, by the last result,

$$\frac{1}{k_\nu} = O\left(\frac{1}{\nu(\log \nu)^4}\right),$$

and the lemma follows immediately, since

$$\sum_{\nu \geq n} \frac{1}{\nu(\log \nu)^4} = O\left((\log n)^{-3}\right).$$

LEMMA 2. *The measure of the set of all ζ in $0 \leq \zeta \leq 1$, such that the denominator B_n of one of the convergents A_n/B_n of ζ is equal to a given integer $k \geq 1$, is not greater than $1/k$.*

This is trivial, since

$$\left| \zeta - \frac{A_n}{B_n} \right| \leq \frac{1}{2B_n^2}.$$

THEOREM 1. *The set of all ζ in $0 \leq \zeta \leq 1$, for which an infinity of indices n exist satisfying*

$$(2) \quad G(B_n) \leq \exp\left(\frac{\log B_n}{20 \log \log B_n}\right),$$

is of measure zero.

Proof. Obviously $B_n \geq B_n'$, where A_n'/B_n' is the convergent of order n of the special continued fraction

$$\frac{1}{|1|} + \frac{1}{|1|} + \frac{1}{|1|} + \dots$$

Now

$$B_n' = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right\},$$

and therefore

$$(3) \quad B_n' \geq \frac{1}{2} \left(\frac{1+\sqrt{5}}{2} \right)^n \quad (n = 1, 2, 3, \dots).$$

Let n be an arbitrary index. Then, by Lemmas 1 and 2, the measure of all ζ in $0 \leq \zeta \leq 1$, for which (2) holds, is not greater than

$$\sum' \frac{1}{k} = O(n^{-3}),$$

where the summation extends over all integers k for which

$$k \geq \frac{1}{2} \left(\frac{1+\sqrt{5}}{2} \right)^n, \quad G(k) \leq \exp\left(\frac{\log k}{20 \log \log k}\right).$$

Therefore the measure of all ζ in $0 \leq \zeta \leq 1$, for which (2) is satisfied for an infinity of indices $n \geq N$, is not greater than

$$O\left(\sum_{n \geq N} n^{-3}\right) = O(N^{-2}) = o(1),$$

and hence the theorem follows immediately.

In particular, from Theorem 1 and (3), for "almost all" ζ in $0 \leq \zeta \leq 1$ and all sufficiently large n , we have

$$G(B_n) \geq \exp\left(\frac{n}{50 \log n}\right).$$

II.

2. In this second paragraph, we give some properties of the set $M(\{P\})$ and the arithmetical function $N(x; \{P\})$ for special classes of irrational numbers ζ .

LEMMA 3. *For every $\epsilon > 0$ and every finite system $\{P\}$ of given prime numbers P_1, \dots, P_t , there is at most a finite number of systems of three integers*

$$X_1 \neq 0, \quad X_2 \neq 0, \quad X_3 = \xi X_3^* \neq 0,$$

such that

$$X_1 - X_2 = X_3, \quad (X_1, X_2) = 1, \quad |X_3^*| \geq \max(|X_1|, |X_2|)^\epsilon,$$

where ξ and X_3^* are integers, and all prime factors of $X_1 X_2 X_3^*$ belong to $\{P\}$.

Proof. Take a prime number n , for which

$$n \geq 5, \quad \frac{1+2\sqrt{(n-1)}}{n} < \epsilon, \quad \frac{2}{\sqrt{n}} < \epsilon.$$

By hypothesis, X_1 and X_2 can be written in the form

$$X_1 = \eta_1 P_1^{h_1} \dots P_t^{h_t}, \quad X_2 = \eta_2 P_1^{k_1} \dots P_t^{k_t} \quad (\eta_1 = \pm 1, \eta_2 = \pm 1),$$

with non-negative exponents. Dividing them by n , we get, say,

$$h_\tau = nh_\tau' + h_\tau'', \quad k_\tau = nk_\tau' + k_\tau'' \quad (\tau = 1, 2, \dots, t),$$

where $h_\tau', h_\tau'', k_\tau', k_\tau''$ are integers, and

$$0 \leq h_\tau'' \leq n-1, \quad 0 \leq k_\tau'' \leq n-1 \quad (\tau = 1, 2, \dots,$$

Put

$$x = P_1^{h_1'} \dots P_t^{h_t'}, \quad y = P_1^{k_1'} \dots P_t^{k_t'}, \quad a = P_1^{h_1''} \dots P_t^{h_t''} \eta_1, \quad b = P_1^{k_1''} \dots P_t^{k_t''} \eta_2.$$

Then there are only $4n^{2t}$ possible sets (a, b) . Also

$$X_1 = ax^n, \quad X_2 = by^n;$$

hence $(x, y) = 1$ and

$$ax^n - by^n = \xi X_3^*.$$

The binary form on the left-hand side is either irreducible, or is the product of an irreducible form of degree $n-1 \geq 3$ and a linear factor. Also there are only a finite number of possible forms. On the right,

all prime factors of X_3^* belong to $\{P\}$. Hence, by the p -adic generalization of the Thue-Siegel theorem[†], we must have

$$\begin{aligned} X^* &= O\left(\max(|x|, |y|)^{\max\{2\sqrt{n}, 1+2\sqrt{(n-1)}\}}\right) \\ &= O\left(\max(|X_1|, |X_2|)^{2/\sqrt{n}, 1+2\sqrt{(n-1)/n}}\right), \end{aligned}$$

except for at most a finite number of solutions. Hence the lemma follows immediately.

THEOREM 2. *Suppose that for an infinity of different indices*

$$(4) \quad n = n_1, \quad n_2, \quad n_3, \quad \dots$$

the denominators B_{n-1} , B_n , B_{n+1} of three consecutive convergents of ζ are divisible by only a finite system of prime numbers $\{P\}$. Then ζ is a Liouville number, and is therefore transcendental.

Proof. Obviously

$$(5) \quad B_{n+1} - B_{n-1} = a_{n+1} B_n.$$

Put $d = (B_{n-1}, B_{n+1})$. Then $(d, B_n) = 1$, since any two consecutive B 's are relatively prime; and so, by (5), d is a divisor of a_{n+1} . Write

$$B_{n+1} = dB_{n+1}^*, \quad B_{n-1} = dB_{n-1}^*, \quad a_{n+1} = da_{n+1}^*.$$

Then $(B_{n-1}^*, B_{n+1}^*) = 1$, and all prime factors of B_{n-1}^* , B_n , B_{n+1}^* belong to $\{P\}$ if n is an element of the sequence (4). From (5) and Lemma 3,

$$B_{n+1}^* - B_{n-1}^* = a_{n+1}^* B_n;$$

hence

$$B_n \leq \max(|B_{n-1}^*|, |B_{n+1}^*|)^e \leq B_{n+1}^e$$

for sufficiently large n ; and this is the defining property of a Liouville number.

3. By the last proof, for all sufficiently large n at least one of any three consecutive indices $n-1$, n , $n+1$ does not belong to $M(\{P\})$, if ζ is not a Liouville number; hence we have the inequality

$$(A) \quad \limsup_{x \rightarrow \infty} N(x; \{P\})/x \leq \frac{2}{3}, \quad \text{if } \log a_{n+1} = O(\log B_n).$$

[†] K. Mahler, *Math. Annalen*, 107 (1933), 691-730, Satz 2, 722.

In a similar way, by considering a sufficiently large number of consecutive indices and applying a lemma similar to Lemma 3, we can prove the following two results:

$$(B) \quad \lim_{x \rightarrow \infty} N(x; \{P\})/x = 0, \quad \text{if } \log a_{n+1} = o(\log B_n),$$

$$(C) \quad N(x; \{P\}) = O(\log x), \quad \text{if all } a_{n+1} \text{ are bounded.}$$

We mention as examples for these theorems:

(A) All real irrational algebraic numbers, the number π , the logarithms of all real rational numbers, the powers e^a with real irrational algebraic exponents.

(B) The powers e^a with rational exponents $a \neq 0$.

(C) All real quadratic irrational numbers.

(For these quadratic irrationals, it is even possible to show that $M(\{P\})$ has only a finite number of elements.)

Since any two consecutive B_n are relatively prime, all B_n cannot be powers of one single prime number. It is, however, easy to construct a Liouville number for which all B_n are only divisible by two arbitrary given prime numbers. On the other hand there are Liouville numbers for which all B_n are prime numbers. Assuming Riemann's hypothesis to be true, it is easy to show that there exist also non-Liouville numbers with this property.

Added 28 October, 1938. With respect to Theorem 2, it may be remarked that there are *transcendental non-Liouville* numbers, for which the greatest prime divisor of q_n is bounded for an infinity of indices n ; e.g.,

$$\zeta = 3^{-1} + 3^{-3} + 3^{-9} + 3^{-27} + 3^{-81} + \dots$$

We can show further that there exist real numbers ζ , for which the greatest prime factor of both p_n and q_n is bounded for an infinity of n . These numbers are necessarily transcendental, and probably they are Liouville numbers; but this we have not yet proved.

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