NOTE ON THE SEQUENCE $\sqrt{n}$ (mod 1)

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In this note, we denote by

$$|\alpha| < \lambda \pmod{1} \text{ and } |\alpha| \geqq \lambda \pmod{1},$$

that there is a rational integer g such that

$$|\alpha + g| < \lambda,$$

respectively, that no such integer exists.

It is well known that the sequence

$$\sqrt{1}, \quad \sqrt{2}, \quad \sqrt{3}, \ldots$$

is uniformly distributed modulo 1 [1]; hence for every real ξ and for every $\varepsilon > 0$, the inequality

$$|\xi - \sqrt{x}| < \varepsilon \pmod{1}$$

has an infinity of integral solutions x. The three following theorems 1, 1a, and 2 give more precise information:

Theorem 1: *To every real irrational number ξ and to every $\varepsilon > 0$, there is an infinity of integral solutions $x > 0$ of*

$$|\xi - \sqrt{x}| < \frac{1 + \varepsilon}{2\sqrt{5x}} \pmod{1}.$$

Theorem 1*a*. *There is a constant $c > 0$, such that for real irrational ξ and real η, the inequality*

$$|\xi - \sqrt{x + \eta}| < \frac{c}{x} \pmod{1}$$

has an infinity of integral solutions $x > 0$.

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[1]) See J. F. Koksma, Diophantische Approximationen, (Berlin 1936), Satz 4, p. 89.

Theorem 2: To every rational number ξ there is a positive number $c = c(\xi)$, such that for all positive integers x either

$$|\xi - \sqrt{x}| \geqq \frac{c}{\sqrt{x}} \pmod{1} \text{ or } \xi - \sqrt{x} \equiv 0 \pmod{1}.$$

Proof of Theorem 1: We use the two trivial identities

$$(\xi+y-\sqrt{x})(\xi+y+\sqrt{x}) = (\xi+y)^2 - x = \xi^2 + 2\xi y + (y^2 - x), \quad (1)$$

$$\xi + y + \sqrt{x} = (\xi + y - \sqrt{x}) + 2\sqrt{x}. \quad (2)$$

By a theorem of A. KHINTCHINE [1]), there is an infinity of pairs of integers x, y with $y \to +\infty$, such that

$$|\xi^2 + 2y\xi + (y^2 - x)| \leqq \frac{1+\frac{1}{2}\varepsilon}{\sqrt{5}\, y}. \quad (3)$$

By (1) and (2), for this sequence

$$\xi + y - \sqrt{x} \to 0,\ \xi + y + \sqrt{x} \sim 2\sqrt{x},\ y \sim \sqrt{x},$$

and therefore for all sufficiently large x of the sequence

$$|\xi + y - \sqrt{x}| < \frac{1+\varepsilon}{\sqrt{5}\,\sqrt{x}\,.\,2\sqrt{x}} = \frac{1+\varepsilon}{2\sqrt{5}x}, \quad \text{q.e.d.}$$

Proof of Theorem 1*a*. By KHINTCHINE's theorem, there are arbitrarily large integers $X > 0$, Y, such that

$$|2\xi X - Y - \eta + \xi^2| < \frac{1+\varepsilon}{\sqrt{5}X}$$

and therefore $Y = O(X)$. Put $x = X^2 + Y$. Then by the binomial-theorem:

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[1]) See KOKSMA, D. A., Footnote p. 76. The theorem says that for irrational θ and arbitrary β, the inequality

$$|\theta x - y - \beta| < \frac{1}{(\sqrt{5} - \varepsilon)x}$$

has integral solutions x, y with arbitrarily large $x > 0$; here ε is an arbitrary constant with $0 < \varepsilon < \sqrt{5}$.

$$\sqrt{x+\eta} = \sqrt{X^2+Y+\eta} = X + \frac{Y+\eta}{2X} - \frac{1}{8}\frac{(Y+\eta)^2}{X^3} + O\left(\frac{1}{X^2}\right) =$$

$$= X + \left\{\xi + \frac{\xi^2}{2X} + O\left(\frac{1}{X^2}\right)\right\} - \frac{1}{8}\left\{\frac{4\xi^2}{X} + O\left(\frac{1}{X^2}\right)\right\} + O\left(\frac{1}{X^2}\right),$$

and so finally

$$\sqrt{x+\eta} = X + \xi + O\left(\frac{1}{X^2}\right), \text{ i.e. } |\xi - \sqrt{x+\eta}| = O\left(\frac{1}{x}\right) \pmod{1}. \text{ Q.e.d.}$$

Proof of Theorem 2: Let $\xi = \frac{p}{q}$, where $(p, q) = 1$. Then the integer

$$q^2 |(\xi + y - \sqrt{x})(\xi + y + \sqrt{x})| = |p^2 - 2pqy + q^2(y^2 - x)| \begin{cases} = 0 \text{ or} \\ \geqq 1. \end{cases}$$

If

$$\xi + y - \sqrt{x} \to 0,$$

then

$$\xi + y + \sqrt{x} \sim 2\sqrt{x}$$

and so for $\varepsilon > 0$ and all sufficiently large x

$$|\xi + y - \sqrt{x}| \begin{cases} = 0 \text{ or} \\ \geqq \dfrac{1-\varepsilon}{2q^2\sqrt{x}} \end{cases} \quad \text{, q.e.d.}$$

From the two theorems 1 and 2, we get the Corrollary: Let α be any number > 2. Then the real number ξ is rational, if and only if the infinite series

$$\sum_{n=1}^{\infty}{}' \left|\frac{1}{x} \operatorname{ctg} \pi (\xi - \sqrt{x})\right|^{\alpha}$$

converges; infinite terms are to be excluded.

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