A PROBLEM OF DIOPHANTINE APPROXIMATION IN QUATERNIONS

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In two previous papers† I studied the upper bound of products of two or three linear polynomials in real or complex integral variables. In the present paper I use the same method to prove the following result:

"*If* α, β, γ, δ, ρ, σ *are constant quaternions such that*

$$\alpha\bar{\alpha}\delta\bar{\delta}+\beta\bar{\beta}\gamma\bar{\gamma}-\alpha\bar{\gamma}\delta\bar{\beta}-\beta\bar{\delta}\gamma\bar{\alpha}=1,$$

then there are two integral quaternions x, y *satisfying*

$$|\alpha x+\beta y+\rho||\gamma x+\delta y+\sigma|\leqslant \tfrac{1}{2}\text{''}.$$

In so far as the proof is based on special properties of the quaternions, I am indebted to a paper of Speiser‡ and of course to Hurwitz's classical book *Zahlentheorie der Quaternionen*.

In the first chapter, upper bounds for the maximum of $|x|$ in certain sets of quaternions are derived; I mention in particular Theorems 2, 8 and 10, which have some interest in themselves. The second chapter deals chiefly with the reduction of Hermitian forms in quaternions; by combining these results with those of the first chapter, the theorem stated is obtained in the usual way.

I am very much indebted to Prof. Mordell for his help with the manuscript.

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† *Journal London Math. Soc.*, 15 (1940), 215–236 and 305–320.

‡ *Journal für Math.*, 167 (1932), 88–97.

CHAPTER I. INEQUALITIES.

1. *Notation.*

Let K be the field of all quaternions

$$x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3,$$

where x_0, x_1, x_2, x_3 are arbitrary real numbers, and let

$$\bar{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3,$$
$$|x| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}},$$
$$S(x) = x + \bar{x} = 2x_0,$$
$$N(x) = x\bar{x} = |x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

be the *conjugate to* x, its *absolute value*, its *trace* and its *norm*. We represent x by a point with rectangular coordinates (x_0, x_1, x_2, x_3) in the four-dimensional Euclidian space K, and call this point also x. Then two points x and y have the *distance* $|x-y|$.

By Hurwitz's definition†, x is an *integral quaternion*, if it can be written in the form

$$x = g_0 j + g_1 i_1 + g_2 i_2 + g_3 i_3,$$

where g_0, g_1, g_2, g_3 are rational integers, and j is the quaternion

$$j = \frac{1 + i_1 + i_2 + i_3}{2}.$$

Hence x is integral if *all numbers* $2x_0$, $2x_1$, $2x_2$, $2x_3$ are *even* or all are *odd* rational integers. The set of all integral quaternions forms a *lattice* L which is generated by the four points j, i_1, i_2, i_3, and which is the four-dimensional analogue to the *centred cube lattice* in ordinary space.

Two points x and y in K are called *congruent*, in symbols $x \equiv y$, if their difference $x-y$ lies in L.

2. *A lemma on linear inequalities.*

The following simple lemma is used repeatedly in this paper:

THEOREM 1. *Suppose that the linear inequalities with real coefficients*

$$l_h'(x) = a_{h1} x_1 + \ldots + a_{hn} x_n + a_h \geqslant 0 \quad (h = 1, 2, \ldots, m)$$

† "Zahlentheorie der Quaternionen", *Math. Abhandlungen*, 2 (1933), 303–330, in particular p. 309.

define a bounded set S of points $x = (x_1, \ldots, x_n)$ in n-dimensional Euclidian space. Then the maximum of

$$f(x) = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}},$$

the distance of x from the origin, is assumed only in points of S in which at least n of the functions $l_h(x)$ vanish simultaneously.

Proof. Since $f(x)$ is continuous and bounded, there is at least one point $x^0 = (x_1^0, \ldots, x_n^0)$ of S in which $f(x^0)$ takes its largest value. If less than n of the numbers $l_h(x)$ vanish, then we can find a positive number ϵ and n numbers $x_1^1, \ldots, x_n^1$ not all zero, such that all points

$$x = (x_1^0 + tx_1^1, \ldots, x_n^0 + tx_n^1) \quad \text{with} \quad -\epsilon \leqslant t \leqslant +\epsilon$$

belong to S. For these points,

$$f(x)^2 = \alpha t^2 + \beta t + \gamma = \phi(t)$$

with real coefficients α, β, γ, of which $\frac{1}{2}\phi''(0) = \alpha = f(x^1)^2 > 0$. Hence this function is not a maximum for $t = 0$, contrary to hypothesis.

COROLLARY. *The theorem remains true when some of the inequalities $l_h(x) \geqslant 0$, defining S, are replaced by the equations $l_h(x) = 0$.*

This follows from the last proof.

3. *The sets Δ and $\Delta(g)$.*

Let Δ be the set of all quaternions $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ such that

(1) $$|x| \leqslant |x - g| \quad \textit{for every integral quaternion } g.$$

This set has the following properties:

THEOREM 2. (*a*) *Δ is determined by the linear inequalities*

(2) $$|x_0| \leqslant \tfrac{1}{2}, \quad |x_1| \leqslant \tfrac{1}{2}, \quad |x_2| \leqslant \tfrac{1}{2}, \quad |x_3| \leqslant \tfrac{1}{2}, \quad |x_0| + |x_1| + |x_2| + |x_3| \leqslant 1.$$

(*b*) *For all points of Δ,*

(3) $$|x| \leqslant \sqrt{\tfrac{1}{2}},$$

with equality if and only if x is one of the 24 points

(4) $$\frac{\mp 1 \mp i_1}{2}, \quad \frac{\mp 1 \mp i_2}{2}, \quad \frac{\mp 1 \mp i_3}{2}, \quad \frac{\mp i_1 \mp i_2}{2}, \quad \frac{\mp i_1 \mp i_3}{2}, \quad \frac{\mp i_2 \mp i_3}{2}.$$

(*c*) *To every quaternion there is a congruent one in Δ.*

Proof. (A) *If x satisfies* (2), *then* (3) *also holds.* The formulae (2) can be written as

$$\mp x_0+\tfrac{1}{2}\geqslant 0,\quad \mp x_1+\tfrac{1}{2}\geqslant 0,\quad \mp x_2+\tfrac{1}{2}\geqslant 0,\quad \mp x_3+\tfrac{1}{2}\geqslant 0,$$

$$\mp x_0\mp x_1\mp x_2\mp x_3+1\geqslant 0$$

Hence, by Theorem 1, the maximum of $|x|$ is attained in a point in which at least *four* of these inequalities hold in the stronger form with the *sign of equality*. This clearly is possible only in the points (4), and these are all of absolute value $\sqrt{\tfrac{1}{2}}$.

(B) *The set* (2) *is identical with* Δ. If $g=g_0j+g_1i_1+g_2i_2+g_3i_3$ is an integral quaternion, then the inequality $|x|\leqslant|x-g|$ can be written as

$$\gamma_0x_0+\gamma_1x_1+\gamma_2x_2+\gamma_3x_3\leqslant\Gamma,$$

where

$$\gamma_0=\frac{g_0}{2|g|},\quad \gamma_1=\frac{g_0+2g_1}{2|g|},\quad \gamma_2=\frac{g_0+2g_2}{2|g|},\quad \gamma_3=\frac{g_0+2g_3}{2|g|},\quad \Gamma=\frac{|g|}{2},$$

and therefore $$\gamma_0^2+\gamma_1^2+\gamma_2^2+\gamma_3^2=1.$$

Hence the distance Γ of the hyperplane

$$\gamma_0x_0+\gamma_1x_1+\gamma_2x_2+\gamma_3x_3=\Gamma$$

from the origin is $\tfrac{1}{2}$ if g is one of the 24 units

$$\epsilon=\mp 1,\ \mp i_1,\ \mp i_2,\ \mp i_3,\ \frac{\mp 1\mp i_1\mp i_2\mp i_3}{2} \tag{5}$$

of K, and it is greater than or equal to $\sqrt{\tfrac{1}{2}}$, if $g\neq 0$ is any other integral quaternion. Therefore those inequalities (1) which are not included in (2) are a consequence of the latter.

(C) If x is an arbitrary quaternion, then there is an integral quaternion h such that

$$|x-h|\leqslant|x-g-h|\quad\text{for all integral quaternions } g.$$

Hence $x'=x-h\equiv x$ satisfies (1) and therefore lies in Δ. This completes the proof.

For any quaternion g, let $\Delta(g)$ be the set of all points x for which $x-g$ belongs to Δ; in particular $\Delta(0)=\Delta$. $\Delta(g)$ can also be defined as the set of all points whose distance from g is not larger than that from any other point congruent to g. If g runs over all elements of L, then the sets $\Delta(g)$ together just fill up the space K without overlapping.

4. *The singular vertices and the transformations of* Δ.

The 24 points (4) on the boundary of Δ are called *singular vertices*, or, briefly, SV; the same name is given to all congruent points. If Σ_0 denotes the special SV

$$\Sigma_0 = \frac{1+i_1}{2},$$

then all SV of Δ given by (4) can be written as

$$\Sigma = \epsilon\Sigma_0,$$

where ϵ denotes the 24 units (5) of K. The 24 transformations of K

$$x \to \epsilon x \tag{6}$$

(*i.e.* the association with the point x of a new point ϵx, or replacing x by ϵx), transform the SV (4) of Δ into one another; they leave all distances and the lattice L invariant, and therefore also the set Δ. More generally, the group of transformations

$$x \to \epsilon x + g, \tag{7}$$

where ϵ is a unit and g an integral quaternion, transforms the lattice L and the set of all SV into themselves, preserves all distances, and changes Δ into $\Delta(g)$.

To every SV, there is a sub-group of order 8 of this group (7) which leaves this point unchanged. For the special SV Σ_0, this group consists of the transformations

$$x - \Sigma_0 \to e(x - \Sigma_0), \quad \text{where} \quad e = \mp 1,\ \mp i_1,\ \mp i_2,\ \mp i_3. \tag{8}$$

By this group, Δ is transformed into the 8 sets

$$\Delta(g), \quad \text{where} \quad g = 0,\ 1,\ i_1,\ 1+i_1,\ \frac{1+i_1 \mp i_2 \mp i_3}{2}, \tag{9}$$

these being the only $\Delta(g)$ which meet in the SV Σ_0.

The 24 SV of Δ can be divided into 3 sets of 8 congruent points, namely

$$\left(\frac{\mp 1 \mp i_1}{2},\ \frac{\mp i_2 \mp i_3}{2}\right),\quad \left(\frac{\mp 1 \mp i_2}{2},\ \frac{\mp i_1 \mp i_3}{2}\right),\quad \left(\frac{\mp 1 \mp i_3}{2},\ \frac{\mp i_1 \mp i_2}{2}\right),$$

and all points of the first set can be written as

(10) $$\Sigma = \Sigma_0 - g,$$

where g is again one of the 8 numbers (9)

Finally, every SV in K is congruent to one of the three SV

$$\frac{1+i_1}{2}, \quad \frac{1+i_2}{2}, \quad \frac{1+i_3}{2},$$

and therefore can be written as

$$\Sigma = \frac{a_0+a_1 i_1+a_2 i_2+a_3 i_3}{2},$$

where the a's are integers of which two are even and two odd.

5. *The sets Δ^* and D.*

Let Δ^* be the sub-set of all those points x of Δ which satisfy the 24 inequalities

(11) $$|x-\Sigma_0| \leqslant |x-\Sigma| \quad \text{for all SV } \Sigma \text{ in (4).}$$

THEOREM 3. (*a*) *The set Δ^* is determined by the inequalities*

(12) $$\begin{cases} 0 \leqslant x_0 \leqslant \frac{1}{2}, \quad 0 \leqslant x_1 \leqslant \frac{1}{2}, \quad \max(|x_2|, |x_3|) \leqslant \min(x_0, x_1), \\ \qquad x_0+x_1+|x_2|+|x_3| \leqslant 1. \end{cases}$$

(*b*) *To every x in Δ there is a unit ϵ of K such that $\epsilon^{-1}x$ lies in Δ^*.*

Proof. The inequalities (11) are linear in the coordinates of x; it is easily shown that those belonging to the 8 SV

$$\Sigma = \frac{1\mp i_2}{2}, \quad \frac{1\mp i_3}{2}, \quad \frac{i_1\mp i_2}{2}, \quad \frac{i_1\mp i_3}{2}$$

imply the other 16 inequalities. On writing down these 8 inequalities and using the definition (2) of Δ, the formulae (12) follow at once. If x lies in Δ, then let $\Sigma = \epsilon\Sigma_0$ be the SV nearest to it; then Σ_0 is the SV nearest to $\epsilon^{-1}x$, and therefore $\epsilon^{-1}x$ lies in Δ^*.

We define a further set D, which is to consist of all points x in K such that

(13) $$|x-\Sigma_0| \leqslant |x-\Sigma| \quad \text{for all SV } \Sigma.$$

THEOREM 4. (*a*) *D is determined by the inequality*

(14) $$\max(|x_0-\tfrac{1}{2}|, |x_1-\tfrac{1}{2}|)+\max(|x_2|, |x_3|) \leqslant \tfrac{1}{2}.$$

(*b*) *The set Δ^* consists of all those points of D which lie in Δ.*

Proof. Put

$$x-\Sigma_0=\xi=\xi_0+\xi_1 i_1+\xi_2 i_2+\xi_3 i_3,$$

$$\Sigma-\Sigma_0=a=\frac{a_0+a_1 i_1+a_2 i_2+a_3 i_3}{2}.$$

Then either a is an integral quaternion, or one of the numbers of each pair of coefficients (a_0, a_1) and (a_2, a_3) is an *even* and one an *odd* integer, so that a is a SV.

The point ξ satisfies the inequalities

(15) $$|\xi|\leqslant|\xi-a| \quad \text{for all quaternions } a;$$

or written explicitly,

(16) $$a_0\xi_0+a_1\xi_1+a_2\xi_2+a_3\xi_3\leqslant\frac{a_0^2+a_1^2+a_2^2+a_3^2}{4}.$$

The inequalities corresponding to integral a imply that ξ lies in Δ; hence, by Theorem 2,

$$|\xi_0|\leqslant\tfrac{1}{2},\quad|\xi_1|\leqslant\tfrac{1}{2},\quad|\xi_2|\leqslant\tfrac{1}{2},\quad|\xi_3|\leqslant\tfrac{1}{2},\quad|\xi_0|+|\xi_1|+|\xi_2|+|\xi_3|\leqslant 1.$$

But these formulae are a consequence of the inequalities

(17) $$|\xi_0|+|\xi_2|\leqslant\tfrac{1}{2},\quad|\xi_0|+|\xi_3|\leqslant\tfrac{1}{2},\quad|\xi_1|+|\xi_2|\leqslant\tfrac{1}{2},\quad|\xi_1|+|\xi_3|\leqslant\tfrac{1}{2},$$

which are the inequalities (15) or (16) if a is one of the 16 SV

$$\frac{\mp1\mp i_2}{2},\quad\frac{\mp1\mp i_3}{2},\quad\frac{\mp i_1\mp i_2}{2},\quad\frac{\mp i_1\mp i_3}{2}.$$

Now (17) implies (15) for all quaternions a; in order to prove this, it suffices to assume that a is a SV. Let a_κ and a_λ be the two *even*, and a_μ and a_ν the two odd coefficients of a; we may suppose without loss of generality that

$$a_\kappa\geqslant a_\lambda\geqslant 0,\quad a_\mu\geqslant a_\nu\geqslant 0.$$

Then $|\xi_\kappa|\leqslant\frac{1}{2}$, $|\xi_\lambda|\leqslant\frac{1}{2}$; hence

$$|a_\kappa\xi_\kappa+a_\lambda\xi_\lambda|\leqslant\frac{a_\kappa+a_\lambda}{2}\leqslant\frac{a_\kappa^2+a_\lambda^2}{4},$$

since a_κ and a_λ are even and therefore

$$a_\kappa^2\geqslant 2a_\kappa,\quad a_\lambda^2\geqslant 2a_\lambda.$$

Further $|\xi_\mu| \leqslant \frac{1}{2}$, $|\xi_\mu|+|\xi_\nu| \leqslant \frac{1}{2}$; hence

$$|a_\mu \xi_\mu + a_\nu \xi_\nu| \leqslant (a_\mu - a_\nu)|\xi_\mu| + a_\nu(|\xi_\mu|+|\xi_\nu|)$$
$$\leqslant \tfrac{1}{2}(a_\mu - a_\nu) + \tfrac{1}{2}a_\nu = \frac{a_\mu}{2} \leqslant \frac{a_\mu^2+1}{4} \leqslant \frac{a_\mu^2+a_\nu^2}{4},$$

since a_μ and a_ν are odd and therefore

$$2a_\mu \leqslant a_\mu^2+1, \quad a_\nu^2 \geqslant 1.$$

If we now replace ξ by $x = \xi + \Sigma_0$, then (17) changes into (14). The second assertion of the theorem is evident from (2), (12) and (14).

6. *The sets* $\delta(\tau)$, $\delta(\tau|\Sigma)$, *and* $\delta^*(\tau)$.

Let τ be a number in the interval

$$(18) \qquad 0 \leqslant \tau \leqslant \tfrac{1}{4},$$

and $\delta(\tau)$ the set of all points x for which

$$(19) \qquad |x_0 - \tfrac{1}{2}| + |x_1 - \tfrac{1}{2}| \leqslant \tau, \quad |x_2| + |x_3| \leqslant \tau.$$

For $\tau = 0$, $\delta(\tau)$ reduces to the single point Σ_0; for $\tau > 0$, Σ_0 is the centre of the set.

The transformations (8) do not change Σ_0 and are easily seen to leave $\delta(\tau)$ invariant. Hence, if the transformation (7) changes Σ_0 into the new SV Σ, then if x lies in $\delta(\tau)$, the point $\epsilon x + g$ lies in a new set $\delta(\tau|\Sigma)$ of centre Σ. This set depends only on τ and Σ but not on the special transformation (7) which changes Σ_0 into Σ. If, in

$$\Sigma = \frac{a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3}{2},$$

a_κ and a_λ are the two even, and a_μ and a_ν the two odd coefficients, then $\delta(\tau|\Sigma)$ is determined by the inequalities

$$\left|x_\kappa - \frac{a_\kappa}{2}\right| + \left|x_\lambda - \frac{a_\lambda}{2}\right| \leqslant \tau, \quad \left|x_\mu - \frac{a_\mu}{2}\right| + \left|x_\nu - \frac{a_\nu}{2}\right| \leqslant \tau.$$

Since we do not use this result, I omit the simple proof.

For $\tau > 0$, the set $\delta(\tau)$ is bounded by 8 hyperplanes. There are 16 sets of 4 of these hyperplanes which intersect in a point; these 16 points of intersection are easily found to be

$$(20) \quad x = \Sigma_0 - 2\tau s, \quad \text{where} \quad s = \frac{\mp 1 \mp i_2}{2}, \quad \frac{\mp 1 \mp i_3}{2}, \quad \frac{\mp i_1 \mp i_2}{2}, \quad \frac{\mp i_1 \mp i_3}{2}.$$

They form the four-dimensional analogue to the vertices of a cube in ordinary space.

Now let $\delta^*(\tau)$ be that part of $\delta(\tau)$ which lies in Δ; this new set is determined by the inequalities

(21) $$x_0 \leqslant \tfrac{1}{2},\quad x_1 \leqslant \tfrac{1}{2},\quad x_0+x_1 \geqslant 1-\tau,\quad x_0+x_1+|x_2|+|x_3| \leqslant 1.$$

From these inequalities, we get

$$x_0 \geqslant (1-\tau)-\tfrac{1}{2} = \tfrac{1}{2}-\tau \geqslant \tau \geqslant 0,\quad x_1 \geqslant \tau \geqslant 0,$$

$$|x_0-\tfrac{1}{2}|+|x_1-\tfrac{1}{2}| = 1-x_0-x_1 \leqslant \tau,$$

$$|x_2|+|x_3| \leqslant 1-(x_0+x_1) \leqslant 1-(1-\tau) = \tau,$$

$$|x_2| \leqslant \tau \leqslant \tfrac{1}{2},\quad |x_3| \leqslant \tau \leqslant \tfrac{1}{2},$$

so that x indeed belongs both to Δ and to $\delta(\tau)$. But the stronger result holds that x is also an element of Δ^*; for from the last inequalities

$$\max(|x_2|, |x_3|) \leqslant \tau \leqslant \min(x_0, x_1).$$

Of the vertices (20) of $\delta(\tau)$, only 8 belong to $\delta^*(\tau)$, namely

(22) $$x = \Sigma_0 - 2\tau s,\quad \text{where}\quad s = \frac{1\mp i_2}{2},\ \frac{1\mp i_3}{2},\ \frac{i_1\mp i_2}{2},\ \frac{i_1\mp i_3}{2}.$$

They all lie on the hyperplane

(23) $$x_0+x_1 = 1-\tau$$

and have the same distance

(24) $$\rho(\tau) = (2\tau^2-\tau+\tfrac{1}{2})^{\frac{1}{2}}$$

from the origin.

The points (22) are the only points of intersection of (23) with sets of 3 of the hyperplanes

$$x_0 = 0 \text{ or } \tfrac{1}{2},\quad x_1 = 0 \text{ or } \tfrac{1}{2},\quad x_0+x_1\mp x_2\mp x_3 = 1.$$

Hence, by Theorem 1, all points of the set

$$0 \leqslant x_0 \leqslant \tfrac{1}{2},\quad 0 \leqslant x_1 \leqslant \tfrac{1}{2},\quad x_0+x_1+|x_2|+|x_3| \leqslant 1,\quad x_0+x_1 \leqslant 1-\tau$$

satisfy the inequality

$$|x| \leqslant \rho(\tau).$$

Therefore this is also true for all points of Δ^* for which

$$x_0+x_1 \leqslant 1-\tau;$$

and conversely, if x *lies in* Δ^* *and satisfies the inequality*

$$|x| \geqslant \rho(\tau),$$

then x *belongs to the set* $\delta^*(\tau)$, *hence also to* $\delta(\tau)$.

From this result, we deduce

THEOREM 5. *Suppose that the point* x *has a distance not less than* $\rho(\tau)$ *from every lattice point. Then there is a* SV Σ, *such that* x *belongs to* $\delta(\tau \mid \Sigma)$.

Proof. Let Σ be the SV nearest to x. By a suitable transformation (7), we can change Σ_0 into Σ, and a certain point x_0 in Δ^* into x. The assertion has already been proved for the point x_0; it therefore is true also for x, since the transformation (7) preserves all distances, and only interchanges the domains $\delta(\tau \mid \Sigma)$.

It is useful to remark that if the equation

$$\rho(\tau)^2 = 2\tau^2-\tau+\tfrac{1}{2}$$

is solved for τ, then, since $0 \leqslant \tau \leqslant \frac{1}{4}$,

$$\tau(\rho) = \frac{1-\sqrt{(8\rho^2-3)}}{4} \quad \text{for} \quad \sqrt{\tfrac{3}{8}} \leqslant \rho \leqslant \sqrt{\tfrac{1}{2}}. \tag{25}$$

Outside this interval for ρ, the value of τ has no meaning for our problem.

7. *An upper bound for the points in* $\delta(\tau \mid \Sigma)$.

Let Σ be one of the 24 SV of Δ, and x a point in $\delta(\tau \mid \Sigma)$. If x lies in Δ, then we know that $|x| \leqslant \sqrt{\frac{1}{2}}$. A more general result is given by

THEOREM 6. *If* Σ *is a singular vertex of* Δ, *then*

$$|x| \leqslant (2\tau^2+\tau+\tfrac{1}{2})^{\frac{1}{2}} \tag{26}$$

for all points of $\delta(\tau \mid \Sigma)$.

Proof. The 24 sets $\delta(\tau \mid \Sigma)$ are interchanged by the transformations (6), and these preserve all distances. Hence, without loss of generality, we may suppose x to be a point in $\delta(\tau)$. Then, by Theorem 1, the maximum of $|x|$ is attained in one of the 16 points (20). The 8 points (22) belong to Δ and therefore have distances not greater than $\sqrt{\frac{1}{2}}$ from the origin. The distances from the origin of the remaining 8 points

$$x = \Sigma_0 - 2\tau s, \quad \text{where} \quad s = \frac{-1 \mp i_2}{2}, \quad \frac{-1 \mp i_3}{2}, \quad \frac{-i_1 \mp i_2}{2}, \quad \frac{-i_1 \mp i_3}{2},$$

are all equal to

$$(2\tau^2 + \tau + \tfrac{1}{2})^{\frac{1}{2}},$$

which is not less than $\sqrt{\frac{1}{2}}$, and is the minimum required.

THEOREM 7. *If x lies in $\delta(\tau)$, then the 8 points*

$$x^* = x - g, \quad \textit{where} \quad g = 0,\ 1,\ i_1,\ 1+i_1,\ \frac{1+i_1 \mp i_2 \mp i_3}{2},$$

satisfy the inequality

$$|x^*| \leqslant (2\tau^2 + \tau + \tfrac{1}{2})^{\frac{1}{2}}.$$

Proof. This is a special case of the preceding theorem, since the points x^* belong to the sets $\delta(\tau \mid \Sigma)$, where $\Sigma = \Sigma_0 - g$ are the 8 SV of Δ congruent to Σ_0.

Evidently

$$2\tau^2 + \tau + \tfrac{1}{2} = \frac{1 + 2\rho^2 - \sqrt{(8\rho^2 - 3)}}{2}, \tag{27}$$

where ρ is connected with τ by the formulae (24) and (25).

8. *The set $d(\Lambda)$.*

Let Λ be a number in the interval

$$0 \leqslant \Lambda \leqslant 1, \tag{28}$$

and $d(\Lambda)$ the set of all quaternions satisfying

$$0 \leqslant x_3 \leqslant x_2 \leqslant x_1 \leqslant x_0 \leqslant \tfrac{1}{2}, \quad x_0 + x_1 + x_2 + x_3 \leqslant 1, \quad x_0 + x_1 \leqslant \Lambda. \tag{29}$$

THEOREM 8. *For all points of $d(\Lambda)$,*

$$|x|^2 \leqslant \frac{\Lambda}{2}. \tag{30}$$

Proof. We begin with the remark that if $x+y=s$ and $x \geqslant s/2$, where s is a constant, then

$$x^2+y^2=\frac{s^2+(2x-s)^2}{2}$$

is a steadily increasing function of x.

Let x be a point of $\mathrm{d}(\Lambda)$, in which the maximum of $|x|$ is attained. Then the following three statements hold:

(31) $$x_0+x_1=\Lambda.$$

(32) $$\textit{Either} \quad x_1=x_2=x_3 \quad \textit{or} \quad x_0+x_1+x_2+x_3=1.$$

(33) $$\textit{If} \quad x_0+x_1+x_2+x_3=1, \quad \textit{then either} \quad x_0=\tfrac{1}{2}, \quad \textit{or} \quad x_1=x_2.$$

For, if $x_0+x_1<\Lambda$ and also $x_0+x_1+x_2+x_3<1$, then $|x|$ is increased when x_0, x_1, x_2, x_3 are replaced by

$$x_0+\epsilon,\ x_1,\ x_2,\ x_3 \quad \text{for} \quad x_0<\tfrac{1}{2},$$

and by $$x_0,\ x_1+\epsilon,\ x_2,\ x_3 \quad \text{for} \quad x_0=\tfrac{1}{2},$$

where $\epsilon>0$ is a sufficiently small number; for in the second case $x_1<\Lambda-\frac{1}{2}\leqslant x_0$. If, however, $x_0+x_1<\Lambda$ and $x_0+x_1+x_2+x_3=1$, so that $x_2+x_3>1-\Lambda\geqslant 0$ and therefore x_2 and x_3 are not both zero, then let ν be the larger index 2 or 3 for which $x_\nu \neq 0$. Now $|x|$ again increases if x_0, x_1, x_2, x_3 are replaced by new numbers x_0', x_1', x_2', x_3', where

$$x_0'=x_0+\epsilon, \quad x_\nu'=x_\nu-\epsilon \quad \text{for} \quad x_0<\tfrac{1}{2},$$

and $$x_1'=x_1+\epsilon, \quad x_\nu'=x_\nu-\epsilon \quad \text{for} \quad x_0=\tfrac{1}{2},$$

while the two other x's are left unchanged.

Also, if (32) is false and therefore $x_1>0$, then we may add ϵ to x_2 or x_3 and so increase $|x|$.

Finally, if (33) does not hold, then we can increase $|x|$ by replacing x_0 by $x_0+\epsilon$ and x_1 by $x_1-\epsilon$.

We now apply Theorem 1 to the set $\mathrm{d}(\Lambda)$. According to this theorem, the maximum of $|x|$ can be assumed only on those points of the set in which at least four of the boundary hyperplanes

$$L_1=x_3=0, \quad L_2=x_2-x_3=0, \quad L_3=x_1-x_2=0, \quad L_4=x_0-x_1=0,$$

$$L_5=x_0-\tfrac{1}{2}=0, \quad L_6=x_0+x_1+x_2+x_3-1=0, \quad L_7=x_0+x_1-\Lambda=0$$

intersect. The results (31)–(33) exclude all possibilities except the following ten cases:

$L_1 = L_2 = L_3 = L_7 = 0,\quad x = \Lambda,\quad |x|^2 = \Lambda^2,\qquad$ valid for $0 \leqslant \Lambda \leqslant \frac{1}{2}$,

$L_2 = L_3 = L_4 = L_7 = 0,\quad x = \frac{1}{2}\Lambda(1+i_1+i_2+i_3),\quad |x|^2 = \Lambda^2,$

valid for $0 \leqslant \Lambda \leqslant \frac{1}{2}$,

$L_2 = L_3 = L_5 = L_7 = 0,\quad x = \frac{1}{2}+(\Lambda-\frac{1}{2})(i_1+i_2+i_3),\quad |x|^2 = 3\Lambda^2-3\Lambda+1,$

valid for $\frac{1}{2} \leqslant \Lambda \leqslant \frac{2}{3}$,

$L_1 = L_3 = L_6 = L_7 = 0,\quad x = (2\Lambda-1)+(1-\Lambda)(i_1+i_2),\ |x|^2 = 6\Lambda^2-8\Lambda+3,$

valid for $\frac{2}{3} \leqslant \Lambda \leqslant \frac{3}{4}$,

$L_1 = L_5 = L_6 = L_7 = 0,\quad x = \frac{1}{2}+(\Lambda-\frac{1}{2})i_1+(1-\Lambda)i_2,\ |x|^2 = 2\Lambda^2-3\Lambda+\frac{3}{2},$

valid for $\frac{3}{4} \leqslant \Lambda \leqslant 1$,

$$L_2 = L_3 = L_6 = L_7 = 0,\quad x = \frac{(3\Lambda-1)+(1-\Lambda)(i_1+i_2+i_3)}{2},$$

$|x|^2 = 3\Lambda^2-3\Lambda+1,\quad$ valid for $\frac{1}{2} \leqslant \Lambda \leqslant$

$$L_2 = L_5 = L_6 = L_7 = 0,\quad x = \frac{1+(2\Lambda-1)i_1+(1-\Lambda)(i_2+i_3)}{2},$$

$|x|^2 = \frac{3}{2}\Lambda^2-2\Lambda+1,\quad$ valid for $\frac{2}{3} \leqslant \Lambda \leqslant 1$

$$L_3 = L_4 = L_6 = L_7 = 0,\quad x = \frac{\Lambda(1+i_1+i_2)+(2-3\Lambda)i_3}{2},$$

$|x|^2 = 3\Lambda^2-3\Lambda+1,\quad$ valid for $\frac{1}{2} \leqslant \Lambda \leqslant \frac{2}{3}$,

$$L_3 = L_5 = L_6 = L_7 = 0,\quad x = \frac{1+(2\Lambda-1)(i_1+i_2)+(3-4\Lambda)i_3}{2},$$

$|x|^2 = 6\Lambda^2-8\Lambda+3,\quad$ valid for $\frac{2}{3} \leqslant \Lambda \leqslant \frac{3}{4}$,

$L_4 = L_5 = L_6 = L_7 = 0,\quad x = \Sigma_0,\quad |x|^2 = \frac{1}{2},\qquad$ valid for $\Lambda = 1$.

The interval conditions for Λ must be satisfied if the points are to lie in the set $d(\Lambda)$. A trivial discussion shows that all 10 points satisfy the inequality (30), which is therefore true for all points of $d(\Lambda)$.

9. *An extremum problem.*

Let $\mathfrak{r}$ be a number in the interval

$$0 \leqslant \mathfrak{r} \leqslant \sqrt{\tfrac{1}{2}}, \tag{34}$$

and x a point in Δ of absolute value $\mathfrak{r}$. Put

$$m(x) = \min_{g,\, h} (|gx+\Sigma_0-h|), \tag{35}$$

where the minimum is extended over the 6 units

$$g = 1,\ i_1,\ \frac{1+i_1\mp i_2\mp i_3}{2}$$

and over all integral quaternions h. Obviously, $m(x)$ is a bounded continuous function of x; it therefore assumes a maximum value

$$M(\mathfrak{r}) = \max_{\substack{|x|=\mathfrak{r} \\ x \text{ in } \Delta}} m(x). \tag{36}$$

THEOREM 9. *The function* $M(\mathfrak{r})$ *satisfies the inequality*

$$M(\mathfrak{r}) \leqslant \sqrt{(\tfrac{1}{2}-\mathfrak{r}^2)}. \tag{37}$$

Proof. Since the g's are units,

$$m(x) = \min_{g,\, h} \left(|x-g^{-1}(h-\Sigma_0)|\right).$$

Now if g and h take all admissible values, then $g^{-1}(h-\Sigma_0)$, as is easily verified, runs over all SV Σ. Hence

$$m(x) = \min_{\Sigma} (|x-\Sigma|),$$

where the minimum sign extends over all SV. Let ϵ be one of the 24 units of K. Then with Σ also $\epsilon^{\mp 1}\Sigma$ runs over all SV; therefore

$$m(\epsilon x) = \min_{\Sigma} (|\epsilon x-\Sigma|) = \min_{\Sigma} (|x-\epsilon^{-1}\Sigma|) = \min_{\Sigma} (|x-\Sigma|) = m(x).$$

Hence, by Theorem 3, Part (*b*), we may restrict x to the set Δ^*, so that

$$M(\mathfrak{r}) = \max_{\substack{|x|=\mathfrak{r} \\ x \text{ in } \Delta^*}} \left\{\min_{\Sigma} (|x-\Sigma|)\right\}.$$

But then, by Theorem 4, Part (*b*), Σ_0 is the SV nearest to x, so that

$$M(\mathfrak{r}) = \max_{\substack{|x|=\mathfrak{r} \\ x \text{ in } \Delta^*}} (|x-\Sigma_0|).$$

Finally, $|x-\Sigma_0|$ remains unchanged if x_0 and x_1, or x_2 and x_3 are interchanged, or if we change the signs of x_2 or x_3 or both. Therefore

$$M(\mathfrak{r}) = \max_{T} (|x-\Sigma_0|),$$

the maximum being taken for the set T of all points x satisfying

$$|x|=\mathfrak{r},\quad 0\leqslant x_3\leqslant x_2\leqslant x_1\leqslant x_0\leqslant \tfrac{1}{2},\quad x_0+x_1+x_2+x_3\leqslant 1.$$

From this formula we get

$$M(\mathfrak{r})^2=\max_T\,(\mathfrak{r}^2-x_0-x_1+\tfrac{1}{2})=\mathfrak{r}^2+\tfrac{1}{2}-\min_T\,(x_0+x_1).$$

The statement follows therefore at once, since, by Theorem 8,

$$x_0+x_1\geqslant 2\mathfrak{r}^2$$

for all points x such that

$$|x|\geqslant \mathfrak{r},\quad 0\leqslant x_3\leqslant x_2\leqslant x_1\leqslant x_0\leqslant \tfrac{1}{2},\quad x_0+x_1+x_2+x_3\leqslant 1.$$

This completes the proof.

It is possible to determine $M(\mathfrak{r})$ explicitly, namely:

$$M(\mathfrak{r})^2=\begin{cases}\mathfrak{r}^2+\frac{1}{2}-\mathfrak{r} & \text{for}\quad 0\leqslant\mathfrak{r}\leqslant\frac{1}{2},\\[4pt] \mathfrak{r}^2+\frac{1}{2}-\dfrac{3+\sqrt{(12\mathfrak{r}^2-3)}}{6} & \text{for}\quad \frac{1}{2}\leqslant\mathfrak{r}\leqslant\sqrt{\frac{1}{3}},\\[4pt] \mathfrak{r}^2+\frac{1}{2}-\dfrac{4+\sqrt{(6\mathfrak{r}^2-2)}}{6} & \text{for}\quad \sqrt{\frac{1}{3}}\leqslant\mathfrak{r}\leqslant\sqrt{\frac{3}{8}},\\[4pt] \mathfrak{r}^2+\frac{1}{2}-\dfrac{3+\sqrt{(8\mathfrak{r}^2-3)}}{4} & \text{for}\quad \sqrt{\frac{3}{8}}\leqslant\mathfrak{r}\leqslant\sqrt{\frac{1}{2}}.\end{cases}$$

I omit the proof, since it requires no new ideas and since we do not make use of the result.

10. *A property of special Hermitian forms.*

We can now prove the following result:

THEOREM 10. *Let a be a number in the interval*

(38) $$1\leqslant a\leqslant\sqrt{2},$$

let λ be the number

(39) $$\lambda=\sqrt{\left(1-\frac{1}{a^2}\right)},$$

and let ξ be a quaternion in Δ of absolute value $|\xi|=\lambda$, and $H(x,y\,|\,\xi)$ the Hermitian form

(40) $$H(x,y)=H(x,y\,|\,\xi)=aN(x-y\xi)+\frac{1}{a}N(y).$$

Then to any two quaternions x_0, y_0 there are two other quaternions x_1 and y_1, such that

$$x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad H(x_1, y_1) \leqslant 1; \tag{41}$$

here the sign "$\leqslant$" may be replaced by "$<$", except in the case

$$x_0 = \Sigma_1, \quad y_0 = \Sigma_2, \quad a = 1, \quad \xi = 0, \quad H(x, y) = N(x) + N(y),$$

where Σ_1 and Σ_2 are two SV.

Proof. For every unit ϵ in K, the congruence $u \equiv v$ implies

$$\epsilon u \equiv \epsilon v \quad \text{and} \quad u\epsilon \equiv v\epsilon.$$

It has been proved in Theorems 2 and 3 that to every quaternion x there is a congruent one in Δ, and to every x in Δ there exists a unit ϵ such that ϵx lies in Δ^*. It is further true that to this x in Δ there is also a unit ϵ' such that $x\epsilon'$ belongs to Δ^*. For with x also the conjugate quaternion $\bar{x}$ lies in Δ. Therefore there is a unit $\bar{\epsilon}^*$ such that $\bar{\epsilon}^* \bar{x}$ belongs to Δ^*; it is easily verified that also

$$(\overline{\bar{\epsilon}^* \bar{x}})\, i_1 = x\epsilon^* i_1 = x\epsilon', \quad \text{where} \quad \epsilon' = \epsilon^* i_1,$$

is an element of Δ^*, and here ϵ' is a unit.

$H(x, y)$ satisfies the identities

$$H(\epsilon x, \epsilon y \mid \xi) = H(x, y \mid \xi),$$

$$H(x\epsilon, y \mid \xi\epsilon) = H(x, y \mid \xi).$$

From this and the preceding we may assume without loss of generality that

$$y_0 \quad \text{and} \quad z_0 = x_0 - y_0 \xi \quad \text{lie in} \quad \Delta^*.$$

Then by Theorem 2,

$$|y_0|^2 \leqslant \tfrac{1}{2} \quad \text{and} \quad |z_0|^2 \leqslant \tfrac{1}{2}.$$

Hence (41) is true in the stronger form with the sign "$<$" instead of "$\leqslant$" for $x_1 = x_0$ and $y_1 = y_0$, if

$$\text{either} \quad |y_0|^2 < \frac{2a - a^2}{2} \quad \text{or} \quad |z_0|^2 < \frac{2a-1}{2a^2}.$$

We exclude these two trivial cases and suppose that

$$|y_0|^2 \geqslant \frac{2a - a^2}{2}, \quad |z_0|^2 \geqslant \frac{2a-1}{2a^2}. \tag{42}$$

By § 6, the first inequality implies that y_0 belongs to $\delta(\tau_1)$, where τ_1 is determined by

$$\rho_1^2 = \frac{2a-a^2}{2}; \quad \text{hence} \quad \tau_1 = \frac{1-\sqrt{(8a-4a^2-3)}}{4}.$$

Therefore by Theorem 7, if g is one of the 6 quaternions

$$g = 1, \; i_1, \; \frac{1+i_1\mp i_2\mp i_3}{2},$$

then all 6 points

$$y_1^* = y_0 - g$$

satisfy the inequality

$$|y_1^*|^2 \leqslant \frac{1+2a-a^2-\sqrt{(8a-4a^2-3)}}{2}.$$

By the change of y_0 into y_1^*, z_0 is transformed into the 6 numbers

$$z_1^* = z_0 + g\xi = (z_0-\Sigma_0)+(\Sigma_0+g\xi).$$

Let g and an integral quaternion h be chosen such that

$$|g\xi+\Sigma_0-h|$$

is a minimum and therefore equal to $m(\xi)$ as defined in (35); then put

$$x_1 = x_0-h, \quad y_1 = y_0-g, \quad z_1 = x_1-y_1\xi = z_0+g\xi-h.$$

Hence, by the inequality for y_1^*,

$$\frac{1}{a}N(y_1) = \frac{1+2a-a^2-\sqrt{(8a-4a^2-3)}}{2a} = A, \text{ say.} \tag{43}$$

As was proved in Theorem 9,

$$m(\xi) = |g\xi+\Sigma_0-h| \leqslant M(\lambda) \leqslant \sqrt{(\tfrac{1}{2}-\lambda^2)} = \sqrt{\left(\frac{1}{a^2}-\frac{1}{2}\right)}.$$

By the second inequality (42), z_0 belongs to the set $\delta(\tau_2)$, where τ_2 is defined by

$$\rho_2^2 = \frac{2a-1}{2a^2}; \quad \text{hence} \quad \tau_2 = \frac{a-\sqrt{(8a-4-3a^2)}}{4a}.$$

Now the 16 points (20) in which sets of 4 of the boundary hyperplanes of $\delta(\tau)$ intersect, all have the same distance $\tau\sqrt{2}$ from the centre Σ_0 of this set. Hence, by Theorem 1, this is the maximum distance of a point of $\delta(\tau)$ from Σ_0. Therefore, in particular,

$$|z_0-\Sigma_0| \leqslant \tau_2\sqrt{2} = \frac{a-\sqrt{(8a-4-3a^2)}}{4a}\sqrt{2}.$$

Hence

$$|z_1| \leqslant |g\xi+\Sigma_0-h|+|z_0-\Sigma_0| \leqslant \sqrt{\left(\frac{1}{a^2}-\frac{1}{2}\right)}+\frac{a-\sqrt{(8a-4-3a^2)}}{4a}\sqrt{2},$$

so that

$$aN(x_1-y_1\xi) \leqslant a\left(\frac{1}{2a}\sqrt{(4-2a^2)}+\frac{a-\sqrt{(8a-4-3a^2)}}{4a}\sqrt{2}\right)^2 \tag{44}$$

$$= B, \quad \text{say.}$$

With the values (43) and (44) for A and B, we have proved that

$$H(x_1, y_1) \leqslant A+B. \tag{45}$$

If $a = 1$ and therefore $\xi = 0$, then $A = B = \frac{1}{2}$, so that (41) is satisfied. By Theorem 2, the sign of equality holds only if $N(x_1) = N(y_1) = \frac{1}{2}$, *i.e.* if both x_0 and y_0 are SV.

Now let $1 < a \leqslant \sqrt{2}$. Then we show that A and B are both less than $\frac{1}{2}$. In the case of A, we have

$$A = \tfrac{1}{2}+\frac{1+a-a^2-\sqrt{(8a-4a^2-3)}}{2a}$$

$$= \tfrac{1}{2}+\frac{(a-1)(a^3-a^2+2a-4)}{2a\{1+a-a^2+\sqrt{(8a-4a^2-3)}\}}.$$

Here

$$a^3-a^2+2a-4 = -2+3(a-1)+2(a-1)^2+(a-1)^3$$

is an increasing function of a; therefore its maximum is assumed for $a = \sqrt{2}$, and is equal to

$$4\sqrt{2}-6 < 0.$$

This proves that $A < \frac{1}{2}$.

The inequality $B < \frac{1}{2}$ is equivalent to

$$\sqrt{(2B)} = \beta = \frac{1}{\sqrt{a}}\sqrt{(2-a^2)}+\frac{a-\sqrt{(8a-4-3a^2)}}{2\sqrt{a}} < 1.$$

Now

$$\sqrt{(2-a^2)}-1 = \frac{1-a^2}{\sqrt{(2-a^2)}+1} \leqslant 1-a^2 = -(a+1)(a-1) \leqslant -2(a-1).$$

Hence
$$\frac{1}{\sqrt{a}}\sqrt{(2-a^2)} < 1-\frac{2(a-1)}{\sqrt{a}}.$$

Also

$$\frac{a-\sqrt{(8a-4-3a^2)}}{2\sqrt{a}} = \frac{4(a-1)^2}{2\sqrt{a}\{a+\sqrt{(8a-4-3a^2)}\}} \leqslant \frac{2(a-1)^2}{a\sqrt{a}}.$$

Therefore finally

$$\beta < \left(1 - \frac{2(a-1)}{\sqrt{a}}\right) + \frac{2(a-1)^2}{a\sqrt{a}} = 1 - \frac{2(a-1)}{a\sqrt{a}} < 1,$$

which concludes the proof.

COROLLARY TO THEOREM 10. *The assertion* (41) *remains true, if* $H(x, y)$ *is replaced by the Hermitian form*

$$\bar{H}(x, y \mid \xi) = aN(x - \xi y) + \frac{1}{a} N(y).$$

For evidently $$\bar{H}(x, y \mid \xi) = H(\bar{x}, \bar{y} \mid \bar{\xi}).$$

CHAPTER II. HERMITIAN FORMS.

11. *Linear transformations.*

The quaternion field K is associative; therefore the composition of linear transformations

$$\begin{aligned} x &= a_{11} x' + a_{12} y', \\ y &= a_{21} x' + a_{22} y', \end{aligned} \quad \text{or in matrix form } \begin{pmatrix} x \\ y \end{pmatrix} = \Omega \begin{pmatrix} x' \\ y' \end{pmatrix}, \text{ where } \Omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

satisfies the ordinary rules. The determinant of commutative algebra, however, loses its meaning, and has to be replaced by the expression†

$$(46) \quad d(\Omega) = a_{11}\bar{a}_{11}a_{22}\bar{a}_{22} + a_{12}\bar{a}_{12}a_{21}\bar{a}_{21} - a_{11}\bar{a}_{21}a_{22}\bar{a}_{12} - a_{12}\bar{a}_{22}a_{21}\bar{a}_{11},$$

which we call the *determinant of* Ω. As we shall see, $d(\Omega)$ *is either positive or zero*; in the first case Ω is called *regular*. For example, if

$$(47) \qquad S(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(\tau) = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix},$$

where λ and $\tau \neq 0$ are quaternions, then

$$d\big(S(\lambda)\big) = 1, \quad d(T) = 1, \quad d\big(U(\tau)\big) = \tau\bar{\tau},$$

so that all three matrices are regular. These matrices have inverses, namely

$$(48) \qquad S(\lambda)^{-1} = S(-\lambda), \quad T^{-1} = T, \quad U(\tau)^{-1} = U(\tau^{-1}).$$

† For the general theory of generalized determinants in the quaternion field see the beautiful paper of E. Study, *Acta Mathematica*, 42 (1920), 1–61.

Every regular matrix can be written as a finite product of matrices S, T, and U. For, since $d(\Omega) \neq 0$, we have either $a_{12} \neq 0$, or $a_{12} = 0$ and $a_{11} \neq 0$, $a_{22} \neq 0$. In the first case

$$\Omega = S(\lambda_1)\, T U(\tau_1)\, T U(\tau_2)\, T S(\lambda_2),$$

where

$$\tau_1 = a_{21}, \quad \tau_2 = a_{12} - a_{11} a_{21}^{-1} a_{22}, \quad \lambda_1 = a_{11} a_{21}^{-1}, \quad \lambda_2 = a_{21}^{-1} a_{22};$$

and in the second case

$$\Omega = U(a_{11})\, S(a_{11}^{-1} a_{12} a_{22}^{-1})\, T U(a_{22})\, T.$$

From this factorization of Ω and from (48), we see that every regular matrix has an inverse, obtained by taking the inverses of the factors in the reversed order.

THEOREM 11. *If Ω_1 and Ω_2 are regular matrices, and $\Omega_3 = \Omega_1 \Omega_2$, then*

$$d(\Omega_3) = d(\Omega_1)\, d(\Omega_2). \tag{49}$$

Proof. It suffices to consider the cases in which

$$\Omega_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is an arbitrary regular matrix, and Ω_2 is one of the three matrices $S(\lambda)$, T or $U(\tau)$. The assertion is obvious in the two last cases. In the first case,

$$\Omega_3 = \Omega_1 S(\lambda) = \begin{pmatrix} a_{11}, & a_{11}\lambda + a_{12} \\ a_{21}, & a_{21}\lambda + a_{22} \end{pmatrix},$$

hence

$$d(\Omega_3) = a + b + c + d,$$

where $d = d(\Omega_1)$ and

$$a = a_{11} \bar{a}_{11} a_{21} \lambda\bar{\lambda} \bar{a}_{21} + a_{11} \lambda\bar{\lambda} \bar{a}_{11} a_{21} \bar{a}_{21} - a_{11} \bar{a}_{21} a_{21} \lambda\bar{\lambda}\, a_{11} - a_{11} \lambda\bar{\lambda} \bar{a}_{21} a_{21} \bar{a}_{11},$$

$$b = a_{11} \bar{a}_{11} a_{21} \lambda \bar{a}_{22} + a_{11} \lambda \bar{a}_{12} a_{21} \bar{a}_{21} - a_{11} \bar{a}_{21} a_{21} \lambda \bar{a}_{12} - a_{11} \lambda \bar{a}_{22} a_{21} \bar{a}_{11},$$

$$c = a_{11} \bar{a}_{11} a_{22} \bar{\lambda} \bar{a}_{21} + a_{12} \bar{\lambda} \bar{a}_{11} a_{21} \bar{a}_{21} - a_{11} \bar{a}_{21} a_{22} \bar{\lambda}\, a_{11} - a_{12} \bar{\lambda} \bar{a}_{21} a_{21} \bar{a}_{11}.$$

Since real factors are commutative, obviously $a = 0$. Further,

$$b + c = N(a_{11})\, S(a_{21} \lambda \bar{a}_{22}) + N(a_{21})\, S(a_{11} \lambda \bar{a}_{12})$$
$$- N(a_{21})\, S(a_{11} \lambda \bar{a}_{12}) - S(a_{11} \lambda \bar{a}_{22} a_{21} \bar{a}_{11}).$$

This expression vanishes, since $S(ab) = S(ba)$, and therefore the last term†

$$S(a_{11}\lambda\bar{a}_{22}\,.\,a_{21}\bar{a}_{11}) = S(a_{21}\bar{a}_{11}\,.\,a_{11}\lambda\bar{a}_{22}) = N(a_{11})\,S(a_{21}\lambda\bar{a}_{22}).$$

From Theorem 11, the inequality $d(\Omega) > 0$ follows at once, since it holds for the matrices S, T, U. We also see that the product of regular matrices is again regular; they therefore form a group.

12. *The modular group.*

Let Γ be the set of all matrices Ω of determinant 1 and with elements which are integral quaternions. By the last theorem, the product of two elements of Γ belongs again to this set. The set contains the unit element $S(0) = E$. As we now prove, it contains with every matrix Ω also the inverse matrix Ω^{-1}, and therefore it is a group, the *modular group in* K.

In order to establish the existence of Ω^{-1}, it suffices to show that Ω can be written as a product of a finite number of factors $S(\lambda)$, T, and $U(\tau)$, where λ is an integral quaternion, and τ a unit. This can be done just as for the modular group in the rational field; all we need know is that the Euclidian algorithm holds in K‡.

If the element a_{21} in Ω vanishes, then, since $d(\Omega) = 1$, both a_{11} and a_{22} are units; hence

$$\Omega = U(a_{11})\,S(a_{11}^{-1}a_{12}a_{22}^{-1})\,TU(a_{22})\,T$$

is a factorization of the required kind.

If, however, $a_{21} \neq 0$, then by Theorem 2 there is an integral quaternion λ such that $a_{21}^{-1}a_{22} - \lambda$ lies in Δ; therefore

$$|a_{21}^{-1}a_{22} - \lambda| \leqslant \sqrt{\tfrac{1}{2}}, \quad \textit{i.e.,} \quad N(a_{22} - a_{21}\lambda) \leqslant \tfrac{1}{2}N(a_{21}).$$

We now have

$$\Omega = \Omega_1\,TS(\lambda), \qquad \Omega_1 = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}, \quad \text{where} \quad N(a'_{21}) \leqslant \tfrac{1}{2}N(a_{21}),$$

since obviously $$a'_{21} = a_{22} - a_{21}\lambda.$$

If still $a'_{21} \neq 0$, then there is an integral quaternion λ' such that

$$\Omega_1 = \Omega_2\,TS(\lambda'), \qquad \Omega_2 = \begin{pmatrix} a''_{11} & a''_{12} \\ a''_{21} & a''_{22} \end{pmatrix}, \quad \text{where} \quad N(a''_{21}) \leqslant \tfrac{1}{2}N(a'_{21}).$$

---

† We note that $S(\alpha\beta) = S(\beta\alpha) = \alpha S(\beta)$, if α is real.

‡ *Loc. cit.*, footnote †, p. 435.

Repeating this process, if necessary, we finally come to a matrix in which the coefficient with indices (21) vanishes, thus to the case already dealt with.

The factorization of Ω allows us to write down the inverse merely by taking the inverses of the factors in the reversed order.

13. *The discriminant of a Hermitian form.*

An expression

$$H(x, y) = \bar{x}Ax + \bar{x}By + \bar{y}\bar{B}x + \bar{y}Cy, \tag{50}$$

where A and C are real numbers and B and $\bar{B}$ are conjugate quaternions, is called a *Hermitian form in* K; we call

$$D = AC - B\bar{B} \tag{51}$$

its *discriminant*. Evidently, $H(x, y)$ can be written as

$$H(x, y) = AN(x-\xi y) + \frac{D}{A} N(y), \quad \text{where} \quad \xi = -\frac{B}{A}. \tag{52}$$

We consider only *positive definite forms* $H(x, y)$, *i.e.* we assume that $H(x, y) > 0$ for $N(x)+N(y) > 0$. The last formula shows that this is the case if, and only if,

$$A > 0 \quad \textit{and} \quad D > 0.$$

A linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \Omega \begin{pmatrix} x \\ y \end{pmatrix} \tag{53}$$

changes $H(x, y)$ into a new positive definite Hermitian form

$$H'(x, y) = \bar{x}A'x + \bar{x}B'y + \bar{y}\bar{B}'x + \bar{y}C'y$$

of discriminant $\quad D' = A'C' - B'\bar{B}'.$

Here

$$D' = Dd(\Omega). \tag{54}$$

For by the matrix factorization in § 11, it suffices to prove this equation for the three matrices $S(\lambda)$, T, $U(\tau)$, and it is easily verified in these three cases.

If the matrix Ω in (53) is an element of Γ, then $H'(x, y)$ is called *equivalent to* $H(x, y)$, in signs $H' \sim H$. Since Γ is a group, this relation has the properties

$$H \sim H;$$

$$\textit{if } H' \sim H, \textit{ then } H \sim H';$$

$$\textit{if } H' \sim H \textit{ and } H'' \sim H', \textit{ then } H'' \sim H.$$

By (54), *equivalent forms have equal discriminants.*

Equivalent forms evidently represent the same numbers when the arguments run over all integral quaternions. It is further clear from the definition of equivalence that, if $H' \sim H$, then both coefficients A' and C' are of the form $H(x, y)$ with integral x and y. Hence the two sets consisting respectively of the first coefficients A', and of the last coefficients C', of all forms $H' \sim H$, contain only *a finite number* of elements less than a given constant.

14. *The reduction of Hermitian forms.*

The form $H(x, y)$ is called *reduced* if

$$H(x, y) \geqslant \begin{cases} A \text{ for integral } x \text{ and } y \text{ with } N(x)+N(y) > 0, \\ C \text{ for integral } x \text{ and } y \text{ with } N(y) > 0. \end{cases} \tag{55}$$

THEOREM 12. *The form $H(x, y)$ is reduced if, and only if,*

$$A \leqslant C, \textit{ and } \xi = -\frac{B}{A} \textit{ is an element of } \Delta. \tag{56}$$

Proof. If (55) holds, then $C = H(0, 1) \geqslant A$; further, by (52), for all integral x,

$$0 \leqslant H(x, 1) - C = AN(x-\xi) - \frac{B\bar{B}}{A} = A\big(N(x-\xi) - N(\xi)\big),$$

so that ξ lies in Δ.

If (56) is true, then

$$H(x, 0) = N(x)H(1, 0) \geqslant A$$

for integral $x \neq 0$. Further, for every unit ϵ,

$$H(x, \epsilon) - C = H(x\epsilon^{-1}, 1) - C = A\big(N(x\epsilon^{-1} - \xi) - N(\xi)\big) \geqslant 0,$$

since ξ lies in Δ; and, for $N(y) \geqslant 2$,

$$\begin{aligned} H(x, y) - C &= AN(x - \xi y) + C\big(N(y) - 1\big) - \frac{B\bar{B}}{A} N(y) \\ &\geqslant 0 + \frac{C}{2} N(y) - AN(\xi)N(y) \geqslant AN(y)\big(\tfrac{1}{2} - N(\xi)\big) \geqslant 0, \end{aligned}$$

since $A \leqslant C$ and $N(\xi) \leqslant \frac{1}{2}$.

THEOREM 13. *If $H(x, y)$ is not reduced, then there is an equivalent form $H^*(x, y)$ such that*

$$\textit{either} \quad A^* < A, \quad \textit{or} \quad A^* = A, \quad A^* \leqslant C^* < C.$$

Proof. If $A > C$, then we may put $H^*(x, y) = H(y, x)$, so that $A^* = C < A$. Suppose therefore that $A \leqslant C$, but that $\xi = -\dfrac{B}{A}$ does not lie in Δ. Then let λ be an integral quaternion such that $\xi - \lambda$ lies in Δ. The transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow S(\lambda) \begin{pmatrix} x \\ y \end{pmatrix}$$

changes $H(x, y)$ into a form $H'(x, y)$ in which

$$A' = A, \quad B' = A\lambda + B, \quad C' = H(\lambda, 1).$$

If now $C' < A$, then we may put $H^*(x, y) = H'(y, x)$, so that

$$A^* = C' < A' = A.$$

If, however, $C' \geqslant A$, then put $H^*(x, y) = H'(x, y)$, so that

$$C^* - C = C' - C = A\big(N(\xi - \lambda) - N(\xi)\big) < 0,$$

since $\xi - \lambda$ but not ξ lies in Δ.

THEOREM 13. *To every form $H(x, y)$ there exists an equivalent form $H^*(x, y)$ which is reduced.*

Proof. If $H(x, y)$ is not itself reduced, then by the last theorem we can find a sequence of equivalent forms

$$H_1(x, y), \; H_2(x, y), \; \ldots$$

where

$$\text{either} \quad A_1 < A, \quad \text{or} \quad A_1 = A, \quad A_1 \leqslant C_1 < C,$$

$$\text{either} \quad A_2 < A_1, \quad \text{or} \quad A_2 = A_1, \quad A_2 \leqslant C_2 < C_1,$$

etc. Since by the last paragraph there are only a finite number of possible values of the first and the last coefficient of a form equivalent to $H(x, y)$ not exceeding a given value, this sequence of forms necessarily terminates after a finite number of terms; its last element is the wanted reduced form $H^*(x, y)$.

15. *The minima of a Hermitian form.*

Let
$$X = \begin{pmatrix} x_1 x_2 \\ y_1 y_2 \end{pmatrix}$$
be a matrix of determinant different from zero, and with elements which are integral quaternions; further let $H(x, y)$ be a positive definite Hermitian form, and $H^*(x, y)$ a reduced form equivalent to $H(x, y)$. The two expressions

$$(57)\qquad \begin{cases} M_1 = \min\limits_{X} \left\{ \min \left(H(x_1, y_1),\ H(x_2, y_2) \right) \right\}, \\ M_2 = \min\limits_{X} \left\{ \max \left(H(x_1, y_1),\ H(x_2, y_2) \right) \right\}, \end{cases}$$

where X takes all admissible values, are called the *first and second minimum of* $H(x, y)$. It is clear that M_1 is also the minimum of $H(x, y)$ when x and y run over all pairs of integral quaternions except the pair $x = y = 0$.

THEOREM 14. *If A^* and C^* are the first and the last coefficient of $H^*(x, y)$, then the two minima of $H(x, y)$ are given by*

$$(58)\qquad M_1 = A^*, \quad M_2 = C^*.$$

Proof. Let
$$\begin{pmatrix} x \\ y \end{pmatrix} = \Omega \begin{pmatrix} x^* \\ y^* \end{pmatrix}$$
be the transformation in Γ for which
$$H(x, y) = H^*(x^*, y^*),$$
and let
$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \Omega^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$
be its inverse. Put
$$X^* = \Omega^{-1} X = \begin{pmatrix} x_1{}^* x_2{}^* \\ y_1{}^* y_2{}^* \end{pmatrix}, \quad \text{so that} \quad X = \Omega X^*.$$

The elements of X^* are also integral quaternions, and the determinant $d(X^*) = d(X) \neq 0$. Hence the elements in one row or column of X^* are not both zero. Evidently,
$$H(x_1, y_1) = H^*(x_1{}^*, y_1{}^*), \quad H(x_2, y_2) = H^*(x_2{}^*, y_2{}^*).$$

Hence, by the definition of reduction,
$$\min \left(H(x_1, y_1), H(x_2, y_2) \right) \geqslant A^*, \quad \max \left(H(x_1, y_1), H(x_2, y_2) \right) \geqslant C^*.$$

If, in particular,

$$X^* = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \Omega,$$

then

$$H(x_1, y_1) = H^*(1, 0) = A^*, \quad H(x_2, y_2) = H^*(0, 1) = C^*,$$

so that the last formulae hold with the sign of equality.

THEOREM 15. *The minima M_1 and M_2 of $H(x, y)$ satisfy the inequalities*

$$(59) \qquad M_1 \leqslant \sqrt{(2D)}, \quad D \leqslant M_1 M_2 \leqslant 2D.$$

In particular, $M_1 = \sqrt{(2D)}$, and also $M_1 M_2 = 2D$, if, and only if, $A^* = C^*$ and ξ is one of the 24 SV *of Δ. Further, $M_1 M_2 = D$ if, and only if, $B^* = 0$.*

Proof. We have

$$D = A^* C^* - B^* \bar{B}^* \leqslant A^* C^*,$$

with equality if, and only if, $B^* = 0$. Further by Theorem 12, $A^* \leqslant C^*$ and $\xi = -\dfrac{B^*}{A^*}$ lies in Δ, so that $N(\xi) \leqslant \frac{1}{2}$ and therefore

$$B^* \bar{B}^* \leqslant \tfrac{1}{2} A^{*2} \leqslant \tfrac{1}{2} A^* C^*,$$

$$D = A^* C^* - B^* \bar{B}^* \geqslant A^* C^* - \tfrac{1}{2} A^* C^* = \tfrac{1}{2} A^* C^*.$$

If the sign of equality is to hold in the last formula, then $A^* = C^*$ and $N(\xi) = \frac{1}{2}$. The assertion follows therefore at once from Theorem 2 and Theorem 14.

Since Δ is invariant for the 24 transformations (6), all forms $H(x, y)$ satisfying $M_1 = \sqrt{(2D)}$ are equivalent to

$$H_0(x, y) = \sqrt{(2D)}\,(\bar{x}x - \bar{x}\,\Sigma_0 y - \bar{y}\,\bar{\Sigma}_0 x + \bar{y}y).$$

This form assumes its minimum value $M_1 = M_2 = \sqrt{(2D)}$ for $9 \times 24 = 216$ different pairs of integral x, y. For let ϵ be one of the 24 units of K; then the minimum is assumed for $(x, y) = (\epsilon, 0)$, and also for $(x, y) = (x, \epsilon)$ where now x is one of the 8 integral quaternions for which

$$N(x - \Sigma_0\, \epsilon) = \tfrac{1}{2}.$$

The part of the last theorem referring to M_1 is due to A. Speiser†, who used an entirely different method in his proof.

† *Journal für Math.*, 167 (1932), 88–97, in particular p. 97.

16. *The geometrical representation of Hermitian forms.*

Let $H(x, y)$ be a Hermitian form of discriminant $D = 1$; put

$$\xi = -\frac{B}{A}, \quad \eta = \frac{1}{A}, \tag{60}$$

so that

$$A = \frac{1}{\eta}, \quad B = -\frac{\xi}{\eta}, \quad \bar{B} = -\frac{\bar{\xi}}{\eta}, \quad C = \frac{\xi\bar{\xi}+\eta^2}{\eta}. \tag{61}$$

We interpret the four real components of ξ together with the positive real number η as the rectangular coordinates of a point (ξ, η) in the part

$$R: \qquad \eta > 0$$

of five-dimensional Euclidian space, and then there is a one-to-one correspondence between the forms $H(x, y)$ and the points (ξ, η) of R.

If $H(x, y)$ of point (ξ, η) is equivalent to $H'(x', y')$ of point (ξ', η'), then these points are also called equivalent. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Omega \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \Omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be the transformation in Γ which changes the first form into the second. Then a simple calculation shows that

$$\begin{cases} \xi' = -\dfrac{(\bar{a}_{11}-\bar{a}_{21}\,\bar{\xi})(a_{12}-\xi a_{22})+\bar{a}_{21}\,a_{22}\,\eta^2}{(\bar{a}_{11}-\bar{a}_{21}\,\bar{\xi})(a_{11}-\xi a_{21})+\bar{a}_{21}\,a_{21}\,\eta^2}, \\[2ex] \eta' = \dfrac{\eta}{(\bar{a}_{11}-\bar{a}_{21}\,\bar{\xi})(a_{11}-\xi a_{21})+\bar{a}_{21}\,a_{21}\,\eta^2}, \end{cases} \tag{62}$$

with analogous formulae involving the elements of Ω^{-1} for the change-over from (ξ', η') to (ξ, η). The second equation shows that *R is transformed into itself by all elements of Γ.* We write (62) for shortness as $(\xi, \eta) = \Omega(\xi', \eta')$ or $(\xi', \eta') = \Omega^{-1}(\xi, \eta)$. In particular, for the generators of Γ,

if $\Omega = S(\lambda)$, then $\xi' = \xi - \lambda$, $\eta' = \eta$,

if $\Omega = T$, then $\xi' = \dfrac{\bar{\xi}}{\xi\bar{\xi}+\eta^2}$, $\eta' = \dfrac{\eta}{\xi\bar{\xi}+\eta^2}$,

if $\Omega = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$, then $\xi' = \epsilon_1^{-1}\,\xi\,\epsilon_2$, $\eta' = \eta$;

in the last formula, ϵ_1 and ϵ_2 are assumed to be units in K.

Now let F be the set of all points (ξ, η) which belong to reduced forms; by Theorems 2 and 12, F is determined by

(63) $$|\xi| \leqslant |\xi - g| \text{ for all integral quaternions } g; \quad \xi\bar{\xi} + \eta^2 \geqslant 1.$$

The point (ξ, η) is an *inner point* of F if

$$|\xi| < |\xi - g| \text{ for all integral quaternions } g \neq 0; \quad \xi\bar{\xi} + \eta^2 > 1;$$

denote by F_0 the set of all inner points of F. The proof of Theorem 12 shows that the form $H(x, y)$ belongs to a point of F_0 if, and only if, for all integral pairs (x, y),

(64) $$H(x, y) > \begin{cases} A & \text{when} \quad (x, y) \neq (0, 0), \quad \neq (\epsilon, 0), \\ C & \text{when} \quad y \neq 0 \quad \text{and} \quad (x, y) \neq (0, \epsilon), \end{cases}$$

where ϵ denotes the 24 units of K; in particular $A < C$.

Suppose now that $H(x, y)$ belongs to an inner point of F and that $H'(x', y')$ is a reduced form equivalent to $H(x, y)$. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Omega \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \Omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

be the transformation in Γ which changes one form into the other. Then by Theorem 14,

$$A = A', \quad C = C';$$

hence

$$A = H(a_{11}, a_{21}), \quad C = H(a_{12}, a_{22}).$$

By (64), the first formula requires that $a_{11} = \epsilon_1$, $a_{21} = 0$, the second one that $a_{12} = 0$, $a_{22} = \epsilon_2$, where ϵ_1 and ϵ_2 are any two units; therefore the transformation is of the type

(65) $$\Omega = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}.$$

It follows that these are the only transformations which change at least one inner point of F into a point of F. Since the inverse of Ω has the same form, all 24×24 transformations (65) leave the set F_0 invariant.

Denote by $\Omega^{-1} F$ (and $\Omega^{-1} F_0$) the sets of all points (ξ', η') for which $(\xi, \eta) = \Omega(\xi', \eta')$ lies in F (in F_0). The last results show that F_0 and the transformed set $\Omega^{-1} F_0$ either are identical, or have no common points. Hence, by the group property of Γ, any two sets $\Omega_1^{-1} F_0$ and $\Omega_2^{-1} F_0$ either coincide or are without common points. Therefore, finally, two sets $\Omega_1^{-1} F$ and $\Omega_2^{-1} F$ either have no common inner points, or they are identical. This shows that the sets $\Omega^{-1} F$ together fill up the space R without overlapping, and also without gaps since to every point there is an equivalent one in F.

If ϵ_1 and ϵ_2 are two units, and λ is an integral quaternion, then the set

$$\Omega^{-1} F, \quad \text{where} \quad \Omega = \begin{pmatrix} \epsilon_1 & \lambda \\ 0 & \epsilon_2 \end{pmatrix},$$

contains points with arbitrarily large η, since this is true for F, and the transformation leaves η invariant. When, however, $\Omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is an element of Γ with $a_{21} \neq 0$, then η is bounded for all points in $\Omega^{-1} F$. There are, however, points with arbitrarily small positive η in the set $\Omega^{-1} F$, as seen from (62), and these satisfy

$$\lim_{\eta \to 0} \xi = a_{21}^{-1} a_{22}. \tag{66}$$

It follows that if a continuous curve in R tends to a point $(\alpha, 0)$ in $\eta = 0$, where α is an irrational quaternion (*i.e.* not of the form $a^{-1} b$ with integral quaternions $a \neq 0$ and b), then this curve passes through an infinity of different sets $\Omega^{-1} F$. For if it passed only through a finite number, then it would ultimately lie in one set and so by (66) tend to a point $(a_{12}^{-1} a_{22}, 0)$.

17. *On special Hermitian forms.*

THEOREM 16. *Let* $\mathrm{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *be a matrix of determinant* $d(\mathrm{A}) = 1$ *with elements in* K, t *a positive parameter,* $F_t(x, y)$ *the Hermitian form of discriminant* 1

$$F_t(x, y) = tN(\alpha x + \beta y) + \frac{1}{t} N(\gamma x + \delta y), \tag{67}$$

and

$$F_t^*(x, y) = \bar{x} A_t x + \bar{x} B_t y + \bar{y} \bar{B}_t x + \bar{y} C_t y \tag{68}$$

the reduced form equivalent to $F_t(x, y)$. *Then, for at least one value of* t,

$$A_t = C_t.$$

Further, if $\alpha^{-1}\beta$ *is not a rational quaternion, then there are arbitrarily large* t *with this property.*

Proof. Since

$$tF_t(x, y) = \bar{x}(\bar{\alpha}\alpha t^2+\bar{\gamma}\gamma)x+\bar{x}(\bar{\alpha}\beta t^2+\bar{\gamma}\delta)y+\bar{y}(\bar{\beta}\alpha t^2+\bar{\delta}\gamma)x+\bar{y}(\bar{\beta}\beta t^2+\bar{\delta}\delta)y,$$

the point (ξ, η) corresponding to $F_t(x, y)$ is given by

$$\xi = -\frac{\bar{\alpha}\beta t^2+\bar{\gamma}\delta}{\bar{\alpha}\alpha t^2+\bar{\gamma}\gamma}, \quad \eta = \frac{t}{\bar{\alpha}\alpha t^2+\bar{\gamma}\gamma}, \quad \xi\bar{\xi}+\eta^2 = \frac{\bar{\beta}\beta t^2+\bar{\delta}\delta}{\bar{\alpha}\alpha t^2+\bar{\gamma}\gamma}.$$

If t assumes all positive values, then (ξ, η) describes the semicircle **C** which is perpendicular to the hyperplane $\eta = 0$ at the two points $(-\alpha^{-1}\beta, 0)$ and $(-\gamma^{-1}\delta, 0)$. For $t \to \infty$, (ξ, η) tends to the first point $(-\alpha^{-1}\beta, 0)$; hence, by the preceding paragraph, if $\alpha^{-1}\beta$ is irrational, then **C** passes through an infinity of different sets $\Omega^{-1}F$. (It is possible that just one of the two numbers α and γ vanishes; then **C** degenerates into a straight line perpendicular to $\eta = 0$.)

Suppose that a point P: $t = \tau$ of **C**, lies in the set $\Omega^{-1}F$. The transformation $\begin{pmatrix} x \\ y \end{pmatrix} \to \Omega \begin{pmatrix} x \\ y \end{pmatrix}$ changes $\Omega^{-1}F$ into F, **C** into a new semicircle or straight line **C*** perpendicular to $\eta = 0$, and the point P on **C** into a point $P^*(\xi^*, \eta^*)$ on **C***. Then P^* lies in F, so that

$$\xi^* \text{ belongs to } \Delta; \quad \xi^*\bar{\xi}^*+\eta^2 \geqslant 1.$$

Evidently P^* corresponds to the reduced form $F_\tau^*(x, y)$. The point divides **C*** into two parts; let c^* be that part of **C*** for whose points

$$\eta \leqslant \eta^*.$$

Let $g = g(\xi)$ be any integral quaternion such that $\xi - g$ lies in Δ. If there is more than one quaternion g with this property, then, by the definition of Δ, $N(\xi-g)$ has the same value for all of them.

Denote now by S the hypersurface in R defined by

$$(\xi-g)(\bar{\xi}-\bar{g})+\eta^2 = 1. \tag{69}$$

Then η is a continuous single-valued function of ξ, and so S is a continuous and connected hypersurface; it is clearly transformed into itself by all translations of R,

$$\begin{pmatrix} x \\ y \end{pmatrix} \to S(\lambda)\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{i.e.} \quad \xi \to \xi+\lambda, \quad \eta \to \eta, \tag{70}$$

where λ runs over all integral quaternions.

The point P^* belongs to F and therefore either lies on S or has a larger coordinate η^* than the point of S with the same value ξ^* of ξ. In the second case the arc c^* of $\mathbf{C}^*$ intersects S in at least one point, since it joins P^* to a point in $\eta = 0$. By a suitable translation (70), this point of c^* changes into one on S and inside F. It follows that there is a transformation in Γ which changes $\mathbf{C}$ into a semicircle or straight line intersecting S in a point of F. If $t = t_0$ is the parameter value belonging to this point of intersection, then, for the reduced form (68), $A_{t_0} = C_{t_0}$, as follows from (61) since $\xi - g$ in (69) has now been replaced by ξ.

If $\alpha^{-1}\beta$ is irrational, then there are an infinity of ways of changing $\mathbf{C}$ into a semicircle or straight line which intersects S in a point of F. For now $\mathbf{C}$ passes through an infinity of different sets $\Omega^{-1}F$ in the neighbourhood of $(\alpha^{-1}\beta, 0)$. A transformation which changes one of these sets into F transforms at most a finite number of the others into sets $S(\lambda)^{-1}F$. This proves the assertion.

18. *The product of two linear polynomials.*

THEOREM 17. *Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a matrix of determinant $d(A) = 1$ with elements in K. Corresponding to any quaternions x_0 and y_0 there are two other quaternions, x_1 and y_1, such that*

$$\text{(A)} \qquad x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad |(\alpha x_1 + \beta y_1)(\gamma x_1 + \delta y_1)| \leqslant \tfrac{1}{2}.$$

If $\alpha^{-1}\beta$ is irrational, then (A) *has a solution with arbitrarily small value of* $|\alpha x_1 + \beta y_1|$.

Proof. Consider the forms (67) and (68). By Theorem 16, there is a value $t = t_0$ for which $A_{t_0} = C_{t_0}$. If $\alpha^{-1}\beta$ is irrational, then t_0 may be taken arbitrarily large. *Let*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Omega \begin{pmatrix} x \\ y \end{pmatrix}, \quad \Omega = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be the transformation in Γ which changes $F_{t_0}(x, y)$ into $F^*_{t_0}(x', y')$, and let

$$x_0' = a_{11}x_0 + a_{12}y_0, \quad y_0' = a_{21}x_0 + a_{22}y_0.$$

By the corollary to Theorem 10 and by (52), there are two quaternions x_1' and y_1' such that

$$x_1' \equiv x_0', \quad y_1' \equiv y_0', \quad F^*_{t_0}(x_1', y_1') \leqslant 1.$$

Then, if x_1 and y_1 are the quaternions defined by

$$x_1' = a_{11}x_1 + a_{12}y_1, \quad y_1' = a_{21}x_1 + a_{22}y_1,$$

obviously

$$x_1 \equiv x_0, \quad y_1 \equiv y_0, \quad F_{t_0}(x_1, y_1) = t_0 N(\alpha x_1 + \beta y_1) + \frac{1}{t_0} N(\gamma x_1 + \delta y_1) \leqslant 1,$$

and so (A) follows by the theorem of the arithmetic and geometric means. For the solution so obtained,

$$N(\alpha x_1 + \beta y_1) \leqslant \frac{1}{t_0},$$

where $\frac{1}{t_0}$ can be made arbitrarily small for irrational $\alpha^{-1}\beta$.

The equality sign holds in (A), if $\alpha = \delta = 1$, $\beta = \gamma = 0$, and both x_0 and y_0 are singular vertices. Theorem 17 is therefore the best possible result.

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