# ON A THEOREM OF MINKOWSKI ON LATTICE POINTS IN NON-CONVEX POINT SETS

K. MAHLER†.

Let $(x_1, \ldots, x_n)$ be rectangular coordinates in $n$-dimensional Euclidean space $R_n$, and let $K$ be a *star body* in $R_n$, *i.e.* a closed bounded point set which

(*a*) contains the origin $O = (0, \ldots, 0)$ of the coordinate system as an *inner* point, and

(*b*) is bounded by a continuous surface which is met by every radius vector from $O$ in just *one* point.

A lattice

$$\Lambda: \qquad x_h = \sum_{k=1}^{n} a_{hk} g_k \quad (h = 1, 2, \ldots, n;\ g_1, \ldots, g_n = 0, \pm 1, \pm 2, \ldots)$$

in $R_n$, of determinant

$$\left| \, | a_{hk} |_{h,k=1,2,\ldots,n} \right| \neq 0,$$

is called *$K$-admissible* if $O$ is its only point which is an *inner* point of $K$. Denote by $\Delta(K)$ the lower bound of the determinants of all $K$-admissible lattices, and by $V(K)$ the volume of $K$.

In 1891, Minkowski found a theorem‡ which states, in effect, that

$$(1) \qquad \Delta(K) \leqslant c_n V(K), \quad \text{where} \quad c_n = \left( E \sum_{v=1}^{\infty} v^{-n} \right)^{-1},$$

and $E$ is 2 or 1 according as $K$ is, or is not, symmetrical in $O$. From this Minkowski obtained an asymptotic formula for Hermite's constant $\gamma_n$ connected with the minimum of a positive definite quadratic form in $n$ variables§.

Minkowski gave a rather difficult proof of (1) in the special case of an $n$-dimensional sphere‖. But he never published a proof of the general theorem, nor was any other proof known until recently. Then, last year, E. Hlawka published a paper¶ which contained a very ingenious analytical proof of (1) and also other interesting results.

† Received 17 July, 1944; read 16 November, 1944.

‡ *Ges. Abh.*, 1 (1911), 265, 270, 277.

§ Hermite, *Oeuvres*, 1 (1905), 103 ff.; Minkowski, *Ges. Abh.*, 1 (1911), 270.

‖ *Ges. Abh.*, 2 (1911), 94–95.

¶ *Math. Zeitschrift*, 49 (1943), 285–312, in particular 288–298.

Before I knew of Hlawka's paper, I had been trying to prove Minkowski's theorem, and I had found a simple geometrical proof of the slightly less exact result,

$$\Delta(K) \leqslant c_n V(K), \quad \text{where} \quad c_n = n/E. \tag{2}$$

This inequality still implies Minkowski's asymptotic formula for $\gamma_n$; it may therefore be useful to publish the proof, which is entirely different from that of Hlawka.

I restrict myself, however, to the *symmetrical* case. The unsymmetrical case may be treated quite similarly, and the method holds also for infinite sets. For symmetrical *convex* bodies, it leads to the better value $c_n = \frac{1}{2}$, but the true value is presumably at most of the order $c_n = O(1/n)$.

I assume, without stating so each time, *that all integrals occurring in this note exist.*

1. *Notation.* Denote by $D$ any number satisfying

$$0 < D < \Delta(K);$$

hence *no lattice of determinant $D$ is $K$-admissible*;

by $K_0$ the intersection of $K$ with the plane $x_n = 0$, so that $K_0$ is an $(n-1)$-dimensional star body symmetrical in $O$;

by $\Lambda_0$ any $K_0$-admissible $(n-1)$-dimensional lattice in the plane $x_n = 0$;

by $P_h = (x_{h1}, \ldots, x_{hn-1}, 0)$, $h = 1, 2, \ldots, n-1$, a basis of $\Lambda_0$;

by $d = \left| |x_{hk}|_{h,k=1,2,\ldots,n-1} \right|$ the determinant of $\Lambda_0$;

by $\xi = (\xi_1, \ldots, \xi_{n-1})$ any point in $(n-1)$-dimensional Euclidean space $R_{n-1}$;

by $W$ the cube $0 \leqslant \xi_1 < 1, \ldots, 0 \leqslant \xi_{n-1} < 1$ in $R_{n-1}$;

by $P_n^*$ and $P_n^{(\xi)}$ the points†

$$P_n^* = (0, \ldots, 0, D/d), \quad P_n^{(\xi)} = \xi_1 P_1 + \cdots + \xi_{n-1} P_{n-1} + P_n^*$$

in $R_n$;

by $\Lambda_\xi$ the lattice in $R_n$ of basis

$$P_1, P_2, \ldots, P_{n-1}, P_n^{(\xi)}$$

and so of determinant $d \times (D/d) = D$; *hence this lattice is not $K$-admissible*;

† Sums of points, or products of points into scalars, have the meaning usual in linear algebra or vector analysis.

by $K_v$, $v = 1, 2, 3, \ldots$, the intersection of $K$ with the plane $x_n = vD/d$;
*when $v$ is sufficiently large, then $K_v$ is the null set, since $K$ is bounded*;

by $\kappa_v$ the $(n-1)$-dimensional volume

$$\kappa_v = \int \ldots \int_{K_v} dx_1 \ldots dx_{n-1}$$

of $K_v$; *hence $\kappa_v = 0$ when $v$ is sufficiently large.*

Further, if $\xi = (\xi_1, \ldots, \xi_{n-1})$ and $\xi^0 = (\xi_1^0, \ldots, \xi_{n-1}^0)$ are any two points in $R_{n-1}$, then we write

$$\xi \equiv \xi^0 \pmod 1$$

as an abbreviation for the $n-1$ congruences

$$\xi_1 \equiv \xi_1^0 \pmod 1, \quad \ldots, \quad \xi_{n-1} \equiv \xi_{n-1}^0 \pmod 1.$$

2. *The fundamental lemma.* Let $P = (x_1, \ldots, x_{n-1}, vD/d)$ describe the set $K_v$, and define the point $\xi = (\xi_1, \ldots, \xi_{n-1})$ in $R_{n-1}$ by the condition

$$P = vP_n^{(\xi)} = v(\xi_1 P_1 + \ldots + \xi_{n-1} P_{n-1} + P_n^*),$$

so that

$$x_k = v \sum_{h=1}^{n-1} \xi_h x_{hk} \qquad (k = 1, 2, \ldots, n-1).$$

Then $\xi$ describes a certain set in $R_{n-1}$, $L_v$ say, and this set is of volume

$$(3) \qquad \lambda_v = \int \ldots \int_{L_v} d\xi_1 \ldots d\xi_{n-1} = \frac{\kappa_v}{dv^{n-1}},$$

since the linear equations connecting the $x$'s with the $\xi$'s are of determinant $v^{n-1}d$.

Next let $M_v$ be the set of all points $(\xi^1 = \xi_1^1, \ldots, \xi_{n-1}^1)$, in the cube $W$, for which there exist $n-1$ integers $u_1, \ldots, u_{n-1}$ such that the point

$$\begin{aligned} P &= u_1 P_1 + \ldots + u_{n-1} P_{n-1} + vP_n^{(\xi^1)} \\ &= (u_1 + v\xi_1^1) P_1 + \ldots + (u_{n-1} + v\xi_{n-1}^1) P_{n-1} + vP_n^* \end{aligned}$$

lies in $K_v$, and let

$$\mu_v = \int \ldots \int_{M_v} d\xi_1^1 \ldots d\xi_{n-1}^1$$

be the volume of this set. Evidently $\xi^1$ belongs to $M_v$ if, and only if, the point $\xi$ defined by

$$v\xi_1 = u_1 + v\xi_1^1, \quad \ldots, \quad v\xi_{n-1} = u_{n-1} + v\xi_{n-1}^1$$

is a point of $L_v$. Since $\xi^1$ lies in $W$, these equations imply that

$$0 \leqslant v\xi_1 - u_1 < v, \quad \ldots, \quad 0 \leqslant v\xi_{n-1} - u_{n-1} < v,$$

and so, for any given point $\xi$ of $L_v$, each of the integers $u_1, \ldots, u_{n-1}$ has just $v$ possible values. Hence to every point $\xi$ of $L_v$ there correspond at most $v^{n-1}$ points $\xi^1$ of $M_v$, obtained by as many translations from $\xi$. Therefore

$$\mu_v \leqslant v^{n-1} \lambda_v,$$

whence, from (3),

$$\mu_v \leqslant \kappa_v / d. \tag{4}$$

LEMMA. *The volumes* $\kappa_v$ *satisfy the inequality*

$$\sum_{v=1}^{\infty} \kappa_v \geqslant d. \tag{5}$$

*Proof.* Let $\xi^1$ be any point in $W$. The lattice $\Lambda_{\xi^1}$ is not $K$ admissible and so contains at least one point

$$P = u_1 P_1 + \ldots + u_{n-1} P_{n-1} + u_n P_n^{(\xi^1)},$$

different from $O$, which is an *inner* point of $K$. *This point cannot lie in the plane* $x_n = 0$. For in this plane $\Lambda_{\xi^1}$ reduces to the $(n-1)$-dimensional lattice $\Lambda_0$ and $K$ to the $(n-1)$-dimensional star body $K_0$, and, by hypothesis, $\Lambda_0$ is $K_0$-admissible.

Since $K$ is symmetrical in $O$, both $P$ and the symmetrical point $-P$ belong to $K$. Hence, without loss of generality, $u_n$, or $v$ say, may be supposed positive. Then $P$ lies in $K_v$, and so $\xi^1$ belongs to $M_v$.

We conclude therefore that the sets $M_1, M_2, M_3, \ldots$ together cover the whole cube $W$, and so are of total volume

$$\sum_{v=1}^{\infty} \mu_v \geqslant 1,$$

since $W$ has unit volume. The assertion now follows from (4).

3. *Conclusion.* Denote by $T_v$ the cone of vertex $O$ and base $K_v$, and hence of volume

$$\frac{1}{n} \times \kappa_v \times v \frac{D}{d}.$$

Further denote by $T'_v$ the part of $T_v$ between the two planes

$$x_n = v D/d \quad \text{and} \quad x_n = (v-1) D/d;$$

then $T'_v$ is of volume not less than

$$\frac{1}{v} \times \left(\frac{1}{n} \times \kappa_v \times v \frac{D}{d}\right) = \frac{D}{nd} \kappa_v.$$

Since $K$ is a star body, and since $K_v$ consists only of points of $K$, the cone $T_v$, and so also the truncated cone $T'_v$, are subsets of $K$, and the same is true for the cones $-T_v$ and $-T'_v$ symmetrical to $T_v$ and $T'_v$ in $O$. But it is obvious that no two of the truncated cones

$$T'_1, T'_2, T'_3, \ldots, \quad -T'_1, -T'_2, -T'_3, \ldots$$

have inner points in common. Therefore, by (5),

$$V(K) \geqslant 2 \sum_{v=1}^{\infty} \frac{D}{nd} \kappa_v = \frac{2D}{nd} \sum_{v=1}^{\infty} \kappa_v \geqslant \frac{2D}{nd} \times d = \frac{2}{n} D.$$

Since $D$ may be any number smaller than $\Delta(K)$, the assertion (2) follows immediately.

*Addition* (*May* 1945). In a paper, "A mean value theorem in geometry of numbers", dated 8 December, 1944, which is to appear in the *Annals of Mathematics*, C. L. Siegel gives a beautiful new proof of the Minkowski-Hlawka theorem. He establishes the intimate connection of this theorem with the reduction of quadratic forms and the arithmetical theory of the group of all linear transformations, just as Minkowski had predicted.

Mathematics Department,
Manchester University.

## ON THE EXISTENCE OF TANGENTS TO RECTIFIABLE CURVES

A. S. BESICOVITCH*.

In this article I give a very simple proof, which is even independent of the theory of measure, of the existence of a tangent at almost all points of a rectifiable curve.

LEMMA 1. *Given two positive numbers $\alpha$ and $\beta$, there is a positive number $\gamma$ such that, if a segment of length $l$ subtends a simple arc $S$, of length less than $(1+\gamma)l$, then the points of $S$ whose joins to other points of $S$ form an angle greater than $\alpha$ with the segment all lie on a finite or enumerable set of arcs of total length at most $\beta l$.*

* Received 6 February, 1945; read 1 March, 1945.