assume that R is different from the ten points $\pm P_1$, $\pm P_2$, $\pm(2P_2-P_1)$, $\pm(P_2-P_1)$, $\pm(P_2-2P_1)$ of Λ_0 on L; for otherwise we may replace R by a neighbouring point on L having this property and lying outside H.

By Theorem 2, there exists a critical lattice Λ of K which contains the point R on L. This lattice is also H-admissible. It is, however, not a critical lattice of H. For Λ contains six points on the boundary L of K, and so at most four points on the boundary of H; and so Λ would be a singular lattice of H. Then the tac-line conditions (see the preface) must be satisfied by the four points on the boundary of H. These four points lie also on the boundary of K, and we have shown in the proof of Theorem 1 that the tac-line conditions never hold for points on the boundary of K. Hence Λ is not a critical lattice of H, and so there exist critical lattices of smaller determinant than $\Delta(K)$, as was to be proved.

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ON LATTICE POINTS IN THE DOMAIN $|xy|\leqslant 1$, $|x+y|\leqslant\sqrt{5}$, AND APPLICATIONS TO ASYMPTOTIC FORMULAE IN LATTICE POINT THEORY (II)

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I. LATTICE POINTS IN THE DOMAIN $|x|^\alpha+|y|^\alpha\leqslant 1$

THEOREM 4. *Let G be the star domain*

$$|x|^\alpha+|y|^\alpha\leqslant 1,$$

where $\alpha>0$. Then, when α tends to zero,

$$\Delta(G)=2^{-2/\alpha}\sqrt{5}\{1+O(\alpha)\}.$$

Proof. The linear substitution

$$x=2^{-1/\alpha}X,\quad y=2^{-1/\alpha}Y$$

changes G into the similar domain

$$(G')\qquad |X|^\alpha+|Y|^\alpha\leqslant 2,$$

and so

$$\Delta(G)=2^{-2/\alpha}\Delta(G').$$

Now

$$|X|^\alpha+|Y|^\alpha=e^{\alpha\log|X|}+e^{\alpha\log|Y|}=2+\alpha\log|XY|+\rho(X,Y),$$

where, by the mean value theorem of the differential calculus,

$$\rho(X,Y)=\tfrac{1}{2}\alpha^2\{e^{\alpha\theta\log|X|}(\log|X|)^2+e^{\alpha\theta\log|Y|}(\log|Y|)^2\}$$

with $0<\theta<1$. Hence, for all points on the boundary of G',

$$\log|XY|=-\frac{\rho(X,Y)}{\alpha},\quad\text{i.e.}\quad |XY|=e^{-\rho(X,Y)/\alpha}\leqslant 1.$$

This means that G' is a subdomain of the domain K' defined by $|XY| \leqslant 1$; and so, by Hurwitz's theorem,

$$(A) \qquad \Delta(G') \leqslant \Delta(K') = \sqrt{5}.$$

On the other hand, if $\alpha > 0$ is sufficiently small, and $c > 0$ is a suitable constant, then the domain

$$(K_\alpha) \qquad |XY| \leqslant (1-c\alpha)^2, \quad |X+Y| \leqslant \sqrt{5}\,(1-c\alpha),$$

is contained in G'. For, if (X, Y) is a point in K_α, then

$$|X-Y| = +\sqrt{\{(X+Y)^2-4XY\}} < +\sqrt{(5+4)} = 3,$$

and so $$\max(|X|, |Y|) = \tfrac{1}{2}\{|X+Y|+|X-Y|\} < \tfrac{1}{2}(3+\sqrt{5}) < e.$$

If now $|X| \leqslant e^{-2}$ or $|Y| \leqslant e^{-2}$, then for sufficiently small positive α,

$$|X|^\alpha + |Y|^\alpha < e^\alpha + e^{-2\alpha} = 1 - \alpha + O(\alpha^2) < 1,$$

and so (X, Y) is an inner point of G'. We may therefore assume that the point (X, Y) in K_α satisfies the inequalities

$$e^{-2} \leqslant |X| \leqslant e, \quad e^{-2} \leqslant |Y| \leqslant e.$$

But then $$0 \leqslant \rho(X, Y)/\alpha \leqslant \tfrac{1}{2}\alpha(e^\alpha . 2^2 + e^\alpha . 2^2) = 4\alpha e^\alpha,$$

and so, if we choose $c = 5$, $$e^{-\rho(X,Y)/\alpha} > e^{-c\alpha} > (1-c\alpha)^2,$$

if $\alpha > 0$ is again sufficiently small. The assertion follows therefore also in this case.

By Theorem 1 of part I,

$$\Delta(K_\alpha) = (1-c\alpha)^2 \Delta(K) = (1-c\alpha)^2\sqrt{5},$$

since K_α is derivable from K by the linear substitution

$$X = x'(1-c\alpha), \quad Y = y'(1-c\alpha)$$

of determinant $(1-c\alpha)^2$. Hence

$$(B) \qquad \Delta(G') \geqslant \Delta(K_\alpha) = (1-c\alpha)^2\sqrt{5}.$$

Theorem 4 is now an immediate consequence of (A) and (B).

II. On positive definite quartic binary forms

1. *The problem.* Let

$$f(x,y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$$

be a positive definite quartic binary form of invariants

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2; \quad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}; \quad G = g_2^3 - 27g_3^2; \quad J = \frac{g_2^3}{G} = 27\frac{g_3^2}{G} + 1:$$

here G is the discriminant, and J the absolute invariant of $f(x,y)$. Denote by K_f the star domain $$f(x,y) \leqslant 1,$$

and by $\Delta(K_f)$ the lower bound of the determinants of all K_f-admissible lattices. Then $\Delta(K_f)$ is an invariant of $f(x,y)$, and so is a function of g_2 and g_3 alone.

Since $f(x,y)$ is positive definite, its discriminant is positive. We shall therefore assume from now on that $G = 1$. Then g_2 and g_3 become functions of J and of the sign

$$\epsilon = \operatorname{sgn} g_3 = +1 (g_3 > 0), \quad = 0 (g_3 = 0), \quad = -1 (g_3 < 0),$$

namely $$g_2 = J^{\frac{1}{3}}, \quad g_3 = \epsilon\left(\frac{J-1}{27}\right)^{\frac{1}{2}},$$

with the convention that all square and higher roots are taken with the positive sign. Hence also

$$\Delta(K_f) = D_\epsilon(J),$$

say, depends only on the values of J and ϵ. Our problem is to obtain an asymptotic formula for $D_\epsilon(J)$ when J tends to infinity, and to study the minimum of $f(x,y)$ for integral x, y, not both zero, when J is large.

2. *The normal form of* $f(x,y)$. Since $f(x,y)$ is positive definite, it is the product of two positive definite binary quadratic forms. By means of a linear substitution of unit determinant, these factors can be reduced to the forms

$$x^2+y^2, \quad ax^2+by^2 \quad (a>0,\ b>0).$$

Hence we may assume that

$$f(x,y) = (x^2+y^2)(ax^2+by^2),$$

where $a>0$, $b>0$ satisfy the condition that $f(x,y)$ is of discriminant

$$G = \tfrac{1}{16}ab(a-b)^4 = 1.$$

This condition $G=1$ is equivalent to

$$a = 2A\{4AB(A-B)^4\}^{-\frac{1}{6}}, \quad b = 2B\{4AB(A-B)^4\}^{-\frac{1}{6}},$$

where A and B are any two positive numbers.

We therefore suppose from now on that

$$f(x,y) = 2\{4AB(A-B)^4\}^{-\frac{1}{6}}(x^2+y^2)(Ax^2+By^2) \quad (A>0,\ B>0).$$

Then the invariants of $f(x,y)$ are

$$g_2 = \frac{A^2+14AB+B^2}{3\{4AB(A-B)^4\}^{\frac{1}{3}}}, \quad g_3 = -\frac{(A+B)(A^2-34AB+B^2)}{27\{4AB(A-B)^4\}^{\frac{1}{2}}},$$

$$G = 1, \quad J = g_2^3 = 27g_3^2+1.$$

Hence J tends to infinity only if A/B tends to either 0 or 1 or ∞.

3. *The case* $A/B \to 1$. Put

$$B = A(1+t),$$

where $t \to 0$. Then

$$g_2 = \frac{16}{3\,.\,4^{\frac{1}{3}}}t^{-\frac{4}{3}}\{1+O(|t|)\}, \quad g_3 = \frac{32}{27t^2}\{1+O(|t|)\}, \quad J = 2(\tfrac{8}{3})^3 t^{-4}\{1+O(|t|)\},$$

and, conversely,

$$t = \pm\, 2^{\frac{1}{4}}(\tfrac{8}{3})^{\frac{3}{4}} J^{-\frac{1}{4}}\{1+O(J^{-\frac{1}{4}})\}.$$

It is clear that $g_3>0$ for all sufficiently small values of $|t|$, i.e. for all sufficiently large values of J; and therefore

$$\epsilon = \operatorname{sgn} g_3 = +1.$$

The quartic $f(x,y)$ takes now the form

$$f(x,y) = \frac{2(x^2+y^2)\{x^2+(1+t)y^2\}}{\{4t^4(1+t)\}^{\frac{1}{6}}};$$

whence

$$\frac{2(1-|t|)}{\{4t^4(1+|t|)\}^{\frac{1}{6}}}(x^2+y^2)^2 \leqslant f(x,y) \leqslant \frac{2(1+|t|)}{\{4t^4(1-|t|)\}^{\frac{1}{6}}}(x^2+y^2)^2.$$

for all real values of x and y. Hence K_f is contained in the circle

$$(C_1) \quad x^2+y^2 \leqslant \frac{\{4t^4(1+|t|)\}^{1/12}}{\{2(1-|t|)\}^{\frac{1}{2}}} = 2^{-\frac{1}{3}}|t|^{\frac{1}{3}}\{1+O(|t|)\} = (\tfrac{4}{3})^{\frac{1}{4}} J^{-1/12}\{1+O(J^{-\frac{1}{4}})\},$$

and itself contains the circle

$$(C_2)\quad x^2+y^2 \leqslant \frac{\{4t^4(1-|t|)\}^{1/12}}{\{2(1+|t|)\}^{\frac{1}{3}}} = 2^{-\frac{1}{6}}|t|^{\frac{1}{3}}\{1+O(|t|)\} = (\tfrac{4}{3})^{\frac{1}{4}}J^{-1/12}\{1+O(J^{-\frac{1}{4}})\}.$$

Now it is well known that, for a circle $x^2+y^2 \leqslant r^2$, the minimum determinant of all admissible lattices is given by $\Delta(C) = \sqrt{(\tfrac{3}{4})}\,r^2.$

Hence
$$\sqrt{\left(\frac{3}{4}\right)}\frac{\{4t^4(1-|t|)\}^{1/12}}{\{2(1+|t|)\}^{\frac{1}{3}}} \leqslant D_\epsilon(J) \leqslant \sqrt{\left(\frac{3}{4}\right)}\frac{\{4t^4(1+|t|)\}^{1/12}}{\{2(1-|t|)\}^{\frac{1}{3}}},$$

and so finally

$$\text{(I)}\qquad D_{+1}(J) = (\tfrac{3}{4})^{\frac{1}{4}}J^{-1/12}\{1+O(J^{-\frac{1}{4}})\}$$

as $J \to \infty$.

4. *The case $A/B \to 0$ or ∞.* Since $f(x,y)$ is symmetrical in A and B, it suffices to consider the case when A/B tends to infinity. Put therefore

$$B = At,$$

where $t>0$ and $t\to 0$. Then

$$g_2 = \frac{1+O(t)}{3(4t)^{\frac{1}{3}}},\quad g_3 = -\frac{1+O(t)}{27(4t)^{\frac{1}{2}}},\quad J = \frac{1+O(t)}{108t},$$

and conversely,
$$t = \frac{1+O(J^{-1})}{108J}.$$

It is clear that $g_3<0$ for all sufficiently small values of t, i.e. for all sufficiently large values of J; and therefore $\epsilon = \operatorname{sgn} g_3 = -1.$

Now $f(x,y)$ takes the form

$$f(x,y) = \frac{2(x^2+y^2)(x^2+ty^2)}{\{4t(1-t)^4\}^{\frac{1}{6}}}.$$

Put
$$x = t^{\frac{1}{6}}X,\quad y = t^{-1/12}Y;$$

these formulae define a linear substitution of determinant $t^{1/12}$. By this substitution, $f(x,y)$ is changed into the new form

$$f(x,y) = F(X,Y) = \left(\frac{16}{(1-t)^4}\right)^{\frac{1}{6}}\{(1+t)X^2Y^2+\sqrt{t}(X^4+Y^4)\},$$

and K_f is changed into the new star domain

$$(K_f')\qquad F(X,Y)\leqslant 1,\quad\text{i.e.}\quad X^2Y^2 \leqslant \frac{1}{1+t}\left\{\left(\frac{(1-t)^4}{16}\right)^{\frac{1}{6}} - \sqrt{t}(X^4+Y^4)\right\}.$$

By the relation connecting K_f with K_f',

$$D_{-1}(J) = \Delta(K_f) = t^{1/12}\Delta(K_f').$$

Now K_f' is contained in the star domain

$$(H_1)\qquad |XY| \leqslant 2^{-\frac{1}{3}}.$$

On the other hand, if $t>0$ is sufficiently small, $c>0$ denotes a suitable absolute constant, and X, Y are bounded independently of t and c, then

$$\frac{1}{1+t}\left\{\left(\frac{(1-t)^4}{16}\right)^{\frac{1}{6}} - \sqrt{t}(X^4+Y^4)\right\} \geqslant 4^{-\frac{1}{3}}(1-c\sqrt{t})^2$$

in K_f'. Hence K_f' contains the star domain

(H_2) $$|XY| \leqslant 2^{-\frac{1}{3}}(1-c\sqrt{t})^2, \quad |X+Y| \leqslant 2^{-\frac{1}{3}}\sqrt{5}(1-c\sqrt{t}),$$

for which obviously X, Y are bounded independently of t and c.

Further, by Hurwitz's theorem,

$$\Delta(H_1) = 2^{-\frac{1}{3}}\sqrt{5},$$

and by Theorem 1 of Part I, $\Delta(H_2) = 2^{-\frac{1}{3}}\sqrt{5}(1-ct^{\frac{1}{2}})^2$.

Hence, finally, $$2^{-\frac{1}{3}}\sqrt{5}(1-ct^{\frac{1}{2}})^2 t^{1/12} \leqslant D_{-1}(J) \leqslant 2^{-\frac{1}{3}}\sqrt{5}\, t^{1/12},$$

and, on replacing t by its value as a function of J,

(II) $$D_{-1}(J) = (\tfrac{25}{12})^{\frac{1}{4}} J^{-1/12}\{1+O(J^{-\frac{1}{2}})\}$$

as $J \to \infty$.

5. *The minimum of $f(x,y)$*. By definition, $D_\epsilon(J)$ is the lower bound of the determinants of the K_f-admissible lattices. Consider now the similar domain

$(K_{f,s})$ $$f(\bar{x},\bar{y}) \leqslant s,$$

where $s > 0$. Since it can be obtained from K_f by the linear substitution

$$\bar{x} = s^{\frac{1}{4}}x, \quad \bar{y} = s^{\frac{1}{4}}y,$$

we find the equation $$\Delta(K_{f,s}) = s^{\frac{1}{2}}D_\epsilon(J).$$

Hence, if in particular $s = D_\epsilon(J)^{-2}$, then $\Delta(K_{f,s}) = 1$, i.e. *every lattice of determinant* 1 *contains at least one point, other than the origin, of the domain*

$$f(x,y) \leqslant D_\epsilon(J)^{-2}.$$

Further, there exist lattices of determinant 1 *which contain no inner points, other than the origin, of this domain.*

All lattices of determinant 1 can be obtained from the lattice of all points with integral coordinates by linear substitutions of unit determinant. Hence, by the invariance of J and ϵ, and by the formulae (I) and (II), we arrive at the following result.

THEOREM 5. *Let $\Sigma_\epsilon(J)$ be the set of all positive definite binary quartic forms $f(x,y)$ of discriminant $G = 1$, absolute invariant J, and of invariant g_3 satisfying*

$$\operatorname{sgn} g_3 = \epsilon.$$

Let further $m(f)$ be the minimum of $f(x,y)$ for all pairs of integers x, y not both zero, and let $M_\epsilon(J)$ be the upper bound of $m(f)$ extended over all forms in $\Sigma_\epsilon(J)$. Then, when J tends to infinity,

$$M_\epsilon(J) = \begin{cases} \sqrt{(\frac{4}{3})}\, J^{\frac{1}{6}}\{1+O(J^{-\frac{1}{2}})\} & \text{if} \quad \epsilon = +1, \\ \sqrt{(\frac{12}{25})}\, J^{\frac{1}{6}}\{1+O(J^{-\frac{1}{2}})\} & \text{if} \quad \epsilon = -1. \end{cases}$$

In both cases there exist forms $f(x,y)$ such that $m(f) = M_\epsilon(J)$.

This theorem shows that $M_\epsilon(J)$ tends to infinity with J. In the other direction, it is not difficult to show that $M_\epsilon(J)$ has a positive lower bound, namely,

$$M_\epsilon(J) \geqslant \tfrac{1}{5}(432)^{\frac{1}{6}}.$$

But this is presumably not the exact lower bound, and there is a possibility that this lower bound is attained if $J = 1$.

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