LATTICE POINTS IN TWO-DIMENSIONAL STAR DOMAINS (III)

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[Received 14 May, 1942.—Read 21 May, 1942.]

If
$$f(x, y) = ax^2+2bxy+cy^2$$
is a positive definite binary quadratic form of determinant
$$ac-b^2 = 1,$$
and E denotes the domain
$$f(x, y) \leqslant 1,$$
bounded by the ellipse $f(x, y) = 1$, then by a classical result†,
$$\Delta(E) = \sqrt{\tfrac{3}{4}}.$$
There exists a continuous infinity of critical lattices Λ. Every such lattice contains just six points $\pm P_1$, $\pm P_2$, $\pm P_3$ on the boundary of E. It is possible to choose the notation such that
$$P_1+P_2+P_3 = O.$$
Conversely, six arbitrary boundary points of this type generate a critical lattice, any two independent points among them forming a basis.

The present fourth chapter of this paper deals with the more complicated domain K obtained by combining two concentric ellipses each of area π. An algorithm is developed for determining $\Delta(K)$, which turns out to be a rather complicated function of the simultaneous invariant of the two ellipses.

A similar method can be applied to all domains obtained by combining two convex domains with centre at O, *e.g.* the star-shaped octagon investigated by Prof. Mordell.

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† Bachmann, *Quadratische Formen*, II (Leipzig und Berlin, 1923), Kap. 5.

CHAPTER IV. THE DOMAIN BOUNDED BY TWO ELLIPSES.

25. *The invariant J.*

Let

(50) $$f_1(x, y) = a_1 x^2 + 2b_1 xy + c_1 y^2 \quad \text{and} \quad f_2(x, y) = a_2 x^2 + 2b_2 xy + c_2 y^2$$

be two positive definite binary quadratic forms of determinants

(51) $$a_1 c_1 - b_1^2 = a_2 c_2 - b_2^2 = 1.$$

Further, let

(52) $$J = a_1 c_2 - 2b_1 b_2 + c_1 a_2$$

be the *simultaneous invariant* of these two forms. If an affine transformation of determinant unity,

(53) $$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y', \quad \text{where} \quad \alpha\delta - \beta\gamma = 1,$$

changes f_1 and f_2 into the new forms

$$f_1'(x', y') = a_1' x'^2 + 2b_1' x' y' + c_1' y'^2$$

and

$$f_2'(x', y') = a_2' x'^2 + 2b_2' x' y' + c_2' y'^2,$$

then by the invariantive property of the determinants and of J,

$$a_1' c_1' - b_1'^2 = a_2' c_2' - b_2'^2 = 1, \quad a_1' c_2' - 2b_1' b_2' + c_1' a_2' = J.$$

It is always possible to choose the transformation (53) so that f_1' and f_2' take the canonical forms

(54) $$f_1'(x', y') = x'^2 + y'^2 \quad \text{and} \quad f_2'(x', y') = \lambda x'^2 + \frac{1}{\lambda} y'^2,$$

where λ is a positive number. In this case

(55) $$J = \lambda + \frac{1}{\lambda}.$$

I assume in this chapter that f_1 and f_2, and so also f_1' and f_2', are *not identical*. Hence $\lambda \neq 1$, and therefore, from (55),

(56) $$J > 2.$$

We may further suppose without loss of generality that $\lambda > 1$.

26. *The domain K.*

Let now K be the domain of all points (x, y) satisfying at least one of the two inequalities

$$f_1(x, y) \leqslant 1 \quad \text{and} \quad f_2(x, y) \leqslant 1.$$

Hence K is formed by combining two concentric ellipses each of area π. It is evident that K is a simple star domain; we can then consider the lower bound $\Delta(K)$.

The affine transformation (53) changes K into a domain K' formed by the points (x', y') satisfying at least one of the inequalities

$$f_1'(x', y') \leqslant 1 \quad \text{and} \quad f_2'(x', y') \leqslant 1.$$

Hence K' is of the same type as K.

We can assert that

$$\Delta(K) = \Delta(K'). \tag{57}$$

For (53) changes K-admissible lattices into K'-admissible lattices, and critical lattices of K into critical lattices of K'; and it leaves the determinant of two points and so also the determinant of a lattice invariant.

Choose the transformation (53) so that f_1, f_2 change into the two forms (54). Then K' becomes the set of all points (x', y') for which at least one of the inequalities

$$x'^2+y'^2 \leqslant 1 \quad \text{and} \quad \lambda x'^2+\frac{1}{\lambda}\, y'^2 \leqslant 1$$

holds. Here λ is determined uniquely as a function of J by

$$\lambda = \frac{J+\sqrt{(J^2-4)}}{2}.$$

Hence the lower bound $\Delta(K) = \Delta(K')$ becomes a function of J, say

$$\Delta(K) = D(J). \tag{58}$$

27. *A property of the critical lattices.*

By the last paragraph, we may assume from now on that

$$f_1(x, y) = x^2+y^2, \quad f_2(x, y) = \lambda x^2+\frac{1}{\lambda}\, y^2.$$

The two ellipses $f_1 = 1$ and $f_2 = 1$ intersect at the four points

$$Q_1: (\mu, \nu), \quad Q_2: (-\mu, \nu), \quad Q_3: (-\mu, -\nu), \quad Q_4: (\mu, -\nu),$$

where

$$\mu = \sqrt{\left(\frac{1}{\lambda+1}\right)}, \quad \nu = \sqrt{\left(\frac{\lambda}{\lambda+1}\right)}.$$

Denote by C_1 and C_2 those arcs of $f_1 = 1$ and $f_2 = 1$, respectively, which together form the boundary $C = C_1 + C_2$ of K. Hence, on describing C in a positive direction, the arc of C

from Q_4 to Q_1 belongs to C_1,

from Q_1 to Q_2 belongs to C_2,

from Q_2 to Q_3 belongs to C_1,

from Q_3 to Q_4 belongs to C_2.

We use the convention of counting every one of the four points Q_1, Q_2, Q_3, Q_4 twice, once in C_1 and once in C_2.

The affine transformation of determinant unity,

$$x \to \lambda^{-\frac{1}{2}} y, \quad y \to \lambda^{\frac{1}{2}} x, \tag{59}$$

evidently transforms K into itself, interchanges the parts C_1 and C_2 of C, and permutes the points Q_1, Q_2, Q_3, Q_4 cyclically, and by the last paragraph it changes critical lattices again into critical lattices. Hence to every critical lattice with just m points on C_1 and n points on C_2 there corresponds a second critical lattice with just n points on C_1 and m points on C_2.

THEOREM 23. *A critical lattice Λ of K has at most six points on C_1. If it contains six points on C_1, then these are of the form $\pm P_1$, $\pm P_2$, $\pm P_3$, where $P_1 + P_2 + P_3 = O$. Further,*

$$\Delta(K) = d(\Lambda) = \sqrt{\tfrac{3}{4}}, \tag{60}$$

and there are also six lattice points of the same type on C_2†.

Proof. The lattice Λ is admissible with respect to the circle $f_1 \leqslant 1$, and so, by the introduction, cannot contain more than six points on its boundary. If it has six points on C_1, then these are of the mentioned form, and the lattice is critical with respect to the circle; hence (60) is satisfied. Then Λ must also be critical with respect to the ellipse $f_2 \leqslant 1$; for otherwise, since $d(\Lambda) = \sqrt{\frac{3}{4}}$, at least one lattice point $P \neq O$ would be an *inner* point of the ellipse and so also an inner point of K. Hence there are also exactly six points of Λ on C_2.

THEOREM 24. *Let Λ be a critical lattice with less than six points on C_1. Then there are just four lattice points $\pm P_1$, $\pm P_2$ on C_1, and four lattice points $\pm P_3$, $\pm P_4$ on C_2†.*

† It is possible for some of the lattice points on C_1 to be identical with lattice points on C_2 This happens when some of the points Q_1, Q_2, Q_3, Q_4 are lattice points

Proof. First, let Λ be a singular lattice. Then, by Theorem 14, its only points on C are Q_1, Q_2, Q_3, Q_4; the assertion is therefore true. Secondly, let Λ be regular; then it has at least six points on C. We may assume, by the last theorem, that there are just four points of Λ on C_1; otherwise we apply the transformation (59) and thus obtain a regular lattice with this property.

Let, then, the four lattice points on C_1 be $\pm P_1$, $\pm P_2$, and assume that there are only two symmetrical lattice points $\pm P_3$ on C_2. Then at most one of the two pairs of symmetrical points Q_1, Q_3 and Q_2, Q_4 belong to Λ. Hence there exists a sufficiently small angle α such that the rotation

$$x \to x \cos\alpha - y \sin\alpha, \quad y \to x \sin\alpha + y \cos\alpha$$

changes Λ into a new lattice Λ^* with only four points $\pm P_1^*$, $\pm P_2^*$ on C_1 and containing no further points $P \neq O$ of K. This lattice is therefore K-admissible, but not critical. Hence there exist lattices of smaller determinants. But this is impossible, since obviously $d(\Lambda^*) = d(\Lambda)$.

By Theorem 11, any two points of Λ on C_1, or any two such points on C_2, form a basis. Hence, if for brevity we write

$$(61) \qquad Y = D(J), \quad \text{then} \quad \sqrt{\tfrac{3}{4}} \leqslant Y \leqslant 1.$$

For K contains the circle $f_1 = 1$; further, $|(P, Q)| \leqslant 1$ for any two points P and Q on C_1, or on C_2.

28. *A sufficient condition for admissible lattices.*

The construction of the critical lattices of K makes use of

THEOREM 25. *Suppose that the lattice Λ of determinant*

$$d(\Lambda) \geqslant \sqrt{\tfrac{3}{4}}$$

has a basis consisting of two points P_1, P_2 on $f_1 = 1$, and a second basis consisting of two points P_3, P_4 on $f_2 = 1$. Then Λ is K-admissible.

Proof. It suffices to show that no lattice point $P \neq O$ is an inner point of $f_2 = 1$; the analogous result for $f_1 \leqslant 1$ is proved similarly.

Every point $P:(x, y)$ can be written as

$$(62) \qquad P = uP_3 + vP_4, \quad \text{where} \quad u = \frac{(P, P_4)}{(P_3, P_4)}, \quad v = -\frac{(P, P_3)}{(P_3, P_4)}.$$

The new coordinates u, v are integers if, and only if, P is a lattice point. The result of replacing x, y by u, v is that f_2 takes the form

(63) $$f_2(x, y) = f_2^*(u, v) = u^2 + 2suv + v^2,$$

since the two points $u = 1$, $v = 0$ and $u = 0$, $v = 1$ lie on $f_2^* = 1$. By the invariance property of the determinant of a quadratic form,

(64) $$1 - s^2 = (P_3, P_4)^2 = d(\Lambda)^2 \geqslant \tfrac{3}{4},$$

so that

(65) $$-\tfrac{1}{2} \leqslant s \leqslant \tfrac{1}{2}.$$

Hence f_2^* is a reduced form†. Its minimum for integral u, v not both zero is then 1, as asserted.

Henceforth let $S(J)$ be the set of lattices Λ with the following properties:

(a) *Λ has a basis P_1, P_2 on $f_1 = 1$, and a basis P_3, P_4 on $f_2 = 1$.*

(b) *The determinant $d(\Lambda) \geqslant \sqrt{\tfrac{3}{4}}$.*

We shall prove later that $S(J)$ *has only a finite number of elements*, say the lattices

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_n.$$

By Theorem 25, these lattices are K-admissible; by Theorems 23 and 24, all critical lattices Λ belong to $S(J)$. Hence

(66) $$D(J) = \min_{\nu = 1, 2, \ldots, n} d(\Lambda_\nu),$$

and so the critical lattices of K are just those elements Λ_ν of $S(J)$ for which $d(\Lambda_\nu)$ assumes the minimum value $D(J)$.

29. *Construction of the set $S(J)$.*

Let Λ be a lattice in $S(J)$. We may assume, without loss of generality, that the two bases

$$P_1 : (x_1, y_1),\ P_2 : (x_2, y_2) \quad \text{and} \quad P_3 : (x_3, y_3),\ P_4 : (x_4, y_4)$$

of Λ satisfy the inequalities

(67) $$(P_1, P_2) > 0 \quad \text{and} \quad (P_3, P_4) > 0;$$

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† See footnote †, page 168.

hence

(68) $\qquad d(\Lambda) = (P_1, P_2) = (P_3, P_4) = x_1 y_2 - x_2 y_1 = x_3 y_4 - x_4 y_3.$

The inequalities (67) remain satisfied if the pair of points P_1, P_2 is replaced by one of the four pairs

$$P_1, P_2, \quad \text{or} \quad P_2, -P_1, \quad \text{or} \quad -P_1, -P_2, \quad \text{or} \quad -P_2, P_1;$$

and if the pair of points P_3, P_4 is replaced by one of the four pairs

$$P_3, P_4, \quad \text{or} \quad P_4, -P_3, \quad \text{or} \quad -P_3, -P_4, \quad \text{or} \quad -P_4, P_3.$$

This gives a set Ω of $4 \times 4 = 16$ pairs of bases of Λ.

By the basis property and by (68), there are four integers α_1, β_1, α_2, β_2 such that

(69) $\qquad P_3 = \alpha_1 P_1 + \beta_1 P_2, \quad P_4 = \alpha_2 P_1 + \beta_2 P_2, \quad \alpha_1 \beta_2 - \alpha_2 \beta_1 = +1.$

When the pair of bases P_1, P_2 and P_3, P_4 is replaced by one of the other pairs in Ω, then α_1, β_1, α_2, β_2 undergo certain permutations and changes of signs, for which I refer to the following table.

The 16 *elements of* Ω.

P_1	P_2	P_3	P_4	α_1	β_1	α_2	β_2	X	Y	u	v	s	1
P_1	P_2	P_4	$-P_3$	α_2	β_2	$-\alpha_1$	$-\beta_1$	X	Y	v	$-u$	$-s$	2
P_1	P_2	$-P_3$	$-P_4$	$-\alpha_1$	$-\beta_1$	$-\alpha_2$	$-\beta_2$	X	Y	$-u$	$-v$	s	3
P_1	P_2	$-P_4$	P_3	$-\alpha_2$	$-\beta_2$	α_1	β_1	X	Y	$-v$	u	$-s$	4
P_2	$-P_1$	P_3	P_4	β_1	$-\alpha_1$	β_2	$-\alpha_2$	$-X$	Y	u	v	s	5
P_2	$-P_1$	P_4	$-P_3$	β_2	$-\alpha_2$	$-\beta_1$	α_1	$-X$	Y	v	$-u$	$-s$	6
P_2	$-P_1$	$-P_3$	$-P_4$	$-\beta_1$	α_1	$-\beta_2$	α_2	$-X$	Y	$-u$	$-v$	s	7
P_2	$-P_1$	$-P_4$	P_3	$-\beta_2$	α_2	β_1	$-\alpha_1$	$-X$	Y	$-v$	u	$-s$	8
$-P_1$	$-P_2$	P_3	P_4	$-\alpha_1$	$-\beta_1$	$-\alpha_2$	$-\beta_2$	X	Y	u	v	s	9
$-P_1$	$-P_2$	P_4	$-P_3$	$-\alpha_2$	$-\beta_2$	α_1	β_1	X	Y	v	$-u$	$-s$	10
$-P_1$	$-P_2$	$-P_3$	$-P_4$	α_1	β_1	α_2	β_2	X	Y	$-u$	$-v$	s	11
$-P_1$	$-P_2$	$-P_4$	P_3	α_2	β_2	$-\alpha_1$	$-\beta_1$	X	Y	$-v$	u	$-s$	12
$-P_2$	P_1	P_3	P_4	$-\beta_1$	α_1	$-\beta_2$	α_2	$-X$	Y	u	v	s	13
$-P_2$	P_1	P_4	$-P_3$	$-\beta_2$	α_2	β_1	$-\alpha_1$	$-X$	Y	v	$-u$	$-s$	14
$-P_2$	P_1	$-P_3$	$-P_4$	β_1	$-\alpha_1$	β_2	$-\alpha_2$	$-X$	Y	$-u$	$-v$	s	15
$-P_2$	P_1	$-P_4$	P_3	β_2	$-\alpha_2$	$-\beta_1$	α_1	$-X$	Y	$-v$	u	$-s$	16
1	2	3	4	5	6	7	8	9	10	11	12	13	

Let a new system of rectangular coordinates U, V be defined by

(70) $$x = x_1 U - y_1 V, \quad y = y_1 U + x_1 V,$$

or conversely, since $x_1^2 + y_1^2 = 1$,

(71) $$U = x_1 x + y_1 y, \quad V = -y_1 x + x_1 y.$$

In this system, P_1 and P_2 have the coordinates

$$U_1 = 1, \ V_1 = 0 \quad \text{and} \quad U_2 = X = x_1 x_2 + y_1 y_2, \ V_2 = Y = x_1 y_2 - x_2 y_1;$$

here

(72) $$X^2 + Y^2 = 1, \quad Y = d(\Lambda) > 0.$$

Further, by (69), the coordinates of P_3 and P_4 are given by

$$U_3 = \alpha_1 + \beta_1 X, \ V_3 = \beta_1 Y \quad \text{and} \quad U_4 = \alpha_2 + \beta_2 X, \ V_4 = \beta_2 Y.$$

Finally, if, as in § 28, we introduce u, v by (62), then

$$U = (\alpha_1 + \beta_1 X) u + (\alpha_2 + \beta_2 X) v,$$
$$V = \qquad \beta_1 X \ u + \qquad \beta_2 Y \ v,$$

and so, on solving for u and v, we have

(73) $$\begin{cases} Yu = +\beta_2 YU - (\alpha_2 + \beta_2 X) V. \\ Yv = -\beta_1 YU + (\alpha_1 + \beta_1 X) V. \end{cases}$$

I refer to the last table for the changes of these numbers α_1, β_1, α_2, β_2, X, Y, u, v, when the pair of bases P_1, P_2 and P_3, P_4 is replaced by another pair in Ω.

By § 28, f_2 takes the form (63) in u and v. By (64) and (72),

(74) $$s = \epsilon X, \quad \text{where} \quad \epsilon = \pm 1.$$

An inspection of the table shows that it is always possible to choose the pair of bases P_1, P_2 and P_3, P_4 in Ω so that the following inequalities are satisfied:

(75) $$X \geqslant 0, \quad s \geqslant 0, \quad \alpha_1 \geqslant 0.$$

Therefore, in particular,

(76) $$s = X.$$

Replace u and v by U and V. Then f_2 changes into

(77) $$f_2(x, y) = F_2(U, V) = AU^2 + 2BUV + CV^2,$$

where, by (63), (73), and (76),

$$(78)\begin{cases} A = \beta_1^2 - 2\beta_1\beta_2 X + \beta_2^2, \\ YB = -\beta_1(\alpha_1+\beta_1 X) + X\{\beta_2(\alpha_1+\beta_1 X)+\beta_1(\alpha_2+\beta_2 X)\} - \beta_2(\alpha_2+\beta_2 X), \\ Y^2 C = (\alpha_1+\beta_1 X)^2 - 2(\alpha_1+\beta_1 X)(\alpha_2+\beta_2 X)X + (\alpha_2+\beta_2 X)^2. \end{cases}$$

Further, since the change from x, y to U, V is an orthogonal transformation,

$$f_1(x, y) = F_1(U, V) = U^2 + V^2.$$

Hence the simultaneous invariant

$$J = A + C,$$

so that, by (72) and (78),

$$(79)\quad (\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-J) - 2(\alpha_1-\beta_2)(\alpha_2-\beta_1)X - \{2(\alpha_1\beta_2+\alpha_2\beta_1)-J\}X^2 = 0.$$

For given J, this is a quadratic equation for X. *It does not reduce to an identity*, for then

$$\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2 = J, \quad 2(\alpha_1\beta_2+\alpha_2\beta_1) = J;$$

hence

$$(\alpha_1-\beta_2)^2 + (\alpha_2-\beta_1)^2 = 0,$$

and since $\alpha_1 \geqslant 0$, $\alpha_1\beta_2 - \alpha_2\beta_1 = 1$,

$$\alpha_1 = \beta_2 = 1, \quad \alpha_2 = \beta_1 = 0, \quad J = 2.$$

This value of J was, however, excluded by § 25.

By the assumption (*b*) in § 28, and by (72) and (75),

$$(80)\qquad 0 \leqslant X \leqslant \tfrac{1}{2}.$$

Suppose now, conversely, that (79) has a solution X satisfying these inequalities. Then the coefficients A, B, C of F are given by (78), with

$$(81)\qquad Y = |\sqrt{(1-X^2)}|.$$

We further obtain the (U, V)-coordinates of P_1, P_2, P_3, P_4 from their expressions as functions of α_1, β_1, α_2, β_2, X, Y. There remains the reduction of $F_1(U, V)$ and $F_2(U, V)$ to the normal form (54) by means of an orthogonal transformation (71); this problem is dealt with in the theory of conics. After this reduction, the (x, y)-coordinates of P_1, P_2, P_3, P_4 and so the lattice Λ are known.

Therefore, in order to construct all elements of $S(J)$, it suffices to solve (79) with respect to X. Here the coefficients α_1, β_1, α_2, β_2 must take all integral values with

$$(82)\qquad \alpha_1 \geqslant 0, \quad \alpha_1\beta_2 - \alpha_2\beta_1 = 1,$$

for which both (79) and (80) can be satisfied.

30. *The finiteness of* $S(J)$.

THEOREM 26. *The set* $S(J)$ *has only a finite number of elements.*

Proof. It suffices to show that the conditions (79) and (80) are solvable for at most a finite number of sets of integers $\alpha_1, \beta_1, \alpha_2, \beta_2$.

The equation (79) can be written as

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = J, \tag{83}$$

where

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{(\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2)-2(\alpha_1-\beta_2)(\alpha_2-\beta_1)X-2(\alpha_1\beta_2+\alpha_2\beta_1)X^2}{1-X^2}.$$

This expression Φ is a *positive definite* quadratic form in $\alpha_1, \beta_1, \alpha_2, \beta_2$; for it can be written as

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{1}{1-X^2}(\alpha_1-X^2\beta_2-X\alpha_2+X\beta_1)^2+(1+X^2)\left(\beta_2+\frac{X}{1+X^2}\alpha_2-\frac{X}{1+X^2}\beta_1\right)^2 + \frac{1}{1+X^2}(\alpha_2+X^2\beta_1)^2+(1-X^2)\beta_1^2.$$

From this identity, by (80),

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) \geqslant (1-X^2)\beta_1^2 \geqslant \tfrac{3}{4}\beta_1^2.$$

Further, from the definition of Φ,

$$\Phi(X;\ \alpha_1, \beta_1, \alpha_2, \beta_2) = \Phi(X;\ \beta_1, \alpha_1, \beta_2, \alpha_2) = \Phi(X;\ \alpha_2, \beta_2, \alpha_1, \beta_1) = \Phi(X;\ \beta_2, \alpha_2, \beta_1, \alpha_1).$$

Hence β_1 may be replaced by $\alpha_1, \beta_1, \alpha_2, \beta_2$ in the last inequality, and so, by (83),

$$\max(\alpha_1^2, \beta_1^2, \alpha_2^2, \beta_2^2) \leqslant \frac{4J}{3}, \tag{84}$$

which proves the assertion.

Let then Λ_ν $(\nu = 1, 2, \ldots, n)$

be the elements of $S(J)$; let

$$\alpha_1^{(\nu)}, \beta_1^{(\nu)}, \alpha_2^{(\nu)}, \beta_2^{(\nu)} \qquad (\nu = 1, 2, \ldots, n)$$

be the sets of four integers; and let

$$\Phi_\nu(X) = \Phi(X;\ \alpha_1^{(\nu)}, \beta_1^{(\nu)}, \alpha_2^{(\nu)}, \beta_2^{(\nu)}) \quad (\nu = 1, 2, \ldots, n)$$

be the functions belonging to these lattices. The following table contains all functions Φ_ν which represent at least one value of J in $2 \leqslant J \leqslant 25$ for an argument X in $0 \leqslant X \leqslant \frac{1}{2}$.

Table of all functions Φ which represent J for J ⩽ 25.

$\Phi(\frac{1}{2})$	$(1-X^2)\,\Phi(X;\alpha_1,\beta_1,\alpha_2,\beta_2)$	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2	α_1	β_1	α_2	β_2
2	$2-2X^2$ †	1	0	0	1												
	$3-4X+2X^2$	0	1	−1	1	0	−1	1	−1	1	−1	1	0				
$\frac{10}{3}$	$2+2X^2$ ‡	0	1	−1	0	0	−1	1	0								
	$3-2X^2$	1	1	0	1	1	0	1	1	1	−1	0	1	1	0	−1	1
	$6-8X+2X^2$	0	1	−1	2	0	−1	1	−2	2	−1	1	0				
	$7-12X+6X^2$	1	−2	1	−1	1	−1	2	−1								
$\frac{22}{3}$	$3+4X+2X^2$	0	1	−1	−1	0	−1	1	1	1	1	−1	0				
	$6-2X^2$	1	2	0	1	1	−2	0	1	1	0	2	1	1	0	−2	1
	$7-6X^2$	1	1	1	2	1	−1	−1	2	2	1	1	1	2	−1	−1	1
	$11-12X+2X^2$	0	1	−1	3	0	−1	1	−3	3	−1	1	0				
	$15-24X+10X^2$	1	−3	1	−2	1	−1	3	−2	2	−3	1	−1	2	−1	3	−1
	$18-32X+14X^2$	1	−2	2	−3	3	−2	2	−1								
14	$6+8X+2X^2$	0	1	−1	−2	0	−1	1	2	2	1	−1	0				
	$11-2X^2$	1	3	0	1	1	−3	0	1	1	0	3	1	1	0	−3	1
	$15-4X-10X^2$	1	2	1	3	1	−1	−2	2	3	1	2	1	3	−2	−1	1
	$18-16X+2X^2$	0	1	−1	4	0	−1	1	4	4	−1	1	0				
	$27-40X+14X^2$	1	−4	1	−3	1	−1	4	−3	3	−4	1	−1	3	−1	4	−1
	$38-72X+34X^2$	2	−3	3	−4	4	−3	3	−2								
$\frac{58}{3}$	$7+12X+6X^2$	1	2	−1	−1	1	1	−2	−1								
	$15+4X-10X^2$	1	−2	−1	3	1	1	2	3	3	2	1	1	3	−1	−2	1
	$18-14X^2$	2	3	1	2	2	−3	−1	2	2	1	3	2	2	−1	−3	2
	$34-48X+18X^2$	2	−5	1	−2	2	−1	5	−2								
	$39-60X+22X^2$	1	−3	2	−5	1	−2	3	−5	5	−3	2	−1	5	−2	3	−1
	$47-84X+38X^2$	3	−5	2	−3	3	−2	5	−3								
$\frac{70}{3}$	$11+12X+2X^2$	0	1	−1	−3	0	−1	1	3	3	1	−1	0				
	$18-2X^2$	1	4	0	1	1	−4	0	1	1	0	4	1	1	0	−4	1
	$27-20X+2X^2$	0	1	−1	5	0	−1	1	−5	5	−1	1	0				
	$27-12X-14X^2$	1	3	1	4	1	−1	−3	4	4	−3	−1	1	4	1	3	1
	$43-60X+18X^2$	1	−5	1	−4	1	−1	5	−4	4	−5	1	−1	4	−1	5	−1
	$66-128X+62X^2$	3	−4	4	−5	5	−4	4	−3								

Excluded case.
Singular lattices.

As this table shows, there are in general two, three, or four systems of integers $\alpha_1^{(\nu)}$, $\beta_1^{(\nu)}$, $\alpha_2^{(\nu)}$, $\beta_2^{(\nu)}$ belonging to the same function Φ_ν and so also an equal number of lattices Λ_ν†. It is easily seen that if there are different critical lattices belonging to the same function Φ_ν, then these are transformed into one another by the *group* G *of order* 4 generated by the following two affine transformations:

The symmetry in the y-axis,

$$\text{A}: \qquad x \to -x, \quad y \to y.$$

The interchange of $f_1 = 1$ *and* $f_2 = 1$,

$$\text{B}: \qquad x \to \lambda^{-\frac{1}{2}} y, \quad y \to \lambda^{\frac{1}{2}} x.$$

For A replaces the integers α_1, β_1, α_2, β_2 by

$$\epsilon\beta_2, \ \epsilon\alpha_2, \ \epsilon\beta_1, \ \epsilon\alpha_1,$$

where $\epsilon = \pm 1$ is such that $\epsilon\beta_2 \geqslant 0$, and B replaces them by

$$\alpha_1, \ -\alpha_2, \ -\beta_1, \ \beta_2.$$

From now on, two critical lattices are considered as equivalent if they are related by an element of this group G; equivalent lattices belong to the same function Φ_ν.

31. *The value of* $D(J)$ *for* $2 \leqslant J \leqslant 25$.

By formula (66) in § 28,

$$D(J) = \min_{\nu = 1, 2, \ldots, n} d(\Lambda_\nu).$$

Hence, if

$$Y = D(J), \ X = |\sqrt{(1-Y^2)}|, \quad \text{and} \quad Y_\nu = d(\Lambda_\nu), \ X_\nu = |\sqrt{(1-Y_\nu^2}|,$$

then

$$\Phi_\nu(X_\nu) = J, \quad 0 \leqslant X_\nu \leqslant \tfrac{1}{2}, \tag{85}$$

$$X = \max_{\nu = 1, 2, \ldots, n} X_\nu. \tag{86}$$

† Two systems of integers

$$0, \ 1, \ -1, \ \beta_2^{(\nu)} \quad \text{and} \quad 0, \ -1, \ 1, \ -\beta_2^{(\nu)}$$

are interchanged by elements of Ω (§ 29) and generate the same lattice.

By a study of the last table I find that for every J in $2 \leqslant J \leqslant 25$ and for every Φ_ν there is *at most one* solution X_ν of (85). Further, most of these solutions X_ν can be ignored for the following reasons.

The rows of the table have been arranged in sets of functions

$$(1-X^2)\,\Phi_\nu(X)$$

so that $\Phi_\nu(\tfrac{1}{2})$ is the same in each set. It was also found possible to arrange the rows according to *increasing values of these functions* for variable values of X; *e.g.*, in the second set,

$$\frac{2+2X^2}{1-X^2} \leqslant \frac{3-2X^2}{1-X^2} \leqslant \frac{6-8X+2X^2}{1-X^2} \leqslant \frac{7-12X+8X^2}{1-X^2} \quad \text{for} \quad 0 \leqslant X \leqslant \tfrac{1}{2}.$$

Hence, for a given value of J in $2 \leqslant J \leqslant 25$, the maximum $X = X_\nu$ belongs to one of those 11 equations

$$\Phi_\nu(X_\nu) = J$$

in which the function Φ_ν is either at the *beginning* or at the *end* of one of the 6 sets of rows of the table. There is no difficulty in deciding which is the largest of these solutions X_ν. The result depends on the value of J, and is given in the following table. This table further contains the minimum determinant

$$D(J) = \Delta(K)$$

and the corresponding critical lattice†.

In the table, the numbers σ_k are defined thus:

$$\sigma_0 = 2, \quad \sigma_1 = \tfrac{10}{3}, \quad \sigma_2 = \tfrac{22}{3}, \quad \sigma_3 = 14, \quad \sigma_4 = \tfrac{58}{3}, \quad \sigma_5 = \tfrac{70}{3};$$

and J_n is defined thus

$$J_1 = \tfrac{34}{15}, \quad J_2 = \frac{3+14\sqrt{3}}{6}, \quad J_3 = 10, \quad J_4 = \frac{178+576\sqrt{14}}{143},$$

$$J_5 = \frac{63+88\sqrt{7}}{14}.$$

† If there exist several critical lattices, then they are all equivalent to the one given, except when J is one of the numbers σ_ν or J_ν.

$D(J)$ and critical lattices for $2 \leqslant J \leqslant 25$.

No.	Interval.	$(1-X^2)\,Y =$	$X =$	$D(J) = Y =$	Critical lattice.
1	$\sigma_0 \leqslant J \leqslant J_1$	$3-4X+2X^2$	$\frac{2-(J^2-J-2)^{\frac{1}{2}}}{J+2}$	$\frac{\{5J+2+4(J^2-J-2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1+P_2$
2	$J_1 \leqslant J \leqslant \sigma_1$	$2+2X^2$	$\left(\frac{J-2}{J+2}\right)^{\frac{1}{2}}$	$2(J+2)^{-\frac{1}{2}}$	$P_3 = P_2$† $P_4 = -P_1$
3	$\sigma_1 \leqslant J \leqslant J_2$	$7-12X+6X^2$	$\frac{6-(J^2-J-6)^{\frac{1}{2}}}{J+6}$	$\frac{\{13J+6+12(J^2-J-6)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+6}$	$P_3 = P_1-2P_2$ $P_4 = P_1-P_2$
4	$J_2 \leqslant J \leqslant \sigma_2$	$3+4X+2X^2$	$\frac{-2+(J^2-J-2)^{\frac{1}{2}}}{J+2}$	$\frac{\{5J+2+4(J^2-J-2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-P_2$
5	$\sigma_2 \leqslant J \leqslant J_3$	$18-32X+14X^2$	$-\frac{J-18}{J+14}$	$\frac{8(J-2)^{\frac{1}{2}}}{J+14}$	$P_3 = P_1-2P_2$ $P_4 = 2P_1-3P_2$
6	$J_3 \leqslant J \leqslant \sigma_3$	$6+8X+2X^2$	$\frac{J-6}{J+2}$	$\frac{4(J-2)^{\frac{1}{2}}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-2P_2$
7	$\sigma_3 \leqslant J \leqslant J_4$	$38-72X+34X^2$	$-\frac{J-38}{J+34}$	$\frac{12(J-2)^{\frac{1}{2}}}{J+34}$	$P_3 = 2P_1-3P_2$ $P_4 = 3P_1-4P_2$
8	$J_4 \leqslant J \leqslant \sigma_4$	$7+12X+6X^2$	$\frac{-6+(J^2-J-6)^{\frac{1}{2}}}{J+6}$	$\frac{\{13J+6+12(J^2-J-6)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+6}$	$P_3 = P_1+2P_2$ $P_4 = -P_1-P_2$
9	$\sigma_4 \leqslant J \leqslant J_5$	$47-84X+38X^2$	$\frac{42-(J^2-9J-22)^{\frac{1}{2}}}{J+38}$	$\frac{\{85J-298+84(J^2-9J-22)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+38}$	$P_3 = 3P_1-5P_2$ $P_4 = 2P_1-3P_2$
10	$J_5 \leqslant J \leqslant \sigma_5$	$11+12X+2X^2$	$\frac{-6+(J^2-9J+14)^{\frac{1}{2}}}{J+2}$	$\frac{\{13J-46+12(J^2-9J+14)^{\frac{1}{2}}\}^{\frac{1}{2}}}{J+2}$	$P_3 = P_2$ $P_4 = -P_1-3P_2$
11	$\sigma_5 \leqslant J \leqslant 25$	$66-128X+62X^2$	$-\frac{J-66}{J+62}$	$\frac{16(J-2)^{\frac{1}{2}}}{J+62}$	$P_3 = 3P_1-4P_2$‡ $P_4 = 4P_1-5P_2$

† Singular lattice.
‡ These values of X and Y remain true for $\sigma_5 \leqslant J \leqslant \frac{206}{7}$.

In the intervals No. 1–11 of the table, the functions $X = X(J)$ and $Y = Y(J)$ behave in the following manner:

$$\left.\begin{matrix} X \\ Y \end{matrix}\right\} \text{ is steadily } \left\{\begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix}\right\} \text{ in the intervals No. 2, 4, 6, 8, 10.}$$

$$\left.\begin{matrix} X \\ Y \end{matrix}\right\} \text{ is steadily } \left\{\begin{matrix} \text{decreasing} \\ \text{increasing} \end{matrix}\right\} \text{ in the intervals No. 1, 3, 5, 7, 9, 11.}$$

Further,

$$X = \tfrac{1}{2}, \quad Y = \frac{\sqrt{3}}{2} \quad \text{for} \quad J = \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5,$$

and

$$X = \tfrac{1}{4}, \qquad Y = \frac{\sqrt{15}}{4} \qquad \text{for} \quad J = J_1,$$

$$X = 2-\sqrt{3}, \qquad Y = \sqrt{\{4\sqrt{(3)}-6\}} \qquad \text{for} \quad J = J_2,$$

$$X = \tfrac{1}{3}, \qquad Y = \frac{2\sqrt{2}}{3} \qquad \text{for} \quad J = J_3,$$

$$X = \frac{21-4\sqrt{14}}{14}, \qquad Y = \sqrt{\left(\frac{24\sqrt{(14)}-67}{28}\right)} \qquad \text{for} \quad J = J_4,$$

$$X = \frac{4-\sqrt{7}}{3} \qquad Y = \sqrt{\left(\frac{8\sqrt{(7)}-14}{9}\right)} \qquad \text{for} \quad J = J_5.$$

The interval No. 2 is particularly interesting, since here K has only a single critical lattice, and this is singular. At the lower end $J = \frac{34}{15}$ of this interval, K has this singular lattice, and also the regular lattice

$$P_3 = P_2, \quad P_4 = -P_1+P_2,$$

and the lattice symmetrical to it in the y-axis.

The table shows that the critical lattices of K have 2, 3, 4, 5, or 6 pairs of symmetrical points on C, depending on the value of J.

The general law of the function $D(J)$ seems to be very complicated. By the table, the graph of $Y = D(J)$ is a saw-like curve for $2 \leqslant J \leqslant 25$, and possibly for all values of J. In the intervals No. 5, 6, 7, and 11, $D(J)$ takes a surprisingly simple form.

One can show that $\frac{\sqrt{3}}{2} \leqslant D(J) \leqslant \frac{\sqrt{15}}{4}$ for all values of J, and that

$$\lim_{J\to\infty} D(J) = \frac{\sqrt{3}}{2};$$

this limit equation was communicated to me by P. Erdös.

I remark finally that the problem and result of this chapter can be extended to a pair of positive definite Hermitian forms; but then the proof is preferably based on the geometrical theory of Picard's group.

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