SIMULTANEOUS DIOPHANTINE APPROXIMATION

By H. Davenport and K. Mahler

1. Hermite said that "La recherche des fractions p'/p, p''/p qui approchent le plus de deux nombres donnés n'a cessé depuis plus de 50 ans de me préoccuper et aussi de me désespérer". Though some progress has been made since the time of Hermite, notably by Minkowski, the problem of simultaneous Diophantine approximation to two given irrational numbers α, β by fractions of the same denominator is still, in most of its aspects, unsolved and apparently intractable. It is well known that there are constants c such that the inequalities

$$\left| \alpha - \frac{p}{r} \right| < \frac{c}{r^{3/2}}, \qquad \left| \beta - \frac{q}{r} \right| < \frac{c}{r^{3/2}}$$

have an infinity of solutions, but the best possible value of c is not known (see [2; 70–72]). In this note we use

$$\left(\alpha - \frac{p}{r}\right)^2 + \left(\beta - \frac{q}{r}\right)^2$$

as a measure of simultaneous approximation, and we show that in this formulation the problem can be substantially solved. We prove:

Theorem 1. (a) *If $c > 2/23^{1/2}$, and α, β are any two irrational numbers, there exist an infinity of fractions p/r, q/r for which*

$$\left(\alpha - \frac{p}{r}\right)^2 + \left(\beta - \frac{q}{r}\right)^2 < \frac{c}{r^3}.$$

(b) *This is false if $c < 2/23^{1/2}$.*

The same methods lead us to the closely related result:

Theorem 2. (a) *If $c > 2/23^{1/2}$, and α, β are any real numbers for which $1, \alpha, \beta$ are linearly independent, then there exist an infinity of sets of integers p, q, r satisfying*

$$| \alpha p + \beta q + r | < \frac{c}{p^2 + q^2} \qquad (p^2 + q^2 > 0).$$

(b) *This is false if $c < 2/23^{1/2}$.*

2. The proof of (a) in each case is based on the application of a principle due to Mahler [3; Theorem 10, Part I] to a theorem of Davenport. We proceed to give the proof of Theorem 1 (a). Let K be the body in three dimensional space defined by

$$(x^2 + y^2) \, | z | \leq 1,$$

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and let Δ denote the lower bound of the determinants of all lattices which have no point other than O in the interior of K. It was proved by Davenport (see [1], [4]) that

$$\Delta = \tfrac{1}{2}(23)^{\frac{1}{2}}. \tag{1}$$

Now let ϵ be any positive number, and let $K(\epsilon)$ be the body defined by

$$(x^2 + y^2) \mid z \mid \leq 1, \qquad x^2 + y^2 \leq \epsilon^2.$$

Let $\Delta(\epsilon)$ be the analogue of Δ for $K(\epsilon)$. If $t > 0$, the transformation

$$x = x'/t, \qquad y = y'/t, \qquad z = t^2 z',$$

which is of determinant 1, transforms $K(\epsilon)$ into $K(t\epsilon)$, hence we have (by the affine invariance of Δ)

$$\Delta(\epsilon) = \Delta(t\epsilon).$$

Now let $t \to \infty$. Each body $\Delta(t\epsilon)$ is contained in K, and on the other hand any bounded portion of K is contained in $K(t\epsilon)$ if t is sufficiently large. By the theorem of Mahler already mentioned (in the notation of that theorem, we are using here the case $F(X) = \{(x^2 + y^2) \mid z \mid\}^{\frac{1}{3}}$, $G(X) = \{(x^2 + y^2)/\epsilon^2\}^{\frac{1}{2}}$),

$$\lim_{t \to \infty} \Delta(t\epsilon) = \Delta.$$

Hence

$$\Delta(\epsilon) = \Delta = \tfrac{1}{2}(23)^{\frac{1}{2}}$$

for any $\epsilon > 0$.

This means that if $\mathfrak{L}$ is any lattice of determinant less than $\frac{1}{2}(23)^{\frac{1}{2}}$, there is a point of $\mathfrak{L}$ other than O which satisfies

$$(x^2 + y^2) \mid z \mid \leq 1, \qquad x^2 + y^2 \leq \epsilon^2.$$

If we apply this result to the lattice defined by

$$x = p - \alpha r, \qquad y = q - \beta r, \qquad z = r/c \qquad (c > 2/23^{\frac{1}{2}}),$$

where p, q, r take all integral values, we deduce that there exist integers p, q, r which satisfy

$$r\{(p - \alpha r)^2 + (q - \beta r)^2\} \leq c,$$
$$(p - \alpha r)^2 + (q - \beta r)^2 \leq \epsilon^2.$$

Since α, β are irrational, it follows from the second of these inequalities that $r \to \infty$ as $\epsilon \to 0$. This proves Theorem 1 (a).

The proof of Theorem 2 (a) is the same, except that in place of the condition $x^2 + y^2 \leq \epsilon^2$ we use $\mid z \mid \leq \epsilon$, and that we apply our result to the lattice defined by

$$x = p, \qquad y = q, \qquad z = (\alpha p + \beta q + r)/c \qquad (c > 2/23^{\frac{1}{2}}).$$

3. It is also possible to prove the parts (a) of our theorems without using Mahler's theorem, by an appropriate modification of Davenport's proof of (1). The underlying idea is the same as that used above, and we content ourselves with a brief indication for the case of Theorem 1 (a). We define M to be the *lower limit*, as $r \to \infty$, of

$$r\{(p - \alpha r)^2 + (q - \beta r)^2\}.$$

By proceeding as in Davenport's paper [1], with minor modifications, we obtain a lattice in x, y, z space which satisfies the conditions (a), (b), (c) on p. 100 of that paper, except that in place of (4) we have the weaker condition

$$\textit{either} \quad \{(x + z)^2 + y^2\} \mid z \mid > 1 - \epsilon \quad \textit{or} \quad x^2 + y^2 > K,$$

where K is arbitrarily large. This alteration does not affect the subsequent work of that paper.

4. We now proceed to the proof of Theorem 1 (b). We have to construct, for any $\epsilon > 0$, two real numbers α, β such that

$$\varliminf_{r\to\infty} r \{(p - \alpha r)^2 + (q - \beta r)^2\} > \frac{2}{23^{\frac{1}{2}}} - \epsilon,$$

where p, q are the integers nearest to αr and βr. The numbers α, β are necessarily irrational, for if one or both of them were rational, the lower limit would be zero.

The construction is based on properties of the cubic field $k(\phi)$, where $\phi = 1.3247\cdots$ is the real root of $t^3 - t - 1 = 0$. The conjugates of ϕ are $\theta = -\frac{1}{2}\phi + i\psi$ and $\bar{\theta} = -\frac{1}{2}\phi - i\psi$, where we can suppose $\psi > 0$. We have

$$(\phi - \theta)(\phi - \bar{\theta})(\theta - \bar{\theta}) = 23^{\frac{1}{2}}i, \tag{2}$$

since the discriminant of the cubic equation is -23. If ξ is any integer of $k(\phi)$, we denote by ξ' and $\bar{\xi}'$ its conjugates in $k(\theta)$ and $k(\bar{\theta})$ respectively. We require the following result:

LEMMA. *For any $\epsilon > 0$ there exist integers λ_1, λ_2, λ_3 of $k(\phi)$, given by*

$$\lambda_1 = u_1 + v_1\phi + w_1\phi^2, \qquad \lambda_2 = u_2 + v_2\phi + w_2\phi^2, \qquad \lambda_3 = u_3 + v_3\phi + w_3\phi^2, \tag{3}$$

where $u_1, \cdots, w_3$ are integers of determinant 1, *such that*

$$\left| \frac{\lambda_2'}{\lambda_1'} - i \right| < \epsilon. \tag{4}$$

Proof. Let N be a large positive integer. Choose u_1, v_1, w_1, u_2, v_2, w_2 as follows:

$$\begin{cases} u_1 = N, \qquad v_1 = 0, \qquad w_1 = 1, \qquad w_2 = 0, \\ v_2 \text{ is the prime nearest to } N/\psi, \\ u_2 \text{ is the integer nearest to } \frac{1}{2}\phi v_2 . \end{cases} \tag{5}$$

As $N \to \infty$, we have $v_2 \sim N/\psi$, $u_2 \sim \frac{1}{2}\phi v_2$. Now

$$\lambda_1' = u_1 + (-\tfrac{1}{2}\phi + i\psi)v_1 + (-\tfrac{1}{2}\phi + i\psi)^2 w_1 ,$$

and so

$$\Re\lambda_1' = u_1 - \tfrac{1}{2}\phi v_1 + (\tfrac{1}{4}\phi^2 - \psi^2)w_1 \sim N,$$

$$\Im\lambda_1' = \psi(v_1 - \phi w_1) = o(N).$$

On the other hand,

$$\Re\lambda_2' = u_2 - \tfrac{1}{2}\phi v_2 + (\tfrac{1}{4}\phi^2 - \psi^2)w_2 = o(N),$$

$$\Im\lambda_2' = \psi(v_2 - \phi w_2) \sim N.$$

Hence (4) is satisfied if N is sufficiently large.

Further, the expressions

$$v_1w_2 - v_2w_1 , \qquad w_1u_2 - w_2u_1 , \qquad u_1v_2 - u_2v_1$$

have no common factor greater than 1. For their values are $-v_2$, u_2 , Nv_2 respectively, and v_2 is a prime, and $0 < u_2 < v_2$ by (5), since $0 < \frac{1}{2}\phi < 1$. Hence there exist integers u_3 , v_3 , w_3 which satisfy

$$u_3(v_1w_2 - v_2w_1) + v_3(w_1u_2 - w_2u_1) + w_3(u_1v_2 - u_2v_1) = 1.$$

This proves the Lemma.

Continuing the proof of Theorem 1 (b), we observe that

$$\text{(6)} \qquad |(\lambda_1p + \lambda_2q + \lambda_3r)(\lambda_1'p + \lambda_2'q + \lambda_3'r)(\bar\lambda_1'p + \bar\lambda_2'q + \bar\lambda_3'r)| \geq 1$$

for all integers p, q, r which are not all zero, since the product is the norm of the integer $\lambda_1p + \lambda_2q + \lambda_3r$ of $k(\phi)$. The determinant of the three linear forms in the product is $23^{\frac{1}{2}}i$, by (2) and (3). We now determine real numbers α, β such that

$$\lambda_1'p + \lambda_2'q + \lambda_3'r = \lambda_1'(p - \alpha r) + \lambda_2'(q - \beta r).$$

This is possible, since λ_1'/λ_2' is not real, and so every complex number, and in particular $-\lambda_3'$, is representable as $\alpha\lambda_1' + \beta\lambda_2'$. We can write (6) as

$$|\lambda_1p + \lambda_2q + \lambda_3r|\{a(p - \alpha r)^2 + 2b(p - \alpha r)(q - \beta r) + c(q - \beta r)^2\} \geq 1,$$

where

$$a = \lambda_1'\bar\lambda_1' > 0, \qquad 2b = \lambda_1'\bar\lambda_2' + \bar\lambda_1'\lambda_2' , \qquad c = \lambda_2'\bar\lambda_2' .$$

We have

$$\frac{c}{a} = \left|\frac{\lambda_2'}{\lambda_1'}\right|^2, \qquad \frac{b}{a} = \Re\left(\frac{\lambda_2'}{\lambda_1'}\right),$$

and therefore, by (4),

$$\left| \frac{c}{a} - 1 \right| < \epsilon_1 , \qquad \frac{|b|}{a} < \epsilon_1 , \tag{7}$$

where ϵ_1 depends only on ϵ, and tends to zero with ϵ. Since the determinant of the linear forms in (6) was $23^{\frac{1}{2}}i$, we have

$$| \lambda_1\alpha + \lambda_2\beta + \lambda_3 | (ac - b^2)^{\frac{1}{2}} = \tfrac{1}{2}(23)^{\frac{1}{2}}. \tag{8}$$

By (7) we have, for any real x, y,

$$\begin{aligned} ax^2 + 2bxy + cy^2 &\leq ax^2 + 2\epsilon_1 a\, | xy | + a(1 + \epsilon_1)y^2 \\ &\leq ax^2 + \epsilon_1 a(x^2 + y^2) + a(1 + \epsilon_1)y^2 \\ &\leq a(1 + 2\epsilon_1)(x^2 + y^2). \end{aligned}$$

Hence, for any integers p, q, r that are not all zero,

$$| \lambda_1 p + \lambda_2 q + \lambda_3 r | \{(p - \alpha r)^2 + (q - \beta r)^2\} \geq \frac{1}{a(1 + 2\epsilon_1)}.$$

In particular, if we make $r \to \infty$, and give p, q the integral values nearest to αr, βr respectively, we deduce

$$\begin{aligned} \varliminf_{r\to\infty} r\{(p - \alpha r)^2 + (q - \beta r)^2\} &\geq \frac{1}{a(1 + 2\epsilon_1)} \lim_{r\to\infty} \frac{r}{| \lambda_1 p + \lambda_2 q + \lambda_3 r |} \\ &= \frac{1}{a(1 + 2\epsilon_1)\, | \lambda_1\alpha + \lambda_2\beta + \lambda_3 |} = \frac{2}{23^{\frac{1}{2}}} \left(\frac{(ac - b^2)^{\frac{1}{2}}}{a(1 + 2\epsilon_1)} \right), \end{aligned}$$

by (8). The fraction in brackets falls short of 1 by as little as we please, by (7), on taking ϵ sufficiently small. This proves Theorem 1 (b).

5. Finally we prove Theorem 2 (b). Let λ_1, λ_2, λ_3 be the numbers of the Lemma, and let

$$\mu_1 = \frac{\lambda_2'\bar{\lambda}_3' - \bar{\lambda}_2'\lambda_3'}{\theta - \bar{\theta}}, \qquad \mu_2 = \frac{\lambda_3'\bar{\lambda}_1' - \bar{\lambda}_3'\lambda_1'}{\theta - \bar{\theta}}, \qquad \mu_3 = \frac{\lambda_1'\bar{\lambda}_2' - \bar{\lambda}_1'\lambda_2'}{\theta - \bar{\theta}}.$$

These are integers of $k(\phi)$, since, by (3),

$$\begin{aligned} \mu_1 &= u_3v_2 - u_2v_3 + (u_3w_2 - u_2w_3)(\theta + \bar{\theta}) + (w_2v_3 - w_3v_2)\theta\bar{\theta} \\ &= -W_1 - V_1\phi - U_1(\phi^2 - 1), \end{aligned} \tag{9}$$

where U_1, V_1, W_1 are the cofactors of u_1, v_1, w_1 in the determinant of $u_1, \cdots, w_3$. The conjugates of μ_1, μ_2, μ_3 are given by

$$\mu_1' = \frac{\bar{\lambda}_2'\lambda_3 - \lambda_2\bar{\lambda}_3'}{\bar{\theta} - \varphi}, \cdots \qquad \text{and} \qquad \bar{\mu}_1' = \frac{\lambda_2\lambda_3' - \lambda_2'\lambda_3}{\varphi - \theta}, \cdots .$$

It follows from the simple properties of minors, applied to the determinant formed by λ_1, λ_2, λ_3 and their conjugates, that

$$\frac{\mu_1'\mu_3 - \mu_1\mu_3'}{\mu_2'\mu_3 - \mu_2\mu_3'} = -\frac{\bar{\lambda}_2'}{\bar{\lambda}_1'},$$

and consequently, by (4),

$$\left| \frac{\mu_1'\mu_3 - \mu_1\mu_3'}{\mu_2'\mu_3 - \mu_2\mu_3'} - i \right| < \epsilon. \tag{10}$$

For all rational integers p, q, r which are not all zero, we have

$$|(\mu_1 p + \mu_2 q + \mu_3 r)(\mu_1' p + \mu_2' q + \mu_3' r)(\bar{\mu}_1' p + \bar{\mu}_2' q + \bar{\mu}_3' r)| \geq 1, \tag{11}$$

since the product is the norm of the integer $\mu_1 p + \mu_2 q + \mu_3 r$ of $k(\phi)$. The determinant of the three linear forms in the product has absolute value $23^{\frac{1}{2}}$, by (9) and (2). We can write the second factor as

$$(\mu_1' - \mu_1\mu_3'/\mu_3)p + (\mu_2' - \mu_2\mu_3'/\mu_3)q + (\mu_1 p + \mu_2 q + \mu_3 r)\mu_3'/\mu_3 .$$

Hence, if we put $\alpha = \mu_1/\mu_3$, $\beta = \mu_2/\mu_3$, we can write (11) as

$$|(\alpha p + \beta q + r)(\rho p + \sigma q + \tau(\alpha p + \beta q + r))(\bar{\rho} p + \bar{\sigma} q + \bar{\tau}(\alpha p + \beta q + r))| \geq 1,$$

where ρ, σ, τ are complex numbers, and σ/ρ satisfies

$$\left| \frac{\sigma}{\rho} - i \right| < \epsilon, \tag{12}$$

by (10). Also from the determinant of these linear forms, we have

$$|\rho\bar{\sigma} - \bar{\rho}\sigma| = 23^{\frac{1}{2}}. \tag{13}$$

Now make p and q tend to infinity in any manner, and give r the integral value nearest to $-\alpha p - \beta q$. Then $\alpha p + \beta q + r$ is bounded, whereas $|\rho p + \sigma q| \to \infty$. Hence

$$\varliminf_{p,q\to\infty} |(\alpha p + \beta q + r)(\rho p + \sigma q)(\bar{\rho} p + \bar{\sigma} q)| \geq 1. \tag{14}$$

By the same method as in the previous proof, it follows from (12), (13), (14) that

$$\varliminf_{p,q\to\infty} |\alpha p + \beta q + r| (p^2 + q^2) > \frac{2}{23^{\frac{1}{2}}} - \epsilon',$$

where ϵ' is arbitrarily small with ϵ. This proves Theorem 2 (b).

Note added in proof. Mahler has since proved that the results of Theorems 1 (a) and 2 (a) are true also when $c = 2/23^{\frac{1}{2}}$, provided we read $\leq$ in place of $<$ in the main inequality. For this, and rather more, see Theorem *O* of Part II of [3].

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