A REMARK ON THE CONTINUED FRACTIONS OF CONJUGATE ALGEBRAIC NUMBERS

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In the theory of continued fractions, two real numbers ξ_0 and ξ_1 are called *equivalent*: $\xi_0 \backsim \xi_1$, if their regular continued fractions

$$\xi_0 = b_0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \ldots, \qquad \xi_1 = b_0' + \frac{1|}{|b_1'} + \frac{1|}{|b_2'} + \ldots$$

are identical except for at most a finite number of terms, i.e. if

$b_k = b'_{k+r}$ *for some* r *and for all sufficiently large* k.

The following theorem has been proved: [1])

The two numbers ξ_0 *and* ξ_1 *are equivalent if and only if there are four integers* $\alpha, \beta, \gamma, \delta$ *of determinant* $\alpha\delta - \beta\gamma = \mp 1$ *such that*

(1): $\quad \xi_1 = s(\xi_0),$ *where* $s(x) = \dfrac{\alpha x + \beta}{\gamma x + \delta} \not\equiv x.$

By means of a classical method of ABEL [2]), we construct in this note all irreducible algebraic equations

(2): $\quad f(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0 \qquad (a_0 \neq 0)$

with integral coefficients and of degree $n \geqslant 2$, with the property of admitting two different real equivalent roots ξ_0 and ξ_1. This construction proceeds as follows:

Denote by

$$s^2(x) = s(s(x)),\ s^3(x) = s^2(s(x)),\ s^4(x) = s^3(s(x)), \ldots$$

$$g(x) = (\gamma x + \delta)^n f(s(x)),$$

$$h(x) = (\gamma x + \delta)\{x - s(x)\} = \gamma x^2 + (\delta - \alpha)x - \beta.$$

By (1), the two equations $f(x) = 0$ and $g(x) = 0$ have the root ξ_0 in common, and so, since $f(x)$ is irreducible, both equations have the same roots,

(3): $\quad \xi_0, \xi_1, \ldots \xi_{n-1}$

say; therefore all numbers

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[1]) O. PERRON, Kettenbrüche, (1929), p. 64—66.

[2]) N. H. ABEL, Oeuvres I (1881), 478 f.

(4): $\xi_\nu,\ s(\xi_\nu),\ s^2(\xi_\nu), \ldots \qquad (\nu = 0, 1, 2, \ldots, n-1)$

belong to the finite set (3).

Next, *none of the numbers* (3) *is a fixpoint of* $s(x)$, i.e. a root of the equation $h(x) = 0$. For then $f(x) = 0$ and $h(x) = 0$ had a root in common, and so we should have $n = 2$, $\gamma \neq 0$, and both equations had the same roots. This would mean, in particular, that ξ_0 were a fixpoint of $s(x)$, i.e. that $\xi_1 = s(\xi_0) = \xi_0$, contrary to the hypothesis that $\xi_1 \neq \xi_0$.

Then, since the numbers (4) form a finite set and none of them is a fixpoint of $s(x)$, the substitution $y = s(x)$ is elliptical and of finite period, m say [1]; thus

(5): $$s^m(x) \equiv x$$

identical in x, but

(6): $$s^\mu(x) \not\equiv x \qquad (\mu = 1, 2, \ldots, m-1).$$

Therefore the substitution $y = s(x)$ can be written as

$$\frac{y-\Theta_0}{y-\Theta_1} = \varkappa \frac{x-\Theta_0}{x-\Theta_1},$$

where

$$\left.\begin{matrix}\Theta_0\\ \Theta_1\end{matrix}\right\} = \frac{\alpha-\delta \mp \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta-\beta\gamma)}}{2\gamma}$$

are the fixpoints of $s(x)$, while

$$\varkappa = \frac{\alpha-\gamma\Theta_0}{\alpha-\gamma\Theta_1} = \frac{\alpha+\delta+\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}{\alpha+\delta-\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}$$

is its multiplicator. By (5) and (6),

$$\varkappa^m = 1, \text{ but } \varkappa^\mu \neq 1 \text{ if } \mu = 1, 2, \ldots, m-1.$$

Therefore either $m = 2$ and

$$\varkappa = -1,\ \alpha+\delta = 0,\ \alpha\delta-\beta\gamma = \mp 1;$$

or $m > 2$ and $\varkappa$ is non-real, hence

$$(\alpha+\delta)^2 - 4(\alpha\delta-\beta\gamma) < 0,$$

whence $m = 3$ and

$$\varkappa = \frac{-1\mp i\sqrt{3}}{2},\ \alpha+\delta = \mp 1,\ \alpha\delta-\beta\gamma = +1.$$

Hence, if the notation is chosen suitably, then only the following two cases arise: Either

[1]) Klein-Fricke, Modulfunctionen I (1890), 164.

(I): $m = 2,\ s(x) = \dfrac{\alpha x + \beta}{\gamma x - \alpha} \not\equiv x,\ s^2(x) \equiv x, \qquad \alpha^2 + \beta\gamma = \mp 1,$

or

(II): $m = 3,\ s(x) = \dfrac{\alpha x + \beta}{\gamma x - (\alpha - 1)} \not\equiv x, s^2(x) = \dfrac{(\alpha - 1)x + \beta}{\gamma x - \alpha} \not\equiv x,$

$$s^3(x) \equiv x,\ (\alpha - 1)\,\alpha + \beta\gamma = -1.$$

The roots (3) are all different, and none is a fixpoint of $s(x)$; therefore n is an integral multiple $n = jm$ of m, and the roots (3) can be distributed among j sets

(7): $\xi_{\nu m},\ \xi_{\nu m+1}, \ldots, \xi_{(\nu+1)m-1} \qquad (\nu = 0, 1, \ldots, j-1)$

of m roots each, such that

in case (I), $\qquad \xi_{\nu m+1} = s(\xi_{\nu m}) \qquad (\nu = 0, 1, \ldots, j-1)$,

in case (II), $\xi_{\nu m+1} = s(\xi_{\nu m}),\ \xi_{\nu m+2} = s^2(\xi_{\nu m})\ (\nu = 0, 1, \ldots, j-1)$.

Put

in case (I), $\eta_\nu = \begin{cases} \xi_{\nu m} + \xi_{\nu m+1} & \text{if } \gamma \neq 0, \\ \xi_{\nu m}\,\xi_{\nu m+1} & \text{if } \gamma = 0, \end{cases}$

in case (II), $\eta_\nu = \xi_{\nu m} + \xi_{\nu m+1} + \xi_{\nu m+2}$,

so that

(8): $\eta_\nu = \Phi(\xi_{\nu m}) = \Phi(\xi_{\nu m+1}) = \ldots = \Phi(\xi_{(\nu+1)m-1}) \quad (\nu = 0, 1, \ldots, j-1)$,

where $\Phi(x)$ denotes the following rational functions:

in case (I), $\Phi(x) = \begin{cases} x + s(x) = \dfrac{\gamma x^2 + \beta}{\gamma x - \alpha} & \text{if } \gamma \neq 0,\ \alpha^2 + \beta\gamma = \mp 1, \\ x\,s(x) = -x(x + \beta) & \text{if } \gamma = 0,\ \alpha = 1, \end{cases}$

in case (II), $\Phi(x) = x + s(x) + s^2(x) =$

$$= \frac{\gamma^2 x^3 - 3(\alpha^2 - \alpha + 1)\,x - (2\alpha - 1)\,\beta}{(\gamma x - \alpha + 1)(\gamma x - \alpha)} \text{ if } \alpha(\alpha - 1) + \beta\gamma = -1.$$

In case (II), γ cannot vanish since the equation $\alpha(\alpha - 1) = -1$ has no integral solutions.

From (8), the sums

$$\sum_{\nu=0}^{j-1} \eta_\nu^\varrho = \sum_{\nu=0}^{j-1} \Phi(\xi_{\nu m})^\varrho = \frac{1}{m} \sum_{\nu=0}^{j-1} \Big(\Phi(\xi_{\nu m})^\varrho + \ldots + \Phi(\xi_{(\nu+1)m-1})^\varrho \Big)$$
$$= \frac{1}{m} \sum_{l=0}^{n-1} \Phi(\xi_l)^\varrho,$$

whe $\varrho = 1, 2, 3, \ldots$, are rational *symmetrical* functions with *rational* coefficients in the roots (3) of $f(x) = 0$. Hence, by classical theorems on symmetrical functions, they are themselves *rational* numbers, and so are the coefficients of the polynomial

$$p(x) = (x - \eta_0)(x - \eta_1) \ldots (x - \eta_{j-1}) = x^j + p_1 x^{j-1} + \ldots + p_j.$$

Moreover, this polynomial is *irreducible*. For otherwise the root η_0 of $p(x) = 0$ would be of degree less than j, and so ξ_0 would be of degree less than $jm = n$, contrary to hypothesis.

Assume, conversely, that $\Phi(x)$ is the function defined above, that $p(x)$ is any irreducible polynomial of degree j with rational coefficients, and that the polynomial $f(x)$ of degree $n = mj$ is defined as the numerator of the rational function $p(\Phi(x))$. Then the roots of $f(x) = 0$ can again be distributed among j sets (7) of m roots each such that the roots in each set are connected with one and the same root η_ν of $p(x) = 0$ by the formulae (8). It is possible that $f(x)$ is reducible; a simple discussion shows, however, that this may not happen unless $f(x)$ is the product of m irreducible factors $f_0(x), \ldots, f_{m-1}(x)$ of degree j satisfying the identities

$$f_\mu(x) = (\gamma x + \delta)^j f_0(s^\mu(x)) \qquad (\mu = 0, 1, \ldots, m-1).$$

If $p(x)$ is chosen suitably, this exceptional case does not arise, and so we conclude that our problem has solutions for every degree n which is divisible by 2 or 3.

By way of example, let $n = m = 3$, and put

$$s(x) = \frac{x+1}{-x},\ s^2(x) = \frac{-1}{x+1},\ \Phi(x) = \frac{x^3 - 3x - 1}{x^2 + x},\ p(x) = x + 1.$$

We then obtain the irreducible equation

$$f(x) = (x^2 + x)\{\Phi(x) + 1\} = x^3 + x^2 - 2x - 1 = 0;$$

its roots

$$\xi_0 = 2\cos\frac{2\pi}{7},\ \xi_1 = 2\cos\frac{6\pi}{7},\ \xi_2 = 2\cos\frac{4\pi}{7}$$

are equivalent since

$$\xi_1 = s(\xi_0),\quad \xi_2 = s^2(\xi_0),$$

and in fact have the continued fractions,

$$\xi_0 = 1 + \frac{1|}{|4} + \frac{1|}{|20} + \frac{1|}{|2} + \frac{1|}{|3} + \ldots,$$

$$\xi_1 = -1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|4} + \frac{1|}{|20} + \frac{1|}{|2} + \frac{1|}{|3} + \ldots,$$

$$\xi_2 = -2 + \frac{1|}{|5} + \frac{1|}{|20} + \frac{1|}{|2} + \frac{1|}{|3} + \ldots$$

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