KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

# On the area and the densest packing of convex domains

BY

K. MAHLER

Reprinted from Proceedings Vol. L, No. 1, 1947

1947
NORTH-HOLLAND PUBLISHING COMPANY
(N.V. Noord-Hollandsche Uitgevers Mij.)
AMSTERDAM

**Mathematics.** — *On the area and the densest packing of convex domains.* By K. MAHLER (Manchester). (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 21, 1946.)

In the preceding paper "On irreducible convex domains" [1]), I studied the critical lattices of convex domains in the $(x_1, x_2)$-plane and proved that every such domain contains an irreducible convex domain of equal determinant.

Of these results, applications are made in the present paper, which deals with two closely allied problems:

**Problem** I: If $V(K)$ and $\Delta(K)$ denote the area and determinant of a convex domain $K$, *to find the lower bound of*

$$Q(K) = \frac{V(K)}{\Delta(K)}$$

*extended over all convex domains K.*

**Problem** II: About every point $P$ of a lattice $\Lambda$ as its centre describe a convex domain $K(P)$ congruent to $K$ and with the same orientation, but assume that no two domains $K(P)$ overlap. Choose $\Lambda$ such that the ratio of the area covered by the domains $K(P)$ to the whole plane assumes its largest value, $q(K)$ say. *To find the lower bound of* $q(K)$ *extended over all convex domains.*

MINKOWSKI established the close connection between $Q(K)$ and $q(K)$ and obtained the upper bounds for $Q(K)$ and $q(K)$, and some lower bound for $Q(K)$. Also Problem II has been considered before [2]), but no solution seems to have so far been given. I have not succeeded in solving either of the two problems. But I show in this paper how they can be reduced to a question in the calculus of variations. I prove further that this variation problem does admit of a best possible solution in form of an irreducible convex domain, and that this solution is *not* an ellipse, contrary to what might be expected.

All the first paragraphs deal with Problem I; the application to Problem II is made at the end of this paper.

## § 1. **Formulation of the problem.**

Let

$$x_1' = \alpha x_1 + \beta x_2, \quad x_2' = \gamma x_1 + \delta x_2$$

be any affine transformation of determinant

$$d = \alpha\delta - \beta\gamma > 0,$$

---

[1]) Quoted as ICD. Compare this paper for all the definitions and lemmas.

[2]) See W. BLASCHKE, Differentialgeometrie II, § 27, problem 17.

and let $K$ be any convex domain [3]) in the $(x_1, x_2)$-plane. When $(x_1, x_2)$ describes $K$, then $(x'_1, x'_2)$ describes a second convex domain $K'$. As is well known [4]), the areas $V(K)$, $V(K')$ and the determinants $\triangle(K)$, $\triangle(K')$ satisfy the equations

$$V(K') = d\,V(K), \quad \triangle(K') = d\,V(K).$$

Hence the quotient,

$$Q(K) = \frac{V(K)}{\triangle(K)},$$

is an absolute invariant,

$$Q(K') = Q(K), \qquad (1)$$

for all affine transformations.

An upper bound for $Q(K)$ is given by MINKOWSKI's classical theorem on lattice points in convex domains, viz.

$$Q(K) \leqslant 4;$$

the equality sign holds only for parallelograms and certain classes of hexagons [5]).

It is the lower bound for $Q(K)$ with which this paper is concerned. A trivial lower bound for $Q(K)$, namely

$$Q(K) \geqslant 1,$$

follows immediately from the obvious inequality $V(K) \geq \triangle(K)$ [6]). Elsewhere, I proved the much better inequality [7]),

$$Q(K) \geqslant \sqrt{12},$$

but this is also not the exact lower bound for $Q(K)$.

In order to obtain the exact lower bound for $Q(K)$ in the set of all convex domains, the following restrictions on $K$ may be imposed without loss of generality:

($A$): *$K$ is not a parallelogram*; for otherwise $Q(K) = 4$, and this is not the smallest possible value for $Q(K)$, since, e.g. for an ellipse

$$Q(K) = \frac{2\pi}{\sqrt{3}} < 4.$$

($B$): $\triangle(K) = 1$; this condition may be enforced by means of a suitable similar transformation, on account of (1).

---

[3]) As in ICD, all convex domains are assumed symmetrical in $O = (0, 0)$.

[4]) The first equation is classical; for the second one see Theorem 16 of my paper "Lattice points in $n$-dimensional star bodies I". Proc. Royal Society A, **187** (1946), 151—187.

[5]) Geometrie der Zahlen, §§ 34—35.

[6]) ICD, § 7.

[7]) See my paper, "On the theorem of MINKOWSKI–HLAWKA", which is to appear in DUKE's Journal.

(C): The boundary $C$ of $K$ contains the six points,

$$\left.\begin{aligned} P'_1=(\sqrt[4]{\tfrac{4}{3}}, 0),\ P'_2=(\sqrt[4]{\tfrac{1}{12}}, \sqrt[4]{\tfrac{3}{4}}),\ P'_3=(-\sqrt[4]{\tfrac{1}{12}}, \sqrt[4]{\tfrac{3}{4}}), \\ P'_4=-P_1,\ P'_5=-P_2,\ P'_6=-P_3, \end{aligned}\right\} \qquad (2)$$

and these points are the points of a critical lattice $\Lambda'$ of $K$ of basis $P'_1$, $P'_2$. For assume that the conditions ($A$) and ($B$) hold, and choose any critical lattice of $K$. Then, by *ICD*, § 3, this lattice has just six points on $C$, such that three of them together with the origin form the vertices of a parallelogram. Since the lattice is of unit determinant, it can be transformed into $\Lambda'$ by means of an affine transformation with $d=1$.

We can now restate our problem as follows:

**Problem 1′**: *To find a convex domain K of minimum area satisfying the three conditions* ($A$), ($B$), ($C$). *Its area gives the required lower bound for* $Q(K)=V(K)$ [8]).

## § 2. Proof that the lower bound is attained.

Denote by $H'$ the hexagon with the six vertices (2), and by $H''$ the polygon

$$P'_1 P''_1 P'_2 P''_2 P'_3 P''_3 P'_4 P''_4 P'_5 P''_5 P'_6 P''_6,$$

where $P''_1, \ldots, P''_6$ are the points

$$\left.\begin{aligned} P''_1=(\sqrt[4]{\tfrac{27}{4}}, \sqrt[4]{\tfrac{3}{4}}),\ P''_2=(0, \sqrt[4]{12}),\ P''_3=(-\sqrt[4]{\tfrac{27}{4}}, \sqrt[4]{\tfrac{3}{4}}), \\ P''_4=-P''_1,\ P''_5=-P''_2,\ P''_6=-P''_3. \end{aligned}\right\}. \qquad (3)$$

If $K$ is any convex domain satisfying the conditions ($A$), ($B$), ($C$), then it contains $H'$ as a subset, and is itself contained in $H''$. Denote by $\Sigma$ the set of all such convex domains.

As already mentioned in § 1, $Q(K) \geq 1$ for all convex domains, and so

$$V(K) \geqq 1$$

for all elements $K$ of $\Sigma$. Hence the lower bound

$$Q = \underset{K \text{ in } \Sigma}{\text{l.b.}}\ V(K)$$

extended over all elements of $\Sigma$ is a positive number, and is in fact also the lower bound of $Q(K)$ extended over all convex domain. Evidently $Q<4$.

**Definition:** *A convex domain K is called extreme if* $Q(K)=Q$.

**Theorem 1:** *There exists an extreme convex domain.*

Proof: Choose an infinite sequence

$$K_1, K_2, K_3, \ldots \qquad (4)$$

of elements of $\Sigma$, not all necessarily different, such that

$$\lim_{n\to\infty} V(K_n) = Q.$$

---

8) That the area of $K$ attains its lower bound, is proved in the next paragraph, and is here already taken for granted.

All these convex domains $K_n$ are subsets of the bounded polygon $H''$. Hence, by the selection theorem of BLASCHKE [9]), it is possible to choose an infinite subsequence,

$$K_{n_1}, K_{n_2}, K_{n_3}, \ldots$$

of (4) which converges to a convex domain, $K$ say. Then, firstly,

$$V(K) = Q.$$

Secondly, it is obvious that $K$ has the properties $(A)$ and $(C)$. Thirdly, it has also the property $(B)$, since [10])

$$\Delta(K) = \lim_{r \to \infty} \Delta(K_{n_r}) = 1.$$

Hence $K$ is an extreme convex domain, and the assertion is proved.

**Theorem 2:** *Every extreme convex domain is irreducible.*

Proof: If $K$ is reducible, then, by *ICD*, Lemma 13, a convex domain $K'$ contained in, but different from, $K$ can be found such that $\Delta(K') = \Delta(K)$. Hence $Q(K) > Q(K') \geq Q$, and so $K$ is not extreme.

## § 3. A parameter representation of $K$.

The last result allows us to restrict the convex domains to be considered still further and to restate the problem as follows:

**Problem 1″:** *To find an irreducible convex domain $K$ of minimum area $Q$ satisfying the three conditions* $(A)$, $(B)$, $(C)$.

For the investigation of this problem, we apply Lemma 9 of *ICD*:

"*Let $K$ be an irreducible convex domain which is not a parallelogram. Then to every point $P_1$ on $C$, there exists a unique critical lattice $\Lambda = \Lambda(P_1)$ containing $P_1$. This lattice has just six points $P_l = P_l(P_1)$ $(l = 1, 2, \ldots, 6)$ on $C$. Let $A_1, \ldots, A_6$ be the six arcs into which these points divide $C$; denote further by $P_1^*$ a variable point on $A_1$, and by $P_l^* = P_l(P_1)$ for $l = 2, \ldots, 6$ the other five points of $\Lambda(P_1^*)$ on $C$. If $P_1^*$ describes $A_1$ continuously in positive direction, then $P_l^*$, for $l = 2, \ldots, 6$, describes $A_l$ in the same manner.*"

This lemma leads to the following parameter representation of the boundary $C$ of $K$:

Let $P = (x_1, x_2)$ be the general point of $C$. Then denote by $t$ a parameter which runs from 0 to $2\pi$ when $P$ runs in positive direction over $C$ from $P_1' = (\sqrt[4]{\tfrac{4}{3}}, 0)$ back to $P_1'$; thus

$$x_1 = x_1(t), \quad x_2 = x_2(t)$$

are functions of $t$ defined for $0 \leq t \leq 2\pi$ in the first instance. So as to simplify the considerations, extend these two functions to all real values of $t$ by the periodicity condition,

$$x_1(t + 2\pi) = x_1(t), \quad x_2(t + 2\pi) = x_2(t).$$

---

[9]) W. BLASCHKE, Kreis und Kugel, 62.

[10]) Theorem 9 of my paper, l.c. [4]).

Denote further by

$$P(t) = (x_1(t), x_2(t))$$

the point on $C$ of parameter $t$, and by $\Lambda(t)$ the critical lattice of $K$ containing $P(t)$. It is clearly possible to choose the parameter $t$ in such a way that the six points of $\Lambda(t)$ on $C$ are just given by

$$P\left(t + \frac{h\pi}{3}\right), \text{ where } h = 0, 1, \ldots, 5;$$

in particular, it is necessary that

$$P\left(\frac{h\pi}{3}\right) = P'_h \qquad (h = 1, 2, \ldots, 6).$$

Since $\Lambda(t)$ is critical, the quadrilateral

$$OP(t)\,P\left(t + \frac{\pi}{3}\right)P\left(t + \frac{2\pi}{3}\right)$$

is a parallelogram of area $\triangle(K) = 1$; hence

$$P(t) - P\left(t + \frac{\pi}{3}\right) + P\left(t + \frac{2\pi}{3}\right) = 0, \qquad \left\{P(t), P\left(t + \frac{\pi}{3}\right)\right\} = 1.$$

The first condition is equivalent to the functional equations,

$$x_1(t) - x_1\left(t + \frac{\pi}{3}\right) + x_1\left(t + \frac{2\pi}{3}\right) = 0,$$

$$x_2(t) - x_2\left(t + \frac{\pi}{3}\right) + x_2\left(t + \frac{2\pi}{3}\right) = 0,$$

which have the general solution,

$$x_1(t) = a_1(t)\cos t + b_1(t)\sin t; \quad x_2(t) = a_2(t)\cos t + b_2(t)\sin t, \tag{5}$$

where

$$a_1(t), b_1(t), a_2(t), b_2(t) \text{ are functions of } t \text{ of period } \frac{\pi}{3}.$$

The second condition is equivalent to the equation,

$$x_1(t)\,x_2\left(t + \frac{\pi}{3}\right) - x_1\left(t + \frac{\pi}{3}\right)x_2(t) = 1;$$

on substituting the expressions (5) and simplifying, this equation takes the form,

$$a_1(t)\,b_2(t) - a_2(t)\,b_1(t) = +\sqrt{\tfrac{4}{3}}. \quad \ldots \ldots \ldots \tag{6}$$

The conditions $P\left(\frac{h\pi}{3}\right) = P'_h$ give the initial values,

$$a_1(0) = b_2(0) = +\sqrt[4]{\tfrac{4}{3}}, \qquad a_2(0) = b_1(0) = 0. \quad \ldots \tag{7}$$

Next, the condition that $C$ is a convex curve is equivalent to the inequality,

$$\begin{vmatrix} a_1(t_1)\cos t_1 + b_1(t_1)\sin t_1, & a_2(t_1)\cos t_1 + b_2(t_1)\sin t_1, & 1 \\ a_1(t_2)\cos t_2 + b_1(t_2)\sin t_2, & a_2(t_2)\cos t_2 + b_2(t_2)\sin t_2, & 1 \\ a_1(t_3)\cos t_3 + b_1(t_3)\sin t_3, & a_2(t_3)\cos t_3 + b_2(t_3)\sin t_3, & 1 \end{vmatrix} \geqq 0 \quad \text{if } 0 \leqq t_1 < t_2 < t_3 < 2\pi. \tag{8}$$

If $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ have second derivatives, then this inequality implies that

$$\begin{vmatrix} \frac{d}{dt}\{a_1(t)\cos t + b_1(t)\sin t\}, & \frac{d}{dt}\{a_2(t)\cos t + b_2(t)\sin t\} \\ \frac{d^2}{dt^2}\{a_1(t)\cos t + b_1(t)\sin t\}, & \frac{d^2}{dt^2}\{a_2(t)\cos t + b_2(t)\sin t\} \end{vmatrix} \geqq 0. \tag{8'}$$

I have not succeeded in expressing either of these two formulae in a more convenient form. (See, however, § 5.)

Finally, an explicit value for the area $V(K)$ of $K$ is found in the following way, under the assumption that $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are differentiable:

In the integral,

$$V(K) = \frac{1}{2}\int_0^{2\pi} \left\{ x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt} \right\} dt,$$

the integrand may be written as,

$$x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt} =$$
$$= \{a_1(t)b_2(t) - a_2(t)b_1(t)\} + \{A(t)\cos^2 t + B(t)\cos t \sin t + C(t)\sin^2 t\},$$

where

$$A(t) = a_1(t)\frac{da_2(t)}{dt} - a_2(t)\frac{da_1(t)}{dt},$$

$$B(t) = a_1(t)\frac{db_2(t)}{dt} - b_2(t)\frac{da_1(t)}{dt} + b_1(t)\frac{da_2(t)}{dt} - a_2(t)\frac{db_1(t)}{dt},$$

$$C(t) = b_1(t)\frac{db_2(t)}{dt} - b_2(t)\frac{db_1(t)}{dt}.$$

Since by (6),

$$\frac{1}{2}\int_0^{2\pi} \{a_1(t)b_2(t) - a_2(t)b_1(t)\}\, dt = \frac{2\pi}{\sqrt{3}},$$

evidently,

$$V(K) = \frac{2\pi}{\sqrt{3}} + \frac{1}{2}\int_0^{2\pi} \left\{ A(t)\cos^2 t + B(t)\cos t \sin t + C(t)\sin^2 t \right\} dt.$$

The integral on the right can be much simplified since $A(t)$, $B(t)$, $C(t)$ are periodic functions of period $\frac{\pi}{3}$. To this purpose, replace $t$ under the integral sign by

$$t,\ t+\frac{\pi}{3},\ t+\frac{2\pi}{3},$$

and take the arithmetical means. Since

$$\cos^2 t+\cos^2\left(t+\frac{\pi}{3}\right)+\cos^2\left(t+\frac{2\pi}{3}\right)=$$
$$=\sin^2 t+\sin^2\left(t+\frac{\pi}{3}\right)+\sin^2\left(t+\frac{2\pi}{3}\right)=\frac{3}{2}$$

and

$$\cos t\sin t+\cos\left(t+\frac{\pi}{3}\right)\sin\left(t+\frac{\pi}{3}\right)+\cos\left(t+\frac{2\pi}{3}\right)\sin\left(t+\frac{2\pi}{3}\right)=0,$$

this leads to the formula,

$$V(K)=\frac{2\pi}{\sqrt{3}}+I(K), \qquad \dots\dots\dots \quad (9)$$

where

$$I(K)=\frac{1}{4}\int_0^{2\pi}\{A(t)+C(t)\}\,dt,$$

hence,

$$I(K)=\frac{1}{4}\int_0^{2\pi}\left\{a_1(t)\frac{da_2(t)}{dt}-a_2(t)\frac{da_1(t)}{dt}+b_1(t)\frac{db_2(t)}{dt}-b_2(t)\frac{db(t)}{dt}\right\}dt. \quad (10)$$

We see then that Problem 1″ is essentially equivalent [11] to the following **Problem 1′**: *To find four functions* $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ *of period* $\frac{\pi}{3}$ *satisfying the conditions* (6), (7), (8), *and giving the integral* $I(K)$ *in* (10) *a smallest value.*

## § 4. The integrals of the EULER differential equations.

There is no difficulty in applying the classical EULER–LAGRANGE method to Problem 1′, omitting, however, the inequality condition (8).

Write $a_h$, $b_h$ instead of $a_h(t)$, $b_h(t)$, and use a dot for the differential coefficients with respect to $t$; denote further by $\lambda$ a suitable function of $t$. Then the EULER equations for the function

$$F(a_1, b_1, a_2, b_2)=\{a_1\dot{a}_2-\dot{a}_1 a_2+b_1\dot{b}_2-\dot{b}_1 b_2\}+\lambda\{a_1 b_2-a_2 b_1-\sqrt{\tfrac{4}{3}}\},$$

[11]) It is not a priori evident that the boundary of an extreme convex domain has everywhere a tangent, thus that $a_1(t)$, $b_1(t)$, $a_2(t)$, $b_2(t)$ are differentiable for all $t$.

i.e. the differential equations,

$$\frac{\partial F}{\partial a_h} - \frac{d}{dt}\frac{\partial F}{\partial \dot{a}_h} = 0, \qquad \frac{\partial F}{\partial b_h} - \frac{d}{dt}\frac{\partial F}{\partial \dot{b}_h} = 0 \qquad (h = 1, 2),$$

are as follows:

$$2\dot{a}_1 + \lambda b_1 = 2\dot{b}_1 - \lambda a_1 = 2\dot{a}_2 + \lambda b_2 = 2\dot{b}_2 - \lambda a_2 = 0 \quad . \quad . \quad . \quad (11)$$

On eliminating $\lambda$ from the first or the last two equations (11), we get

$$a_1 \dot{a}_1 + b_1 \dot{b}_1 = 0, \qquad a_2 \dot{a}_2 + b_2 \dot{b}_2 = 0,$$

whence, on integrating for $t$,

$$a_1^2 + b_1^2 = \gamma_1^2, \qquad a_2^2 + b_2^2 = \gamma_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $\gamma_1, \gamma_2$ are independent of $t$. If, on the other hand, $\lambda$ is eliminated from the first and the third equation, then

$$\frac{\dot{a}_1}{b_1} = \frac{\dot{a}_2}{b_2}, \quad \text{or} \quad \frac{\dot{a}_1}{\sqrt{\gamma_1^2 - a_1^2}} = \frac{\dot{a}_2}{\sqrt{\gamma_2^2 - a_2^2}},$$

whence, on integrating again for $t$,

$$\cos^{-1}\frac{a_1}{\gamma_1} = \cos^{-1}\frac{a_2}{\gamma_2} - \Gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $\Gamma$ is a further number independent of $t$.

The two equations (12) and (13) imply that there is an angle $\Theta$ such that

$$\left.\begin{array}{ll} a_1 = \gamma_1 \cos\Theta, & b_1 = \gamma_1 \sin\Theta, \\ a_2 = \gamma_2 \cos(\Theta + \Gamma), & b_2 = \gamma_2 \sin(\Theta + \Gamma). \end{array}\right\} \quad . \quad . \quad . \quad . \quad (14)$$

On substituting these values in (6),

$$a_1 b_2 - a_2 b_1 = \gamma_1 \gamma_2 \sin\Gamma = +\sqrt{\tfrac{4}{3}}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Further, from (5),

$$x_1 = a_1 \cos t + b_1 \sin t = \gamma_1 \cos(t - \Theta),$$
$$x_2 = a_2 \cos t + b_2 \sin t = \gamma_2 \cos(t - \Theta - \Gamma),$$

and so,

$$\gamma_2 x_1 \cos\Gamma + \gamma_2 \sqrt{\gamma_2^2 - x_1^2} \sin\Gamma = \gamma_1 x_2,$$

whence from (15),

$$\gamma_2^2 x_1^2 - 2\gamma_1\gamma_2 x_1 x_2 \cos\Gamma + \gamma_1^2 x_2^2 = \tfrac{4}{3}. \quad . \quad . \quad . \quad . \quad (16)$$

Since

$$\sin\Gamma \neq 0, \quad |\cos\Gamma| < 1.$$

this is the equation of an ellipse $E$ which evidently has the properties,

$$\triangle(E) = 1, \quad V(E) = Q(E) = \frac{2\pi}{\sqrt{3}}.$$

For instance, the circle $Z$,

$$x_1^2 + x_2^2 = \sqrt{\tfrac{4}{3}}, \quad . \; . \; . \; . \; . \; . \; . \quad (17)$$

obtained for

$$\gamma_1 = \gamma_2 = \sqrt[4]{\tfrac{4}{3}}, \; \Gamma = \frac{\pi}{2},$$

is of this kind; it passes through the six points $P'_h$.

## § 5. A property of ellipses.

**Theorem 3:** *No ellipse is an extreme domain.*

Proof: By affine invariance, it suffices to prove the assertion for the circle $Z$ defined in (17), i.e. for the functions

$$a_1(t) \equiv b_2(t) \equiv 1, \; a_2(t) \equiv b_1(t) \equiv 0 \text{ identically in } t.$$

Denote by $\varepsilon$ a small positive number, and consider the neighbouring domain $K_\varepsilon$ belonging to the functions,

$$\left.\begin{array}{ll} a_1(t) = \sqrt[4]{\tfrac{4}{3}}\,(1 + \varepsilon \sin 6t)^{-1}, & b_1(t) = -\varepsilon(\cos 6t - 1), \\ a_2(t) = 0, & b_2(t) = \sqrt[4]{\tfrac{4}{3}}\,(1 + \varepsilon \sin 6t). \end{array}\right\} \quad . \quad (18)$$

These functions satisfy both the identity (6) and the initial conditions (7). Further, on substituting in (8′), this determinant can be developed into a power series

$$\sqrt{\tfrac{4}{3}} + \sum_{n=1}^{\infty} u_n(t)\,\varepsilon^n$$

in $\varepsilon$ which converges absolutely and uniformly in $t$ if $\varepsilon$ is sufficiently small. Moreover, the coefficients $u_n(t)$ are continuous functions of $t$. The determinant is therefore positive for sufficiently small positive $\varepsilon$, and so $K_\varepsilon$ is then a convex domain.

On substituting the functions (18) into the integral (10) for $I(K_\varepsilon)$, this integral becomes,

$$I(K_\varepsilon) = \sqrt[4]{\tfrac{27}{4}}\,\varepsilon \int_0^{2\pi} \{-\varepsilon + \varepsilon \cos 6t - \sin 6t\}\,dt = -\sqrt[4]{108}\,\pi\varepsilon^2 < 0.$$

Therefore from (9),

$$V(K_\varepsilon) < \frac{2\pi}{\sqrt{3}} = V(Z), \; Q(K_\varepsilon) < Q(Z),$$

as asserted.

**Corollary:** *The lower bound of $Q(K)$ extended over all convex domains $K$ is smaller than* $\dfrac{2\pi}{\sqrt{3}}$.

## § 6. Another form of the variation problem.

Problem 1 can be expressed in many other ways as a problem in the calculus of variations. One particularly simple formulation is as follows: Assume that

$$\Delta(K) = 2,$$

and that the boundary $C$ of $K$ passes through the six points,

$$p_1 = (2, 0),\ p_2 = (1, 1),\ p_3 = (-1, 1),\ p_4 = -p_1,\ p_5 = -p_2,\ p_6 = -p_3;$$

this is permitted since

$$p_1 + p_3 = p_2, \quad \{p_1, p_2\} = 2.$$

Denote by

$$P_1 = (x_1', x_2'), \quad P_2 = (x_1, x_2), \quad P_3 = (x_1''', x_2''')$$

three points of $C$ on the arcs $p_1p_2$, $p_2p_3$, $p_3p_1$, respectively, which belong to the same critical lattice of $K$. Then $x_2$ is a single-valued continuous function of $x_1$ for $-1 \leq x_1 \leq 1$ such that

$$x_2 = 1 \text{ for } x_1 = -1 \text{ and } x_1 = 1.$$

The conditions,

$$P_1 + P_3 = P_2, \quad \{P_1, P_2\} = 2$$

are satisfied by chosing,

$$P_1 = \left(\frac{2}{x_2} + \frac{1-\varrho}{2} x_1, \frac{1-\varrho}{2} x_2\right), \quad P_3 = \left(-\frac{2}{x_2} + \frac{1+\varrho}{2} x_1, \frac{1+\varrho}{2} x_2\right),$$

where $\varrho = \varrho(x_1)$ is a continuous function of $x_1$; on identifying $P_2$ with $p_2$ or $p_3$, one finds that

$$\varrho(-1) = -1, \quad \varrho(1) = 1.$$

There are further some rather complicated conditions involving the first and second derivatives of $x_2(x_1)$ and $\varrho(x_1)$ which express that $C$ is convex.

A simple calculation leads now to the integral

$$V(K) = \int_{-1}^{+1} \left\{ \frac{3+\varrho^2}{2} (x_2 - x_1 x_2') + 4 \frac{x_2'}{x_2} \varrho + 2\varrho' \right\} dx_1 \qquad \left(x_2' = \frac{dx_2}{dx_1}\right)$$

for $V(K)$. I omit the discussion of EULER's equations which gives the same results as the other method.

Final remark: It seems highly probable from the convexity condition, that the boundary of an extreme convex domain consists of line segments and arcs of hyperbolae. So far, however, I have not succeeded in proving this assertion.

## § 7. The relation to Problem II.

Let $K$ be a convex domain, and let $\Lambda$ and $\lambda = 2\Lambda$ be two lattices such that

$\lambda$ consists of the points $2P$ where $P$ belongs to $\Lambda$.

Denote by $K(P)$ the convex domain of all points

$$P+X \text{ where } X \text{ belongs to } K,$$

by

$$\Sigma = \sum_{P \text{ in } \lambda} K(P)$$

the join of all domains

$$K(P) \text{ where } P \text{ belongs to } \lambda,$$

and by $\Sigma_R$ the set of all points $X=(x_1, x_2)$ of $\Sigma$ which belong to the square $Z_R$:

$$|x_1| \leqslant R, \ |x_2| \leqslant R.$$

By MINKOWSKI [12]), the following results hold:

(1) The ratio,

$$\frac{V(\Sigma_R)}{V(Z_R)} = \frac{V(\Sigma_R)}{4R^2}$$

of the areas of $\Sigma_R$ and $Z_R$ tends of a limit, $q(K, \Lambda)$ say, as $R$ tends to infinity.

(2) When $P$ and $P'$ run over all pairs of different elements of $\lambda$, then no two domains $K(P)$ and $K(P')$ are overlapping if and only if $\Lambda$ is $K$-admissible.

(3) If $\Lambda$ is $K$-admissible, then,

$$q(K, \Lambda) = \frac{V K)}{d(\lambda)} = \frac{V(K)}{4\, d(\Lambda)}.$$

Since

$$d(\Lambda) \geqslant \Delta(K)$$

for all $K$-admissible lattices, with equality only if $K$ is critical, the lower bound of $q(K, \Lambda)$ extended over all admissible lattices $\Lambda$, say $q(K)$, is thus given by

$$q(K) = \frac{V(K)}{4\,\Delta(K)} = \tfrac{1}{4} Q(K).$$

*Hence the two problems I and II are completely equivalent.*

We see, in particular, from the results proved earlier that there exists a convex domain (viz., an extreme domain) such that

$$q(K) < \frac{\pi}{\sqrt{12}},$$

and that this domain is *not* an ellipse.

*Mathematics Department, Manchester University.*

September 29, 1946.

---

[12]) Diophantische Approximationen, 82—90. MINKOWSKI considers the case of three dimensions; but the ideas are the same for the plane. His notation is different from the one used here.