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ON THE ADJOINT OF A REDUCED POSITIVE DEFINITE TERNARY QUADRATIC FORM

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Let

$$f(x) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + 2 b_1 x_2 x_3 + 2 b_2 x_3 x_1 + 2 b_3 x_1 x_2$$

be a positive definite ternary quadratic form of determinant

$$D = a_1 a_2 a_3 - a_1 b_1^2 - a_2 b_2^2 - a_3 b_3^2 + 2 b_1 b_2 b_3 .$$

The adjoined form

$$F(x) = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + 2 B_1 x_2 x_3 + 2 B_2 x_3 x_1 + 2 B_3 x_1 x_2$$

of coefficients

$$A_1 = a_2 a_3 - b_1^2, \qquad A_2 = a_3 a_1 - b_2^2, \qquad A_3 = a_1 a_2 - b_3^2,$$

$$B_1 = b_2 b_3 - a_1 b_1, \qquad B_2 = b_3 b_1 - a_2 b_2, \qquad B_3 = b_1 b_2 - a_3 b_3$$

is also positive definite, and its determinant is

$$D^2 = A_1 A_2 A_3 - A_1 B_1^2 - A_2 B_2^2 - A_3 B_3^2 + 2 B_1 B_2 B_3.$$

One shows easily that

$$a_1 a_2 a_3 \geq D, \qquad A_1 A_2 A_3 \geq D^2,$$

but that $a_1 a_2 a_3/D$ and $A_1 A_2 A_3/D^2$ are not bounded above.

Let, however, f(x) be restricted to the *reduced forms in the sense of Seeber and Minkowski,* i.e. let it belong to the set R of all forms satisfying

$$(R): \quad \begin{gathered} 0 < a_1 \leq a_2 \leq a_3, \quad 0 \leq b_1 \leq \frac{a_2}{2}, \quad \left| b_2 \right| \leq \frac{a_1}{2}, \quad 0 \leq b_3 \leq \frac{a_1}{2}, \\ b_1 - b_2 + b_3 \leqq \frac{a_1 + a_2}{2}. \end{gathered}$$

Then

$$(I): \qquad a_1 a_2 a_3 \leqq 2 D$$

by the theorem of Gauss[1], and this is the best-possible result since the equality sign holds if f(x) is, e.g. the form

$$f_0(x) = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \text{ of determinant } D=1/2.$$

In this note, I prove an analogous inequality for the adjoined of a reduced form, namely

$$(II): \qquad A_1 A_2 A_3 \leq \frac{9}{4} D^2.$$

Also this inequality is *best possible*, because the equality sign holds, e.g. for the form $F_0(x)$ adjoined to $f_0(x)$,

$$F_0(x) = \begin{pmatrix} \frac{3}{4} & 1 & \frac{3}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \qquad \text{of determinant } D^2 = 1/4.$$

The assertion is equivalent to the statement that if f(x) runs over all elements of R, then the rational function

$$Q(f) = \frac{A_1 A_2 A_3}{D^2} = \frac{(a_2 a_3 - b_1^2)(a_3 a_1 - b_2^2)(a_1 a_2 - b_3^2)}{(a_1 a_2 a_3 - a_1 b_1^2 - a_2 b_2^2 - a_3 b_3^2 + 2 b_1 b_2 b_3)^2}$$

of the coefficients of f(x) has the *upper bound* 9/4, and that this upper bound is *attained*, hence is the *maximum* of Q(f). To prove this assertion, we shall first show the existence of the maximum, and we shall then evaluate the maximum by studying the cases $b_2 \geq 0$ and $b_2 < 0$ separately.

§ 1. THE EXISTENCE OF THE MAXIMUM.

LEMMA 1: *Let* f(x) *be a reduced form, and let*

$$l_1 = 1 - \frac{a_1}{a_2}, \quad l_2 = 1 - \frac{a_2}{a_3}, \quad \textit{so that} \quad l_1 \geq 0, \quad l_2 \geq 0.$$

Then

$$a_1 a_2 a_3 \leq \frac{4D}{2 + l_1 + l_2}.$$

Proof: Put

$$\lambda = a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 - 2 b_1 b_2 b_3.$$

I have shown elsewhere that[2]

$$\lambda \leq \frac{a_1 a_2^2 + a_1^2 a_3}{4}.$$

for every reduced form. Hence, by the hypothesis,

$$\lambda \leq \frac{a_1 a_2 \cdot (1 - l_2) a_3 + a_1 \cdot (1 - l_1) a_2 \cdot a_3}{4} = \frac{2 - l_1 - l_2}{4} a_1 a_2 a_3,$$

whence

$$D = a_1 a_2 a_3 - \lambda \geq a_1 a_2 a_3 - \frac{2 - l_1 - l_2}{4} a_1 a_2 a_2 = \frac{2 + l_1 + l_2}{4} a_1 a_2 a_3,$$

as asserted.

LEMMA 2: *Denote by* Q^* *the upper bound of* $Q(f)$ *extended over all reduced forms* $f(x)$. *Then there exists at least one such form such that* $Q(f)=Q^*$.

Proof: By the homogeneity of $Q(f)$ in the coefficients of $f(x)$, it suffices to prove the assertion for reduced forms of determinant

$$D=1;$$

for these forms,

$$Q(f)=(a_2 a_3-b_1^2)(a_3 a_1-b_2^2)(a_1 a_2-b_3^2).$$

Denote by Σ_1 the set of all reduced forms of unit determinant satisfying the inequality

$$a_3 \leq 16 a_1,$$

by Σ_2 the set of all such forms satisfying the inequality

$$a_3 > 16 a_1.$$

If, firstly, $f(x)$ belongs to Σ_1, then by (R) and (I),

$$2 \geqq a_1 a_2 a_3 \geqq a_1^2 a_3 \geqq \left(\frac{a_3}{16}\right)^2 a_3=\frac{a_3^3}{256},$$

$$1 \leq a_1 a_2 a_3 \leq a_1 a_3^2 \leq a_1 (16 a_1)^2=256 a_1^3,$$

whence

$$2^{-8/3} \leq a_1 \leq a_2 \leqq a_3 \leq 8, \qquad 0 \leqq b_1 \leqq 4, \qquad |b_2| \leq 4, \qquad 0 \leqq b_3 \leqq 4.$$

The set Σ_1 is therefore *bounded,* and it is also *closed*; moreover, $Q(f)$ is a *continuous function* in this set. Hence, by the theorem of Weierstrass, $Q(f)$ assumes its maximum value in Σ_1. Since the form

$$f_1(x)=\sqrt[3]{2}\; f_0(x)=\begin{pmatrix} \sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{2} \\ \sqrt[3]{\frac{1}{4}} & 0 & \sqrt[3]{\frac{1}{4}} \end{pmatrix}$$

of unit determinant gives

$$Q(f_1)=9/4,$$

this maximum value cannot be less than 9/4.

Secondly, let $f(x)$ belong to Σ_2. Then at least one of the inequalities

$$\frac{a_1}{a_2}<\frac{1}{4} \qquad \text{or} \qquad \frac{a_2}{a_3}<\frac{1}{4}$$

is satisfied, hence also at least one of the inequalities

$$l_1 = 1 - \frac{a_1}{a_2} > \frac{3}{4} \qquad \text{or} \qquad l_2 = 1 - \frac{a_2}{a_3} > \frac{3}{4} .$$

Therefore by Lemma 1,

$$a_1 a_2 a_3 < \frac{4}{2 + \frac{3}{4} + 0} = \frac{16}{11},$$

whence

$$Q(f) \le (a_1 a_2 a_3)^2 < \left(\frac{16}{11}\right)^2 < \frac{9}{4} .$$

On combining the results for Σ_1 and Σ_2, the assertion follows.

DEFINITION: *A reduced form* f(x) *satisfying*

$$Q(f) = Q^*$$

is called a maximum form.

§ 2. THE CASE $b_2 \geq 0$.

LEMMA 3: *If* f(x) *is a reduced form with* $b_2 \geq 0$, *then* Q(f) $\leq$ 9/4.

Proof: By the hypothesis, all three coefficients b_1, b_2, b_3 are non-negative, hence

$$B_1 = b_2 b_3 - a_1 b_1 \geq -a_1 b_1, \qquad B_2 = b_3 b_1 - a_2 b_2 \geq -a_2 b_2,$$
$$B_3 = b_1 b_2 - a_3 b_3 \geq -a_3 b_3 .$$

At least one of the coefficients B_1, B_2, B_3 must be negative, since otherwise

$$b_2 b_3 \geq a_1 b_1, \qquad b_3 b_1 \geq a_2 b_2, \qquad b_1 b_2 \geq a_3 b_3,$$

and

$$a_1 a_2 a_3 \leq b_1 b_2 b_3 \leq \frac{a_2}{2} \frac{a_1}{2} \frac{a_1}{2} \leq \frac{a_1 a_2 a_3}{8},$$

which is impossible. Hence at least one of the three inequalities

$$|B_1| \leq a_1 b_1, \qquad |B_2| \leq a_2 b_2, \qquad |B_3| \leq a_3 b_3$$

is satisfied, say the second one. Then

$$\begin{aligned} A_1 A_2 A_3 &= A_2 \cdot A_1 A_3 \\ &= (a_3 a_1 - b_2^2)(D a_2 + B_2^2) \\ &\leq (a_3 a_1 - b_2^2)(D a_2 + a_2^2 b_2^2) \\ &= (a_1 a_2 a_3 - a_2 b_2^2)(D + a_2 b_2^2) \qquad \text{, hence by (I),} \\ &\leq (2D - a_2 b_2^2)(D + a_2 b_2^2) \\ &= \frac{9}{4} D^2 - \left\{ \frac{D - 2 a_2 b_2^2}{2} \right\}^2 \leq \frac{9}{4} D^2, \end{aligned}$$

as asserted.

§ 3. THE CASE $b_2 < 0$. FIRST SIMPLIFICATION.

In this and the next paragraphs, only reduced forms

$$f(x) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

with $b_2 < 0$ are considered; hence

$$b_2' = -b_2$$

is *positive.* If no reduced form of this kind is a maximum form, then (II) follows at once from Lemmas 2 and 3; there is therefore no loss of generality in assuming from now on that a maximum form with $b_2 < 0$ does exist. We shall study such maximum forms and show that their coefficients satisfy certain restrictive conditions.

LEMMA 4: *Let*

$$f(x) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & -b_2' & b_3 \end{pmatrix}, \qquad \textit{where } b_2' > 0,$$

be a maximum form. Then

$$\text{(III):} \qquad a_2 = a_3, \quad b_1 + b_2' + b_3 = \frac{a_1 + a_2}{2}.$$

Proof: Assume, firstly, that

$$b_1 + b_2' + b_3 < \frac{a_1 + a_2}{2},$$

so that at least one of the inequalities

$$b_1 < \frac{a_2}{2}, \qquad b_2' < \frac{a_1}{2}, \qquad b_3 < \frac{a_1}{2}$$

is satisfied, say the first one. On solving

$$D = a_1 a_2 a_3 - a_1 b_1^2 - a_2 b_2'^2 - a_3 b_3^2 - 2 b_1 b_2' b_3$$

for a_3, this coefficient becomes the function

$$\text{(a):} \qquad a_3 = \frac{a_1 b_1^2 + a_2 b_2'^2 + 2 b_1 b_2' b_3 + D}{a_1 a_2 - b_3^2}, = \phi(a_1, a_2, b_1, b_2', b_3, D) \qquad \text{say,}$$

of a_1, a_2, b_1, b_2', b_3, D, and $Q(f)$ may be written as

$$\text{(b):} \quad Q(f) = 1 + D^{-1}(a_1 b_1^2 + a_2 b_2'^2 + 2 b_1 b_2' b_3) + D^{-2} \frac{\{a_1 b_1^2 b_3 + a_2 b_2'^2 b_3 + (a_1 a_2 + b_3^2) b_1 b_2' + D b_3\}^2}{a_1 a_2 - b_3^2}.$$

Consider now a neighbouring form

$$f^*(x) = \begin{pmatrix} a_1 & a_2 & a_3^* \\ b_1^* & -b_2' & b_3 \end{pmatrix}, \qquad a_3^* = \phi(a_1, a_2, b_1^*, b_2' b_3, D),$$

of determinant D in which b_1 is replaced by a number $b_1^* > b_1$ sufficiently near to b_1. By (a),

$$a_3^* > a_3,$$

and so $f^*(x)$ is also reduced. Since further, by (b),

$$Q(f^*) > Q(f),$$

$f(x)$ cannot be a maximum form.

Assume, secondly, that

$$a_2 < a_3, \qquad b_1 + b_2' + b_3 = \frac{a_1 + a_2}{2}.$$

Then denote by $t > 1$ a number sufficiently near to 1, and by

$$f^{**}(x) = \begin{pmatrix} t\,a_1 & t\,a_2 & a_3^{**} \\ t\,b_1 & -t\,b_2' & t\,b_3 \end{pmatrix}, \qquad a_3^{**} = \phi\,(t\,a_1, t\,a_2, t\,b_1, t\,b_2', t\,b_3, D),$$

a neighbouring form of determinant D. By the continuity of ϕ and by (a), this form is still reduced, and is it clear from (b) that

$$Q\,(f^{**}) > Q\,(f)\,.$$

Hence f(x) is also in this case not a maximum form. On combining the results of the two cases, the assertion follows.

§ 4. THE CASE $b_2 < 0$. SECOND SIMPLIFICATION.

The function Q(f) is homogeneous of zero dimension in the coefficients of f(x); hence, by $a_1 > 0$, it suffices to consider forms with $a_2 = 2$. Hence, by Lemma 4, every maximum form with $b_2 < 0$ may be assumed of the normal form

(c): $$f(x) = \begin{pmatrix} 2 & 2\xi & 2\xi \\ \xi - \eta - \zeta + 1 & -\eta & \zeta \end{pmatrix}$$

where, by the conditions of reduction,

(d): $$\xi \geq 1\,, \qquad 0 \leq \eta \leq 1\,, \qquad 0 \leq \zeta \leq 1\,, \qquad \eta + \zeta \geq 1\,.$$

It is obvious, that the second form

$$g\,(x) = \begin{pmatrix} 2 & 2\xi & 2\xi \\ \xi - \eta - \zeta + 1 & -\zeta & \eta \end{pmatrix}$$

is then also a maximum form; hence there is no restriction of generality in imposing the further conditions

(e): $$0 < \eta \leq \zeta\,,$$

since, by Lemma 3, $Q(f) \leq 9/4$ if $\eta = 0$.

LEMMA 5. *If the form* f(x) *defined by* (c) *is a maximum form, and if its coefficients satisfy the inequalities* (d) *and* (e), *then*

$$\zeta = 1\,.$$

Proof: For shortness, put

$$\eta + \zeta = u\,, \qquad 4\,\xi + \eta\,\zeta = v\,,$$

so that

$$Q(f) = \frac{\{4\,\xi^2 - (\xi - u + 1)^2\}\,\{v^2 - 4\,\xi\,u^2\}}{4\,\{(\xi + u - 1)\,v - (\xi + u - 1)^2 - \xi\,u^2\}^2}\,.$$

Consider ξ and u as constants, but allow v to vary. Then $Q(f) = \varphi(v)$ becomes a function of v of derivative

$$\frac{d\,\varphi\,(v)}{dv} = \varrho\,\Delta\,(v)\,,$$

where

$$\Delta\,(v) = -\{(\xi + u - 1)^2 + \xi\,u^2\}\,v + 4\,\xi\,u^2\,(\xi + u - 1)\,,$$

while ϱ is a positive number independent of v. The expression $\Delta(v)$ vanishes at

$$v_0 = \frac{4\,\xi\,u^2\,(\xi + u - 1)}{(\xi + u - 1)^2 + \xi\,u^2}\,,$$

and is positive for smaller and negative for larger v. Now

$$v_0 - 4\,\xi = 4\,\xi\,\frac{u^2\,(u - 1) - (\xi + u - 1)^2}{(\xi + u - 1)^2 + \xi\,u^2}\,,$$

and by (d),

$$1 \leq u \leq 2\,, \qquad \xi \geq 1\,,$$

hence

$$v_0 - 4\,\xi \leq 4\,\xi\,\frac{u^2\,(u - 1) - u^2}{(\xi + u - 1)^2 + \xi\,u^2} = 4\,\xi\,\frac{u^2\,(u - 2)}{(\xi + u - 1)^2 + \xi\,u^2} \leq 0\,,$$

whence

$$v_0 \leq 4\,\xi \leq v = 4\,\xi + \eta\,\zeta\,.$$

Therefore, for fixed ξ and $u = \eta + \zeta$, Q(f) assumes its maximum if v, that is, if $\eta\,\zeta$, is as small as possible. Since

$$\eta\,\zeta = \frac{u^2 - (\zeta^2 - \eta)^2}{4}\,,$$

this requires that η is as small as possible and ζ is as large as possible. By (e), must remain positive; hence the assertion follows from (d).

§ 5. The case $b_2 < 0$. Conclusion of the Proof.

By Lemma 5, a maximum form $f(x)$ with $b_2 < 0$ may be written as

$$f(x) = \begin{pmatrix} 2 & 2\xi & 2\xi \\ \xi - \eta & -\eta & 1 \end{pmatrix},$$

where

$$\xi \geq 1, \qquad 0 < \eta \leq 1.$$

Then

$$Q(f) = \frac{(4\xi - 1)(4\xi - \eta^2)\{4\xi^2 - (\xi - \eta)^2\}}{4\xi^2\{(3\xi - 1) + \eta - \eta^2\}^2},$$

hence

$$\frac{9}{4} - Q(f) = \frac{\Lambda(\xi, \eta)}{4\xi^2\{(3\xi - 1) + \eta - \eta^2\}^2},$$

where

$$\Lambda(\xi, \eta) = 9\xi^2\{(3\xi - 1) + \eta - \eta^2\}^2 - (4\xi - 1)(4\xi - \eta^2)\{4\xi^2 - (\xi - \eta)^2\},$$

that is,

$$\Lambda(\xi, \eta) = (9\xi^2 - 4\xi + 1)\eta^4 - (10\xi^2 + 2\xi)\eta^3 - (42\xi^3 - 40\xi^2 + 4\xi)\eta^2 +$$
$$+ (22\xi^3 - 10\xi^2)\eta + (33\xi^4 - 42\xi^3 + 9\xi^2).$$

Replace in this expression ξ by

$$\xi = 1 + \tau, \qquad \text{so that } \tau > 0.$$

Then

$$\Lambda(\xi, \eta) = (24\tau + 81\tau^2 + 90\tau^3 + 33\tau^4) + (12 + 46\tau + 56\tau^2 + 22\tau^3)\eta +$$
$$- (6 + 42\tau + 86\tau^2 + 2\tau^3)\eta^2 - (12 + 22\tau + 10\tau^2)\eta^3 +$$
$$+ (6 + 14\tau + 9\tau^2)\eta^4,$$

or

$$\Lambda(\xi, \eta) = 6\eta(1 - \eta^2)(2 - \eta) + (24 + 46\eta - 42\eta^2 - 22\eta^3 + 14\eta^4)\tau +$$
$$+ (81 + 56\eta - 86\eta^2 - 10\eta^3 + 9\eta^4)\tau^2 + (90 + 22\eta - 2\eta^2)\tau^3 + 33\tau^4.$$

Since

$$\tau \geq 0, \qquad 0 < \eta \leq 1$$

this expression can never be negative; and it vanishes only if

$$\tau = 0, \qquad \eta = 1.$$

This completes the proof of the inequality (II).

§ 6. Further Inequalities.

The inequalities (I) and (II) are not the only ones satisfied by the coefficients of a reduced form and its adjoint. I give here a few further inequalities which are all best possible:

$$D \leq a_1 a_2 a_3 - 4\, b_1 b_2 b_3 \leq 2\, D.$$

$$D \leq a_k A_k \leq 2\, D \qquad (k = 1, 2, 3).$$

$$\frac{3}{4} a_2 a_3 \leq A_1 \leq a_2 a_3, \qquad \frac{3}{4} a_3 a_1 \leq A_2 \leq a_3 a_1, \qquad \frac{3}{4} a_1 a_2 \leq A_3 \leq a_1 a_2.$$

$$A_2 \geq \frac{3}{4} A_3, \qquad A_1 \geq \frac{3}{4} A_3, \qquad A_1 \geq \frac{3}{4} A_2.$$

$$\left| B_1 \right| \leq \frac{2}{3} A_2, \qquad \left| B_2 \right| \leq \frac{2}{3} A_1, \qquad \left| B_3 \right| \leq \frac{2}{3} A_1.$$

In the first line, "4" cannot be replaced by a larger number.

I remark finally that the proof of (II) given in this note does not seem to me very satisfactory; *it would be of interest to find a simpler one.*

References

1. Gauss, *Werke,* Bd. 2, 188-196.
2. K. Mahler, *Journ. Lond. Math. Soc.,* **15** (1940), 193—195.