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On lattice points in polar reciprocal convex domains

BY

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K. MAHLER: *On lattice points in polar reciprocal convex domains.*

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Let k and K be two symmetrical convex domains in the (x, y)-plane with centre at the origin $O = (0, 0)$, and assume that k and K are polar-reciprocal with respect to the unit circle

$$C: \; x^2 + y^2 = 1;$$

i.e. the boundary points of each domain are the poles of the tac-lines (Stützlinien) of the other domain with respect to C.

Define $\Delta(k)$ as the minimum determinant of k, i.e. as the lower bound of the determinants $d(\Lambda)$ of all k-admissible lattices Λ, and define $\Delta(K)$ analogously [1]. I prove in this note that

$$\tfrac{1}{2} \leqslant \Delta(k)\,\Delta(K) \leqslant \tfrac{3}{4},$$

and here both inequalities are best possible [2]. The proof is elementary and depends on similar inequalities for the areas of polar-reciprocal symmetrical convex hexagons.

Troughout this note, $(P_1, P_2) = x_1 y_2 - x_2 y_1$ denotes the determinant of two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, and $-P = (-x, -y)$ is the point symmetrical to $P = (x, y)$ in $O = (0, 0)$.

§ 1. Let h be a symmetrical convex hexagon of vertices $\mp P_1, \mp P_2, \mp P_3$, where $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$. Choose the notation such that these vertices follow one another in the order

$$P_1, P_2, P_3, -P_1, -P_2, -P_3,$$

when the boundary of h is described in the positive direction. Then the three determinants

$$a_1 = (P_2, P_3), \;\; a_2 = (P_1, P_3), \;\; a_3 = (P_1, P_2)$$

represent twice the areas of the triangles

$$T_1 = OP_2P_3, \quad T_2 = OP_3(-P_1), \quad T_3 = OP_1P_2,$$

respectively, and so are all positive,

$$a_1 > 0, \;\; a_2 > 0, \;\; a_3 > 0. \quad \ldots\ldots\ldots \quad (1)$$

[1]) For the terminology used, see my paper, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 331—343 (1946).

[2]) An analogous formula holds for polar-reciprocal symmetrical convex bodies k and K in n-dimensions, viz.

$$c_n \leqslant \Delta(k)\,\Delta(K) \leqslant C_n,$$

where $c_n > 0$ and $C_n > 0$ depend only on n; but it seems a difficult problem to find the best values of these two constants.

The triangles OP_1P_3, $OP_2(-P_1)$, $OP_3(-P_2)$ are proper subsets of the quadrilaterals $OP_1P_2P_3$, $OP_2P_3(-P_1)$, and $OP_3(-P_1)(-P_2)$, respectively; hence also the following inequalities hold,

$$-a_1 + a_2 + a_3 > 0,\ a_1 - a_2 + a_3 > 0,\ a_1 + a_2 - a_3 > 0. \quad . \quad . \quad (2)$$

Finally, h is twice the sum of the three triangles T_1, T_2, and T_3, and so h is of area,

$$a = a_1 + a_2 + a_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When h degenerates into a parallelogram, then some of the signs "$>$" in (1) and (2) are replaced by the equality sign.

It is useful to notice that if a_1, a_2, a_3 are any three numbers satisfying the conditions (1) and (2), then there exists a hexagon h for which these numbers are the double areas of T_1, T_2, and T_3, respectively. For the hexagon of vertices $P_1 = (1, 0)$, $P_2 = (0, a_3)$, $P_3 = (-a_1/a_3, a_2)$, $-P_1$, $-P_2$, $-P_3$, has this property.

§ 2. Let H be the symmetrical convex hexagon which is polar-reciprocal to h with respect to the unit circle C. The sides of H are therefore,

$$x_j\, x + y_j\, y = \mp 1 \qquad (j = 1, 2, 3),$$

and its vertices are the points Q_{12}, Q_{23}, Q_{31}, where

$$Q_{12} = \text{pole of the line } P_1 P_2 \quad = \left(\frac{y_2 - y_1}{a_3}, \frac{x_1 - x_2}{a_3}\right),$$

$$Q_{23} = \text{pole of the line } P_2 P_3 \quad = \left(\frac{y_3 - y_2}{a_1}, \frac{x_2 - x_3}{a_1}\right),$$

$$Q_{31} = \text{pole of the line } P_3(-P_1) = \left(\frac{-y_1 - y_3}{a_2}, \frac{x_1 + x_3}{a_2}\right).$$

A simple calculation shows that the determinants

$$A_1 = (Q_{12}, Q_{31}), \quad A_2 = (Q_{12}, Q_{23}), \quad A_3 = (Q_{23}, Q_{31}),$$

corresponding to a_1, a_2, a_3 in the case of h, have the values

$$A_1 = \frac{-a_1 + a_2 + a_3}{a_2\, a_3},\ A_2 = \frac{a_1 - a_2 + a_3}{a_1\, a_3},\ A_3 = \frac{a_1 + a_2 - a_3}{a_1\, a_2}. \qquad (4)$$

Again the inequalities

$$A_1 > 0,\ A_2 > 0,\ A_3 > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$-A_1 + A_2 + A_3 > 0,\ A_1 - A_2 + A_3 > 0,\ A_1 + A_2 - A_3 > 0 \quad . \quad (6)$$

hold; this is proved just as in § 1, and can also be seen from (1) and (2) since, e.g.

$$A_1 = \frac{-a_1 + a_2 + a_3}{a_2\, a_3},\ -A_1 + A_2 + A_3 = \frac{(a_1 - a_2 + a_3)(a_1 + a_2 - a_3)}{a_1\, a_2\, a_3}.$$

Finally, H is of area,

$$A = A_1 + A_2 + A_3. \quad (7)$$

§ 3. Denote by

$$\Pi = a A \quad (8)$$

the product of the areas of the two polar-reciprocal hexagons h and H. It is easily verified from (3), (4), and (7), that

$$\Pi - 8 = \frac{(-a_1 + a_2 + a_3)(a_1 - a_2 + a_3)(a_1 + a_2 - a_3)}{a_1 a_2 a_3} \quad (9)$$

and

$$9 - \Pi = \frac{(a_2 + a_3 - a_1)(a_2 - a_3)^2 + (a_3 + a_1 - a_2)(a_3 - a_1)^2 + (a_1 + a_2 - a_3)(a_1 - a_2)^2}{2 a_1 a_2 a_3}. \quad (10)$$

Hence

$$8 \leqslant a A \leqslant 9. \quad (A)$$

Here the equality sign on the left cannot hold for proper hexagons, but only for hexagons degenerated into parallelograms. The equality sign on the right demands that

$$a_1 = a_2 = a_3,$$

a condition satisfied for the hexagons which are affine equivalent to a regular hexagon.

§ 4. A theorem of K. REINHARDT [3]), recently rediscovered by myself without knowledge of his earlier work [4]), states:

Theorem 1: *Let K be a symmetrical convex domain, and let $\mathfrak{U}_K$ be the set of all circumscribed hexagons H of K. Then $\Delta(K)$ is equal to a quarter times the lower bound of the areas of all elements H of $\mathfrak{U}_K$.*

On applying this result to the polar-reciprocal hexagons h and H considered in §§ 1—3, we find that

$$\Delta(h) = a/4, \quad \Delta(H) = A/4,$$

since a hexagon coincides with its smallest circumscribed hexagon. The inequality (A) of the last paragraph leads then to the following result:

Theorem 2: *If the two symmetrical convex hexagons h and H are polar-reciprocal with respect to the unit circle, then*

$$1/2 \leqslant \Delta(h)\,\Delta(H) \leqslant 9/16. \quad (B)$$

§ 5. From now on, let k and K be any two symmetrical convex domains which are polar-reciprocal with respect to the unit circle, and let $V(k)$ and $V(K)$ be their areas. In order to generalize (B) to this case,

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3) Abh. Math. Sem. Hamburg, **10**, 216—230 (1934).

4) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 692—703 (1947).

we apply a theorem of MINKOWSKI [5]) and one of myself [6]), as follows:

Theorem 3: *Let K be a symmetrical convex domain, and let I_K be the set of inscribed convex hexagons H which have their six vertices $\mp P_1$, $\mp P_2$, $\mp P_3$ on the boundary of K such that $P_1 + P_3 = P_2$. Then $\Delta(K)$ is equal to a third times the lower bound of the areas of all elements H of I_K.*

Theorem 4: *If the two symmetrical convex domains k and K are polar-reciprocal with respect to the unit circle, then*

$$V(k)\, V(K) \geqq 8.$$

From these two theorems, together with Theorem 2, we obtain:

Theorem 5: *If the two symmetrical convex domains k and K are polar-reciprocal with respect to the unit circle, then*

$$1/2 \leqq \Delta(k)\, \Delta(K) \leqq 3/4. \quad \ldots \ldots \ldots \quad (C)$$

Proof of the upper bound: Inscribe into k a hexagon h, the six vertices $\mp P_1$, $\mp P_2$, $\mp P_3$ of which lie on the boundary of k and satisfy the equation $P_1 + P_3 = P_2$, and which is of smallest area a; hence

$$\Delta(k) = a/3.$$

Denote by H the hexagon polar-reciprocal to h with respect to the unit circle, and by A its area. By polarity, H is circumscribed to K, and so by Theorem 1,

$$\Delta(K) \leqq \Delta(H) = A/4.$$

Hence by Theorem 2,

$$\Delta(k)\, \Delta(K) \leqq a/3 \cdot A/4 \leqq 9 \cdot 1/12 = 3/4,$$

as asserted.

Proof of the lower bound: By MINKOWSKI's theorem on convex domains [7]),

$$\Delta(k) \geqq V(k)/4, \quad \Delta(K) \geqq V(K)/4.$$

Therefore,

$$\Delta(k)\, \Delta(K) \geqq V(k)\, V(K)/16 \geqq 8/16 = 1/2,$$

as asserted.

Both formulae (C) are best possible, since the left-hand equality sign holds when k and K are the squares

$$k: |x| \leqq 1, \ |y| \leqq 1, \quad \text{and} \quad K: |x| + |y| \leqq 1,$$

and the right-hand equality sign holds when both k and K become the unit circle.

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[5]) See, e.g. my paper l.c. [4]), Lemma 2 and Formula (I).

[6]) See my paper "Ein Minimalproblem für konvexe Polygone", Mathematica B (Zutphen), 7 (1938—1939).

[7]) These two inequalities follow also from Theorem 1, since the area of k or K is not larger than that of any circumscribed hexagon.