Mathematics. — *On the minimum determinant of a special point set.* By K. MAHLER (Manchester). (Communicated by Prof. J. G. VAN DER CORPUT.) *)

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In a preceding paper [1]) C. A. ROGERS proves the inequality

$$\lambda_1 \lambda_2 \ldots \lambda_n \triangle (K) \leqslant 2^{\frac{n-1}{2}} d(\Lambda)$$

for the successive minima $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an arbitrary point set K for a lattice Λ. In the present paper, I shall construct a point set for which this formula holds with the *equality sign*. I prove, moreover, that there exist *bounded star bodies* for which the quotient of the two sides of ROGERS's inequality approaches arbitrarily near to 1. *The constant* $2^{\frac{n-1}{2}}$ *of* ROGERS *is therefore best-possible, even in the very specialized case of a bounded star body.*

1) Let $\mathfrak{R}_n$ be the n-dimensional Euclidean space of all points

$$X = (x_1, x_2, \ldots, x_n)$$

with real coordinates. For $k = 1, 2, \ldots, n$, denote by Γ_k the set of all points

$$(g_1, g_2, \ldots, g_k, 0, \ldots, 0)$$

with integral coordinates satisfying [2])

$$g_k \neq 0, \quad gcd(g_1, g_2, \ldots, g_k) = 1,$$

and by C_k the set of all points

$$X = tP, \text{ where } t \geq 2^{\frac{n-k}{n}} \text{ and } P \varepsilon \Gamma_k.$$

Further write

$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$

for the union of $C_1, C_2, \ldots, C_n$, and

$$K = \mathfrak{R}_n - C$$

for the set of all points in $\mathfrak{R}_n$ which do not belong to C.

Although K is not a bounded set, it is of the finite type. For the lattice Λ_0 consisting of the points

$$(2g_1, 2g_2, \ldots, 2g_{n-1}, g_n),$$

*) This article has been sent to J. G. VAN DER CORPUT on February 12, 1949.

1) C. A. ROGERS, The product of the minima and the determinant of a set. These Proceedings **52**, 256—263 (1949).

2) $gcd(g_1, g_2, \ldots, g_k)$ means the greatest common divisor of $g_1, g_2, \ldots, g_k$, and similarly in other cases.

where $g_1, g_2, \ldots, g_n$ run over all integers, is evidently K-admissible, and so

$$\Delta(K) \leqslant d(\Lambda) = 2^{n-1} \quad \ldots \ldots \ldots \quad (1)$$

Our aim is to find the exact value of $\Delta(K)$.

2) The origin $O = (0, 0, \ldots, 0)$ is an inner point of K, and K is of the finite type; therefore [3]) K possesses at least one critical lattice, the lattice Λ say. By (1),

$$d(\Lambda) \leqslant 2^{n-1}. \quad \ldots \ldots \ldots \quad (2)$$

For $k = 1, 2, \ldots, n$, let Π_k be the parallelepiped

$$|x_h| \begin{cases} < 1 \text{ if } 1 \leqslant h \leqslant n,\ h \neq k; \\ \leqslant 2^{n-1} \text{ if } h = k. \end{cases}$$

By (2) and by MINKOWSKI's theorem on linear forms, each parallelepiped Π_k contains a point $Q_k \neq O$ of Λ. Since Λ is K-admissible, and from the definition of K, this point belongs to C; hence only the k-th coordinate of Q_k, η_k say, is different from zero and may be assumed positive:

$$Q_k = (0, \ldots, \eta_k, \ldots, 0), \qquad \text{where } \eta_k > 0. \quad \ldots \quad (3)$$

The point

$$Q = Q_1 + Q_2 + \ldots + Q_n = (\eta_1, \eta_2, \ldots, \eta_n)$$

also belongs to Λ and therefore to C. Since $\eta_n > 0$, Q necessarily lies in C_n. From the definition of this set, there exist then a positive number η and n positive integers $q_1, q_2, \ldots, q_n$ such that

$$\eta_k = \eta\, q_k \qquad (k = 1, 2, \ldots, n). \quad \ldots \ldots \quad (4)$$

3) The n lattice points

$$Q_1, Q_2, \ldots, Q_n$$

do not necessarily form a basis of Λ; they are, however, linearly independent, and so they generate a sublattice of Λ. Hence there exists a fixed positive integer, q say, such that every point P of Λ can be written in the form

$$P = \frac{1}{q}\{p_1 Q_1 + p_2 Q_2 + \ldots + p_n Q_n\} = \left(\frac{\eta}{q} p_1 q_1, \frac{\eta}{q} p_2 q_2, \ldots, \frac{\eta}{q} p_n q_n\right)$$

with integral coefficients $p_1, p_2, \ldots, p_n$ depending on P. For shortness, put

$$\xi = \frac{\eta}{q}, \text{ so that } \xi > 0. \quad \ldots \ldots \ldots \quad (5)$$

By MINKOWSKI's method of reduction [4]), we can now select a basis

$$P_1, P_2, \ldots, P_n$$

[3]) See my paper, *On the critical lattices of an arbitrary point set,* Canadian Journal of Mathematics, I (1949), 78—87.

[4]) Geometrie der Zahlen (1910), § 46.

of Λ such that each basis point P_k, where $k = 1, 2, \ldots, n$, is a linear combination of $Q_1, Q_2, \ldots, Q_k$, hence of the form

$$P_k = (\xi p_{k1}, \xi p_{k2}, \ldots, \xi p_{kk}, 0, \ldots, 0) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$p_{k1}, p_{k2}, \ldots, p_{kk} \text{ are integers, and } p_{kk} > 0 \quad . \quad . \quad . \quad . \quad (7)$$

It may, moreover, be assumed that

$$0 \leqslant p_{kl} < p_{ll} \text{ for all pairs of indices } k, l \text{ satisfying } 1 \leqslant k < l \leqslant n. \ (8)$$

4) **Lemma:** *Let*

$$L_h(x) = \sum_{k=1}^{n} a_{hk} x_k \qquad (h = 1, 2, \ldots, m)$$

be m *linear forms in* n *variables* $x_1, x_2, \ldots, x_n$, *with integral coefficients* a_{hk} *not all zero. Denote by*

$$a = gcd\, a_{hk}$$

the greatest common divisor of these coefficients, and by

$$L(x) = gcd\, L_h(x)$$

the greatest common divisor of the numbers $L_h(x)$, *where* $h = 1, 2, \ldots, m$. *Then there exist integers* $x_1, x_2, \ldots, x_n$ *such that*

$$L(x) = a.$$

Proof: By the theory of elementary divisors [5]), two integral unimodular square matrices

$$(b_{gh}) \text{ and } (c_{kl})$$

of m^2 and n^2 elements, respectively, can be found such that the product matrix

$$(b_{gh})\,(a_{hk})\,(c_{kl}), \qquad = (d_{gl}) \text{ say,}$$

of mn elements is a diagonal matrix, viz.

$$d_{gl} = 0 \text{ if } g \neq l.$$

Put

$$r = \min(m, n)$$

and

$$x_k = \sum_{l=1}^{n} c_{kl} x'_l, \qquad L_g(x') = \sum_{h=1}^{m} b_{gh} L_h(x),$$

so that

$$L_g(x') = \begin{cases} d_{gg} x'_g & \text{if } g \leqslant r, \\ 0 & \text{if } g > r. \end{cases}$$

Then evidently

$$a = gcd\,(d_{11}, d_{22}, \ldots, d_{rr})$$

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5) See e.g. B. L. VAN DER WAERDEN, Moderne Algebra, Vol. 2 (1931), § 106.

and

$$L(x) = gcd\, L_g(x') = gcd\,(d_{11}x'_1, d_{22}x'_2, \ldots, d_{rr}x'_r),$$

and the assertion follows on putting

$$x'_1 = x'_2 = \ldots = x'_r = 1.$$

5) Every point P of Λ can be written as

$$P = x_1P_1 + x_2P_2 + \ldots + x_nP_n$$

with integral coefficients $x_1, x_2, \ldots, x_n$. Therefore P has the coordinates

$$P = (\xi L_1(x), \xi L_2(x), \ldots, \xi L_n(x)), \quad \ldots\ldots \quad (9)$$

where, for shortness,

$$L_h(x) = \sum_{g=h}^{n} p_{gh}\, x_g \qquad (h = 1, 2, \ldots, n). \quad \ldots\ldots \quad (10)$$

Let now d_k, for $k = 1, 2, \ldots, n$, be the greatest common divisor of the coefficients

$$p_{gh} \text{ with } 1 \leq h \leq g \leq k.$$

From this definition, it is obvious that

$$d_k \text{ is divisible by } d_{k+1} \text{ for } k = 1, 2, \ldots, n-1. \quad \ldots \quad (11)$$

Since the matrix of the n forms $L_1(x), L_2(x), \ldots, L_n(x)$ is triangular, d_k may also be defined as the greatest common divisor of the coefficients of

$$x_1, x_2, \ldots, x_k$$

in the forms

$$L_1(x), L_2(x), \ldots, L_k(x).$$

It follows therefore, for $k = 1, 2, \ldots, n$, from the lemma in 4) that there exist integers

$$x_{k1}, x_{k2}, \ldots, x_{kk}$$

not all zero such that the greatest common divisor of the k numbers

$$g_{hk} = \sum_{g=h}^{k} p_{gh}\, x_{kg} \qquad (h = 1, 2, \ldots, k)$$

is equal to d_k.

The point

$$R_k = x_{k1}P_1 + x_{k2}P_2 + \ldots + x_{kk}P_k \neq O \quad \ldots\ldots \quad (12)$$

belongs to Λ and has the coordinates

$$R_k = (\xi g_{1k}, \xi g_{2k}, \ldots, \xi g_{kk}, 0, \ldots, 0) \quad \ldots\ldots \quad (13)$$

which are not all zero and satisfy the equation

$$gcd\,(g_{1k}, g_{2k}, \ldots, g_{kk}) = d_k. \quad \ldots\ldots \quad (14)$$

Since R_k is not an inner point of K, it belongs to one of the sets $C_1, C_2, \ldots, C_k$. We conclude therefore, from the definition of these sets, that

$$\xi d_k \geqslant 2^{\frac{n-k}{n}} \qquad (k = 1, 2, \ldots, n). \quad \ldots\ldots \quad (15)$$

6) Next let ζ be the positive real number for which

$$\zeta \min_{k=1,2,\ldots,n} 2^{-\frac{n-k}{n}} d_k = 1, \text{ whence } 0 < \zeta \leqq \xi. \quad . \quad . \quad . \quad (16)$$

There is then an index $\varkappa$ with $1 \leq \varkappa \leq n$ such that

$$\zeta d_k \begin{cases} \geqq 2^{\frac{n-k}{n}} \text{ for } k = 1, 2, \ldots, n, \\ = 2^{\frac{n-\varkappa}{n}} \text{ for } k = \varkappa. \end{cases} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From these formulae (17):

$$d_k \geqq \zeta^{-1} \cdot 2^{\frac{n-k}{n}} = 2^{\frac{\varkappa-k}{n}} d_\varkappa \qquad (k = 1, 2, \ldots, n).$$

Hence, if $k < \varkappa$, then

$$d_k \geqq 2^{\frac{1}{n}} d_\varkappa,$$

whence, by (11),

$$d_k \geqq 2 d_\varkappa \qquad \text{for } k = 1, 2, \ldots, \varkappa - 1 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If, however, $k \geq \varkappa$, then

$$d_k \geqq 2^{\frac{\varkappa-n}{n}} d_\varkappa > \tfrac{1}{2} d_\varkappa$$

and (11) implies now that

$$d_k = d_\varkappa \qquad \text{for } k = \varkappa, \varkappa + 1, \ldots, n. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

On combining (18) and (19), we obtain the further inequality,

$$\xi^n d_1 d_2 \ldots d_n \geqq \zeta^n d_1 d_2 \ldots d_n \geqq 2^{\varkappa-1} (\zeta d_\varkappa)^n = 2^{n-1}. \quad . \quad . \quad (20)$$

7) The critical lattice Λ we have been considering, has the basis $P_1, P_2, \ldots, P_n$ of the form (6). Its determinant is therefore

$$d(\Lambda) = \xi^n p_{11} p_{22} \ldots p_{nn}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

since all factors on the right-hand side of this equation are positive. From the definition of d_k,

$$p_{kk} \text{ is divisible by } d_k \qquad \text{for } k = 1, 2, \ldots, n. \quad . \quad . \quad . \quad (22)$$

Hence by (20) and (21),

$$d(\Lambda) \geqq \xi^n d_1 d_2 \ldots d_n \geqq 2^{n-1}$$

whence

$$\triangle(K) \geqq 2^{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The same right-hand side was, by (1), also a lower bound of $\triangle(K)$; hence the final result

$$\triangle(K) = 2^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

is obtained.

8) By means of the last formulae, all critical lattices of K can be obtained as follows.

It is clear, from the previous discussion, that to any critical lattice Λ, there is a unique index $\varkappa$ with $1 \leq \varkappa \leq n$ such that

$$d_k = \begin{cases} 2\,d_\varkappa \text{ for } k = 1, 2, \ldots, \varkappa - 1, \\ \;\; d_\varkappa \text{ for } k = \varkappa, \varkappa + 1, \ldots, n, \end{cases} \quad \ldots \quad (24)$$

and that further

$$\zeta = \xi, \quad \ldots \quad (25)$$

$$p_{11} = d_1, \; p_{22} = d_2, \ldots, p_{nn} = d_n; \quad \ldots \quad (26)$$

for otherwise $d(\Lambda)$ would be larger than 2^{n-1}. Since we may, if necessary, replace ξ by $d_\varkappa \xi$, there is no loss of generality in assuming that

$$d_\varkappa = 1, \quad \ldots \quad (27)$$

whence, by (17):

$$\xi = 2^{\frac{n-\varkappa}{n}}. \quad \ldots \quad (28)$$

The basis points $P_1, P_2, \ldots, P_n$ become,

$$P_k = \left(2^{\frac{n-\varkappa}{n}} p_{k1}, 2^{\frac{n-\varkappa}{n}} p_{k2}, \ldots, 2^{\frac{n-\varkappa}{n}} p_{kk}, 0, \ldots, 0\right)$$

with integral p_{kl}. By (7), (8), and (24)—(28), moreover

$$p_{kk} = \begin{cases} 2 \text{ if } k = 1, 2, \ldots, \varkappa - 1, \\ 1 \text{ if } k = \varkappa, \varkappa + 1, \ldots, n, \end{cases} \quad \ldots \quad (29)$$

and

$$p_{kl} = \begin{cases} 0 & \text{if } 1 \leqslant l < k \leqslant \varkappa - 1, \\ 0 & \text{if } \varkappa \leqslant l < k \leqslant n, \\ 0 \text{ or } 1 & \text{if } \varkappa \leqslant k \leqslant n, \; 1 \leqslant l \leqslant \varkappa - 1. \end{cases} \quad \ldots \quad (30)$$

It is also clear that different choices of $\varkappa$ and of the integers p_{kl} lead to different critical lattices. Since for exactly

$$(\varkappa-1)\,(n-\varkappa+1)$$

coefficients p_{kl} there is the alternative $p_{kl} = 0$ or 1, there are then for each $\varkappa$ just

$$2^{(\varkappa-1)\,(n-\varkappa+1)}$$

different critical lattices. We find therefore, on summing over $\varkappa$, that *the total number $N(n)$ of different critical lattices of K is given by the formula*

$$N(n) = \sum_{\varkappa=1}^{n} 2^{(\varkappa-1)\,(n-\varkappa+1)}. \quad \ldots \quad (B)$$

Thus $N(n) = 3, 9, 33, 161, 1089, \ldots$ for $n = 2, 3, 4, 5, 6, \ldots$.

9) *We next determine the successive minima*

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

of K in the lattice Λ_1 of all points with integral coordinates.

Denote by λK, for $\lambda > 0$, the set of all points λX where X belongs to K. The first minimum λ_1 of K for Λ_1 is defined as the lower bound of all $\lambda > 0$ such that λK contains a point of Λ_1 different from O; if further $k = 2, 3, \ldots, n$, then the n-th minimum λ_k of K for Λ_1 is defined as the lower bound of all $\lambda > 0$ such that λK contains k linearly independent points of Λ_n. We find these minima as follows.

Consider an arbitrary point

$$P = (g_1, g_2, \ldots, g_n) \neq O$$

of Λ_1; here $g_1, g_2, \ldots, g_n$ are integers. Put

$$d = gcd\,(g_1, g_2, \ldots, g_n), \text{ so that } d \geq 1,$$

and assume, say, that

$$g_k \neq 0, \text{ but } g_{k+1} = \ldots = g_n = 0,$$

for some integer k with $1 \leq k \leq n$. Then P/d belongs to Γ_k, and tP belongs to C_k if and only if

$$t \geqslant 2^{\frac{n-k}{n}} d^{-1}.$$

Therefore λK, for $\lambda > 0$, contains P if, and only if,

$$\lambda > 2^{-\frac{n-k}{n}} d.$$

We deduce that if

$$\lambda \leqslant 2^{-\frac{n-1}{n}},$$

then λK contains no lattice point except O; if, however,

$$2^{-\frac{n-k}{n}} < \lambda \leqslant 2^{-\frac{n-k-1}{n}}, \quad \ldots \ldots \ldots \quad (31)$$

where $k = 1, 2, \ldots, n$, then λK contains just the points of the k sets

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_k.$$

Hence, if (31) holds, then λK contains k, and not more, linearly independent points of Λ_1. The successive minima of K for Λ_1 are therefore given by the equations,

$$\lambda_k = 2^{-\frac{n-k}{n}} \qquad (k = 1, 2, \ldots, n). \quad \ldots \ldots \quad (32)$$

By (A), this implies that

$$\lambda_1 \lambda_2 \ldots \lambda_n \triangle(K) = 2^{-\sum_{k=1}^{n} \frac{n-k}{n}} 2^{n-1} = 2^{\frac{n-1}{2}} = 2^{\frac{n-1}{2}} d(\Lambda_1). \quad . \quad . \quad (C)$$

We have thus proved that *in the special case of the point set K and the* *lattice Λ_1, the sign of equality holds in* ROGERS's *inequality for the successive minima of a point set* [6]).

10) The point set K is neither bounded nor a star body. It can, however, be approximated by a bounded star body of nearly the same minimum determinant and with the same successive minima, as follows.

Let ε be a small positive number. If X is any point different from O, then denote by $S_\varepsilon(X)$ the open set consisting of all points

$$tX + \varepsilon(t-1)Y$$

where t runs over all numbers with

$$t > 1,$$

and Y runs over all points of the open unit sphere

$$|Y| < 1;$$

evidently $S_\varepsilon(X)$ is a cone open towards infinity with vertex at X and axis on the line through O and X. Let further S_ε be the closed sphere of radius $1/\varepsilon$ which consists of all points Z satisfying

$$|Z| \leq 1/\varepsilon.$$

We now define K_ε as the set of all those points of K which belong to S_ε, but to none of the cones

$$S_\varepsilon\left(2^{\frac{n-k}{n}} X\right), \text{ where } X \varepsilon \Gamma_k \text{ and } k = 1, 2, \ldots, n.$$

Since only a finite number of the cones contains points of S_ε, it is clear that K_ε is a bounded star body.

Let $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ be the successive minima of K_ε for Λ. Since K_ε is a subset of K, necessarily

$$\lambda'_k \geqslant \lambda_k \qquad (k = 1, 2, \ldots, n).$$

We can in the present case replace these inequalities immediately by the equations

$$\lambda'_k = \lambda_k \qquad (k = 1, 2, \ldots, n) \quad \ldots\ldots \quad (33)$$

because the n boundary points

$$\left(2^{\frac{n-1}{n}}, 0, \ldots, 0\right), \left(0, 2^{\frac{n-2}{n}}, \ldots, 0\right), \ldots, (0, 0, \ldots, 1),$$

in which the successive minima of K for Λ_1 are attained, are still boundary points of K_ε provided ε is sufficiently small.

11) We further show that

$$\lim_{\varepsilon \to 0} \Delta(K_\varepsilon) = \Delta(K). \quad \ldots\ldots \quad (34)$$

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[6]) See l.c. [1]).

Let this equation be false. There exists then a sequence of positive numbers

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \qquad (\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots > 0)$$

tending to zero such that

$$\lim_{r \to \infty} \triangle (K_{\varepsilon_r})$$

exists, but is different from $\triangle(K)$. But then

$$\lim_{r \to \infty} \triangle (K_{\varepsilon_r}) < \triangle (K), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

since each K_{ε_r} is a subset of K. As a bounded star body, each K_{ε_r} possesses at least one critical lattice, Λ_r say; by the last formula, it may be assumed that

$$d(\Lambda_r) = \triangle (K_{\varepsilon_r}) \leqslant \triangle (K) \qquad (r = 1, 2, 3, \ldots).$$

Moreover, all sets K_{ε_r} contain a fixed neighbourhood of the origin O as subset. The sequence of lattices

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots$$

is therefore bounded, and so, on possibly replacing this sequence by a suitable infinite subsequence, we may assume that the lattices Λ_r tend to a limiting lattice, Λ say. By (35),

$$d(\Lambda) = \lim_{r \to \infty} d(\Lambda_r) = \lim_{r \to \infty} \triangle (K_{\varepsilon_r}) < \triangle (K), \quad . \quad . \quad . \quad . \quad (36)$$

and therefore Λ cannot be K-admissible. Hence there exists a point $P \neq O$ of Λ which is an inner point of K. This means that P, for sufficiently small $\varepsilon > 0$, is also an inner point of K_ε.

We can now select in each lattice Λ_r a point $P_r \neq O$ such that the sequence of points

$$P_1, P_2, P_3, \ldots$$

tends to P. Hence, for any fixed sufficiently small $\varepsilon > 0$, all but a finite number of these points are inner points of K. Now, since

$$\varepsilon_r > \varepsilon_{r+1},$$

each star body K_{ε_r} is contained in all the following bodies

$$K_{\varepsilon_{r+1}}, K_{\varepsilon_{r+2}}, K_{\varepsilon_{r+3}}, \ldots.$$

Therefore, when r is sufficiently large, then the point P_r is an inner point of K_{ε_r}, contrary to the hypothesis that Λ_r is a critical, hence also an admissible lattice of K_{ε_r}. This concludes the proof of (34).

12) The two formulae (33) and (34) imply that

$$\lim_{\varepsilon \to 0} \lambda_1' \lambda_2' \ldots \lambda_n' \triangle (K_\varepsilon) = 2^{\frac{n-1}{2}} d(\Lambda_1).$$

Hence if $\delta > 0$ is an arbitrarily small number, then there exists a positive number ε such that the successive minima $\lambda'_1, \lambda'_2, \dots, \lambda'_n$ of K_ε satisfy the inequality,

$$\lambda'_1 \lambda'_2 \dots \lambda'_n \triangle (K_\varepsilon) > (1 - \delta) 2^{\frac{n-1}{2}} d(\Lambda_1),$$

where Λ_1 is the lattice of all points with integral coordinates.

We have therefore proved that the constant $2^{\frac{n-1}{2}}$ in ROGERS's inequality is best-possible *even for bounded star bodies*. This is very surprising as this inequality applies to general sets.

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Postscript (May 16, 1949): In a note in the C.R. de l'Academie des Sciences (Paris), 228 (March 7, 1949), 796—797, Ch. Chabauty announces the main result of this paper, but does not give a detailed proof.