ON A THEOREM OF LIOUVILLE IN FIELDS OF POSITIVE CHARACTERISTIC

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A classical theorem of J. Liouville[1] states that if z is a real algebraic number of degree $n \geq 2$, then there exists a constant $c > 0$ such that

$$\left| z - \frac{a}{b} \right| \geq \frac{c}{|b|^n}$$

for every pair of integers a, b with $b \neq 0$.

This theorem has an analogue in function fields. Let k be an arbitrary field, x an indeterminate, $k[x]$ the ring of all polynomials in x with coefficients in k, $k(x)$ the field of all rational functions in x with coefficients in k, and $k<x>$ the field of all formal series

$$z = a_f x^f + a_{f-1} x^{f-1} + a_{f-2} x^{f-2} + \dots$$

in x where the coefficients $a_f, a_{f-1}, a_{f-2}, \dots$ are in k. Thus $k(x)$ is the quotient field of $k[x]$ and a subfield of $k<x>$.

A valuation $|z|$ in $k<x>$ is now defined by putting $|0| = 0$; but $|z| = e^f$ if $z = a_f x^f + a_{f-1} x^{f-1} + a_{f-2} x^{f-2} + \dots$ and $a_f \neq 0$. If z lies in $k[x]$, then $\log |z|$ is simply the degree of z.

With this notation, the analogue to Liouville's theorem states:

THEOREM 1. *If the element z of $k<x>$ is algebraic of degree $n \geq 2$ over $k(x)$, then there exists a constant $c > 0$ such that*

$$\left| z - \frac{a}{b} \right| \geq \frac{c}{|b|^n}$$

for all pairs of elements a and $b \neq 0$ of $k[x]$.

Proof. Denote by

$$f(y) = a_0 y^n + a_1 y^{n-1} + \dots + a_n, \qquad \text{where } a_0 \neq 0,$$

a polynomial in y with coefficients in $k[x]$ which is irreducible over $k(x)$ and vanishes for $y = z$; further put

$$g(y) = a_0 y^{n-1} + (a_0 z + a_1) y^{n-2} + (a_0 z^2 + a_1 z + a_2) y^{n-3} + \dots + (a_0 z^{n-1} + a_1 z^{n-2} + \dots + a_{n-1}).$$

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[1] *C.R. Acad. Sci. Paris*, vol. 18 (1844), 883-885, 910-911.

Then

$$\frac{f(y)}{y-z} = \frac{f(y)-f(z)}{y-z} = g(y)$$

identically in y, and therefore

$$y - z = \frac{f(y)}{g(y)}.$$

Put

$$\max(|a_0|, |a_1|, \ldots, |a_n|) = c_1, \quad \max(1, |z|) = c_2$$

and take

$$y = \frac{a}{b}$$

where a and $b \neq 0$ are in $k[x]$.

If

$$\left|\frac{a}{b}\right| > c_2 = |z|,$$

then

(1) $$\left|z - \frac{a}{b}\right| = \left|\frac{a}{b}\right| > c_2 \geq \frac{c_2}{|b|^n}, \qquad \text{since } |b| \geq 1.$$

Next let

$$\left|\frac{a}{b}\right| \leq c_2,$$

so that

$$\left|g\left(\frac{a}{b}\right)\right| \leq c_1 c_2^{n-1}.$$

The expression

$$b^n f\left(\frac{a}{b}\right) = a_0 a^n + a_1 a^{n-1} b + \ldots + a_n b^n$$

lies in $k[x]$ and does not vanish since $f(y)$ is irreducible and at least of the second degree. Therefore

$$\left|b^n f\left(\frac{a}{b}\right)\right| \geq 1, \quad \left|f\left(\frac{a}{b}\right)\right| \geq |b|^{-n},$$

whence

(2) $$\left|z - \frac{a}{b}\right| = \left|\frac{f\left(\frac{a}{b}\right)}{g\left(\frac{a}{b}\right)}\right| \geq \frac{1}{c_1 c_2^{n-1} |b|^n}.$$

If we now put

$$c = \min\left(c_2, \frac{1}{c_1 c_2^{n-1}}\right),$$

then the assertion of the theorem is contained in (1) and (2).

In the case of a real algebraic number of degree $n \geq 3$, Liouville's theorem is not the best-possible, and it was first improved by A. Thue,[2] who showed that, for every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that

$$\left| z - \frac{a}{b} \right| \geq \frac{c(\epsilon)}{|b|^{\frac{n}{2}+1+\epsilon}}$$

for all pairs of integers a and $b \neq 0$. Still better inequalities were given by C. L. Siegel[3] and F. J. Dyson.[4] A similar improvement is possible in the case of the analogue of Liouville's theorem for algebraic functions, if the constant field k is the field of all complex numbers, or, more generally, any field of characteristic 0, as was proved by B. P. Gill.[5]

It is then of some interest to note that *the analogue of Liouville's theorem for algebraic functions cannot be improved if the ground field k is of characteristic p where p is a positive prime number.* Indeed, the following result holds.

THEOREM 2. *Let k be any field of characteristic p, x an indeterminate, and z the element*

$$z = x^{-1} + x^{-p} + x^{-p^2} + x^{-p^3} + \dots$$

of $k\langle x\rangle$. *Then z is of exact degree p over* $k(x)$, *and there exists an infinite sequence of pairs of elements* a_n *and* $b_n \neq 0$ *of* $k[x]$ *such that*

$$\left| z - \frac{a_n}{b_n} \right| = |b_n|^{-p}, \quad \lim_{n \to \infty} |b_n| = \infty.$$

Proof. If $a, b, c, \dots$ are elements of $k\langle x\rangle$, then

$$(a + b + c + \dots)^p = a^p + b^p + c^p + \dots,$$

by a well-known property of fields of characteristic p. Hence, in particular,

$$z = x^{-1} + (x^{-p} + x^{-p^2} + x^{-p^3} + \dots) = x^{-1} + (x^{-1} + x^{-p} + x^{-p^2} + \dots)^p$$

and so z is a root of the algebraic equation[6]

$$z^p - z + x^{-1} = 0 \tag{3}$$

of degree p over $k(x)$.

Put, for $n = 1, 2, 3, \dots,$

$$a_n = x^{p^{n-1}}(x^{-1} + x^{-p} + \dots + x^{-p^{n-1}}), \quad b_n = x^{p^{n-1}}$$

[2] *Norske Vid. Selsk. Scr.* (1908), Nr. 7.

[3] *Math. Zeit.*, vol. 10 (1921), 173-213.

[4] *Acta Math.*, vol. 79 (1947), 225-240.

[5] *Ann. of Math.* (2) 31 (1930), 207-218.

[6] I am indebted to E. Artin for the remark that z is algebraic if k is of characteristic p. If k is of characteristic 0, then z is, of course, transcendental over $k(x)$.

so that

$$|b_n| = e^{p^{n-1}}, \text{ and } \left|z - \frac{a_n}{b_n}\right| = |x^{-p^n} + x^{-p^{n+1}} + \ldots| = e^{-p^n} = |b_n|^{-p}.$$

The assertion will therefore be proved if we can show that z is of exact degree p. But, by Theorem 1, z cannot be of lower degree than p, unless it is of degree 1 and lies in $k(x)$. Suppose then that

$$z = \frac{A}{B},$$

where A and $B \neq 0$ are elements of $k[x]$. Since the fractions a_n/b_n are all different,

$$\frac{a_n}{b_n} \neq z, \quad Ab_n - a_nB \neq 0, \quad |Ab_n - a_nB| \geq 1,$$

for all sufficiently large n. But then

$$|b_n|^{-p} = \left|z - \frac{a_n}{b_n}\right| = \left|\frac{A}{B} - \frac{a_n}{b_n}\right| = \left|\frac{Ab_n - a_nB}{Bb_n}\right| \geq \frac{1}{|B|\,|b_n|},$$

whence

$$|B| \geq |b_n|^{p-1},$$

contrary to the fact that

$$\lim_{n\to\infty} |b_n| = \infty.$$

It would be of interest to investigate whether the analogue of Liouville's theorem remains still the best-possible for elements $k<x>$ not in $k(x)$ which are of a degree *less than* p over $k(x)$.

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