On the continued fractions of quadratic and cubic irrationals.

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Summary. - *Let* ζ *be a quadratic or cubic irrational, and let* $\frac{p_n}{q_n}$ *be the* n-*th approximation of its regular continued fraction. It ist proved that the greatest prime factor of* q_n *tends to infinity with* n.

A number of years ago, I applied a method due to TH. SCHNEIDER [1] to prove the following result [2]:

Let ζ *be a real irrational algebraic number; let*

$$\zeta = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots,$$

where a_0, $a_1 \geq 1$, $a_2 \geq 1, \ldots$ *are integers, be its continued fraction; and let* $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ *be its approximations. Then the greatest prime factor of* q_n (hence also that of p_n) *is unbounded.*

A method recently given by F. J. DYSON [3] allows us now to prove the following more special, but stronger result:

If ζ *is a quadratic or cubic real irrational number, then the greatest prime factor of* q_n (hence also that of p_n) *tends to infinity with* n.

This result follows immediately from Theorem 3 of this paper, viz.:

If ζ *is a real algebraic number of degree* n, *and if*

$$\left|\frac{p}{q} - \zeta\right| < q^{-\mu}$$

has infinitely many solutions in fractions $\frac{p}{q}$ *where the greatest prime factor of* $q \geq 1$ *is bounded, then* $\mu \leq \sqrt{n}$.

As the method of this paper may possibly have other applications, I have tried to give all details of the proof.

(1) « Journal f. d. r. u. ang. Mathematik », **175** (1936).

(2) « Akad. v. Wetensch. te Amsterdam », Proc. **39**, 633-640, 729-737 (1936).

(3) « Acta Mathematica », **79**, 225-240 (1947). See also A. O. GELFOND, « Vestnik MGU », **9**, 3 (1948) and TH. SCHNEIDER, « Math. Nachr. », **2**, 288-295 (1949).

1. The Main Lemma.

[1] In order to stress the generality of the main lemma proved in this chapter, all polynomials occurring are allowed to have their coefficients in an arbitrary, but fixed, field K of characteristic zero. As usual, $K[x]$ and $K[x, y]$ denote then the rings of all polynomials in x, or in x and y, respectively, with coefficients in K.

[2] Let r and s be two fixed positive integers such that

$$r \geq s,$$

and let $R(x, y)$ be a fixed polynomial in $K[x, y]$ of the form

$$R(x, y) = \sum_{\substack{h \geq 0 \\ \frac{h}{r} + \frac{k}{s} \leq 1}} \sum_{k \geq 0} R_{hk} x^h y^k \not\equiv 0. \tag{1}$$

If we write

$$R(x, y) = \sum_{k=0}^{s} p_k(x) y^k, \tag{2}$$

then, from the definition, $p_k(x)$ is an element of $K[x]$ of the form

$$p_k(x) = \sum_{h=0}^{\left[r\left(1-\frac{k}{s}\right)\right]} R_{hk} x^h \qquad (k = 0, 1, 2, \ldots, s). \tag{3}$$

hence is of degree not higher than

$$\left[r\left(1 - \frac{k}{s}\right)\right]$$

in x.

[3] The polynomials

$$p_0(x),\ p_1(x), \ldots, p_s(x) \tag{4}$$

need not be all independent (i.e. linearly independent over K), and some of them may be identically zero. The following algorism enables us to obtain an independent subsystem of the same rank.

Denote by

$$u_0(x) = p_{k_0}(x), \qquad \text{where } k_0 \leq s,$$

that polynomial $p_k(x)$ which is of largest index k_0 and does not vanish identically; such a polynomial exists since $R(x, y) \not\equiv 0$. Denote, similarly, by

$$u_1(x) = p_{k_1}(x), \qquad \text{where } k_1 < k_0,$$

that polynomial $p_k(x)$ which is of largest index k_1 and is independent of $p_{k_0}(x)$; by

$$u_2(x) = p_{k_2}(x), \qquad \text{where } k_2 < k_1,$$

that polynomial $p_k(x)$ which is of largest index k_2 and is independent of $p_{k_0}(x)$ and $p_{k_1}(x)$; and continuing in the same way, finally by

$$u_{l-1}(x) = p_{k_{l-1}}(x), \qquad \text{where } k_{l-1} < k_{l-2},$$

that polynomial $p_k(x)$ which is of largest index k_{l-1}, is independent of

$$p_{k_0}(x),\ p_{k_1}(x), \dots,\ p_{k_{l-2}}(x),$$

and has the property that all polynomials (4) are dependent on

(5) $$u_0(x) = p_{k_0}(x),\ u_1(x) = p_{k_1}(x), \dots,\ u_{l-1}(x) = p_{k_{l-1}}(x).$$

Then

(6) $$1 \leq l \leq s+1$$

and

(7) $$s \geq k_0 > k_1 > k_2 \dots > k_{l-1} \geq 0.$$

Put

(8) $$r_\lambda = \left[r\left(1 - \frac{k_\lambda}{s}\right)\right] \qquad (\lambda = 0,\ 1, \dots,\ l-1);$$

then $u_\lambda(x)$ is at most of degree r_λ; moreover,

(9) $$0 \leq r_0 < r_1 < r_2 < \dots < r_{l-1} \leq r$$

since $r \geq s$.

[4] To simplify formulae, put

$$k_{-1} = s+1, \qquad k_l = -1.$$

Then, to every index

$$k = 0,\ 1,\ 2, \dots,\ s,$$

there exists a unique integer

$$\varkappa = \varkappa(k) \quad \text{with} \quad 0 \leq \varkappa \leq l$$

such that

(10) $$k_\varkappa < k \leq k_{\varkappa-1}.$$

By the construction in [3],

(11) *$p_k(x)$ is dependent on $u_0(x),\ u_1(x), \dots,\ u_{\varkappa-1}(x)$ if $k_\varkappa < k \leq k_{\varkappa-1}$.*

(If $\varkappa = 0$, then this means that $p_k(x)$ is identically zero). Moreover, there are elements $\alpha_{k\lambda}$ of K such that

(12) $$p_k(x) = \sum_{\lambda=0}^{\varkappa(k)-1} \alpha_{k\lambda} u_\lambda(x) \qquad (k = 0,\ 1, \dots,\ s)$$

identically in x. In particular, when

$$k = k_{\varkappa-1} \qquad (\varkappa = 1,\ 2, \dots,\ l),$$

then

(13) $$\alpha_{k_{\varkappa-1}\lambda} = 0 \quad \text{if} \quad 0 \leq \lambda \leq \varkappa-2, \qquad = 1 \quad \text{if} \quad \lambda = \varkappa - 1.$$

[5] By (2) and (12), $R(x, y)$ can be written as

$$R(x, y) = \sum_{k=0}^{s} \sum_{\lambda=0}^{\varkappa(k)-1} \alpha_{k\lambda} u_\lambda(x) y^k,$$

or

$$R(x, y) = \sum_{\lambda=0}^{l-1} u_\lambda(x) v_\lambda(y) \tag{14}$$

where $v_\lambda(y)$ is the polynomial in y defined by

$$v_\lambda(y) = \sum_{k=0}^{k_\lambda} \alpha_{k\lambda} y^k \qquad (\lambda = 0, 1, \dots, l-1). \tag{15}$$

By construction, the polynomials (5) are independent. We can now add the fact that also the polynomials

$$v_0(y),\ v_1(y), \dots,\ v_{l-1}(y) \tag{16}$$

are independent. For $v_\lambda(y)$ is of the form

$$v_\lambda(y) = y^{k_\lambda} + \text{terms in lower powers of } y,$$

and all exponents k_λ are different.

[6] Denote by $U(x)$ and $V(y)$ the two WRONSKI determinants

$$U(x) = \left| \frac{d^\varkappa u_\lambda(x)}{dx^\varkappa} \right|_{\varkappa, \lambda = 0, 1, \dots, l-1}; \qquad V(y) = \left| \frac{d^\varkappa v_\lambda(y)}{dy^\varkappa} \right|_{\varkappa, \lambda = 0, 1, \dots, l-1}, \tag{17}$$

where the differential coefficients are defined in a purely formal way. A well-known theorem states that *the* WRONSKI *determinant of a finite set of independent polynomials in one variable is not identically zero*: this theorem remains true even when the constants field K is an arbitrary field of characteristic zero (but not, if it is of positive characteristic). Therefore

$$U(x) \not\equiv 0, \qquad V(y) \not\equiv 0. \tag{18}$$

[7] Upper bounds for the degrees of $U(x)$ and $V(y)$ in x and y, respectively, are obtained as follows.

The determinant $U(x)$ may be written as a sum of $l\,!$ terms

$$\sum_{(i)} \pm \frac{d^{i_0} u_0(x)}{dx^{i_0}} \frac{d^{i_1} u_1(x)}{dx^{i_1}} \cdots \frac{d^{i_{l-1}} u_{l-1}(x)}{dx^{i_{l-1}}}$$

where the summation extends over all permutations $i_0, i_1, \dots, i_{l-1}$ of $0, 1, \dots, l-1$. By the definition of $u_\lambda(x)$ the general term of this sum is of degree not greater than

$$(r_0 - i_0) + (r_1 - i_1) + \dots + (r_{l-1} - i_{l-1}),$$

hence at most of degree

$$u = \sum_{\lambda=0}^{l-1} r_\lambda - \frac{l(l-1)}{2}, \tag{19}$$

because

$$i_0 + i_1 + \ldots + i_{l-1} = 0 + 1 + \ldots + (l-1) = \frac{l(l-1)}{2}.$$

We deduce that also $U(x)$ is at most of degree u in x. In the same way, we can show that $V(y)$ is at most of degree

$$v = \sum_{\lambda=0}^{l-1} k_\lambda - \frac{l(l-1)}{2}. \tag{20}$$

in y.

[8] The two bounds u and v contain the integers k_λ and r_λ defined in [3]. Now, by (8),

$$r_\lambda = \left[r\left(1 - \frac{k_\lambda}{s}\right)\right] \leq r - \frac{r}{s} k_\lambda,$$

and so we obtain the basic inequality

$$\frac{u}{r} + \frac{v}{s} \leq l - \frac{l(l-1)}{2}\left(\frac{1}{r} + \frac{1}{s}\right), \tag{21}$$

where these integers no longer occur on the right-hand side.

[9] From now on, let

$$\xi_0, \xi_1, \ldots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \ldots, \eta_n \tag{22}$$

be two systems each of $n+1$ numbers in K, where n is a positive integer, and where no two numbers of the same system are equal. Let further

$$\Theta_0, \Theta_1, \ldots, \Theta_n$$

be $n+1$ real numbers satisfying

$$0 < \Theta_f \leq 1 \qquad (f = 0, 1, \ldots, n), \tag{23}$$

and assume that $R(x, y)$ satisfies simultaneously the equations

$$\left.\frac{\partial^{i+j} R(x, y)}{\partial x^i \partial y^j}\right|_{\substack{x=\xi_f \\ y=\eta_f}} = 0 \quad \text{if} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta_f,\ f = 0, 1, \ldots, n. \tag{24}$$

[10] The two WRONSKI determinants $U(x)$ and $V(y)$ can be factorized in the forms,

$$U(x) = U^*(x) \prod_{f=0}^{n} (x - \xi_f)^{u_f}, \qquad V(y) = V^*(y) \prod_{f=0}^{n} (y - \eta_f)^{v_f}, \tag{25}$$

where the exponents u_f and v_f are certain non-negative integers, and where $U^*(x)$ and $V^*(y)$ do not vanish at any one of the points $x = \xi_f$ or $y = \eta_f$, respectively. Since $U(x)$ ist ast most of degree u in x, and $V(y)$ is at most of degree v in y, the two inequalities

$$\sum_{f=0}^{n} u_f \leq u, \qquad \sum_{f=0}^{n} v_f \leq v \tag{26}$$

hold.

[11] Denote by $W(x, y)$ the further determinant

$$W(x, y) = \left| \frac{\partial^{i+j} R(x, y)}{\partial x^i \partial y^j} \right|_{i, j = 0, 1, \ldots, l-1}. \tag{27}$$

We deduce, from (14), that

$$\frac{\partial^{i+j} R(x, y)}{\partial x^i \partial y^j} = \sum_{\lambda=0}^{l-1} \frac{d^i U_\lambda(x)}{dx^i} \frac{d^j V_\lambda(x)}{dy^j};$$

hence, by the multiplication rule for determinants,

$$W(x, y) = U(x) V(y) \tag{28}$$

identically in x and y.

[12] Let f be one of the indices $0, 1, 2, \ldots, n$, and let $P(x, y)$ be any element in $K[x, y]$. If z is a further variable, then the expression,

$$P(\xi_f + xz^s, \ \eta_f + yz^r), \qquad = P«z» \text{ say,}$$

can be written as a power series in z since, by TAYLOR's formula,

$$P«z» = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i}{i!} \frac{y^j}{j!} z^{rs\left(\frac{i}{r} + \frac{j}{s}\right)} \frac{\partial^{i+j} P(x, y)}{\partial x^i \partial y^j} \Bigg|_{\substack{x = \xi_f \\ y = \eta_f}}. \tag{29}$$

This series does not vanish identically in z unless $P(x, y)$ is identically zero as function of x and y.

Hence there is a unique non-negative number Θ such that $P«z»$ is divisible by $z^{rs\Theta}$, but not by $z^{rs\Theta'}$ for $\Theta' > \Theta$; if $P(x, y) \equiv 0$, then Θ may be taken to mean $+\infty$. We write for shortness,

$$D_f P(x, y) = \Theta.$$

By (29), the number Θ has the property that

$$\frac{\partial^{i+j} P(x, y)}{\partial x^i \partial y^j} \Bigg|_{\substack{x = \xi_f \\ y = \eta_f}}$$

vanishes for all pairs of integers i, j *satisfying*

$$i \geq 0, \qquad j \geq 0, \qquad \frac{i}{r} + \frac{i}{s} < \Theta,$$

but is different from z ro for at least one pair of integers i, j *with*

$$i \geq 0, \qquad j \geq 0, \qquad \frac{i}{r} + \frac{i}{s} = \Theta.$$

(The second assertion has no meaning if $\Theta = +\infty$).

From the definition of $D_f P(x, y)$, the following relations follow immediately:

(30) $$D_f \left\{ \prod_{\lambda=0}^{l-1} P_\lambda(x, y) \right\} = \sum_{\lambda=0}^{l-1} D_f P_\lambda(x, y),$$

and

(31) $$D_f \left\{ \sum_{\lambda=0}^{l-1} P_\lambda(x, y) \right\} \geq \min_{\lambda=0, 1, ..., l-1} D_f P_\lambda(x, y)$$

if $P_0(x, y)$, $P_1(x, y), ...,$ $P_{l-1}(x. y)$ is any finite set of elements of $K[x, y]$. From the connection with partial derivatives, it is further clear that

(32) $$D_f \left\{ \frac{\partial^{i+j} P(x, y)}{\partial x^i \partial y^j} \right\} \geq \max\left(0,\ D_f P(x, y) - \frac{i}{r} - \frac{j}{s}\right).$$

[13] The $n+1$ expressions

(33) $$D_f R(x, y) = \Theta_f \qquad (f = 0, 1, ..., n)$$

can be estimated in the following way.

First, by (25), we have

$$D_f U(x) = \frac{u_f}{r}, \qquad D_f V(y) = \frac{v_f}{s} \qquad (f = 0, 1, ..., n)$$

whence, by (28) and (30), we obtain the values,

(34) $$D_f W(x, y) = \frac{u_f}{r} + \frac{v_f}{s} \qquad (f = 0, 1, ..., n).$$

Secondly, we find lower bounds for the expressions (33), as follows. From its definition as a determinant, $W(x, y)$ may be written as a sum of $l!$ terms

$$W(x, y) = \sum_{(i)} \pm \frac{\partial^{i_0+0} R(x, y)}{\partial x^{i_0} \partial y^0} \frac{\partial^{i_1+1} R(x, y)}{\partial x^{i_1} \partial y^1} \cdots \frac{\partial^{i_{l-1}+(l-1)} R(x, y)}{\partial x^{i_{l-1}} \partial y^{l-1}},$$

where the summation extends over all permutations $i_0 .\ i_1, ..., i_{l-1}$ of $0, 1. ..., l-1$. But, by (32) and (33),

$$D_f \left\{ \frac{\partial^{i_\lambda+\lambda} R(x, y)}{\partial x^{i_\lambda} \partial y^\lambda} \right\} \geq \max\left(0,\ \Theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right);$$

the general rules (31) and (32) imply therefore that

(35) $$D_f W(x, y) \geq \min_{(i)} \sum_{\lambda=0}^{l-1} \max\left(0,\ \Theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right),$$

where the minimum extends again over all permutations $i_0, i_1, ..., i_{l-1}$ of $0, 1, ..., l-1$.

Next

$$\max\left(0,\ \Theta_f-\frac{i_\lambda}{r}-\frac{\lambda}{s}\right)\geq\max\left(-\frac{i_\lambda}{r},\ \Theta_f-\frac{i_\lambda}{r}-\frac{\lambda}{s}\right)=\max\left(0,\ \Theta_f-\frac{\lambda}{s}\right)-\frac{i_\lambda}{r}$$

and

$$\sum_{\lambda=0}^{l-1}\frac{i_\lambda}{r}=\frac{l(l-1)}{2r},$$

and so (35) implies the simpler inequality

$$D_f W(x,\ y)\geq\sum_{\lambda=0}^{l-1}\max\left(0,\ \Theta_f-\frac{\lambda}{s}\right)-\frac{l(l-1)}{2r}\qquad (f=0,\ 1,\dots,\ n). \tag{36}$$

Comparing now the results (34) and (36) for $D_f W(x,\ y)$, we obtain

$$\sum_{\lambda=0}^{l-1}\max\left(0,\ \Theta_f-\frac{\lambda}{s}\right)\leq\frac{u_f}{r}+\frac{v_f}{s}+\frac{l(l-1)}{2r}\qquad (f=0,\ 1,\dots,\ n).$$

Finally, on adding these inequalities over $f=0,\ 1,\dots,\ n$, and making use of (21) and (26), we obtain the basic inequality,

$$\sum_{f=0}^{n}\sum_{\lambda=0}^{l-1}\max\left(0,\ \Theta_f-\frac{\lambda}{s}\right)\leq l-\frac{l(l-1)}{2}\left(\frac{1}{r}+\frac{1}{s}\right)+(n+1)\frac{l(l-1)}{2r}. \tag{37}$$

[14] In order to simplify this inequality, put

$$X=\frac{l}{s}, \tag{38}$$

and

$$X_f=\min(\Theta_f,\ X),\qquad \Lambda_f=\min([\Theta_f s])+1,\ l)\qquad (f=0,\ 1,\dots,\ n). \tag{39}$$

Then

$$\Lambda_f-1\leq X_f s\leq\Lambda_f$$

so that

$$\sum_{\lambda=0}^{l-1}\max\left(0,\ \Theta_f-\frac{\lambda}{s}\right)=\sum_{\lambda=0}^{\Lambda_f-1}\left(\Theta_f-\frac{\lambda}{s}\right)=\frac{1}{2}\Lambda_f\left(2\Theta_f-\frac{\Lambda_f-1}{s}\right)\geq\frac{s}{2}X_f(2\Theta_f-X_f);$$

the left-hand side of (37) is therefore not less than

$$\frac{s}{2}\sum_{f=0}^{n}X_f(2\Theta_f-X_f). \tag{40}$$

We also need a simple estimate for the right-hand side of (37). To this purpose, denote by δ a real number satisfying

$$0<\delta\leq 1, \tag{41}$$

and assume that r and s have the lower bounds given by

$$r\geq\frac{5n}{3\delta}s,\qquad s\geq\frac{5}{\delta}\geq 5. \tag{42}$$

Therefore, by (6),

$$1 \leq l \leq s+1, \qquad 0 < X \leq 1 + \frac{1}{5} \leq \frac{6}{5}, \ \frac{1}{2-X} \leq \frac{5}{4},$$

and

$$\frac{1}{2-X}\left(\frac{1}{s} + n\frac{l-1}{r}\right) \leq \frac{5}{4}\left(\frac{1}{s} + \frac{ns}{r}\right) \leq \frac{5}{4}\left(\frac{\delta}{5} + \frac{3\delta}{5}\right) = \delta.$$

Because now the right-hand side of (37) can be written in the form

$$\left(l - \frac{l^2}{2s}\right) + \left(\frac{l}{2s} + n\frac{l(l-1)}{2r}\right) = \frac{s}{2}(2X - X^2)\left\{1 + \frac{1}{2-X}\left(\frac{1}{s} + n\frac{(l-1)}{r}\right)\right\},$$

it is at most equal to

(43) $$\frac{s}{2}(2X - X^2)(1+\delta).$$

[15] On substituting the lower and upper bounds (40) and (43) in (37), we obtain the very much simpler inequality,

(44) $$\sum_{f=0}^{n} X_f(2\Theta_f - X_f) \leq (2X - X^2)(1+\delta).$$

From this, an even simpler inequality may be obtained which contains only the numbers Θ_f, but not the numbers X_f.

For if, first,

$$X \geq \Theta_f, \quad \text{then} \quad X_f = \Theta_f,$$

and so

(45) $$X_f(2\Theta_f - X_f) = \Theta_f^2 \geq \Theta_f^2(2X - X^2),$$

because

$$0 < 2X - X^2 = 1 - (1-X)^2 \leq 1$$

from $0 < X \leq 6/5$.

Let, secondly,

$$X < \Theta_f, \quad \text{thus} \quad X_f = X.$$

Then, from the special form (1) of $R(x, y)$, necessarily

$$\Theta_f \leq 1 \qquad (f = 0, 1, \ldots, n),$$

since otherwise $R(x, y)$ would vanish identically. Therefore

$$X_f(2\Theta_f - X_f) = X(2\Theta_f - X) = \Theta_f^2(2X - X^2) + X\{2\Theta_f(1-\Theta_f) + (1-\Theta_f^2)X\},$$

whence

(46) $$X_f(2\Theta_f - X_f) \geq \Theta_f^2(2X - X^2).$$

The inequalities (44), (45), (46) together imply that

$$\sum_{f=0}^{n} \Theta_f^2(2X - X^2) \leq \sum_{f=0}^{n} X_f(2\Theta_f - X_f) \leq (2X - X^2)(1+\delta),$$

that is,

(47) $$\sum_{f=0}^{n} \Theta_f^2 \leq 1 + \delta.$$

Our investigation has thus let to the following result:

Theorem 1. - *Let* K *be an arbitrary field of characteristic zero; let* δ *be a real number satisfying*

$$0 < \delta \leq 1;$$

and let r *and* s *be two positive integers for which*

$$r \geq \frac{5n}{3\delta} s, \qquad s \geq \frac{5}{\delta},$$

n *being an arbitrary positive integer. Denote further by* R(x, y) *a polynomial of the form*

$$R(x, y) = \sum_{\substack{h \geq 0 \quad k \geq 0 \\ \frac{h}{r} + \frac{k}{s} \leq 1}} \sum R_{hk} x^h y^k \not\equiv 0$$

with coefficients in K; *let*

$$\xi_0, \xi_1, \ldots, \xi_n \quad and \quad \eta_0, \eta_2, \ldots, \eta_n$$

be two systems of n + 1 *elements of* K *each such that each system has only different elements; and let* $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ *be* n + 1 *non-negative real numbers.*

Then if, for f = 0, 1, ..., n, *the relations*

$$\left. \frac{\partial^{i+j} R(x, y)}{\partial x^i \partial y^j} \right|_{\substack{x=\xi_f \\ y=\eta_f}} = 0$$

hold for all indices i, j *with*

$$i \geq 0, \quad j \geq 0, \quad \frac{i}{r} + \frac{j}{s} < \vartheta_f,$$

necessarily

$$\sum_{f=0}^{n} \vartheta_f^2 \leq 1 + \delta.$$

For, by the definition of $D_f R(x, y) = \Theta_f$ in [13], it is clear that

$$\Theta_f \geq \vartheta_f \qquad (f = 0, 1, \ldots, n),$$

and so the assertion is contained in (47).

2. Construction of the Aproximation Polynomial.

[16] In this and the next chapter, we consider polynomials $F(x, y, z, \ldots)$ in one or more variables, and in most cases with integral coefficients. Such a polynomial is said to be of degree $n \geq 0$ in x if it can be written in the form

$$F(x, y, z, \ldots) = a_0(y, z, \ldots) x^n + a_1(y, z, \ldots) x^{n-1} + \ldots + a_n(y, z, \ldots),$$

where $a_0(y, z, \ldots)$, $a_1(y, z, \ldots), \ldots, a_n(y, z, \ldots)$ are polynomials in $y, z, \ldots$ alone. The polynomial is said to be of *exact* degree $n \geq 0$ if the highest coefficient $a_0(y, z, \ldots)$ does not vanish identically in $y, z, \ldots$. Every polynomial of negative degree is identically zero.

By $\overline{|F(x, y, z, \ldots)|}$ we denote the maximum of the absolute values of all the numerical coefficients of $F(x, y, z, \ldots)$.

[17] LEMMA 1. - *Let*

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n \quad \textit{and} \quad g(x) = b_0x^m + b_1x^{m-1} + \dots + b_m$$

be two polynomials with integral coefficients. Then there exist two further polynomials

$$q(x) = c_0x^{m-n} + c_1x^{m-n-1} + \dots + c_{m-n} \quad \textit{and} \quad r(x) = d_0x^{n-1} + d_1x^{n-2} + \dots + d_{n-1}$$

with integral coefficients such that

$$a_0^{\max(0,\, m-n+1)}g(x) = f(x)q(x) + r(x), \quad \overline{|r(x)|} \leq \{2\,\overline{|f(x)|}\}^{\max(0,\, m-n+1)}\,\overline{|g(x)|}.$$

Proof: If $m \leq n-1$, then the assertion is satisfied with

$$\max(0 \ \ m-n+1) = 0, \quad q(x) = 0, \quad r(x) = g(x).$$

Let therefore from now on

$$m \geq n, \qquad \text{so that} \qquad s = \max(0, m-n+1) \geq 1, \quad s-1 = \max(0, m-n) \geq 0;$$

we assume that the assertion has already been proved for all polynomials $g(x)$ of degree less than m.

Write $a_k = 0$ if $k > n$, and put

$$g^*(x) = a_0g(x) - b_0x^{m-n}f(x) =$$
$$= (a_0b_1 - a_1b_0)x^{m-1} + (a_0b_2 - a_2b_0)x^{m-2} + \dots + (a_0b_m - a_mb_0).$$

Then $g^*(x)$ is of degree $m-1$ and has integral coefficients satisfying

$$\overline{|g^*(x)|} \leq 2\,\overline{|f(x)|}\,\overline{|g(x)|}.$$

By the induction hypothesis, there exist two polynomials $q^*(x)$ of degree $m-n-1$ and $r(x)$ of degree $n-1$, both with integral coefficients and such that

$$a_0^{s-1}g^*(x) = f(x)q^*(x) + r(x), \quad \overline{|r(x)|} \leq \{2\,\overline{|f(x)|}\}^{s-1}\overline{|g^*(x)|} \leq \{2\,\overline{|f(x)|}\}^{s}\,\overline{|g(x)|}.$$

The first formula implies that

$$a_0^s g(x) = f(x)q(x) + r(x) \quad \text{where} \quad q(x) = b_0x^{m-n} + q^*(x).$$

Since $q(x)$ is of degree $m-n$ and has integral coefficients, this proves the assertion.

[18] LEMMA 2. - *Let* r *and* s *be two positive integers, and let* Θ *be a positive number. Denote by* $N(\Theta)$ *the number of solutions in integers* h, k *of the inequalities*

$$(1) \qquad h \geq 0, \qquad k \geq 0, \qquad \frac{h}{r} + \frac{k}{s} < \Theta.$$

Then

$$\frac{1}{2}\Theta^2 rs \leq N(\Theta) \leq \frac{1}{2}\left(\Theta + \frac{1}{r} + \frac{1}{s}\right)^2 rs.$$

Proof: For every pair of integers h, k satisfying (1), let Q_{hk} be the square of all real points (x, y) for which

$$h \leq x < h+1, \qquad k \leq y < k+1,$$

and denote by

$$Q(\Theta) = \bigcup Q_{hk}$$

the join of all these squares. Every point (x, y) of $Q(\Theta)$ belongs to a pair of integers h, k satisfying (1), and so

$$\frac{x}{r} + \frac{y}{s} < \frac{h+1}{r} + \frac{k+1}{s} < \Theta + \frac{1}{r} + \frac{1}{s};$$

hence $Q(\Theta)$ is contained in the triangle

$$x \geq 0, \quad y \geq 0, \quad \frac{x}{r} + \frac{y}{s} < \Theta + \frac{1}{r} + \frac{1}{s}$$

of area

$$\frac{1}{2}\left(\Theta + \frac{1}{r} + \frac{1}{s}\right)^2 rs.$$

On the other hand, the triangle

$$x \geq 0, \quad y \geq 0, \quad \frac{x}{r} + \frac{y}{s} < \Theta$$

of area

$$\frac{1}{2}\Theta^2 rs$$

is clearly contained in $Q(\Theta)$. Since $Q(\Theta)$ is of area

$$N(\Theta) \cdot 1,$$

this proves the assertion.

[19] In what follows,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n, \qquad \text{where } a_0 \neq 0,$$

is a fixed polynomial of exact degree $n \geq 2$ in x with integral coefficients; we assume that the equation

$$f(x) = 0$$

has no multiple roots, but allow $f(x)$ to be reducible in the rational field.

We denote by $\varepsilon > 0$ a fixed constant, and by r and s two positive integers on which further on certain inequality conditions will be imposed. We further denote by A a positive integer to be chosen later, and consider the set, $S(A)$ say, of all polynomials

$$P(x, y) = \sum_{\substack{h \geq 0 \\ \frac{h}{r} + \frac{k}{s} < 1}} \sum_{k \geq 0} P_{hk} x^h y^k$$

with integral coefficients P_{hk} satisfying

$$\overline{|P(x, y)|} = \max_{h,k} |P_{hk}| \leq A.$$

Each coefficient P_{hk} of $P(x, y)$ has $2A+1$ possible values; moreover, by Lemma 2, $P(x, y)$ has at least $\frac{1}{2} rs$ coefficients. The set $S(A)$ contains therefore at least

$$N_1 = (2A+1)^{\frac{1}{2} rs}$$

polynomials.

[20] For any two non-negative integers i and j put

$$P^{(i,j)}(x, y) = \frac{\partial^{i+j} P(x, y)}{i!\, j!\, \partial x^i \partial y^j},$$

so that

$$P^{(i,j)}(x, y) = \sum_{\substack{h\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} \sum_{k\geq 0} \binom{h}{i}\binom{k}{j} P_{hk} x^{h-i} y^{k-j},$$

and in particular

$$P^{(i,j)}(x, x) = \sum_{\substack{h\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} \sum_{k\geq 0} \binom{h}{i}\binom{k}{j} P_{hk} x^{h+k-i-j}.$$

We see therefore that

$$P^{(i,j)}(x, x) \quad \textit{is of degree} \quad r+s \quad \textit{in} \quad x.$$

Upper bounds for $\overline{|P^{(i,j)}(x, y)|}$ and $\overline{|P^{(i,j)}(x, x)|}$ are obtained in the following way:

Since

$$\binom{h}{i} \leq \sum_{i=0}^{h} \binom{h}{i} = 2^h \leq 2^r, \qquad \binom{k}{j} \leq \sum_{j=0}^{k} \binom{k}{j} = 2^k \leq 2^s,$$

it is at once clear that

$$\overline{|P^{(i,j)}(x, y)|} \leq 2^{r+s} A.$$

We further find that all coefficients of $P^{(i,j)}(x, x)$ are of absolute value not greater than

$$2^{r+s} A \sum_{\substack{h\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} \sum_{k\geq 0} 1 \leq 2^{r+s} A \cdot \frac{1}{2}\left(1 + \frac{1}{r} + \frac{1}{s}\right)^2 rs,$$

as follows from Lemma 2. Next for all positive integers r and s,

$$r \leq 2^{r-1}, \qquad s \leq 2^{s-1},$$

and if we assume from now on that

$$\boxed{r \geq 2 \quad \text{and} \quad s \geq 2},$$

we have

$$1 + \frac{1}{r} + \frac{1}{s} \leq 2.$$

Hence

$$\overline{|P^{(i,j)}(x, x)|} \leq 2^{r+s} A \cdot \frac{1}{2} 2^2 2^{r-1} 2^{s-1}.$$

We find therefore the inequalities

$$\overline{|P^{(i,j)}(x, x)|} \leq \frac{1}{2} 4^{r+s} A \qquad (i \geq 0,\ j \geq 0)$$

for all polynomials $P(x, y)$ in $S(A)$.

[21] Divide now each polynomial $P^{(i,j)}(x, x)$ by $f(x)$. By Lemma 1, we obtain the formula

$$a_0^{\max(0,\, r+s-n+1)} P^{(i,j)}(x, x) = Q^{(i,j)}(x) f(x) + R^{(i,j)}(x),$$

where both polynomials $Q^{(i,j)}(x)$ and $R^{(i,j)}(x)$ have integral coefficients, $Q^{(i,j)}(x)$ is of degree $r + s - n$, and $R^{(i,j)}(x)$ is of degree $n - 1$, while

$$\overline{|R^{(i,j)}(x)|} \leq \left\{ 2 \overline{|f(x)|} \right\}^{\max(0,\, r+s-n+1)} \overline{|P^{(i,j)}(x, x)|}.$$

Assume from now on that

$$\boxed{r + s \geq n - 1 \geq 1},$$

and put

$$8 \overline{|f(x)|} = \alpha, \qquad \text{so that } \alpha \geq 8 > 2.$$

We find then that

$$\overline{|R^{(i,j)}(x)|} \leq \frac{1}{2} 4^{r+s} A \left\{ 2 \overline{|f(x)|} \right\}^{r+s} = \frac{1}{2} \alpha^{r+s} A$$

for every element $P(x, y)$ of $S(A)$ and for all integers $i \geq 0$, $j \geq 0$.

[22] From now on put

$$\Theta = \sqrt{\frac{1 - 2\varepsilon}{n}} \qquad \text{where } 0 < \varepsilon < \frac{1}{2}.$$

Then consider, for every element $P(x, y)$ of $S(A)$, the set of all remainder polynomials

$$R^{(i,j)}(x) \quad \text{where} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta.$$

By Lemma 2, there are at most

$$\frac{1}{2}\left(\Theta+\frac{1}{r}+\frac{1}{s}\right)^2 rs$$

such polynomials. Here

$$\left(\Theta+\frac{1}{r}+\frac{1}{s}\right)^2=\frac{1-2\varepsilon}{n}+\left(\frac{1}{r}+\frac{1}{s}\right)\left(2\Theta+\frac{1}{r}+\frac{1}{s}\right)$$

is not greater than

$$\frac{1-\varepsilon}{n},$$

provided

$$\left(\frac{1}{r}+\frac{1}{s}\right)\left(2\Theta+\frac{1}{r}+\frac{1}{s}\right)\leq\frac{\varepsilon}{n}.$$

But, by hypothesis, $r\geq 2$ and $s\geq 2$, and further $\Theta\leq 1$ from the definition of Θ; hence

$$2\Theta+\frac{1}{r}+\frac{1}{s}\leq 2+\frac{1}{2}+\frac{1}{2}=3.$$

The last inequality is therefore certainly satisfied if we make from now on the additional assumption that

$$\boxed{\frac{1}{r}+\frac{1}{s}\leq\frac{\varepsilon}{3n}}\,.$$

Under this condition, we are then considering at most

$$\frac{(1-\varepsilon)rs}{2n}$$

such remainder polynomials $R^{(i,j)}(x)$. Each such polynomial has n coefficients as it is of degree $n-1$, and each coefficient has at most

$$2\cdot\frac{1}{2}\alpha^{r+s}A+1\leq 2\alpha^{r+s}A$$

possibilities. The total system of remainder polynomials

$$R^{i,j}(x)\quad\text{where}\quad i\geq 0,\quad j\geq 0,\quad \frac{i}{r}+\frac{j}{s}<\Theta$$

has therefore at most

$$(2\alpha^{r+s}A)^{n\cdot\frac{1-\varepsilon}{2n}rs}<\alpha^{(r+s)\frac{1-\varepsilon}{2}rs}(2A+1)^{\frac{1-\varepsilon}{2}rs},\quad =N_2\ \text{say},$$

possibilities.

[23] Determine now the integer A by the condition that

$$2A+3>\alpha^{(r+s)\frac{1-\varepsilon}{\varepsilon}}\geq 2A+1;$$

there is just one integer A of this kind. Then

$$\frac{N_2}{N_1} > \frac{\alpha^{(r+s)\frac{1-\varepsilon}{2}rs}(2A+1)^{\frac{1-\varepsilon}{2}rs}}{(2A+1)^{\frac{1}{2}rs}} = \left\{\frac{\alpha^{(r+s)\frac{1-\varepsilon}{2}}}{(2A+1)^{\frac{\varepsilon}{2}}}\right\}^{rs} \geq 1,$$

that is

$$N_2 > N_1.$$

Hence amongst the at least N_1 polynomials in $S(A)$ there are two different ones,

$$P_{\mathrm{I}}(x, y) \quad \text{and} \quad P_{\mathrm{II}}(x, y) \qquad \text{say,}$$

for which the corresponding sets of polynomials $R_{\mathrm{I}}^{(i,j)}(x)$ and $R_{\mathrm{II}}^{(i,j)}(x)$ satisfy the identities

$$R_{\mathrm{I}}^{(i,j)}(x) \equiv R_{\mathrm{II}}^{(i,j)}(x) \quad \text{if} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta.$$

Put

$$S(x, y) = P_{\mathrm{I}}(x, y) - P_{\mathrm{II}}(x, y).$$

Then $S(x, y) \not\equiv 0$, and this polynomial is of the form

$$S(x, y) = \sum_{\substack{h\geq 0\ k\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} S_{hk}x^h y^k$$

with integral coefficients satisfying

$$\overline{|S(x, y)|} = \max(|S_{hk}|) \leq 2A < \alpha^{(r+s)\frac{1-\varepsilon}{\varepsilon}}.$$

By applying the proof in [20] to $S(x, y)$ instead of $P(x, y)$, we get

$$\overline{\left|\frac{\partial^{i+j}S(x, y)}{i!\,j!\,\partial x^i \partial y^j}\right|} < 2^{r+s}\cdot\alpha^{(r+s)\frac{1-\varepsilon}{\varepsilon}} < \alpha^{r+s}\alpha^{(r+s)\frac{1-\varepsilon}{\varepsilon}} = \alpha^{\frac{r+s}{\varepsilon}} \quad \text{for } i, j = 0, 1, 2, \ldots.$$

It is further clear from the definition of $S(x, y)$ that the derivatives

$$S_{ij}(x) = \left.\frac{\partial^{i+j}S(x, y)}{i!\,j!\,\partial x^i \partial y^j}\right|_{x=y}, \quad \text{where} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta,$$

are divisible by $f(x)$.

[24] By hypothesis, the n roots of the equation $f(x) = 0$,

$$\zeta_1, \zeta_2, \ldots, \zeta_n \quad \text{say,}$$

are all different; by [23], they satisfy the equations

$$S_{ij}(\zeta_f) = 0 \quad \text{if} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta,\ f = 1, 2, \ldots, n.$$

Let ξ, η be two numbers different from $\zeta_1, \zeta_2, \dots, \zeta_n$. Let further Θ_0 be a positive number and δ a number satisfying

$$0 < \delta \leq 1,$$

and assume that

$$r \geq \frac{5n}{3\delta} s, \quad s \geq \frac{5}{\delta}.$$

Then, by Theorem 1, the additional equations

$$\left.\frac{\partial^{i+j} S(x, y)}{\partial x^i \partial y^j}\right|_{x=\xi,\ y=\eta} = 0 \quad \text{for} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta_0$$

cannot hold unless

$$n\Theta^2 + \Theta_0^2 \leq 1 + \delta,$$

that is,

$$\Theta_0^2 \leq 1 + \delta - n\Theta^2 = 1 + \delta - n\,\frac{1 - 2\varepsilon}{n} = 2\varepsilon + \delta.$$

Take for δ the value

$$\delta = \frac{\varepsilon}{n}.$$

Since $n \geq 2$ and $0 < \varepsilon < \frac{1}{2}$, this is permitted and implies that

$$0 < \delta \leq \frac{\varepsilon}{2}, \quad 2\varepsilon + \delta < 3\varepsilon.$$

The inequality assumptions for r and s take then the from

$$r \geq \frac{5n^2}{3\varepsilon} s, \quad s \geq \frac{5n}{\varepsilon}$$

and imply that

$$s \geq \frac{5\cdot 2}{3\cdot\frac{1}{2}} > 6, \quad r \geq \frac{5\cdot 4}{3\cdot\frac{1}{2}}\cdot 6 = 80$$

and

$$r + s > r \geq \frac{5\cdot 2\cdot 6}{3\cdot\frac{1}{2}} n > n - 1, \quad \frac{1}{r} + \frac{1}{s} = \frac{\varepsilon}{n}\left(\frac{3}{5ns} + \frac{1}{5}\right) < \frac{\varepsilon}{n}\left(\frac{3}{5\cdot 2\cdot 6} + \frac{1}{5}\right) < \frac{\varepsilon}{3n},$$

so that the conditions for r and s in [20], [21], and [22], are satisfied.

We therefore have obtained the following result:

Theorem 2. – *Let*

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n, \qquad \textit{where } a_0 \neq 0,$$

be a polynomial of exact degree $n \geq 2$, *with integral coefficients and such that the equation* $f(x) = 0$ *has no multiple root; put*

$$\alpha = 8\,|\overline{f(x)}|.$$

Let ε *be a real number in the interval*

$$0 < \varepsilon < \frac{1}{2},$$

let r *and* s *be two positive integers satisfying*

$$r \geq \frac{5n^2}{3\varepsilon} s, \qquad s \geq \frac{5n}{\varepsilon},$$

and let

$$\Theta = \sqrt{\frac{1-2\varepsilon}{n}}.$$

Then there exists a polynomial

$$S(x, y) = \sum_{\substack{h \geq 0 \\ \frac{h}{r} + \frac{k}{s} < 1}} \sum_{k \geq 0} S_{hk} x^h y^k \not\equiv 0$$

with integral coefficients, with the following properties:

a) $$\left|\frac{\partial^{i+j} S(x, y)}{i!\, j!\, \partial x^i \partial y^j}\right| < \alpha^{\frac{r+s}{\varepsilon}} \quad \textit{for} \quad i, j = 0, 1, 2, \dots;$$

b) $$\left.\frac{\partial^{i+j} S(x, y)}{i!\, j!\, \partial x^i \partial y^j}\right|_{x=y=\zeta} = 0 \quad \textit{for} \quad f(\zeta) = 0,\ i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{i}{s} < \Theta;$$

c) *If* ξ *and* η *are two numbers such that* $f(\xi) \neq 0$, $f(\eta) \neq 0$, *and if*

$$\left.\frac{\partial^{i+j} S(x, y)}{i!\, j!\, \partial x^i \partial y^j}\right|_{x=\xi,\, y=\eta} = 0 \quad \textit{for} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{i}{s} < \Theta_0,$$

then

$$\Theta_0 < \sqrt{3\varepsilon}.$$

3. Conclusion of the Proof.

[25] From now on ζ will denote one fixed real root of $f(x)$, the polynomial defined in Theorem 2. We assume that there exists a real number

$$\mu > \sqrt{n}$$

and an infinite sequence Σ of rational numbers

$$\frac{p}{q} = \frac{p_1}{q_1}, \quad \frac{p_2}{q_2}, \quad \frac{p_3}{q_3}, \dots$$

with the following properties:

a) *The numerators* p_r *and the denominators* q_r *are integers, and the denominators* q_r *are not less than* 2 *and tend to infinity with* r.

b) *Each denominator* q_r *is divisible at most by a given finites et of prime numbers* $P_1, P_2, \dots, P_t$.

c) $$\left|\frac{p_r}{q_r} - \zeta\right| < q_r^{-\mu} \qquad (r = 1, 2, 3, \dots).$$

[26] The last hypothesis can be replaced by a simpler one. Denote by σ a small positive number to be chosen later, and select an integer φ satisfying the inequality

(1) $$\varphi\sigma \geq t.$$

If $\frac{p}{q}$ is any element of Σ, then the denominator q may be written as

(2) $$q = P_1^{g_1} P_2^{g_2} \dots P_t^{g_t}$$

where $g_1, g_2, \dots, g_t$ are non-negative integers. There are then t uniquely determined non-negative integers $a_1, a_2, \dots, a_t$ satisfying

(3) $$q^{\frac{a_\tau - 1}{\varphi}} < P_\tau^{g_\tau} \leq q^{\frac{a_\tau}{\varphi}} \qquad (\tau = 1, 2, \dots, t),$$

so that

$$q^{\sum_{\tau=1}^{t} \frac{a_\tau - 1}{\varphi}} < \prod_{\tau=1}^{t} P_\tau^{g_\tau} = q \leq q^{\sum_{\tau=1}^{t} \frac{a_\tau}{\varphi}}$$

Therefore

$$\sum_{\tau=1}^{t} (a_\tau - 1) < \varphi \leq \sum_{\tau=1}^{t} a_\tau,$$

whence by (1),

$$\sum_{\tau=1}^{t} a_\tau < \varphi + t \leq (1 + \sigma)\varphi.$$

Since t, σ, φ are fixed, and since the a's are non-negative integers, this inequality implies that the system of integers

$$(a_1, a_2, \dots, a_t)$$

has only a finite number of possibilities.

Now every infinite subsequence of Σ has the three properties a), b), c), just as Σ itself has. Hence, without loss of generality, we may assume, from now on, that the system ot integers $(a_1, a_2, \dots, a_t)$ isf ixed for all elements $\frac{p}{q}$ of Σ.

[27] We consider now the polynomial

$$S(x, y) = \sum_{\substack{h\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} \sum_{k\geq 0} S_{hk} x^h y^k$$

given by Theorem 2, and study its derivatives

$$S_{ij}(x, y) = \frac{\partial^{i+j} S(x, y)}{i!\, j!\, \partial x^i \partial y^j} \qquad (i \geq 0,\ j \geq 0)$$

for

$$x = \frac{p}{q}, \qquad y = \frac{p'}{q'}$$

where $\frac{p}{q}$ and $\frac{p'}{q'}$ are two elements of Σ, which will be selected later. We can write

$$S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) = \frac{U_{ij}}{V_{ij}}$$

where U_{ij} and V_{ij} are integers, and where $V_{ij} > 0$.

Denote now by V the least common multiple of the products

$$q^h q'^k, \quad \text{where} \quad h \geq 0,\ k \geq 0,\ \frac{h}{r} + \frac{k}{s} < 1.$$

Since

$$S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) = \sum_{\substack{h\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}} \sum_{k\geq 0} S_{hk} \binom{h}{i}\binom{k}{j}\left(\frac{p}{q}\right)^{h-i}\left(\frac{p'}{q'}\right)^{k-j},$$

all denominators V_{ij} may be put equal to V. An upper bound for V is now obtained as follows.

By [26], q and q' may be written as

$$q = P_1^{g_1} P_2^{g_2} \ldots P_t^{g_t}, \qquad q' = P_1^{g'_1} P_2^{g'_2} \ldots P_t^{g'_t}$$

where the $g's$ are non-negative integers satisfying

$$q^{\frac{a_\tau - 1}{\varphi}} < P_\tau^{g_\tau} \leq q^{\frac{a_\tau}{\varphi}}, \qquad q'^{\frac{a_\tau - 1}{\varphi}} < P^{\frac{g'_\tau}{\tau}} \leq q'^{\frac{a_\tau}{\varphi}} \qquad (\tau = 1, 2, \ldots, t).$$

Then

$$q^h q'^k = P_1^{hg_1 + kg'_1} P_2^{hg_2 + kg'_2} \ldots P_t^{hg_t + k\,'_t},$$

and here

$$P_\tau^{hg_\tau + kg'_\tau} \leq q^{\frac{ha_\tau}{\varphi}} q'^{\frac{ka_\tau}{\varphi}} = (q^h q'^k)^{\frac{a_\tau}{\varphi}} \qquad (\tau = 1, 2, \ldots, t).$$

Let us now assume that q, q', r, and s, are connected by

(4) $$r = \left[s \frac{\log q'}{\log q}\right].$$

Since h and k assume only values for which

$$\frac{h}{r}+\frac{k}{s}<1,$$

we have then

$$h<r\left(1-\frac{k}{s}\right)\leq s\frac{\log q'}{\log q}\left(1-\frac{k}{s}\right),$$

whence

$$q^h q'^k \leq e^{(s-k)\log q' + k\log q'} = q'^s.$$

The least common multiple V has, however, at most the prime factors P_1, $P_2, \ldots, P_t$; it therefore satisfies the inequality

$$V\leq \prod_{\tau=1}^{t} (q^h q'^k)^{\frac{a_\tau}{\varphi}} \leq q'^{\,s\sum_{\tau=1}^{t}\frac{a_\tau}{\varphi}} < q'^{(1+\sigma)s}, \qquad \text{since } \sum_{\tau=1}^{t} a_\tau < (1+\sigma)\varphi.$$

We have so found an upper bound for V, hence also for the denominator V_{ij} of the rational number $S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right)$. This bound immediately implies that either

(5) $$S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right)=0, \quad \text{or} \quad \left|S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right)\right| \geq \frac{1}{V} > q'^{-(1+\sigma)s}.$$

[28] We must obtain also an upper bound for $\left|S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right)\right|$. This is done as follows.

By hypothesis, ζ satisfies the equation $f(x)=0$; we have put

$$\alpha = 8\,\overline{|f(x)|}.$$

Since $f(x)$ has integral coefficients, it is easily seen that

$$\alpha\geq 8, \qquad |\zeta|\leq\frac{\alpha}{4}.$$

We apply now the upper bound

$$\overline{|S_{ij}(x, y)|} < \alpha^{\frac{r+s}{\varepsilon}}$$

given in Theorem 2; then

$$|S_{ij}(\zeta, \zeta)| < \alpha^{\frac{r+s}{\varepsilon}} \sum_{\substack{h\geq 0 \quad k\geq 0 \\ \frac{h}{r}+\frac{k}{s}<1}}\sum |\zeta|^{h+k}.$$

Further

$$\sum_{\substack{h\geq 0\ k\geq 0\\ \frac{h}{r}+\frac{k}{s}<1}} |\zeta|^{h+k} \leq \sum_{h=0}^{r}\sum_{k=0}^{s}\left(\frac{\alpha}{4}\right)^{h+k} = \frac{\left(\frac{\alpha}{4}\right)^{r+1}-1}{\frac{\alpha}{4}-1}\ \frac{\left(\frac{\alpha}{4}\right)^{s+1}-1}{\frac{\alpha}{4}-1}$$

$$\leq \frac{\left(\frac{\alpha}{4}\right)^{r+1}-1}{\frac{\alpha}{4}-\frac{\alpha}{8}}\ \frac{\left(\frac{\alpha}{4}\right)^{s+1}-1}{\frac{\alpha}{4}-\frac{\alpha}{8}} = 4\left(\frac{\alpha}{4}\right)^{r+s},$$

so that

$$|S_{ij}(\zeta,\ \zeta)| < 4^{1-r-s}\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)}.$$

Next, by the theorem,

$$S_{ij}(\zeta,\ \zeta) = 0 \quad \text{if} \quad i\geq 0,\ k\geq 0,\ \frac{h}{r}+\frac{k}{s}<\Theta.$$

Therefore, by TAYLOR's formula,

$$S(x,\ y) = \sum_{\substack{i\geq 0\ j\geq 0\\ \Theta\leq\frac{i}{r}+\frac{j}{s}<1}} S_{ij}(\zeta,\ \zeta)(x-\zeta)^i(y-\zeta)^j,$$

since

$$S_{ij}(x,\ y)\equiv 0 \quad \text{if} \quad \frac{i}{r}+\frac{j}{s}\geq 1.$$

On replacing the summation indices i, j by h, k, and differentiating repeatedly, this gives

$$S_{ij}(x,\ y) = \sum_{\substack{h\geq i\ k\geq j\\ \Theta\leq\frac{h}{r}+\frac{k}{s}<1}} S_{hk}(\zeta,\ \zeta)\binom{h}{i}\binom{k}{j}(x-\zeta)^{h-i}(y-\zeta)^{k-j}.$$

Let us now assume that

$$i\geq 0,\ j\geq 0,\ \frac{i}{r}+\frac{j}{s}<\Theta_0<\Theta,$$

and use Lemma 2 and the inequalities in [20]. By these,

$$\binom{h}{i}\binom{k}{j}\leq 2^{r+s},$$

and the sum

$$\sum_{\substack{h\geq i\ k\geq i\\ \Theta\leq\frac{h}{r}+\frac{k}{s}<1}}$$

has not more than

$$\frac{1}{2}\left(1+\frac{1}{r}+\frac{1}{s}\right)^2 rs\leq 2rs\leq 2^{r+s-1}$$

terms. We obtain therefore the inequality

$$|S_{ij}(x,\ y)| \leq 4^{i-r-s}\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)} \times 2^{r+s-i} \times 2^{r+s}\lambda$$

where

$$\lambda = \max_{\substack{h \geq i,\ k \geq j \\ \Theta \leq \frac{h}{r}+\frac{k}{s}<1}} (|x-\zeta|^{h-i}\,|y-\zeta|^{k-j}).$$

Replace now $h-i$ by ρ and $k-j$ by σ; then

$$\lambda \leq \max_{\substack{\rho \geq 0,\ \sigma \geq 0 \\ \Theta-\Theta_0<\frac{\rho}{r}+\frac{\sigma}{s}<1}} (|x-\zeta|^{\rho}\,|y-\zeta|^{\sigma}).$$

[29] In the last formulae we now put

$$x = \frac{p}{q}, \quad y = \frac{p'}{q'}, \qquad\qquad \frac{p}{q}, \frac{p'}{q'} \in \Sigma,$$

where r, s, q, and q', satisfy the relation (4). Since

$$\left|\frac{p}{q}-\zeta\right| < q^{-\mu}, \quad \left|\frac{p'}{q'}-\zeta\right| < q'^{-\mu},$$

we obtain

$$\lambda \leq \max_{\substack{\rho \geq 0,\ \sigma \geq 0 \\ \Theta-\Theta_0<\frac{\rho}{r}+\frac{\sigma}{s}<1}} \{(q^{\rho}q'^{\sigma})^{-\mu}\}.$$

The conditions

$$\rho \geq 0,\ \sigma \geq 0,\ \Theta-\Theta_0 < \frac{\rho}{r}+\frac{\sigma}{s} < 1$$

imply that either

$$\rho > r\left(\Theta-\Theta_0-\frac{\sigma}{s}\right), \quad 0 \leq \sigma \leq (\Theta-\Theta_0)s,$$

or that

$$\rho \geq 0, \quad (\Theta-\Theta_0)s < \sigma < s.$$

In the first case,

$$q^{\rho}q'^{\sigma} \geq q^{r(\Theta-\Theta_0)} \cdot \left(q^{-\frac{r}{s}}q'\right)^{\sigma},$$

and in the second case,

$$q^{\rho}q'^{\sigma} \geq q'^{s(\Theta-\Theta_0)}.$$

Now

$$r = \left[s\frac{\log q'}{\log q}\right] \leq s\frac{\log q'}{\log q},$$

hence

$$q^r \leq q'^s, \quad q^{-\frac{r}{s}} q' \geq 1,$$

and we find therefore that in both cases

$$q^\rho q'^\sigma \geq q^{r(\Theta-\Theta_0)},$$

whence

$$\lambda \leq q^{-r(\Theta-\Theta_0)\mu}.$$

We substitute this value in the inequality

$$\left| S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) \right| \leq 2\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)} \lambda$$

and so obtain the inequality

$$(6) \qquad \left| S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) \right| \leq 2\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)} q^{-r(\Theta-\Theta_0)\mu} \quad \text{for} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta_0.$$

[30] We now combine the last results with the assumptions made in Theorem 2.

By the theorem, if

$$(7) \qquad 0 < \varepsilon < \frac{1}{2}, \quad \Theta = \sqrt{\frac{1-2\varepsilon}{n}}, \quad \Theta_0 = 2\sqrt{\varepsilon}$$

then at least one of the numbers

$$S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right), \quad \text{where} \quad i \geq 0,\ j \geq 0,\ \frac{i}{r} + \frac{j}{s} < \Theta_0$$

is different from zero, provided

$$(8) \qquad r \geq \frac{5n^2}{3\varepsilon} s, \qquad s \geq \frac{5n}{\varepsilon}.$$

Let us then assume that (7) and (8) hold; we shall immediately satisfy (8) by choosing s, q, and q' suitably.

Select i and j such that

$$i \geq 0, \qquad j \geq 0, \qquad \frac{i}{r} + \frac{j}{s} < \Theta_0$$

and that $S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) \neq 0$, hence that by (5) and (6),

$$q'^{-(1+\sigma)s} < \left| S_{ij}\left(\frac{p}{q}, \frac{p'}{q'}\right) \right| \leq 2\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)} q^{-r(\Theta-\Theta_0)\mu}.$$

Since $r > s$ and $r + s > 1$, evidently

$$2\alpha^{\left(1+\frac{1}{\varepsilon}\right)(r+s)} < 2^{\frac{4}{\varepsilon}r}\alpha^{\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\right)(r+r)} = (2\alpha)^{\frac{4}{\varepsilon}r},$$

and so the last inequality implies that

(9) $$q'^{-(1+\sigma)s} < (2\alpha)^{\frac{4}{\varepsilon}r} q^{-r(\Theta-\Theta_0)\mu}.$$

Denote, from now on, by s the integer defined by

(10) $$\frac{5n}{\varepsilon} \leq s < \frac{5n}{\varepsilon} + 1;$$

then the second condition (8) is satisfied. Assume further that

(11) $$\frac{\log q}{\log q} \geq \frac{5n^2}{2\varepsilon}.$$

Since

$$\frac{3}{2} - \frac{3\varepsilon^2}{25n^2} \geq \frac{3}{2} - \frac{3\cdot\left(\frac{1}{2}\right)^2}{25\cdot 2^2} > 1,$$

we have then

$$r = \left[s\frac{\log q'}{\log q}\right] \geq s\left(\frac{\log q'}{\log q} - \frac{1}{s}\right) \geq s\left(\frac{5n^2}{2\varepsilon} - \frac{\varepsilon}{5n}\right) = \frac{5n^2}{3\varepsilon}s\left(\frac{3}{2} - \frac{3\varepsilon^2}{25n^2}\right) > \frac{5n^2}{3\varepsilon}s,$$

and so the second condition (8) also holds.

Next

$$\frac{2\varepsilon}{25n^3 - 2\varepsilon^2} \leq \frac{2\cdot\frac{1}{2}}{25\cdot 2^3 - 2\cdot\left(\frac{1}{2}\right)^2} < 1,$$

hence

$$\left(1 - \frac{2\varepsilon^2}{25n^3}\right)^{-1} = 1 + \frac{2\varepsilon^2}{25n^3 - 2\varepsilon^2} < 1 + \varepsilon,$$

whence

$$\frac{s\log q'}{r\log q} \leq \frac{s\log q'}{s\left(\frac{\log q'}{\log q} - \frac{1}{s}\right)\log q} = \left(1 - \frac{1}{s}\frac{\log q}{\log q'}\right)^{-1} \leq \left(1 - \frac{\varepsilon}{5n}\cdot\frac{2\varepsilon}{5n^2}\right)^{-1}$$

$$= \left(1 - \frac{2\varepsilon^2}{25n^3}\right)^{-1} < 1 + \varepsilon.$$

Therefore

$$q'^{-s} > q^{-(1+\varepsilon)r},$$

and so the inequality (9) implies that

$$q^{-(1+\sigma)(1+\varepsilon)r} < (2\alpha)^{\frac{4}{\varepsilon}r} q^{-(\Theta-\Theta_0)\mu r},$$

or more simply

$$(12)\qquad q^c < (2\alpha)^{\frac{4}{\varepsilon}}, \quad \text{where} \quad c = (\Theta - \Theta_0)\mu - (1+\sigma)(1+\varepsilon) = \left(\sqrt{\frac{1-2\varepsilon}{n}} - 2\sqrt{\varepsilon}\right)\mu - (1+\sigma)(1+\varepsilon).$$

We put now

$$\sigma = \varepsilon.$$

Then, as ε tends to zero through positive values, evidently

$$\lim c = \frac{\mu}{n} - 1 > 0.$$

We can thus find a sufficiently small positive number ε for which

$$c > 0.$$

Having made this choice, take q so large that

$$q^c \geq (2\alpha)^{\frac{4}{\varepsilon}},$$

and then select q' so as to satisfy (11). Then (12) gives a contradiction.

[31] The hypothesis in [25] is therefore not allowed, and the following theorem has been proved:

THEOREM 3. - *If* ζ *is a real algebraic number of degree* $n \geq 2$; *if* P, $P_2, \ldots, P_t$ *is a finite set of different primes; and if the inequality*

$$\left|\frac{p}{q} - \zeta\right| < q^{-\mu}$$

has infinitely many solutions in rational numbers $\frac{p}{q}$ *where* p *and* $q \geq 1$ *are relatively prime integers, and where* q *is divisible by no prime different from* $P_1, P_2, \ldots, P_t$*: then* $\mu \leq \sqrt{n}$.

This theorem allows of an interesting application. Let

$$\zeta = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \ldots,$$

where a_0, $a_1 \geq 1$, $a_2 \geq 1, \ldots$ are integers, be the regular continued fraction for ζ, and let $\frac{p_n}{q_n}$, for $n = 0, 1, 2, \ldots$, be the n-th approximation of this continued fraction. It is well known that then

$$\left|\frac{p_n}{q_n} - \zeta\right| < q_n^{-2}.$$

Since $\sqrt{n} < 2$ for $n = 2$ and $n = 3$, we conclude immediately that *the greatest prime factor of* q_n (and of course also that of p_n) *tends to infinity with increasing* n, *if* ζ *is a real quadratic or cubic irrational number.*