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ON THE COMPOSITION OF PSEUDO-VALUATIONS

BY

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To Prof. A. OSTROWSKI
on his 60th birthday

In this paper we consider pseudo-valuations [1]) on commutative rings R with a unit-element, and define certain processes for obtaining new pseudo-valuations from given ones or from real functions on R. For each type of operation there are two definitions according to whether the resulting pseudo-valuation is, or is not, required to be non-archimedean. Given a real non-negative function φ on R we derive in Ch. I a pseudo-valuation which may be described as 'the greatest pseudo-valuation majorised by φ'. We then define the *products* and *compounds* of two pseudo-valuations and obtain some of their properties in Ch. II. These operations are illustrated in Ch. III by examples from algebraic number fields and rings of algebraic integers. In Ch. IV the connexion between the different operations, and their invariance properties, are established. The operations of forming the product and compound correspond to forming the sum and product of two ideals, just as the sum of pseudo-valuations corresponds to the intersection of ideals (P.I. 19). This suggests a number of relations connecting the different operations, which are in fact found to hold when we restrict ourselves to bounded non-archimedean pseudo-valuations. On the other hand, we can prove that certain laws such as the distributive law for multiplication (or compounding) and addition do not hold generally (Ch. V).

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[1]) Cf. the definition in § 1, and K. MAHLER, Über Pseudobewertungen I, Acta Mathematica 66 (1935) 79–119. This paper will be referred to as P. I, followed by the number of the paragraph.

I.

1. In all that follows R is a commutative ring with the unit-element 1. A pseudo-valuation of R is a real-valued function $W(a)$ defined for all a in R and having the properties

i) $W(0) = 0, \quad W(a) \geqslant 0$;

ii) $W(a-b) \leqslant W(a) + W(b)$;

iii) $W(ab) \leqslant W(a) . W(b)$.

We call a real-valued function on R *admissible*, if it satisfies i), *subadditive*, if it satisfies ii) and *submultiplicative*, if it satisfies iii). A real-valued function $\varphi(a)$ which satisfies the inequality

ii)′ $\varphi(a-b) \leqslant \max\{\varphi(a), \varphi(b)\}$

is called *non-archimedean*. Thus a function which satisfies i), ii)′ and iii) will be called a *non-archimedean pseudo-valuation*, which agrees with the usual terminology (as e.g. in P. I). For the sake of distinction the ordinary pseudo-valuations will sometimes be called *subadditive*.

As examples of pseudo-valuations we have the functions

$$U(a) \equiv 0 \text{ and } W_0(a) = \begin{cases} 0 \text{ if } a = 0 \\ 1 \text{ if } a \neq 0. \end{cases}$$

They are called the *improper pseudo-valuation* and the *trivial pseudo-valuation* of R, respectively.

An admissible submultiplicative function φ always satisfies

$$\varphi(1) \geqslant 1 \tag{1}$$

unless it is identically zero; for if $\varphi(a) \neq 0$ for some a in R, then

$$0 < \varphi(a) = \varphi(1.a) \leqslant \varphi(1)\varphi(a),$$

from which (1) follows on dividing by $\varphi(a)$. Hence we have

LEMMA 1.1. *An admissible submultiplicative function φ, which is such that $\varphi(1) < 1$, is identically zero.*

In particular, a pseudo-valuation W is improper if $W(1) < 1$.

2. In P.I. 7 it was shown that any finite set of pseudo-valuations $W_1, \ldots, W_n$ defines two new pseudo-valuations

$$W_\Sigma(a) = W_1(a) + \ldots + W_n(a)$$

and

$$W_\Sigma^*(a) = \max\{W_1(a), \ldots, W_n(a)\},$$

which are in fact equivalent [2]).

[2]) Cf. P.I. 8, or § 14 below, for the definition of 'equivalent'.

If we have an infinite set of pseudo-valuations: W_λ $(\lambda \in \Lambda)$, then we can similarly define the functions

$$W_\Sigma(a) = \Sigma_{\lambda \in \Lambda} W_\lambda(a); \qquad W_\Sigma^*(a) = \sup\{W_\lambda(a)\},$$

where the expressions on the right may assume the value $+\infty$. If W_Σ or W_Σ^* is finite for all a in R, it will again define a pseudo-valuation; the proof is exactly as in the finite case. However, W_Σ will not be finite unless almost all the W_λ are improper. For every proper pseudo-valuation satisfies $W(1) \geqslant 1$, and so

$$W_\Sigma(1) \geqslant \Sigma_{W_\lambda \text{ proper}} 1.$$

Hence the sum W_Σ gives nothing new, and we shall only consider the second type of sum.

We have the following obvious criterion for deciding when $\sup(W_\lambda(a))$ is a pseudo-valuation:

LEMMA 2.1. *If W_λ $(\lambda \in \Lambda)$ is a non-empty family of pseudo-valuations on R and if there is a real function φ on R such that*

$$W_\lambda(a) \leqslant \varphi(a) \qquad \textit{for all } a \in R,\ \lambda \in \Lambda, \tag{2}$$

then $\sup(W_\lambda(a))$ *is a pseudo-valuation.*

For the condition (2) ensures that $\sup(W_\lambda(a))$ shall be finite for all values of a.

3. The criterion of Lemma 2.1 suggests considering the pseudo-valuations which are majorised by a given real function φ on R. Such pseudo-valuations exist if and only if $\varphi(a) \geqslant 0$ for all $a \in R$. Let Ω_R be the set of all pseudo-valuations on R. Then if φ is any real non-negative function on R, we can put

$$W_\varphi(a) = \sup\{W(a) \mid W \in \Omega_R,\ W \leqslant \varphi\}, \tag{3}$$

where $W \leqslant \varphi$ means $W(a) \leqslant \varphi(a)$ for all $a \in R$. By Lemma 2.1, W_φ is a pseudo-valuation; in fact it is the greatest pseudo-valuation majorised by φ. Similarly we can define the greatest *non-archimedean* pseudo-valuation majorised by φ as

$$W_{\bar\varphi}(a) = \sup\{W(a) \mid W \in \Omega_R,\ W \text{ non-archimedean and } W \leqslant \varphi\}. \tag{4}$$

In particular, if we take as our non-negative function φ a pseudo-valuation V, then $W_{\bar V}$ defined by (4) is the largest non-archimedean pseudo-valuation majorised by V. Of course this may be the improper pseudo-valuation U; e.g. if V is the ordinary absolute value on the field of rational numbers, then there is no proper non-archimedean pseudo-valuation majorised by V, and so $W_{\bar{V}} = U$ in this case. On the other hand, if V is again taken to be the absolute value, this time on the ring of rational integers, then the largest non-archimedean pseudo-valuation majorised by V is W_0, the trivial pseudo-valuation. This is not difficult to prove and also follows from the alternative definitions given later.

Let φ be any real non-negative function on R. We can think of the set of pseudo-valuations W majorised by φ as determined by a set of conditions, one for each $a \in R$, viz. $W(a) \leqslant \varphi(a)$. Only the condition for $a = 0$ is vacuous, since $\varphi(0) \geqslant W(0) = 0$ holds in any case. Thus the value $\varphi(0)$ does not affect our set of pseudo-valuations and we may suppose from the outset that $\varphi(0) = 0$, i.e. that φ is admissible. Then we can sum up the results of this paragraph in

THEOREM 3.1. *If φ is any admissible function on R, then there is i) a uniquely determined greatest pseudo-valuation W_φ which is majorised by φ and ii) a uniquely determined greatest non-archimedean pseudo-valuation $W_{\bar{\varphi}}$ which is majorised by φ.*

Clearly

$$0 \leqslant W_{\bar{\varphi}} \leqslant W_\varphi \leqslant \varphi.$$

Further, W_φ coincides with φ if and only if φ is a pseudo-valuation, and $W_{\bar{\varphi}}$ coincides with φ if and only if φ is a non-archimedean pseudo-valuation. It is also clear that $W_{\bar{\varphi}}$ is the greatest non-archimedean pseudo-valuation majorised by W_φ.

4. The above definitions of W_φ and $W_{\bar{\varphi}}$ were non-constructive. We shall now give a constructive definition of these functions. For every admissible function φ we define the function

$$\varphi^\times(a) = \inf_{\Pi x_i = a} \Pi_i \, \varphi(x_i), \tag{5}$$

where the greatest lower bound is extended over all factorisations of a in R [3]).

The function $\varphi^\times$ is again admissible. For the lower bound of a set of non-negative numbers exists and is non-negative, and since there are factorisations of a (e.g. $a = a$), the set over which the lower bound is extended is not empty. Since $\varphi(0) = 0$, we have $\varphi^\times(0) = 0$ and so $\varphi^\times$ is admissible.

[3]) The greatest lower bound (inf) is sometimes loosely referred to as "the lower bound", when no confusion is possible. Similarly for the least upper bound (sup).

Further $\varphi^\times$ is submultiplicative. To prove this, let $a, b \in R$. Then for any factorisations $a = \Pi_i x_i$, $b = \Pi_j y_j$,

$$\varphi^\times(ab) \leqslant \Pi_i \varphi(x_i) \, \Pi_j \varphi(y_j).$$

Hence, by taking the lower bound over all such factorisations of a and b, we get

$$\begin{aligned}\varphi^\times(ab) &\leqslant \inf \Pi\varphi(x_i) \, . \inf \Pi\varphi(y_j) \\ &= \varphi^\times(a) \, \varphi^\times(b)\end{aligned}$$

as asserted.

The operation $\varphi \to \varphi^\times$ is *monotone*, i.e. if $\varphi_1 \leqslant \varphi_2$, then

$$\varphi_1^\times \leqslant \varphi_2^\times,$$

as is immediate from the definition.

If we take the factorisation $a = a$ in (5), we see that $\varphi^\times(a) \leqslant \varphi(a)$ for all $a \in R$. Thus $\varphi^\times \leqslant \varphi$. Equality holds here if (and of course only if) φ is submultiplicative. For then

$$\varphi(a) \leqslant \Pi_i \varphi(x_i)$$

for all factorisations of a, hence

$$\varphi(a) \leqslant \inf \Pi_i \varphi(x_i) = \varphi^\times(a).$$

5. Next we define for every admissible function φ a second function φ^+ by the rule

$$\varphi^+(a) = \inf_{\Sigma x_i = a} \max_i \varphi(\pm x_i), \tag{6}$$

where the lower bound is taken over all additive decompositions $\Sigma x_i = a$ and over all possible distributions of signs before the x_i. As for $\varphi^\times$ we can verify that φ^+ is admissible. It is also non-archimedean, for if $\Sigma x_i = a$ and $\Sigma y_j = b$, then

$$\varphi^+(a - b) \leqslant \max \{\max_i \varphi(\pm x_i), \max_j \varphi(\pm y_j)\},$$

whence, on taking the lower bound,

$$\begin{aligned}\varphi^+(a - b) &\leqslant \max \{\inf \max_i \varphi(\pm x_i), \inf \max_j \varphi(\pm y_j)\} \\ &= \max \{\varphi^+(a), \varphi^+(b)\}.\end{aligned}$$

Similarly we define

$$\varphi^\oplus(a) = \inf_{\Sigma x_i = a} (\Sigma_i \, \varphi(\pm x_i)) \tag{7}$$

and show that $\varphi^\oplus$ is admissible and subadditive.

We note that the operations $\varphi \to \varphi^{+}$ and $\varphi \to \varphi^{\oplus}$ are both monotone. Further $\varphi^{+} \leqslant \varphi^{\oplus} \leqslant \varphi$, and $\varphi = \varphi^{+}$ or $= \varphi^{\oplus}$ if and only if φ is non-archimedean or subadditive, respectively. This follows in the same way as for the function $\varphi^{\times}$.

LEMMA 5.1. *If φ is an admissible submultiplicative function, then φ^{+} and $\varphi^{\oplus}$ are likewise submultiplicative.*

Proof. If $\Sigma\, x_i = a$, $\Sigma\, y_j = b$ are additive decompositions of a and b respectively, then $\Sigma\, x_i y_j = ab$. Hence, by the definition of φ^{+} and because φ is submultiplicative,

$$\begin{aligned}\varphi^{+}(ab) &\leqslant \max_{i\,j} \varphi(\pm\, x_i y_j)\\ &\leqslant \max_{i\,j} \varphi(\pm\, x_i)\, \varphi(\pm\, y_j)\\ &= (\max_i \varphi(\pm\, x_i))\, (\max_j \varphi(\pm\, y_j)).\end{aligned}$$

Taking the lower bound over all decompositions of a and b we obtain

$$\varphi^{+}(ab) \leqslant \varphi^{+}(a)\, \varphi^{+}(b),$$

which shows that φ^{+} is submultiplicative. The proof for $\varphi^{\oplus}$ is analogous.

We note that the roles of $+$ and $\times$ cannot be interchanged in this lemma, i.e. if φ is an admissible non-archimedean (or subadditive) function, it does not follow that $\varphi^{\times}$ is again non-archimedean (or subadditive). In fact the proof of the lemma depends essentially on the distributive law, and this does not remain true if we interchange $+$ and $\times$.

6. If φ is any admissible function, $\varphi^{\times}$ is both admissible and submultiplicative. Both these properties are preserved in $\varphi^{\times\oplus}$ which, moreover, is subadditive and hence is a pseudo-valuation. Similarly $\varphi^{\times +}$ is a non-archimedean pseudo-valuation.

THEOREM 6.1. *The function $\varphi^{\times\oplus}$ coincides with W_{φ}, the greatest pseudo-valuation majorised by φ. Similarly $\varphi^{\times +}$ equals $W_{\bar{\varphi}}$, the greatest non-archimedean pseudo-valuation majorised by φ.*

Proof. We have seen that $\varphi^{\times\oplus}$ is a pseudo-valuation. Since $\varphi^{\times\oplus} \leqslant \varphi^{\times} \leqslant \varphi$, it is majorised by φ; hence $\varphi^{\times\oplus} \leqslant W_{\varphi}$, because W_{φ} is the greatest pseudo-valuation majorised by φ. Conversely, since W_{φ} is a pseudo-valuation and $W_{\varphi} \leqslant \varphi$,

$$W_{\varphi} = W_{\varphi}{}^{\times\oplus} \leqslant \varphi^{\times\oplus},$$

whence $W_{\varphi} = {}^{\times\oplus}$. A similar argument shows that $W_{\bar{\varphi}} = \varphi^{\times +}$.

We note that $\varphi^{\times +}$ and $\varphi^{\times\oplus}$ may be defined directly by

$$\varphi^{\times +}(a) = \inf \max_i \Pi_{\nu} \varphi(\pm\, x_{i\nu}),$$

$$\varphi^{\times\oplus}(a) = \inf \Sigma_i\, \Pi_{\nu}\, \varphi(\pm\, x_{i\nu}),$$

where in each case the lower bound is taken over all decompositions $a = \Sigma_i\, \Pi_{\nu} x_{i\nu}$ and all distributions of signs.

II.

7. We now turn to the study of certain binary operations on Ω_R, the set of pseudo-valuations on R. Their interpretation in the ring R will be considered in Ch. IV, with the help of the results of Ch. I.

Let a be any element of R. Since $a = a.1$, there always exist decompositions

$$a = x_1 y_1 + \ldots + x_n y_n \tag{1}$$

of a, where $x_i, y_i \in R$, and n is arbitrary. Hence, for any two pseudo-valuations W_1, W_2 of R, we can form the lower bounds

$$W_I(a) = W_1 \,.\, W_2(a) = \inf \max_i (W_1(x_i) W_2(y_i)), \qquad (2)_1$$

$$W_{II}(a) = W_1 \odot W_2(a) = \inf \Sigma_i\, W_1(x_i) W_2(y_i), \qquad (2)_2$$

$$W_{III}(a) = W_1 \times W_2(a) = \inf \max_i (W_1(x_i), W_2(y_i)), \qquad (2)_3$$

$$W_{IV}(a) = W_1 \otimes W_2(a) = \inf \Sigma_i\, (W_1(x_i) + W_2(y_i)), \qquad (2)_4$$

extended over all decompositions (1) of a. As we now show, the functions W_I–W_{IV} thus defined are pseudo-valuations on R.

8. As in the case of $\varphi^{\times}$ (§ 4) we can prove that the functions W_N, where $N = I, II, III, IV$, exist and are admissible. To prove that they are subadditive, let b be a second element of R, and let ε be any positive number. In each of the four cases $N = I, II, III, IV$ we select some decomposition (1) of a and some decomposition

$$b = \xi_1 \eta_1 + \ldots + \xi_m \eta_m \tag{3}$$

of b such that

$$W_I(a) > \max_i (W_1(x_i) W_2(y_i)) - \varepsilon$$

and $\qquad (4)_1$

$$W_I(b) > \max_j (W_1(\xi_j) W_2(\eta_j)) - \varepsilon,$$

or

$$W_{II}(a) > \Sigma_i\, (W_1(x_i)W_2(y_i)) - \varepsilon$$

and $$(4)_2$$

$$W_{II}(b) > \Sigma_j\, (W_1(\xi_j)W_2(\eta_j)) - \varepsilon,$$

or

$$W_{III}(a) > \max_i(W_1(x_i),\ W_2(y_i)) - \varepsilon$$

and $$(4)_3$$

$$W_{III}(b) > \max_j(W_1(\xi_j),\ W_2(\eta_j)) - \varepsilon,$$

or

$$W_{IV}(a) > \Sigma_i\, (W_1(x_i) + W_2(y_i)) - \varepsilon$$

and $$(4)_4$$

$$W_{IV}(b) > \Sigma_j\, (W_1(\xi_j) + W_2(\eta_j)) - \varepsilon,$$

respectively.

By (1) and (3), $a - b$ admits the decomposition

$$a - b = x_1y_1 + \ldots + x_ny_n + (-\xi_1)\eta_1 + \ldots + (-\xi_m)\eta_m;$$

further $W_1(-\xi_j) = W_1(\xi_j)$. Hence, from the definition of $W_N(a-b)$ and from the inequalities (4) for $W_N(a)$ and $W_N(b)$,

$$\begin{aligned}
W_I(a-b) &\leqslant \max\{\max_i(W_1(x_i)W_2(y_i)),\ \max_j(W_1(\xi_j)W_2(\eta_j))\}\\
&< \max(W_I(a),\ W_I(b)) + \varepsilon,\\
W_{II}(a-b) &\leqslant \Sigma_i\, (W_1(x_i)W_2(y_i)) + \Sigma_j\, (W_1(\xi_j)W_2(\eta_j))\\
&< W_{II}(a) + W_{II}(b) + 2\varepsilon,\\
W_{III}(a-b) &\leqslant \max\{\max_i(W_1(x_i),\ W_2(y_i)),\ \max_j(W_1(\xi_j),\ W_2(\eta_j))\}\\
&< \max(W_{III}(a),\ W_{III}(b)) + \varepsilon,\\
W_{IV}(a-b) &\leqslant \Sigma_i\, (W_1(x_i) + W_2(y_i)) + \Sigma_j\, (W_1(\xi_j) + W_2(\eta_j))\\
&< W_{IV}(a) + W_{IV}(b) + 2\varepsilon.
\end{aligned}$$

In the limit, as $\varepsilon \to 0$,

$$\begin{aligned}
W_I(a-b) &\leqslant \max\,(W_I(a),\ W_I(b)),\\
W_{II}(a-b) &\leqslant W_{II}(a) + W_{II}(b),\\
W_{III}(a-b) &\leqslant \max\,(W_{III}(a),\ W_{III}(b)),\\
W_{IV}(a-b) &\leqslant W_{IV}(a) + W_{IV}(b).
\end{aligned}$$

This proves that W_{II} and W_{IV} are subadditive, and W_I and W_{III} are non-archimedean.

To prove W_N submultiplicative, let the decompositions (1) of a and (3) of b be chosen so as to satisfy (4). From (1) and (3) we obtain the decomposition

$$ab = \Sigma_i\Sigma_j(x_i\xi_j)\ (y_i\eta_j)$$

of ab. Therefore, since W_1 and W_2, as pseudo-valuations, are submultiplicative,

$$\begin{aligned}
W_I(ab) &\leqslant \max_{i,j}(W_1(x_i)W_1(\xi_j)W_2(y_i)W_2(\eta_j))\\
&= \max_i(W_1(x_i)W_2(y_i)).\max_j(W_1(\xi_j)W_2(\eta_j))\\
&< (W_I(a) + \varepsilon)\ (W_I(b) + \varepsilon),\\
W_{II}(ab) &\leqslant \Sigma_i\Sigma_j(W_1(x_i)W_1(\xi_j)W_2(y_i)W_2(\eta_j))\\
&= \Sigma_i(W_1(x_i)W_2(y_i)).\Sigma_j(W_1(\xi_j)W_2(\eta_j))\\
&< (W_{II}(a) + \varepsilon)\ (W_{II}(b) + \varepsilon),\\
W_{III}(ab) &\leqslant \max_{i,j}(W_1(x_i)W_1(\xi_j),\ W_2(y_i)W_2(\eta_j))\\
&\leqslant \max_i(W_1(x_i),\ W_2(y_i)).\max_j(W_1(\xi_j),\ W_2(\eta_j))\\
&< (W_{III}(a) + \varepsilon)\ (W_{III}(b) + \varepsilon),\\
W_{IV}(ab) &\leqslant \Sigma_i\Sigma_j(W_1(x_i)W_1(\xi_j) + W_2(y_i)W_2(\eta_j))\\
&\leqslant \Sigma_i(W_1(x_i) + W_2(y_i)).\Sigma_j(W_1(\xi_j) + W_2(\eta_j))\\
&< (W_{IV}(a) + \varepsilon)\ (W_{IV}(b) + \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we see that W_N is in fact submultiplicative. Thus the following result has been proved:

THEOREM 8.1. *If W_1 and W_2 are arbitrary pseudo-valuations on R, then $W_1.W_2$, $W_1 \odot W_2$, $W_1 \times W_2$ and $W_1 \otimes W_2$ are likewise pseudo-valuations on R. Moreover $W_1.W_2$ and $W_1 \times W_2$ are non-archimedean.*

The functions $W_1.W_2$ and $W_1 \odot W_2$ will be called the *non-archimedean product* and the *subadditive product*, and the functions $W_1 \times W_2$ and $W_1 \otimes W_2$ will be called the *non-archimedean compound* and the *subadditive compound*, of W_1 and W_2 respectively.

In establishing Theorem 8.1 we have not used the full force of the hypothesis. Instead of taking W_1 and W_2 to be pseudo-valuations it is enough to assume that they are admissible and submultiplicative and satisfy $W_k(-a) = W_k(a)$ $(k = 1,2)$. The last condition can also be dropped provided we modify the definitions (2) by allowing $-x_i$ and $-y_i$ as arguments on the right and taking the lower bound over all distributions of signs.

9. Just as the letter N was used to denote the four indices I, II, III, IV in $W_N(a)$, we let the symbol $\bigcirc$ stand for any one of the four signs $.$, $\odot$, $\times$, or $\otimes$. Then it follows from the definitions and the commutativity of R that all four operations are commutative:

$$W_1 \bigcirc W_2 = W_2 \bigcirc W_1. \tag{5}$$

Here (5) is an abbreviation for $W_1 \bigcirc W_2(a) = W_2 \bigcirc W_1(a)$ identically in a, and similarly in later cases.

Next we prove that the four operations are associative. For this purpose we make use of

LEMMA 9.1. *If W_1, W_2, W_3 are any three pseudo-valuations of R (or indeed any admissible functions), then the function*

$$(W_1 \bigcirc W_2) \bigcirc W_3$$

(where $\bigcirc$ is $.$, $\odot$, $\times$ or $\otimes$) is symmetric in W_1, W_2, W_3.

Assuming the truth of this lemma for the moment, we have by (5),

$$\begin{aligned}(W_1 \bigcirc W_2) \bigcirc W_3 &= (W_2 \bigcirc W_3) \bigcirc W_1 \\ &= W_1 \bigcirc (W_2 \bigcirc W_3),\end{aligned}$$

and hence

THEOREM 9.2. *The operations* $.$, $\odot$, $\times$ and $\otimes$ *are commutative and associative.*

10. It remains to prove Lemma 9.1. We do this by writing $(W_1 \bigcirc W_2) \bigcirc W_3(a)$ as the lower bound of expressions involving certain types of ternary decomposition of a. For $N = I, II, III, IV$ write

$$W_N^*(a) = W_1 \bigcirc W_2(a),$$

$$V_N(a) = W_N^* \bigcirc W_3(a) = (W_1 \bigcirc W_2) \bigcirc W_3(a).$$

An upper bound for $V_N(a)$ in terms of ternary decompositions is obtained as follows. Let

$$a = x_1y_1z_1 + \ldots + x_ny_nz_n. \tag{6}$$

Then

$$V_I(a) \leqslant \max_i W_I^*(x_iy_i)W_3(z_i) \leqslant \max_i W_1(x_i)W_2(y_i)W_3(z_i),$$

$$V_{II}(a) \leqslant \Sigma_i W_{II}^*(x_iy_i)W_3(z_i) \leqslant \Sigma_i W_1(x_i)W_2(y_i)W_3(z_i),$$

$$V_{III}(a) \leqslant \max_i(W_{III}^*(x_iy_i), W_3(z_i)) \leqslant \max_i(W_1(x_i), W_2(y_i), W_3(z_i)).$$

For the subadditive compound V_{IV} we need decompositions of the more general type

$$a = \Sigma_{i=1}^n \Sigma_{\varrho=1}^{r_i} \Sigma_{\sigma=1}^{s_i} \Sigma_{\tau=1}^{t_i} x_{i\varrho\sigma}y_{i\varrho\tau}z_{i\sigma\tau}. \tag{7}$$

Then

$$\begin{aligned}V_{IV}(a) &\leqslant \Sigma_{i\sigma\tau}\{W_{IV}^*(\Sigma_\varrho x_{i\varrho\sigma}y_{i\varrho\tau}) + W_3(z_{i\sigma\tau})\} \\ &\leqslant \Sigma_{i\varrho\sigma\tau}\{W_1(x_{i\varrho\sigma}) + W_2(y_{i\varrho\tau}) + W_3(z_{i\sigma\tau})\}.\end{aligned}$$

By taking the lower bounds of the right-hand sides for all ternary decompositions of the type (6) for $N = I, II, III$, and of the type (7) for $N =$ IV, we find that

$$\begin{aligned}V_I(a) &\leqslant \inf \max_i W_1(x_i)W_2(y_i)W_3(z_i), \\ V_{II}(a) &\leqslant \inf \Sigma_i W_1(x_i)W_2(y_i)W_3(z_i), \\ V_{III}(a) &\leqslant \inf \max_i \{W_1(x_i), W_2(y_i), W_3(z_i)\}, \\ V_{IV}(a) &\leqslant \inf \Sigma_{i\varrho\sigma\tau}\{W_1(x_{i\varrho\sigma}) + W_2(y_{i\varrho\tau}) + W_3(z_{i\sigma\tau})\}.\end{aligned} \tag{8}$$

We now show that equality holds in each case. The Lemma will follow from this, since the right-hand sides are obviously symmetric in the W's.

I. *The non-archimedean product.*

If $\varepsilon > 0$, there exists a decomposition

$$a = u_1z_1 + \ldots + u_nz_n \tag{9}$$

of a such that

$$W_I^*.W_3(a) > \max W_I^*(u_i)W_3(z_i) - \varepsilon. \tag{10}$$

Write

$$\omega = \max(W_3(z_i), 1),$$

so that ω is a finite constant not less than 1, which only depends on the decomposition (9) of a.

Next choose decompositions

$$u_i = x_{i1}y_{i1} + \ldots x_{i\nu_i}y_{i\nu_i} \quad (i = 1, \ldots, n) \tag{11}$$

of $u_1, \ldots, u_n$ such that

$$W_I^*(u_i) = W_1.W_2(u_i) > \max_{1\leqslant\varrho\leqslant\nu_i} W_1(x_{i\varrho})W_2(y_{i\varrho}) - \frac{\varepsilon}{\omega} \tag{12}$$

$$(i = 1, \ldots, n).$$

In general the numbers ν_i will depend on i, but by adding, if necessary, a number of zero terms 0.0 to the decomposition (11) of u_i we can arrange that the decompositions (11) all have the same number of terms, so that $\nu_1 = \nu_2 = \ldots = \nu_n, = \nu$ say.

In order to unify the notation we put $z_{i\varrho} = z_i$ ($\varrho = 1, \ldots, \nu$; $i = 1, \ldots, n$). Then by (9) and (11),

$$a = \Sigma_{i=1}^n z_i \Sigma_{\varrho=1}^{\nu} x_{i\varrho} y_{i\varrho}$$
$$= \Sigma_{i=1}^n \Sigma_{\varrho=1}^{\nu} x_{i\varrho} y_{i\varrho} z_{i\varrho}.$$

This is a ternary decomposition of the form (6), as becomes clear when we replace the index-pair $i\varrho$ by a simple index μ running from 1 to $m = n\nu$.

By (10) and (12)

$$V_I(a) > \max_i \max_\varrho \left\{ W_1(x_{i\varrho}) W_2(y_{i\varrho}) - \frac{\varepsilon}{\omega} \right\} W_3(z_{i\varrho}) - \varepsilon,$$

and from the definition of ω

$$\frac{\varepsilon}{\omega} W_3(z_{i\varrho}) \leqslant \varepsilon.$$

Hence

$$V_I(a) > \max_i \max_\varrho \{W_1(x_{i\varrho}) W_2(y_{i\varrho}) W_3(z_{i\varrho})\} - 2\varepsilon,$$

or, on changing to the single index μ,

$$V_I(a) > \max_\mu \{W_1(x_\mu) W_2(y_\mu) W_3(z_\mu)\} - 2\varepsilon.$$

Letting $\varepsilon \to 0$ and combining the result with (8), we find that

$$(W_1 . W_2) . W_3(a) = \inf \max_\mu W_1(x_\mu) W_2(y_\mu) W_3(z_\mu), \qquad (13)$$

where the lower bound is extended over all decompositions (6) of a. This proves the assertion for the case of the non-archimedean product.

The proofs in the other three cases are similar and we can therefore be more concise. In each case ε is a fixed but arbitrary positive number.

II. *The subadditive product.*

We choose a decomposition (9) of a such that

$$W_{II}^* \odot W_3(a) > \Sigma_i W_{II}^*(u_i) W_3(z_i) - \varepsilon.$$

Next put

$$\omega = \Sigma_i W_3(z_i) + 1,$$

and choose decompositions (11) of $u_1, \ldots, u_n$ such that

$$W_{II}^*(u_i) = W_1 \odot W_2(u_i) > \Sigma_\varrho W_1(x_{i\varrho}) W_2(y_{i\varrho}) - \frac{\varepsilon}{\omega} \; (i = 1, \ldots, n).$$

Then

$$V_{II}(a) > \Sigma_i \left(\Sigma_\varrho W_1(x_{i\varrho}) W_2(y_{i\varrho}) - \frac{\varepsilon}{\omega} \right) W_3(z_i) - \varepsilon,$$

and hence

$$V_{II}(a) > \Sigma_{i\varrho} W_1(x_{i\varrho}) W_2(y_{i\varrho}) W_3(z_{i\varrho}) - 2\varepsilon.$$

Changing the notation as before, we obtain the formula

$$(W_1 \odot W_2) \odot W_3(a) = \inf \Sigma_\mu W_1(x_\mu) W_2(y_\mu) W_3(z_\mu),$$

where the lower bound is again extended over all decompositions (6) of a.

III. *The non-archimedean compound.*

We now choose the decomposition (9) of a so that

$$W_{III}^* \times W_3(a) > \max_i (W_{III}^*(u_i), W_3(z_i)) - \varepsilon,$$

and decompositions (11) of $u_1, \ldots, u_n$ such that

$$W_{III}^*(u_i) = W_1 \times W_2(u_i) > \max_\varrho (W_1(x_{i\varrho}), W_2(y_{i\varrho})) - \varepsilon \qquad (i = 1, \ldots, n).$$

Then

$$V_{III}(a) > \max_i \{\max_\varrho (W_1(x_{i\varrho}), W_2(y_{i\varrho})) - \varepsilon, W_3(z_i)\} - \varepsilon,$$

whence

$$V_{III}(a) > \max_i \max_\varrho (W_1(x_{i\varrho}), W_2(y_{i\varrho}), W_3(z_{i\varrho})) - 2\varepsilon,$$

and we obtain the result

$$(W_1 \times W_2) \times W_3(a) = \inf \max_\mu (W_1(x_\mu), W_2(y_\mu), W_3(z_\mu))$$

where the lower bound is again taken over all decompositions (6) of a.

IV. *The subadditive compound.*

Again we choose a decomposition (9) of a such that

$$W_{IV}^* \otimes W_3(a) > \Sigma_i \left(W_{IV}^*(u_i) + W_3(z_i)\right) - \varepsilon,$$

and next decompositions (11) of $u_1, \ldots, u_n$ such that

$$W_{IV}^*(u_i) > \Sigma_\varrho \left(W_1(x_{i\varrho}) + W_2(y_{i\varrho})\right) - \frac{\varepsilon}{n}.$$

Then

$$V_{IV}(a) > \Sigma_i \left\{\Sigma^\varrho (W_1(x_{i\varrho}) + W_2(y_{i\varrho})) - \frac{\varepsilon}{n} + W_3(z_i)\right\} - \varepsilon$$
$$= \Sigma_{i\varrho} \left(W_1(x_{i\varrho}) + W_2(y_{i\varrho}) + W_3(z_i)\right) - 2\varepsilon.$$

Now the decomposition

$$a = \Sigma_{i\varrho}\, x_{i\varrho} y_{i\varrho} z_i$$

is of the form (7) (the range of each of the suffixes σ and τ is 1 for every i and they have therefore been omitted). Hence

$$(W_1 \otimes W_2) \otimes W_3(a) = \inf \Sigma_{i\varrho\sigma\tau} \left(W_1(x_{i\varrho\sigma}) + W_2(y_{i\varrho\tau}) + W_3(z_{i\sigma\tau})\right),$$

where this time the lower bound is taken over all decompositions (7) of a.

This completes the proof of Lemma 9.1.

From the proof of the Lemma it is clear why the different types of ternary decomposition (6) and (7) have to be considered. By renaming the suffixes we can regard any decomposition (7) of a as being of the form (6) [4], but then we must count certain factors repeatedly. For the products and the non-archimedean compound this is of no importance, and it is therefore sufficient to consider decompositions (6) when taking the lower bound. In the case of the subadditive compound we cannot make this simplification but have to consider all decompositions (7) when calculating the lower bound. Though we actually use only a decomposition of the form $a = \Sigma\, x_{i\varrho} y_{i\varrho} z_i$, it is not enough to take such decompositions, because they are not symmetric in the three sets of factors. The decomposition (7) is in fact the simplest type of ternary decomposition which suffices for our purpose. It is not the most general type, as we might have a decomposition

$$a = \Sigma\, x_{i\varrho\sigma\lambda} y_{i\varrho\tau\mu} z_{i\sigma\tau\nu},$$

[4]) This is precisely what we did when we replaced the suffix pair $i\varrho$ by μ in order to obtain (13).

where each of $i, \varrho, \sigma, \tau, \lambda, \mu, \nu$ runs over some range depending on the suffixes preceding it in $x_{i\varrho\sigma\lambda}, y_{i\varrho\tau\mu}, z_{i\sigma\tau\nu}$. However, this will not be required.

III.

11. Before considering the products and compounds in greater detail we shall illustrate the definitions by a few examples.

First take R to be an algebraic number field K of finite degree n over the rational field P. Suppose that of the n conjugate fields

$$K^{(1)}, K^{(2)}, \ldots, K^{(n)}$$

of K the first r_1 are real and the remaining $2r_2 = n - r_1$ fields

$$K^{(r_1+1)} \text{ and } K^{(r_1+r_2+1)},\ K^{(r_1+2)} \text{ and } K^{(r_1+r_2+2)},\ \ldots, K^{(r_1+r_2)} \text{ and } K^{(r_1+2r_2)}$$

are complex conjugate in pairs. If a is any element of K, denote by $a^{(h)}$ the number conjugate to it in $K^{(h)}$. Then $a^{(h)}$ is real for $h = 1, \ldots, r_1$, while $a^{(r_1+h)}$ and $a^{(r_1+r_2+h)}$ are complex conjugate for $h = 1, \ldots, r_2$.

The field K has exactly $r_1 + r_2$ inequivalent absolute values, viz.

$$\Omega^{(h)}(a) = |a^{(h)}| \qquad (h = 1, \ldots, r_1 + r_2).$$

Further let $\mathfrak{p}$ be an arbitrary prime ideal in the ring J of integers of K. To $\mathfrak{p}$ there corresponds a $\mathfrak{p}$-adic valuation of K (unique to within equivalence) which we denote by $\Omega_\mathfrak{p}(a)$. This valuation is fully determined if we know that

$$\Omega_\mathfrak{p}(a) = c_\mathfrak{p}, \qquad \text{where } 0 < c_\mathfrak{p} < 1,$$

for all elements a of K which have a denominator prime to $\mathfrak{p}$ and a numerator divisible by the first and no higher power of $\mathfrak{p}$.

It is known that every pseudo-valuation of K not equivalent to U or W_0 is equivalent to the sum of a finite number of valuations $\Omega^{(h)}$ and $\Omega_\mathfrak{p}$ [5]). The tables which follow give the results of the four operations $W_1 \bigcirc W_2$ applied to the functions U, W_0, $\Omega^{(h)}$ and $\Omega_\mathfrak{p}$. In these tables h and k are distinct indices $1, 2, \ldots, r_1 + r_2$, provided that $r_1 + r_2 \geqslant 2$; in the exceptional case $r_1 + r_2 = 1$ the row and column belonging to $\Omega^{(k)}$ are to be omitted [6]). Similarly $\mathfrak{p}$ and $\mathfrak{q}$ denote two distinct prime ideals of J.

[5]) Cf. K. Mahler, Über Pseudobewertungen II, Acta Mathematica 67 (1936), 51—80.

[6]) This is the case when K is the rational field or an imaginary quadratic field.

Table 1

$W_1 . W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_\mathfrak{p}$	$\Omega_\mathfrak{q}$
U	U	U	U	U	U	U
W_0	U	W_0	U	U	U	U
$\Omega^{(h)}$	U	U	U	U	U	U
$\Omega^{(k)}$	U	U	U	U	U	U
$\Omega_\mathfrak{p}$	U	U	U	U	$\Omega_\mathfrak{p}$	U
$\Omega_\mathfrak{q}$	U	U	U	U	U	$\Omega_\mathfrak{q}$

Table 2

$W_1 \odot W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_\mathfrak{p}$	$\Omega_\mathfrak{q}$
U	U	U	U	U	U	U
W_0	U	W_0	U	U	U	U
$\Omega^{(h)}$	U	U	$\Omega^{(h)}$	U	U	U
$\Omega^{(k)}$	U	U	U	$\Omega^{(k)}$	U	U
$\Omega_\mathfrak{p}$	U	U	U	U	$\Omega_\mathfrak{p}$	U
$\Omega_\mathfrak{q}$	U	U	U	U	U	$\Omega_\mathfrak{q}$

Table 3

$W_1 \times W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_\mathfrak{p}$	$\Omega_\mathfrak{q}$
U	U	W_0	U	U	U	U
W_0	W_0	W_0	W_0	W_0	W_0	W_0
$\Omega^{(h)}$	U	W_0	U	U	U	U
$\Omega^{(k)}$	U	W_0	U	U	U	U
$\Omega_\mathfrak{p}$	U	W_0	U	U	$V_\mathfrak{p}$	U
$\Omega_\mathfrak{q}$	U	W_0	U	U	U	$V_\mathfrak{q}$

Table 4

$W_1 \otimes W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_\mathfrak{p}$	$\Omega_\mathfrak{q}$
U	U	W_0	U	U	U	U
W_0	W_0	$2W_0$	W_0	W_0	W_0	W_0
$\Omega^{(h)}$	U	W_0	$2\Omega^{(h)1/2}$	U	U	U
$\Omega^{(k)}$	U	W_0	U	$2\Omega^{(k)1/2}$	U	U
$\Omega_\mathfrak{p}$	U	W_0	U	U	$V^*_\mathfrak{p}$	U
$\Omega_\mathfrak{q}$	U	W_0	U	U	U	$V^*_\mathfrak{q}$

The pseudo-valuations $V_\mathfrak{p}$ and $V^*_\mathfrak{p}$ occurring in the Tables 3 and 4 are defined by

$$V_\mathfrak{p}(a) = c_\mathfrak{p}^{[\nu/2]}$$

and

$$V^*_\mathfrak{p}(a) = c_\mathfrak{p}^{[\nu/2]} + c_\mathfrak{p}^{\nu-[\nu/2]}$$

where ν is the integer determined by $\Omega_\mathfrak{p}(a) = c_\mathfrak{p}^\nu$. Clearly both $V_\mathfrak{p}$ and $V^*_\mathfrak{p}$ are equivalent to the valuation $\Omega_\mathfrak{p}$.

12. Most of the entries in these Tables are an immediate consequence of

LEMMA 12.1. *Let R be a field, $W(a)$ a pseudo-valuation and $\Omega(a)$ a valuation of R. If there exists an element a of R such that*

$$W(a) < 1 < \Omega(a), \tag{1}$$

then

$$W . \Omega = W \odot \Omega = W \times \Omega = W \otimes \Omega = U.$$

In the case of the products (. *and* $\odot$) *only one inequality in* (1) *need be strict.*

Proof. Since $\Omega(a) > 1$, a is different from zero, and so the unit-element 1 admits the decompositions

$$1 = a^m . a^{-m} \qquad (m = 1, 2, \ldots).$$

By the hypothesis

$$0 \leqslant \lim_{m\to\infty} W(a^m) \leqslant \lim_{m\to\infty} W(a)^m = 0,$$

$$\lim_{m\to\infty} \Omega(a^{-m}) = \lim_{m\to\infty} \Omega(a)^{-m} = 0,$$

hence $W \odot \Omega\ (1) = 0$, and the Lemma follows by Lemma 1.1.

We shall discuss briefly the entries in the four Tables. Since the commutative law holds, we need only consider the entries on or above the main diagonal.

Table 1. We can use Lemma 12.1 for the non-diagonal elements, since all the argument functions except U are valuations and all except W_0 are non-trivial.
$\Omega^{(h)} . \Omega^{(h)} = U$. This follows by using the decomposition

$$1 = \tfrac{1}{2}.1 + \tfrac{1}{2}.1 \tag{2}$$

of 1 and applying Lemma 1.1.
$\Omega_\mathfrak{p} . \Omega_\mathfrak{p} = \Omega_\mathfrak{p}$. If

$$a = x_1 y_1 + \ldots + x_m y_m \tag{3}$$

is any decomposition of a, then

$$\Omega_\mathfrak{p}(a) = \Omega_\mathfrak{p}(\Sigma\, x_i y_i) \leqslant \max_i \{\Omega_\mathfrak{p}(x_i)\Omega_\mathfrak{p}(y_i)\},$$

hence $\Omega_\mathfrak{p} \leqslant \Omega_\mathfrak{p} . \Omega_\mathfrak{p}$, and equality is established by using the decomposition $a = a.1$.

The same proof shows that $W_0 . W_0 = W_0$, while it is evident that $U.U = U$.

Table 2. As for Table 1, except that $\Omega^{(h)} \odot \Omega^{(h)} = \Omega^{(h)}$.
Clearly

$$\Omega^{(h)}(a) = \Omega^{(h)}(\Sigma\, x_i y_i) \leqslant \Sigma\, \Omega^{(h)}(x_i)\Omega^{(h)}(y_i),$$

for any decomposition (3) of a. Hence $\Omega^{(h)} \leqslant \Omega^{(h)} \odot \Omega^{(h)}$, and equality again follows by writing $a = a.1$.

Table 3. $W_0 \times W = W_0$, where W is any pseudo-valuation occurring as argument in the Table. For if (3) is any decomposition of $a \neq 0$, then at least one x_i is different from zero; therefore $\max_i\{W_0(x_i), W(y_i)\} \geqslant 1 = W_0(a)$, and equality is attained for the decomposition $a = a.1$.
$U \times W = U$ $(W \neq W_0)$. Let $a \neq 0$ be such that $W(a) < 1$, then

$$U \times W(a) \leqslant \max\{U(1/a), W(a)\} < 1,$$

hence $U \times W = U$.

The remaining non-diagonal elements follow from Lemma 12.1.
$\Omega^{(h)} \times \Omega^{(h)} = U$. As in Table 1, using the decomposition $1 = \tfrac{1}{2}.\tfrac{1}{2} + \tfrac{1}{2}.\tfrac{1}{2} + \tfrac{1}{2}.\tfrac{1}{2} + \tfrac{1}{2}.\tfrac{1}{2}$ of 1.
$\Omega_\mathfrak{p} \times \Omega_\mathfrak{p} = V_\mathfrak{p}$. If (3) is any decomposition of $a \neq 0$, then

$$\max_i \Omega_\mathfrak{p}(x_i y_i) \geqslant \Omega_\mathfrak{p}(a).$$

Let $i = \mu$ be a suffix for which the maximum is attained, then

$$\Omega_\mathfrak{p}(x_\mu)\Omega_\mathfrak{p}(y_\mu) \geqslant \Omega_\mathfrak{p}(a).$$

Hence, by the definition of $V_\mathfrak{p}$,

$$\max\{\Omega_\mathfrak{p}(x_\mu),\ \Omega_\mathfrak{p}(y_\mu)\} \geqslant V_\mathfrak{p}(a),$$

and so $V_\mathfrak{p} \leqslant \Omega_\mathfrak{p} \times \Omega_\mathfrak{p}$. The reverse inequality follows by choosing x and y in R such that both $xy = a$ and

$$\max\{\Omega_\mathfrak{p}(x),\ \Omega_\mathfrak{p}(y)\} = V_\mathfrak{p}(a).$$

Table 4. $U \otimes W = U$ $(W \neq W_0)$. As for Table 3.
$W_0 \otimes W = W_0$ $(W \neq W_0)$. Again, for $a \neq 0$,

$$\Sigma_i \{W_0(x_i) + W(y_i)\} \geqslant 1 = W_0(a).$$

To obtain equality, take a decomposition $a = xy$ with $W(y) \to 0$. The proof that $W_0 \otimes W_0 = 2W_0$, is similar.

The remaining non-diagonal elements follow again from Lemma 12.1.
$\Omega^{(h)} \otimes \Omega^{(h)} = 2\Omega^{(h)^{1/2}}$. If (3) is any decomposition of a, then by the theorem of the arithmetic and geometric means and by Jensen's inequality,

$$\begin{aligned}\Sigma_i \{\Omega^{(h)}(x_i) + \Omega^{(h)}(y_i)\} &\geqslant 2\, \Sigma_i\, \Omega^{(h)}(x_i y_i)^{1/2} \\ &\geqslant 2\, \{\Sigma_i\, \Omega^{(h)}(x_i y_i)\}^{1/2} \\ &\geqslant 2\, \Omega^{(h)}(a)^{1/2}.\end{aligned}$$

In this inequality the difference between the left- and righthand sides can be made arbitrarily small by choosing a decomposition $a = xy$ for which $|\, \Omega^{(h)}(x) - \Omega^{(h)}(y)\, |$ is sufficiently small.
$\Omega_\mathfrak{p} \otimes \Omega_\mathfrak{p} = V_\mathfrak{p}^*$. This follows as for the corresponding entry in Table 3.

13. As a second example consider the ring J of all integers in the algebraic number field K discussed in §§ 11—12. Let $\Omega^{(h)}$ and $\Omega_\mathfrak{p}$ have the same meaning as before; if $\mathfrak{a}$ is an ideal in J, denote by $W_\mathfrak{a}(a)$ the pseudo-valuation defined by

$$W_\mathfrak{a}(a) = \begin{cases} 0 \text{ if } a \equiv 0 \pmod{\mathfrak{a}}, \\ 1 \text{ otherwise.} \end{cases}$$

It has been proved [7]) that every pseudo-valuation of J not equivalent to U or W_0 is equivalent to the sum of a finite number of

[7]) Cf. K. Mahler, Über Pseudobewertungen III, Acta Mathematica 67 (1936) 283—328.

absolute values $\Omega^{(h)}$, a finite number of $\mathfrak{p}$-adic valuations $\Omega_{\mathfrak{p}}$, and a residue-class pseudo-valuation $W_{\mathfrak{a}}$.

When $r_1 + r_2 = 1$, J is the ring of rational integers or the ring of integers of an imaginary quadratic field; in these cases there is only one absolute value, which is moreover equivalent to W_0. We omit this case in the discussion which follows; thus in Tables 5—8 we suppose that $r_1 + r_2 \geqslant 2$. As before, h and k are two distinct integers $1, 2, \ldots, r_1 + r_2$, and $\mathfrak{p}$ and $\mathfrak{q}$ are two distinct prime ideals of J. Further $\mathfrak{a}$ and $\mathfrak{b}$ are any two distinct ideals of J. Write $\mathfrak{p}^r \mid| \mathfrak{a}$ to indicate that $\mathfrak{a}$ is divisible by $\mathfrak{p}^r$, but not by $\mathfrak{p}^{r+1}$. If $\mathfrak{p}^r \mid| \mathfrak{a}$, the pseudo-valuation $V_{\mathfrak{p}}^{(\mathfrak{a})}(a)$ is defined by

$$V_{\mathfrak{p}}^{(\mathfrak{a})}(a) = \begin{cases} 0 \text{ if } a \equiv 0 \pmod{\mathfrak{p}^r}, \\ \Omega_{\mathfrak{p}}(a) \text{ otherwise.} \end{cases}$$

Clearly $V_{\mathfrak{p}}^{(\mathfrak{a})}$ is equivalent to $W_{\mathfrak{p}^r}$, unless $\mathfrak{a} = 0$.

We observe that W_0 and U are the limiting cases of the residue-class pseudo-valuation $W_{\mathfrak{a}}$ with $\mathfrak{a} = 0$ or J respectively. Therefore the proofs given for $W_{\mathfrak{a}}$ will hold for W_0 and U if correctly interpreted.

Table 5

$W_1 . W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
U	U	U	U	U	U	U	U	U
W_0	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega^{(h)}$	U	U	U	U	U	U	U	U
$\Omega^{(k)}$	U	U	U	U	U	U	U	U
$\Omega_{\mathfrak{p}}$	U	$\Omega_{\mathfrak{p}}$	U	U	$\Omega_{\mathfrak{p}}$	U	$V_{\mathfrak{p}}^{(\mathfrak{a})}$	$V_{\mathfrak{p}}^{(\mathfrak{b})}$
$\Omega_{\mathfrak{q}}$	U	$\Omega_{\mathfrak{q}}$	U	U	U	$\Omega_{\mathfrak{q}}$	$V_{\mathfrak{q}}^{(\mathfrak{a})}$	$V_{\mathfrak{q}}^{(\mathfrak{b})}$
$W_{\mathfrak{a}}$	U	$W_{\mathfrak{a}}$	U	U	$V_{\mathfrak{p}}^{(\mathfrak{a})}$	$V_{\mathfrak{q}}^{(\mathfrak{a})}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{a}+\mathfrak{b}}$
$W_{\mathfrak{b}}$	U	$W_{\mathfrak{b}}$	U	U	$V_{\mathfrak{p}}^{(\mathfrak{b})}$	$V_{\mathfrak{q}}^{(\mathfrak{b})}$	$W_{\mathfrak{a}+\mathfrak{b}}$	$W_{\mathfrak{b}}$

Table 5. It is clear that $W_{\mathfrak{a}} . W_{\mathfrak{b}} = W_{\mathfrak{a}+\mathfrak{b}}$ [8]), and this holds even if $\mathfrak{a} = \mathfrak{b}$.

$\Omega^{(h)} . W = U$. Let e be a unit of J such that $\Omega^{(h)}(e) < 1$ and let $\omega_1, \ldots, \omega_n$ be a basis of J over the ring of rational integers. Then $e^{-m} = \Sigma\, \xi_i \omega_i$, with rational integral ξ_i, can be written as a sum of a finite number of terms $\pm\, \omega_i$; hence $1 = e^m . e^{-m}$ is a sum of terms $e^m(\pm\, \omega_i)$, and

$$\Omega^{(h)} . W\,(1) \leqslant \max_i \{\Omega^{(h)}(e^m) W(\pm\, \omega_i)\} < 1$$

for sufficiently large m. Now the result follows by Lemma 1.1.

$\Omega_{\mathfrak{p}} . \Omega_{\mathfrak{p}} = \Omega_{\mathfrak{p}}$ follows as for Table 1.

$\Omega_{\mathfrak{p}} . \Omega_{\mathfrak{q}} = U$. There exist elements ξ and η in J such that

$$\xi + \eta = 1,\ \xi \equiv 0 \pmod{\mathfrak{p}},\ \eta \equiv 0 \pmod{\mathfrak{q}}. \tag{4}$$

Hence

$$\Omega_{\mathfrak{p}} . \Omega_{\mathfrak{q}}\,(1) \leqslant \max\{c_{\mathfrak{p}}, c_{\mathfrak{q}}\} < 1.$$

$\Omega_{\mathfrak{p}} . W_{\mathfrak{a}} = V_{\mathfrak{p}}^{(\mathfrak{a})}$. Let $\mathfrak{p}^r \mid| \mathfrak{a}$ and write $\mathfrak{a} = \mathfrak{p}^r \mathfrak{c}$. Then $\mathfrak{p}^m$ and $\mathfrak{q}^m$ are relatively prime for each m, so there exist π and γ in J such that

$$\pi + \gamma = 1,\ \pi \equiv 0 \pmod{\mathfrak{p}^m},\ \gamma \equiv 0 \pmod{\mathfrak{c}}. \tag{5}$$

If $\mathfrak{p}^r | a$, then, since $a = a\pi + a\gamma$,

$$\Omega_{\mathfrak{p}} . W_{\mathfrak{a}}(a) \leqslant \max\{c_{\mathfrak{p}}^m,\ W_{\mathfrak{a}}(a\gamma)\} \to 0 \text{ as } m \to \infty.$$

On the other hand, if $\mathfrak{p} \nmid a$, then $x_i y_i \not\equiv 0 \pmod{\mathfrak{p}^r \mathfrak{c}}$ for at least one term of any decomposition (3) of a in the ring J, and so

$$\Omega_{\mathfrak{p}}(x_i) W_{\mathfrak{a}}(y_i) = \Omega_{\mathfrak{p}}(x_i) \geqslant \Omega_{\mathfrak{p}}(a).$$

Evidently equality holds for the decomposition $a = 1.a$. In the limiting cases $\mathfrak{a} = 0$, J the relation reduces to $\Omega_{\mathfrak{p}} . W_0 = \Omega_{\mathfrak{p}}$, $\Omega_{\mathfrak{p}} . U = U$, respectively.

Table 6

$W_1 \odot W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
U	U	U	U	U	U	U	U	U
W_0	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega^{(h)}$	U	U	$\Omega^{(h)}$	U	U	U	U	U
$\Omega^{(k)}$	U	U	U	$\Omega^{(k)}$	U	U	U	U
$\Omega_{\mathfrak{p}}$	U	$\Omega_{\mathfrak{p}}$	U	U	$\Omega_{\mathfrak{p}}$	U	$V_{\mathfrak{p}}^{(\mathfrak{a})}$	$V_{\mathfrak{p}}^{(\mathfrak{b})}$
$\Omega_{\mathfrak{q}}$	U	$\Omega_{\mathfrak{q}}$	U	U	U	$\Omega_{\mathfrak{q}}$	$V_{\mathfrak{q}}^{(\mathfrak{a})}$	$V_{\mathfrak{q}}^{(\mathfrak{b})}$
$W_{\mathfrak{a}}$	U	$W_{\mathfrak{a}}$	U	U	$V_{\mathfrak{p}}^{(\mathfrak{a})}$	$V_{\mathfrak{q}}^{(\mathfrak{a})}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{a}+\mathfrak{b}}$
$W_{\mathfrak{b}}$	U	$W_{\mathfrak{b}}$	U	U	$V_{\mathfrak{p}}^{(\mathfrak{b})}$	$V_{\mathfrak{q}}^{(\mathfrak{b})}$	$W_{\mathfrak{a}+\mathfrak{b}}$	$W_{\mathfrak{b}}$

Table 6. The proofs are as for Table 5 with the following exceptions:

$\Omega^{(h)} \odot \Omega^{(h)} = \Omega^{(h)}$ follows as in Table 2.

[8]) Cf. also § 21.

$\Omega^{(h)} \odot \Omega^{(k)} = U$. There is a unit e of J such that $\Omega^{(h)}(e) < 1$, but $\Omega^{(k)}(e) > 1$; then the result follows with the help of Lemma 1.1.

$\Omega^{(h)} \odot W = U$ $(W = \Omega_{\mathfrak{p}}$ or $W_{\mathfrak{a}})$. Take a unit e such that $\Omega^{(h)}(e) < 1$ and write $e^{-m} = \sum \xi_i \omega_i$, where ω_i is a basis of J and the ξ_i are rational integers. Then, since W is non-archimedean,

$$\begin{aligned}\Omega^{(h)} \odot W(1) &\leqslant \textstyle\sum_i \Omega^{(h)}(e^m) W(\xi_i \omega_i)\\ &\leqslant \Omega^{(h)}(e)^m \,.\, (\textstyle\sum_i W(\omega_i))\\ &< 1\end{aligned}$$

for sufficiently large m.

For Table 7 we define $\Sigma_{\mathfrak{p},\mathfrak{q}}$ as $\max(\Omega_{\mathfrak{p}}, \Omega_{\mathfrak{q}})$ and put

$$W_{\mathfrak{p}}^{(\mathfrak{a})}(a) = \begin{cases} c_{\mathfrak{p}}^{-r}\ \Omega_{\mathfrak{p}}(a) & \text{if } a \equiv 0 \pmod{\mathfrak{a}},\\ 1 & \text{otherwise,}\end{cases}$$

where r is determined by $\mathfrak{p}^r \,||\, \mathfrak{a}$. By e we understand again a unit of J such that $\Omega^{(h)}(e) < 1$, $\Omega^{(k)}(e) > 1$.

Table 7

$W_1 \times W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
U	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
W_0	W_0	W_0	W_0	W_0	W_0	W_0	W_0	W_0
$\Omega^{(h)}$	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega^{(k)}$	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}}$	W_0	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}}$	$V_{\mathfrak{p}}$	$\Sigma_{\mathfrak{p},\mathfrak{q}}$	$W_{\mathfrak{p}}^{(\mathfrak{a})}$	$W_{\mathfrak{p}}^{(\mathfrak{b})}$
$\Omega_{\mathfrak{q}}$	$\Omega_{\mathfrak{q}}$	W_0	$\Omega_{\mathfrak{q}}$	$\Omega_{\mathfrak{q}}$	$\Sigma_{\mathfrak{p},\mathfrak{q}}$	$V_{\mathfrak{q}}$	$W_{\mathfrak{q}}^{(\mathfrak{a})}$	$W_{\mathfrak{q}}^{(\mathfrak{b})}$
$W_{\mathfrak{a}}$	$W_{\mathfrak{a}}$	W_0	$W_{\mathfrak{a}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{p}}^{(\mathfrak{a})}$	$W_{\mathfrak{q}}^{(\mathfrak{a})}$	$W_{\mathfrak{a}^2}$	$W_{\mathfrak{a}\mathfrak{b}}$
$W_{\mathfrak{b}}$	$W_{\mathfrak{b}}$	W_0	$W_{\mathfrak{b}}$	$W_{\mathfrak{b}}$	$W_{\mathfrak{p}}^{(\mathfrak{b})}$	$W_{\mathfrak{q}}^{(\mathfrak{b})}$	$W_{\mathfrak{a}\mathfrak{b}}$	$W_{\mathfrak{b}^2}$

Table 7. $\Omega^{(h)} \times W_{\mathfrak{a}} = W_{\mathfrak{a}}$. If $a \equiv 0 \pmod{\mathfrak{a}}$, then, since $a = e^m . e^{-m}a$,

$$\Omega^{(h)} \times W_{\mathfrak{a}}(a) \leqslant \max\{\Omega^{(h)}(e^m),\ W_{\mathfrak{a}}(e^{-m}a)\} \to 0,$$

as $m \to \infty$. If $a \not\equiv 0 \pmod{\mathfrak{a}}$, then, in every decomposition (3) of a (in J), $x_i y_i \not\equiv 0 \pmod{\mathfrak{a}}$ for some i, hence

$$\max\{\Omega^{(h)}(x_i),\ W_{\mathfrak{a}}(y_i)\} \geqslant 1;$$

and the lower bound 1 is attained for the decomposition

$$a = e^m . e^{-m}a.$$

$\Omega^{(h)} \times \Omega_{\mathfrak{p}} = \Omega_{\mathfrak{p}}$. For every decomposition (3) of a (in J) $\Omega_{\mathfrak{p}}(a) \leqslant \max_i \Omega_{\mathfrak{p}}(x_i)$, hence

$$\Omega_{\mathfrak{p}}(a) \leqslant \Omega^{(h)} \times \Omega_{\mathfrak{p}}(a).$$

Conversely, since $a = e^m . e^{-m}a$,

$$\begin{aligned}\Omega^{(h)} \times \Omega_{\mathfrak{p}}(a) &\leqslant \max\{\Omega^{(h)}(e^m),\ \Omega_{\mathfrak{p}}(e^{-m}a)\}\\ &\leqslant \Omega_{\mathfrak{p}}(e^{-m}a) = \Omega_{\mathfrak{p}}(a),\end{aligned}$$

when m is sufficiently large.

$\Omega^{(h)} \times \Omega^{(h)} = U$. Write $1 = e^m . e^m e^{-2m}$ and express e^{-2m} as a sum of terms $\pm\, \omega_i$.

$\Omega^{(h)} \times \Omega^{(k)} = U$. $\Omega^{(h)} \times \Omega^{(k)}(1) \leqslant \max\{\Omega^{(h)}(e^m),\ \Omega^{(k)}(e^{-m})\} < 1$ $(m \geqslant 1)$.

$\Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{p}} = V_{\mathfrak{p}}$. As for Table 3.

$\Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{q}} = \Sigma_{\mathfrak{p},\mathfrak{q}}$. For any decomposition (3), $\Omega_{\mathfrak{p}}(a) \leqslant \max_i \Omega_{\mathfrak{p}}(x_i)$, $\Omega_{\mathfrak{q}}(a) \leqslant \max \Omega_{\mathfrak{q}}(y_i)$, hence

$$\Sigma_{\mathfrak{p},\mathfrak{q}} = \max\{\Omega_{\mathfrak{p}}(a),\ \Omega_{\mathfrak{q}}(a)\} \leqslant \max_i\{\Omega_{\mathfrak{p}}(x_i),\ \Omega_{\mathfrak{q}}(y_i)\},$$

and so $\Sigma_{\mathfrak{p},\mathfrak{q}} \leqslant \Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{q}}$. For the converse, let m be any positive integer and choose ξ and η in J such that $\xi + \eta = 1$, $\xi \equiv 0 \pmod{\mathfrak{p}^m}$, $\eta \equiv 0 \pmod{\mathfrak{q}^m}$.

Then

$$\begin{aligned}\Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{q}}(a) &\leqslant \max\{\Omega_{\mathfrak{p}}(\xi), \Omega_{\mathfrak{q}}(a), \Omega_{\mathfrak{q}}(\eta), \Omega_{\mathfrak{p}}(a)\}\\ &\leqslant \max\{\Omega_{\mathfrak{p}}(a), \Omega_{\mathfrak{q}}(a)\},\end{aligned}$$

when m is sufficiently large.

$\Omega_{\mathfrak{p}} \times W_{\mathfrak{a}} = W_{\mathfrak{p}}^{(\mathfrak{a})}$. Clearly $\Omega_{\mathfrak{p}} \times W_{\mathfrak{a}}(a) = 1$, if $a \not\equiv 0 \pmod{\mathfrak{a}}$, so suppose $a \equiv 0 \pmod{\mathfrak{a}}$. Let $\mathfrak{p}^r \,||\, \mathfrak{a}$ and $\mathfrak{p}^s \,||\, a$ and put $\mathfrak{a} = \mathfrak{p}^r\mathfrak{c}$, so that $s \geqslant r$ and $\mathfrak{p} \nmid \mathfrak{c}$. It suffices to consider decompositions (3) of a in which all $y_i \equiv 0 \pmod{\mathfrak{a}}$. Then at least one x_i satisfies $\mathfrak{p}^{s-r+1} \nmid x_i$, whence

$$\Omega_{\mathfrak{p}} \times W_{\mathfrak{a}}(a) \geqslant c_{\mathfrak{p}}^{-r}\, \Omega_{\mathfrak{p}}(a).$$

To show that equality is attained we construct a special decomposition of a as follows. Let $u_1, \ldots, u_n$ be a basis of $\mathfrak{p}^r$ and $v_1, \ldots, v_n$ a basis of $\mathfrak{p}^{s-r}$. Then a can be written as

$$a = \textstyle\sum_{i=1}^n \sum_{j=1}^n \xi_{ij}\, u_i v_j,$$

where the coefficients ξ_{ij} are rational integers. Now for any positive integer m determine π and γ in J such that $\pi + \gamma = 1$, $\pi \equiv 0 \pmod{\mathfrak{p}^m}$, $\gamma \equiv 0 \pmod{\mathfrak{c}}$.

Then

$$a = \pi . a + \gamma . a = \pi . a + \textstyle\sum_{ij} \xi_{ij} v_j . \gamma u_i.$$

By hypothesis $W_{\mathfrak{a}}(a)$ and $W_{\mathfrak{a}}(\gamma u_i)$ vanish, $\Omega_{\mathfrak{p}}(\Sigma_j \xi_{ij} v_j) \leqslant c^{-r}\Omega_{\mathfrak{p}}(a)$ and $\Omega_{\mathfrak{p}}(\pi) \to 0$ as $m \to \infty$; hence $\Omega_{\mathfrak{p}} \times W_{\mathfrak{a}}(a) \leqslant c_{\mathfrak{p}}^{-r}\Omega_{\mathfrak{p}}(a)$. We note again the limiting cases: $\Omega_{\mathfrak{p}} \times W_0 = W_0$, $\Omega_{\mathfrak{p}} \times U = \Omega_{\mathfrak{p}}$.
$W_{\mathfrak{a}} \times W_{\mathfrak{b}} = W_{\mathfrak{ab}}$. Clear (Cf. § 21).

For Table 8 we define $W_{\mathfrak{a},\mathfrak{b}}$ as $W_{\mathfrak{ab}} + \min(W_{\mathfrak{a}}, W_{\mathfrak{b}})$; further we abbreviate $\Omega^{(h)} \otimes \Omega^{(h)}$ by $\Phi^{(h)}$. This function is specified in greater detail in the discussion following the Table.

Table 8.

$W_1 \otimes W_2$	U	W_0	$\Omega^{(h)}$	$\Omega^{(k)}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
U	U	W_0	U	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
W_0	W_0	$2W_0$	W_0	W_0	$W_0 + \Omega_{\mathfrak{p}}$	$W_0 + \Omega_{\mathfrak{q}}$	$W_0 + W_{\mathfrak{a}}$	$W_0 + W_{\mathfrak{b}}$
$\Omega^{(h)}$	U	W_0	$\Phi^{(h)}$	U	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega^{(k)}$	U	W_0	U	$\Phi^{(k)}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{q}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{b}}$
$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}}$	$W_0 + \Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}}$	$V^*_{\mathfrak{p}}$	$\Omega_{\mathfrak{p}} + \Omega_{\mathfrak{q}}$	$W^{(\mathfrak{a})}_{\mathfrak{p}} + W_{\mathfrak{a}}$	$W^{(\mathfrak{b})}_{\mathfrak{p}} + W_{\mathfrak{b}}$
$\Omega_{\mathfrak{a}}$	$\Omega_{\mathfrak{q}}$	$W_0 + \Omega_{\mathfrak{q}}$	$\Omega_{\mathfrak{q}}$	$\Omega_{\mathfrak{q}}$	$\Omega_{\mathfrak{p}} + \Omega_{\mathfrak{q}}$	$V^*_{\mathfrak{q}}$	$W^{(\mathfrak{a})}_{\mathfrak{q}} + W_{\mathfrak{a}}$	$W^{(\mathfrak{b})}_{\mathfrak{q}} + W_{\mathfrak{b}}$
$W_{\mathfrak{a}}$	$W_{\mathfrak{a}}$	$W_0 + W_{\mathfrak{a}}$	$W_{\mathfrak{a}}$	$W_{\mathfrak{a}}$	$W^{(\mathfrak{a})}_{\mathfrak{p}} + W_{\mathfrak{a}}$	$W^{(\mathfrak{a})}_{\mathfrak{q}} + W_{\mathfrak{a}}$	$W_{\mathfrak{a},\mathfrak{a}}$	$W_{\mathfrak{a},\mathfrak{b}}$
$W_{\mathfrak{b}}$	$W_{\mathfrak{b}}$	$W_0 + W_{\mathfrak{b}}$	$W_{\mathfrak{b}}$	$W_{\mathfrak{b}}$	$W^{(\mathfrak{b})}_{\mathfrak{p}} + W_{\mathfrak{b}}$	$W^{(\mathfrak{b})}_{\mathfrak{q}} + W_{\mathfrak{b}}$	$W_{\mathfrak{a},\mathfrak{b}}$	$W_{\mathfrak{b},\mathfrak{b}}$

Table 8. The proofs are as for Table 7 with the following exceptions:
$\Omega^{(h)} \otimes \Omega^{(h)} = \Phi^{(h)}$. Taking this as the definition of $\Phi^{(h)}$, we find as for Table 4 that

$$\Phi^{(h)} \geqslant 2\Omega^{(h)1/2}.$$

Let $\Omega^{(h)}(e) = \alpha < 1$. Then there exists an integer m such that

$$\alpha < \frac{\Omega^{(h)}(e^m a)}{\Omega^{(h)}(e^{-m})} \leqslant \frac{1}{\alpha}.$$

It follows from the decomposition $a = e^m a . e^{-m}$ that

$$\Phi^{(h)}(a) \leqslant \Omega^{(h)}(e^m a) + \Omega^{(h)}(e^{-m}) \leqslant \frac{2}{\alpha^{1/2}}\, \Omega^{(h)}(a)^{1/2},$$

and so

$$2\Omega^{(h)1/2} \leqslant \Phi^{(h)} \leqslant \frac{2}{\alpha^{1/2}}\Omega^{(h)1/2}.$$

Thus the pseudo-valuation $\Phi^{(h)}$ is equivalent to $\Omega^{(h)}$ [9]).

[9]) In the terminology of § 14 we can say that $\Phi^{(h)}$ is strongly equivalent to $\Omega^{(h)1/2}$. We do not attempt a more precise determination of $\Phi^{(h)}$, as this is immaterial to our purpose, but we remark that in the special case when there exists a unit e such that $\Omega^{(h)}(e)$ is arbitrarily near to 1 without being equal to 1, this proof shows that $\Phi^{(h)}(a) = 2\Omega^{(h)}(a)^{1/2}$.

$\Omega_{\mathfrak{p}} \otimes \Omega_{\mathfrak{p}} = V^*_{\mathfrak{p}}$. As for Table 4.
$\Omega_{\mathfrak{p}} \otimes W_{\mathfrak{a}} = W^{(\mathfrak{a})}_{\mathfrak{p}} + W_{\mathfrak{a}}$. Suppose first that $a \not\equiv 0 \pmod{\mathfrak{a}}$. Then in every decomposition (3) of a $\Omega_{\mathfrak{p}}(x_i) \geqslant \Omega_{\mathfrak{p}}(a)$ for at least one suffix i, and $W_{\mathfrak{a}}(y_j) = 1$ for at least one suffix j, hence $\Omega_{\mathfrak{p}} \otimes W_{\mathfrak{a}}(a) \geqslant \Omega_{\mathfrak{p}}(a) + 1$, and equality may hold, as the decomposition $a = a.1$ shows.

Now let $\mathfrak{a} \mid a$. In every decomposition (3) of a either $y_i \neq 0 \pmod{\mathfrak{a}}$ for at least one i, and so $W_{\mathfrak{a}}(y_i) = 1$, whence

$$\Omega_{\mathfrak{p}} \otimes W_{\mathfrak{a}}(a) \geqslant 1 \geqslant c_{\mathfrak{p}}^{-r}\, \Omega_{\mathfrak{p}}(a);$$

or $y_i \equiv 0 \pmod{\mathfrak{a}}$ for all i, and then at least one x_i satisfies the inequality $\Omega_{\mathfrak{p}}(x_i) \geqslant c_{\mathfrak{p}}^{-r}\Omega_{\mathfrak{p}}(a)$, so that in any case

$$\Omega_{\mathfrak{p}} \otimes W_{\mathfrak{a}}(a) \geqslant W^{(\mathfrak{a})}_{\mathfrak{p}}(a) + W_{\mathfrak{a}}(a). \tag{6}$$

To show that equality holds in (6), we construct again a special decomposition of a. Let $\mathfrak{p}^r \mid\mid \mathfrak{a}$, and denote by p the positive rational prime divisible by $\mathfrak{p}$. Let $\alpha \neq 0$ be an arbitrary element of $\mathfrak{a}$, and m any positive integer. Then there exists a second element β of $\mathfrak{a}$ such that

$$\mathfrak{a} = (\alpha\, p^m, \beta). \tag{7}$$

Since $a \in \mathfrak{a}$, there exist μ and ν in J such that

$$a = \mu\, p^m . \alpha + \nu . \beta. \tag{8}$$

It follows from this decomposition of a that

$$\Omega_{\mathfrak{p}} \otimes W_{\mathfrak{a}}(a) \leqslant \Omega_{\mathfrak{p}}(\mu\, p^m) + W_{\mathfrak{a}}(\alpha) + \Omega_{\mathfrak{p}}(\nu) + W_{\mathfrak{a}}(\beta). \tag{9}$$

By (7) $\mathfrak{p}^r \mid\mid \beta$ and by (8) $\Omega_{\mathfrak{p}}(a) = \Omega_{\mathfrak{p}}(\nu\beta)$ for all sufficiently large m, hence $\Omega_{\mathfrak{p}}(\nu) = c_{\mathfrak{p}}^{-r}\Omega_{\mathfrak{p}}(a)$. Inserting this in (9) and letting m tend to infinity we obtain the required result.
$W_{\mathfrak{a}} \otimes W_{\mathfrak{b}} = W_{\mathfrak{a},\mathfrak{b}}$. Clear.

IV.

14. Our next object is to establish a connexion between the binary operations defined in Ch. II and the unary operations defined in Ch. I. Whereas the formal laws proved in Ch. II (commutativity, associativity) took the form of equalities, we cannot hope to establish equalities now, but only some form of equivalence, since there is a certain amount of arbitrariness in the definitions.

We shall say that two admissible functions φ_1, φ_2 are *equivalent*, $\varphi_1 \sim \varphi_2$, if, for any sequence a_n in R, $\varphi_1(a_n) \to 0$ if and only if $\varphi_2(a_n) \to 0$. The functions will certainly be equivalent if there exists a constant k such that

$$\varphi_1 \leqslant k\varphi_2 \text{ and } \varphi_2 \leqslant k\varphi_1.$$

In this case we say that φ_1 and φ_2 are *strongly equivalent* and write $\varphi_1 \approx \varphi_2$. We use the signs $\nsim$ and $\not\approx$ to indicate the negation of $\sim$ and $\approx$ respectively.

The concept of equivalence is of greater theoretical importance than that of strong equivalence, because it determines the largest class of pseudo-valuations which define the same topology on R [10]. However, we shall often use the notion of strong equivalence, as it is easier to handle. This is largely due to the fact (which is easily verified) that a linear operation is always $\approx$-invariant.

15. The relations $\sim$ and $\approx$ are evidently reflexive, symmetric and transitive; thus $\sim$ defines a partition of the set of all admissible functions on R into equivalence-classes, the $\sim$-classes, and each such class is further subdivided into $\approx$-classes. We shall say that an operation $*: \varphi \to \varphi^*$ on admissible functions is $\sim$-invariant, if

$$\varphi_1 \sim \varphi_2 \text{ implies } \varphi_1^* \sim \varphi_2^*.$$

The operation $*$ is called *$\approx$-invariant*, if

$$\varphi_1 \approx \varphi_2 \text{ implies } \varphi_1^* \approx \varphi_2^*.$$

A binary operation is called *$\sim$-invariant* (*$\approx$-invariant*), if it is $\sim$-invariant ($\approx$-invariant) with respect to each argument. Thus a $\sim$-invariant operation is one which can be defined in a natural way on the $\sim$-classes, and similarly for $\approx$-invariance.

We note that of the two properties, $\sim$-invariance and $\approx$-invariance, neither implies the other. Thus the operation which associates with each pseudo-valuation W the pseudo-valuation V defined by

$$V(a) = [W(a)]^{1/2},$$

is $\sim$-invariant but not always $\approx$-invariant. On the other hand, the non-archimedean product on a field is $\approx$-invariant but not always $\sim$-invariant, as we shall see later (§ 19).

We have the obvious

LEMMA 15.1. *The operations* $+$ *and* $\oplus$ *are* $\approx$*-invariant.* This is clear, since both operations are linear.

On the other hand, the operation $\times$ defined in § 4. is not $\approx$-invariant,

[10]) Cf. P.I.8. An admissible function on R does not in general define a topology compatible with the ring structure of R: e.g. consider the function φ on the ring of rational integers defined by $\varphi(0) = 0$, $\varphi(n) = |1/n|$ for $n \neq 0$. For this reason the notion of equivalence will chiefly be used for pseudo-valuations.

nor are the combined operations $\times +$ or $\times \oplus$. For, consider any proper non-archimedean pseudo-valuation W, and define φ by

$$\varphi(a) = \frac{1}{2W(1)} W(a).$$

Then φ is admissible and non-archimedean, and $\varphi \approx W$, but $\varphi^\times = U$ by Lemma 1.1, because $\varphi^\times(1) \leqslant \varphi(1) = \frac{1}{2}$. Hence $\varphi^\times \nsim W$ and a fortiori $\varphi^\times \not\approx W$. Since $\varphi^\times$ and W are non-archimedean pseudo-valuations, it follows also that $\times +$ and $\times \oplus$ are not $\approx$-invariant. The argument shows further that $\times$ is not $\sim$-invariant.

16. Let W_1 and $W_2 \in \Omega_R$. We shall prove in this § that the greatest non-archimedean (subadditive) pseudo-valuation majorised by W_1 and W_2 is strongly equivalent to the non-archimedean (subadditive) product of W_1 and W_2. We simply follow the construction of §§ 4—6.

LEMMA 16.1. *Let W_1 and W_2 be any two submultiplicative admissible functions and put*

$$\varphi(a) = \min(W_1(a), W_2(a)). \tag{1}$$

Then

$$\varphi^\times(a) \approx \inf_{xy=a} W_1(x)W_2(y).$$

Proof. Write

$$\Psi(a) = \inf_{xy=a} W_1(x)W_2(y); \tag{2}$$

we have to show that $\varphi^\times \approx \Psi$. From the factorisations $a = a.1 = 1.a$ we see that $\Psi(a) \leqslant W_1(a)W_2(1)$ and $\Psi(a) \leqslant W_1(1)W_2(a)$, hence $\Psi(a) \leqslant k\varphi(a)$, where $k = \max(W_1(1), W_2(1))$. Therefore, if we can prove that Ψ is submultiplicative it will follow that

$$\Psi(a) \leqslant k\varphi^\times(a).$$

Now

$$\Psi(ab) = \inf_{xy=ab} W_1(x)W_2(y).$$

Hence, for any factorisations $a = x'x''$, $b = y'y''$ of a and b we have, since $x'y'x''y'' = ab$,

$$\begin{aligned}\Psi(ab) &\leqslant W_1(x'y')W_2(x''y'') \\ &\leqslant W_1(x')W_1(y')W_2(x'')W_2(y'').\end{aligned}$$

Taking the lower bound with respect to all such factorisations, we obtain

$$\begin{aligned}\Psi(ab) &\leqslant \inf_{x'x''=a}\{W_1(x')W_2(x'')\} \inf_{y'y''=b}\{W_1(y')W_2(y'')\} \\ &= \Psi(a)\,\Psi(b).\end{aligned}$$

Hence Ψ is submultiplicative, and it follows that $\Psi \leqslant k\varphi^{\times}$.

Conversely, given any factorisation $a = xy$,

$$\begin{aligned}\varphi^{\times}(a) = \varphi^{\times}(xy) &\leqslant \varphi^{\times}(x)\varphi^{\times}(y)\\ &\leqslant \varphi(x)\varphi(y)\\ &\leqslant W_1(x)W_2(y);\end{aligned}$$

hence $\varphi^{\times}(a) \leqslant \inf_{xy=a} W_1(x)W_2(y) = \Psi(a)$. Thus $\varphi^{\times} \leqslant \Psi \leqslant k\varphi^{\times}$ and the lemma follows.

LEMMA 16.2. *If* $W_1, W_2 \in \Omega_R$ *and* $\Psi(a) = \inf_{xy=a} W_1(x)W_2(y)$,

$$\Psi^{+} = W_1 . W_2, \quad \Psi^{\oplus} = W_1 \odot W_2.$$

COR. 1. $W_1 . W_2 \leqslant W_1 \odot W_2$.

COR 2. *If* $W_1 \odot W_2$ *is non-archimedean, then* $W_1 \odot W_2 = W_1 . W_2$.

Proof. By definition

$$\begin{aligned}\Psi^{+}(a) &= \inf_{\Sigma z_i=a} \max_i \Psi(\pm z_i)\\ &= \inf_{\Sigma z_i=a} \max_i \inf_{x_iy_i=\pm z_i} W_1(x_i)W_2(y_i)\\ &= \inf_{\Sigma z_i=a} \max_i \inf_{x_iy_i=z_i} W_1(\pm x_i)W_2(y_i),\end{aligned}$$

or

$$\Psi^{+}(a) = \inf_{\Sigma z_i=a} \max_i \inf_{x_iy_i=z_i} W_1(x_i)W_2(y_i), \tag{3}$$

since W_1 is a pseudo-valuation. To complete the proof we use the following lemma on real functions in R.

LEMMA 16.3. *If* $z_1, \ldots, z_n$ *are fixed elements of* R *and* F *is a real-valued function of two variables in* R, *then*

$$\max_{i=1,\ldots,n} \inf_{x_iy_i=z_i} F(x_i, y_i) = \inf^{x_iy_i=z_i}_{i=1,\ldots,n} \max_{j=1,\ldots,n} F(x_j, y_j). \tag{4}$$

For when we consider a fixed i, then

$$\inf_{x_iy_i=z_i} F(x_i, y_i) \leqslant \inf_{x_iy_i=z_i} \max_j F(x_j, y_j),$$

and hence

$$\max_i \inf_{x_iy_i=z_i} F(x_i, y_i) \leqslant \inf^{x_iy_i=z_i}_{i=1,\ldots,n} \max_j F(x_j, y_j).$$

To obtain the reverse inequality, denote the value of the left-hand side by K. Given $\varepsilon > 0$, we can choose x_i, y_i $(i = 1, \ldots, n)$ such that $F(x_i, y_i) < K + \varepsilon$ $(i = 1, \ldots, n)$, and this shows that

$$\inf_{x_iy_i=a} \max_j F(x_j, y_j) < K + \varepsilon.$$

Letting ε tend to zero we obtain (4), and this proves the lemma.

We apply this lemma to the decomposition (3) with $F(x, y) = W_1(x)W_2(y)$ and get

$$\Psi^{+}(a) = \inf_{\Sigma x_iy_i=a} \max_i W_1(x_i)W_2(y_i) = W_1 . W_2(a).$$

Similarly we have, by definition,

$$\begin{aligned}\Psi^{\oplus}(a) &= \inf_{\Sigma z_i=a} \Sigma_i \Psi(\pm z_i)\\ &= \inf_{\Sigma z_i=a} \Sigma_i \inf_{x_iy_i=z_i} W_1(x_i)W_2(y_i),\end{aligned}$$

whence

$$\Psi^{\oplus}(a) = \inf_{\Sigma x_iy_i=a} \Sigma_i W_1(x_i)W_2(y_i) = W_1 \odot W_2(a),$$

and this completes the proof of Lemma 16.2. The two corollaries now follow from the properties of $+$ and $\oplus$ (§ 5).

Since $\Psi \approx \varphi^{\times}$, where $\varphi(a) = \min(W_1(a), W_2(a))$, it follows from Lemma 15.1 that $\varphi^{\times +} \approx W_1 . W_2$ and $\varphi^{\times\oplus} \approx W_1 \odot W_2$. By Theorem 6.1 we can express this as

THEOREM 16.4. *If* $W_1, W_2 \in \Omega_R$, *then the greatest pseudo-valuation (non-archimedean pseudo-valuation) majorised by* $\min(W_1, W_2)$ *is strongly equivalent to the subadditive product* $W_1 \odot W_2$ *(resp. the non-archimedean product* $W_1 . W_2$*).*

If $W_1(1), W_2(1) \leqslant 1$, then the strong equivalence in Theorem 16.4 can be replaced by equality and in this form the Theorem provides an alternative proof of the entries in the Tables 1, 2, 5 and 6 of Ch. III.

17. There is no analogous interpretation of the compound of two pseudo-valuations, but we can now define a function

$$\Psi(a) = \inf_{xy=a} \max\{W_1(x), W_2(y)\} \tag{5}$$

and show that

$$\Psi^{+} = W_1 \times W_2, \tag{6}$$

$$\Psi^{\oplus} \approx W_1 \otimes W_2. \tag{7}$$

By defining Ψ as $\inf(W_1(x) + W_2(y))$ instead of using (5) we can strengthen (7) to an equality but only at the cost of weakening (6) to an equivalence. We shall not go into the proof which is very similar to the case of the product given in the previous §, but only mention the results which here correspond to the corollaries of Lemma 16.2:

THEOREM 17.1 $W_1 \times W_2 \leqslant W_1 \otimes W_2$.

THEOREM 17.2. *If* $W_1 \otimes W_2$ *is non-archimedean, then* $W_1 \otimes W_2 \approx$ $\approx W_1 \times W_2$.

18. We now come to the question of the invariance of the different operations. For the sake of a later application we prove the invariance properties not with respect to equivalence, but with

respect to the quasi-ordering which defines the equivalence. For this purpose we recall the definition of P.I. 8:

If $W_1, W_2 \in \Omega_R$, then W_1 is said to be *contained* in W_2: $W_1 \subset W_2$, if to every $\varepsilon_1 > 0$ there corresponds a number ε_2 such that

$$W_1(a) < \varepsilon_1 \text{ holds whenever } W_2(a) < \varepsilon_2. \tag{8}$$

More briefly, the condition (8) states that W_1 is small whenever W_2 is small. Similarly we say that W_1 is *strongly contained* in W_2 : $W_1 \Subset W_2$, if $W_1 \leqslant kW_2$ for some k. It is clear that $W_1 \Subset W_2$ implies $W_1 \subset W_2$. Further $W_1 \sim W_2$ if and only if $W_1 \subset W_2 \subset W_1$; $W_1 \approx W_2$ if and only if $W_1 \Subset W_2 \Subset W_1$.

LEMMA 18.1 *If* $W_i \Subset W_i'$ $(i = 1,2)$, *then*

$$W_1 \bigcirc W_2 \Subset W_1' \bigcirc W_2',$$

where $\bigcirc$ *is* ., $\odot$, $\times$ *or* $\otimes$.

Proof. Let $W_1 \leqslant kW_1'$, $W_2 \leqslant kW_2'$, then it is clear from the definitions that

$$W_1 \bigcirc W_2 \leqslant lW_1' \bigcirc W_2',$$

where $l = \max(k^2, 1)$.

By applying this Lemma first as it stands, and then with W_i and W_i' interchanged $(i = 1,2)$, we obtain

THEOREM 18.2. *The operations* ., $\odot$, $\times$ *and* $\otimes$ *are* $\approx$*-invariant.*

19. The results on $\sim$-invariance are less complete. For the non-archimedean operations they are given by the following theorems.

THEOREM 19.1. *The non-archimedean compound is* $\sim$*-invariant.*

THEOREM 19.2. *The non-archimedean product is* $\sim$*-invariant when applied to bounded pseudo-valuations.*

We prove these Theorems by a Lemma analogous to Lemma 18.1:

LEMMA 19.3. *If* $W_i \subset W_i'$ $(i = 1,2)$, *then* $W_1 \times W_2 \subset W_1' \times W_2'$; *and further if* W_i *and* W_i' *are bounded, then* $W_1 . W_2 \subset W_1' . W_2'$.

Proof. Consider first $\times$. We shall prove: If $W_1 \subset W_1'$, then $W_1 \times W_2 \subset W_1' \times W_2$. From this the first part of the Lemma will follow by the commutativity of $\times$.

Assume that

$$W_1' \times W_2(a) < \alpha \qquad (\alpha > 0); \tag{9}$$

then there is a decomposition $a = x_1y_1 + \ldots + x_ny_n$ of a such that $\max_i\{W_1'(x_i), W_2(y_i)\} < 2\alpha$, or

$$W_1'(x_i) < 2\alpha,\; W_2(y_i) < 2\alpha \quad (i = 1, \ldots, n). \tag{10}$$

Let $\varepsilon > 0$ be fixed. Then there is a $\delta > 0$ such that

$$W_1'(z) < \delta \text{ implies } W_1(z) < \varepsilon \quad \text{where } z \in R. \tag{11}$$

Now choose $\alpha = \frac{1}{2}\min\{\delta, \varepsilon\}$ in (9); then, by (10),

$$W_2(y_i) < 2\alpha \leqslant \varepsilon \text{ and } W_1'(x_i) < 2\alpha \leqslant \delta,$$

so that $W_1(x_i) < \varepsilon$ and therefore

$$W_1 \times W_2(a) \leqslant \max_i\{W_1(x_i),\; W_2(y_i)\} < \varepsilon.$$

Hence $W_1 \times W_2$ is small whenever $W_1' \times W_2$ is small and this proves the first part of the Lemma.

The proof for the non-archimedean product is similar. We suppose now that W_1 and W_2 are bounded, say that

$$W_1(z) < \omega,\; W_2(z) < \omega \qquad \text{for all } z \in R.$$

If

$$W_1' . W_2(a) < \alpha \qquad (\alpha > 0), \tag{12}$$

then there is a decomposition $a = x_1y_1 + \ldots + x_ny_n$ of a such that $\max_i\{W_1'(x_i)W_2(y_i)\} < 2\alpha$. Therefore for each $i = 1, \ldots, n$,

$$\text{either } W_1'(x_i) < \sqrt{(2\alpha)} \text{ or } W_2(y_i) < \sqrt{(2\alpha)}. \tag{13}$$

Let $\delta = \delta(\varepsilon)$ be as in (11) and choose $\alpha = \frac{1}{2}\min\{(\varepsilon/\omega)^2, \delta(\varepsilon/\omega)^2\}$ in (12). Then, by (13), either $W_1'(x_i) < \sqrt{(2\alpha)} < \delta(\varepsilon/\omega)$ and hence $W_1(x_i) < \varepsilon/\omega$, or $W_2(y_i) < \sqrt{(2\alpha)} < \varepsilon/\omega$. In either case $W_1(x_i)W_2(y_i) < \varepsilon/\omega . \omega = \varepsilon$, whence

$$W_1 . W_2(a) \leqslant \max_i\{W_1(x_i)W_2(y_i)\} < \varepsilon.$$

This completes the proof of the Lemma.

We note that in the second part of the Lemma it is enough to assume that W_1 and W_2 are bounded for the conclusion to hold.

Theorems 19.1 and 19.2 are an immediate consequence of the Lemma. The following example shows that the boundedness condition in Theorem 19.2 cannot be omitted.

Let Ω_p be the p-adic valuation on the rational field. Then $\Omega_p^{1/2}$ is again a valuation and is equivalent to Ω_p. By the remark after Theorem 16.4, or by Table 1,

$$\Omega_p . \Omega_p = \Omega_p.$$

On the other hand

$$\Omega_p^{1/2} . \Omega_p = U.$$

For $\Omega_p(p) < \Omega_p^{1/2}(p)$, and so

$$\Omega_p^{1/2}.\Omega_p(1) \leqslant \Omega_p\left(\frac{1}{p}\right)^{1/2}\Omega_p(p)$$

$$= \frac{\Omega_p(p)}{\Omega_p(p)^{1/2}} < 1.$$

Thus $\Omega_p^{1/2}.\Omega_p \not\sim \Omega_p.\Omega_p$, although $\Omega_p^{1/2} \sim \Omega_p$.

Similarly one can show that $\Omega_p^{1/2} \odot \Omega_p = U$, and therefore $\Omega_p^{1/2} \odot \Omega_p \not\sim \Omega_p \odot \Omega_p$. It will be proved later (§ 22) that if only bounded and non-archimedean pseudo-valuations are considered, then the subadditive product is equivalent to the non-archimedean product and is therefore $\sim$-invariant. The example just given shows that the boundedness condition is again essential. The question of the $\sim$-invariance of the subadditive compound remains open.

V.

20. With every ideal $\mathfrak{a}$ of the ring R a pseudo-valuation $W_\mathfrak{a}$ can be associated by defining

$$W_\mathfrak{a}(a) = \begin{cases} 0 \text{ if } a \equiv 0 \pmod{\mathfrak{a}}, \\ 1 \text{ otherwise}^{11)} \end{cases} \tag{1}$$

This is consistent with the notation W_0 for the trivial pseudo-valuation, which can be regarded as the pseudo-valuation associated with the zero-ideal. Similarly the improper pseudo-valuation U is associated with the whole ring.

In P.I. 3 it was shown that the set of elements for which a given pseudo-valuation vanishes is an ideal, and so we have the converse that every pseudo-valuation W which takes only the values 0 and 1 must be the pseudo-valuation associated with an ideal $\mathfrak{a}$; here $\mathfrak{a}$ is uniquely determined as the set where W vanishes.

Let us call a pseudo-valuation of the form (1) *special*. It is clear that two special pseudo-valuations which are equivalent must in fact be equal, so that the relation $\sim$ defines a partial ordering on the set of special pseudo-valuations. This corresponds to the partial ordering by inclusion of the ideals of R in the sense that

$$W_\mathfrak{a} \subset W_\mathfrak{b} \text{ if and only if } \mathfrak{a} \supseteq \mathfrak{b}. \tag{2}$$

11) For the special case of the ring of algebraic integers this pseudo-valuation occurred already in Ch. III. Cf. also P.I.3.

Therefore every relation between ideals in R corresponds to a relation between special pseudo-valuations and conversely. We investigate some of these relations and in particular give another interpretation of the operations defined in Ch. II.

21. Let $\mathfrak{a}$, $\mathfrak{b}$ be any ideals of R and $W_\mathfrak{a}$, $W_\mathfrak{b}$ the pseudo-valuations associated with them. Consider the subadditive product $V = W_\mathfrak{a} \odot W_\mathfrak{b}$. From the definition it is clear that V takes only the values 0 or 1 and is therefore special; and $V(c) = 0$ if and only if there is a decomposition $c = x_1y_1 + \dots + x_ny_n$ for which $W_\mathfrak{a}(x_i)W_\mathfrak{b}(y_i) = 0$ $(i = 1, \dots, n)$, that is, if and only if $c \in \mathfrak{a} + \mathfrak{b}$. Hence

$$W_\mathfrak{a} \odot W_\mathfrak{b} = W_{\mathfrak{a}+\mathfrak{b}}.$$

In the same way (or by Lemma 16.2 Cor. 2) it follows that

$$W_\mathfrak{a}.W_\mathfrak{b} = W_{\mathfrak{a}+\mathfrak{b}},$$

so that the product of special pseudo-valuations corresponds to the sum of ideals.

Consider next $V' = W_\mathfrak{a} \times W_\mathfrak{b}$. Again V' takes only the values 0 and 1. Further $V'(c) = 0$ if and only if there is a decomposition $c = x_1y_1 + \dots + x_ny_n$ of c such that $\max\{W_\mathfrak{a}(x_i), W_\mathfrak{b}(y_i)\} = 0$, which is the case if and only if $x_i \in \mathfrak{a}$, $y_i \in \mathfrak{b}$, i.e. if $c \in \mathfrak{ab}$. Hence $V' = W_{\mathfrak{ab}}$. If $c \in \mathfrak{ab}$, it is clear that $W_\mathfrak{a} \otimes W_\mathfrak{b}(c)$ also vanishes; combining this fact with the relation $W_\mathfrak{a} \times W_\mathfrak{b} \leqslant W_\mathfrak{a} \otimes W_\mathfrak{b}$ (Theorem 17.1) we see that the subadditive compound is equivalent to $W_{\mathfrak{ab}}$. Now $W_\mathfrak{a} \otimes W_\mathfrak{b}$ is bounded by the constant 2, as we see by using the decomposition $c = c.1$; hence $W_\mathfrak{a} \otimes W_\mathfrak{b}$ is strongly equivalent to $W_{\mathfrak{ab}}$. Summing up, we have

THEOREM 21.1. *If $W_\mathfrak{a}$, $W_\mathfrak{b}$ are the pseudo-valuations associated with two ideals $\mathfrak{a}$, $\mathfrak{b}$ of R, then*

$$W_\mathfrak{a} \odot W_\mathfrak{b} = W_\mathfrak{a}.W_\mathfrak{b} = W_{\mathfrak{a}+\mathfrak{b}},$$
$$W_\mathfrak{a} \otimes W_\mathfrak{b} \approx W_\mathfrak{a} \times W_\mathfrak{b} = W_{\mathfrak{ab}}.$$

The strong equivalence in the last line cannot be improved to equality since e.g. $W_0 \otimes W_0(1) = 2$. In fact, as we saw in Ch. III, $W_\mathfrak{a} \otimes W_\mathfrak{b} = W_{\mathfrak{ab}} + \min(W_\mathfrak{a}, W_\mathfrak{b})$.

For the sake of comparison we recall the result proved in P.I. 11 that $\max(W_\mathfrak{a}, W_\mathfrak{b}) = W_{\mathfrak{a}\cap\mathfrak{b}}$. We restate this in a more general form as

THEOREM 21.2. *If* $\mathfrak{a}_\lambda$ $(\lambda \in \Lambda)$ *is any non-empty family of ideals and their intersection is* $\mathfrak{a}$, *then*

$$\sup\{W\mathfrak{a}_\lambda \mid \lambda \in \Lambda\} = W_\mathfrak{a}.$$

This Theorem, together with (2), implies that the set of ideals of R and the set of special pseudo-valuations form isomorphic complete lattices [12]).

22. The interpretation of the product of two special pseudo-valuations given in § 21 can be extended to the case of bounded non-archimedean pseudo-valuations.

Let W_1, W_2 be two such pseudo-valuations and consider an element a of R for which $W_1.W_2(a)$ is small. If $W_1.W_2(a) < \alpha$, where $\alpha > 0$, then there is a decomposition $a = x_1y_1 + \ldots + x_ny_n$ of a such that

$$W_1(x_i)W_2(y_i) < 2\alpha \qquad (i = 1, \ldots, n)$$

Let ω be an upper bound for W_1 and W_2, then for each $i = 1, \ldots, n$ either $W_1(x_i) < \sqrt{(2\alpha)}$ or $W_2(y_i) < \sqrt{(2\alpha)}$, and so either

$$W_1(x_iy_i) \leqslant W_1(x_i)W_1(y_i) < \omega\sqrt{(2\alpha)} \qquad (3)$$

or

$$W_2(x_iy_i) \leqslant W_2(x_i)W_2(y_i) < \omega\sqrt{(2\alpha)}. \qquad (4)$$

Denote by b the sum of the terms x_iy_i for which (3) holds, and by c the sum of the remaining terms. Then $a = b + c$, and by (3) and (4)

$$W_1(b) < \omega\sqrt{(2\alpha)},\ W_2(c) < \omega\sqrt{(2\alpha)},$$

since W_1 and W_2 are non-archimedean. Thus if $\varepsilon > 0$ is fixed, and $\delta < \frac{1}{2}(\varepsilon/\omega)^2$, then for any $a \in R$ such that $W_1.W_2(a) < \delta$, we can find b and $c \in R$ such that $b + c = a$ and $W_1(b) < \varepsilon$, $W_2(c) < \varepsilon$. More concisely, we can say that if $W_1.W_2(a)$ is small, then $a = b + c$, where $W_1(b)$ and $W_2(c)$ are small.

Conversely, if $W_1(b) < \alpha$, $W_2(c) < \alpha$, then by Theorem 16.4

$$\begin{aligned} W_1.W_2(b+c) &\leqslant \max(W_1.W_2(b), W_1.W_2(c)) \\ &\leqslant k \max(W_1(b), W_2(c)) \\ &< k\,\alpha, \end{aligned}$$

where k depends on W_1 and W_2 only. By taking $\delta < \varepsilon/k$, we see that if $W_1(b)$, $W_2(c) < \delta$, then $W_1.W_2(b+c) < \varepsilon$, and so we have obtained the following result:

[12]) Cf. G. BIRKHOFF, Lattice Theory, (2nd ed.), New York, 1948, Ch. IV, Theorem 2.

THEOREM 22.1. *If* W_1 *and* W_2 *are bounded non-archimedean pseudo-valuations on* R, *then* $W_1.W_2(a)$ *is small if and only if* a *can be written in the form* $a = b + c$, *where* $W_1(b)$ *and* $W_2(c)$ *are small.*

We note that this defines $W_1.W_2$ to within equivalence when W_1 and W_2 are given, since the neighbourhoods of zero and hence the topology defined by a pseudo-valuation are determined by a knowledge of the regions where the pseudo-valuation is small. In particular this gives another proof of the $\sim$-invariance of the non-archimedean product (Theorem 19.2), this time restricted to bounded non-archimedean pseudo-valuations.

Suppose again that W_1 and W_2 are non-archimedean pseudo-valuations and that $W_1.W_2(a)$ is small. Then by Theorem 22.1 $a = b + c$, where $W_1(b)$ and $W_2(c)$ are small. By Theorem 16.4

$$\begin{aligned} W_1 \odot W_2(a) &\leqslant W_1 \odot W_2(b) + W_1 \odot W_2(c) \\ &\leqslant k(W_1(b) + W_2(c)), \end{aligned}$$

where k depends on W_1 and W_2 only. Therefore $W_1 \odot W_2(a)$ is small. Conversely, by the inequality $W_1.W_2 \leqslant W_1 \odot W_2$ of Lemma 16.2 Cor. 1, if $W_1 \odot W_2(a)$ is small then $W_1.W_2(a)$ is small. Hence $W_1 \odot W_2(a)$ is small if and only if $W_1.W_2(a)$ is small, or in other words, $W_1 \odot W_2 \sim W_1.W_2$. Thus we have proved

THEOREM 22.2. *If* W_1, W_2 *are non-archimedean bounded pseudo-valuations, then*

$$W_1 \,.\, W_2 \sim W_1 \odot W_2.$$

COR. *The operation* $\odot$ *is* $\sim$*-invariant when applied to non-archimedean bounded pseudo-valuations.*

For then it is equivalent to . which is $\sim$-invariant by Theorem 19.2.

A similar $\sim$-invariant characterization of the non-archimedean compound $\times$ of non-archimedean bounded pseudo-valuations is possible, viz. that $W_1 \times W_2(a)$ is small if and only if a has the form $a = x_1y_1 + \ldots + x_ny_n$ where $W_1(x_i)$ and $W_2(y_i)$ are small. But we do not know whether this is also true for the subadditive compound.

23. The correspondence between ideal operations and operations on special pseudo-valuations established in § 21 shows that to every identity holding between ideals there corresponds an identity between special pseudo-valuations, and inequalities correspond in a similar way. In many cases these identities and equalities still hold, with $=$ and $\supseteq$ replaced by $\sim$ and $\subset$ respectively, for any bounde

non-archimedean pseudo-valuations. As examples we consider the ideal relations

1) $$\mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b},$$

2) $$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c},$$

3) $$\mathfrak{a}\mathfrak{b} \supseteq (\mathfrak{a} \cap \mathfrak{b})(\mathfrak{a} + \mathfrak{b}).$$

The corresponding relations for pseudo-valuations are given by

THEOREM 23.1. *If* W_1, W_2, W_3 *are bounded non-archimedean pseudo-valuations, then*

1) $$W_1 + W_2 \subset W_1 \times W_2,$$

2) $$W_1 \times (W_2 . W_3) \sim (W_1 \times W_2) . (W_1 \times W_3),$$

3) $$W_1 \times W_2 \subset (W_1 + W_2) \times (W_1 . W_2).$$

Proof. 1) Let ω be an upper bound for W_1 and W_2. If $W_1 \times W_2(a) < \alpha$ $(\alpha > 0)$, then there is a decomposition $a = x_1y_1 + \ldots + x_ny_n$ such that $W_1(x_i)$, $W_2(y_i) < 2\alpha$. Hence

$$W_1(a) \leqslant \max_i\{W_1(x_i)W_1(y_i)\} < 2\alpha\omega.$$

Similarly $W_2(a) < 2\alpha\omega$, and 1) follows.

2) Since $W_2 . W_3 \Subset W_2$, it follows by Lemma 18.1 that $W_1 \times (W_2 . W_3) \Subset W_1 \times W_2$ and similarly $W_1 \times (W_2 . W_3) \Subset W_1 \times W_3$; in other words

$$W_1 \times (W_2 . W_3) \leqslant k(W_1 \times W_2),\ k(W_1 \times W_3),$$

where k is some constant, which may without loss of generality be taken to be greater than 1. Then $k(W_1 \times W_i)$ is again a pseudo-valuation, hence by Theorem 16.4

$$W_1 \times (W_2 . W_3) \leqslant [k(W_1 \times W_2)] . [k(W_1 \times W_3)]$$
$$\Subset (W_1 \times W_2) . (W_1 \times W_3);$$

therefore the left-hand side of 2) is small if the right-hand side is small.

Conversely, if $W_1 \times (W_2 . W_3)\ (a)$ is small, then a can be written as $a = x_1y_1 + \ldots x_ny_n$, where $W_1(x_i)$ and $W_2 . W_3(y_i)$ are small. Hence y_i has the form $y_i = y_i' + y_i''$,where $W_2(y_i')$ and $W_3(y_i'')$ are small. Then $W_1 \times W_2(\Sigma\, x_iy_i')$ and $W_1 \times W_3(\Sigma\, x_iy_i'')$ are small, and since $a = \Sigma x_iy_i' + \Sigma x_iy_i''$, it follows that $(W_1 \times W_2) . (W_1 \times W_3)\ (a)$ is small, and this proves 2).

3) By Lemma 19.3, since W_1, $W_2 \subset W_1 + W_2$,

$$W_1 \times W_2 \subset (W_1 + W_2) \times W_1,$$
$$W_1 \times W_2 \subset (W_1 + W_2) \times W_2;$$

hence by Theorem 16.4

$$W_1 \times W_2 \subset [(W_1 + W_2) \times W_1] . [(W_1 + W_2) \times W_2]$$
$$\subset (W_1 + W_2) \times (W_1 . W_2)$$

by 2). This completes the proof.

24. It would be of interest to decide whether any relation holding between ideals in a commutative ring with a unit-element always has its analogue for non-archimedean bounded pseudo-valuations. The converse is obviously true: If a relation holds generally for non-archimedean bounded pseudo-valuations, then the corresponding relation for ideals must also hold, since all we need is to consider the special pseudo-valuations in order to effect the change-over to ideals. As an illustration of this principle we prove that the product or the compound of pseudo-valuations is not distributive with respect to addition.

THEOREM 24.1. *The equivalences*

$$W_1\ .\ (W_2 + W_3) \sim W_1\ .\ W_2 + W_1\ .\ W_3, \tag{5}$$
$$W_1 \times (W_2 + W_3) \sim W_1 \times W_2 + W_1 \times W_3, \tag{6}$$

where the W_i *are non-archimedean bounded pseudo-valuations, do not hold generally.*

Proof. Suppose that (5) and (6) are true generally in any commutative ring with a unit-element. On taking the W_i to be special and rewriting the resulting equations in terms of ideals we obtain the equations

$$\mathfrak{a} + \mathfrak{b} \cap \mathfrak{c} = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}),$$
$$\mathfrak{a} . (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a}\mathfrak{b}) \cap (\mathfrak{a}\mathfrak{c}),$$

which must hold identically in the ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. It is therefore enough to give examples of rings in which these equations are false.

i) Let R be an algebra over any field with basis 1, u, v, and multiplication-table

	1	u	v
1	1	u	v
u	u	0	0 ;
v	v	0	0

R is associative and commutative and has a unit-element. Denote by $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ the ideals generated by $u + v$, u and v respectively. Then

$$\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = (u + v) \neq (u, v) = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}).$$

ii) Let R be the ring of polynomials in two indeterminates x, y over a field, and put $\mathfrak{a} = (x, y)$, $\mathfrak{b} = (x^2)$, $\mathfrak{c} = (y^2)$.

Then $$\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) = (x^3y^2, x^2y^3) \neq (x^2y^2) = (\mathfrak{a}\mathfrak{b}) \cap (\mathfrak{a}\mathfrak{c}).$$

Therefore the equivalences (5) and (6) do not hold universally. This completes the proof.

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