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On a problem in the geometry of numbers

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Let K be a bounded convex body in n-dimensional space R_n which is symmetrical in the origin $O = (0, 0, \ldots, 0)$ of the coordinate system; let further $s > 0$. The subset K_s of K defined by

$$X = (x_1, x_2, \ldots, x_n) \in K, \quad |x_n| \leq s$$

is also a bounded convex body symmetrical in O.

It follows easily from the well-known theorem of Brunn and Minkowski [1] that, if $V(s)$ denotes the volume of K_s, the quotient

$$\Phi(s) = V(s)/s$$

is a decreasing function of s,

$$\text{(1)} \qquad \Phi(s) \geq \Phi(t) \quad \text{if} \quad 0 < s < t.$$

This note deals with an analogous property of K_s from the geometry of numbers. Denote by $\Delta(s) = \Delta(K_s)$ the lattice determinant of K_s and put

$$\Psi(s) = \Delta(s)/s.$$

It lies then near to conjecture that, in analogy to (1), also $\Psi(s)$ is a decreasing function,

$$\text{(2)} \qquad \Psi(s) \geq \Psi(t) \quad \text{if} \quad 0 < s < t.$$

[1] See T. Bonnesen and W. Fenchel, *Theorie der konvexen Korper,* (Berlin, 1934), 87 ff.

However, I have not so far succeeded in finding a general proof. In this note, only the two-dimensional case has been solved. It would have some interest to decide whether the conjecture (2) remains true for more than two dimensions.

1. Let K be a bounded symmetric convex domain in the (x, y) plane, and let K_s, for $s > 0$, denote the subset of those points of K for which $|y| \leq s$. Thus K_s is of the same kind as K and, in fact, is identical with K when s is sufficiently large. The boundary, C_s say, of K_s consists, (i) of an arc A_s on the boundary C of K, together with its image $-A_s$ in O, and (ii) of a line segment L_s on $y = s$, together with its image $-L_s$ in O. It is obvious that the length of A_s increases with increasing s; on the other hand, the length of L_s is a decreasing function of s because K is symmetric and convex.

Select any critical lattice Λ_s of K_s with exactly three pairs of points $\mp P_s, \mp Q_s, \mp R_s$ on C_s; such a lattice always exists. The notation may be chosen such that

$$Q_s = P_s + R_s. \tag{3}$$

Let again $0 < s < t$ and define $\Psi(s)$ as in the introduction. It is obvious that if L_s consists of a single point or is the null set, then $K_s = K_t$ so that the assertion (2) is certainly satisfied. This case may thus be excluded, and we shall from now on assume that L_s is a proper line segment with inner points. Several cases will be distinguished, according to the number of lattice points on L_s.

2. Case A: No point of Λ_s is an inner point of L_s.

Hence all six points $\mp P_s, \mp Q_s, \mp R_s$ lie on the two arcs $\mp A_s$ of C_s. Denote by p_s, q_s, r_s tac-lines of K at P_s, Q_s, R_s, respectively, and by $-p_s, -q_s, -r_s$ the parallel tac-lines at $-P_s$, $-Q_s$, $-R_s$. The hexagon H_s bounded by the six tac-lines $\mp p_s$, $\mp q_s$, $\mp r_s$ contains both K_s and K as subsets because its sides are tac-lines of K. Further Λ_s remains a critical lattice of H_s.

If now $0 < s < t$, then $K_s \subset K_t \subset K \subset H_s$. Hence $\Delta(t) = \Delta(H_s) = \Delta(s)$, and the assertion (2) becomes evident.

3. Case B: Just one point of Λ_s is an inner point of L_s.

We may assume that this is the point Q_s, and that P_s lies on A_s and R_s on $-A_s$. Denote by p_s and r_s tac-lines of K at these two points, by $-p_s$ and $-r_s$ the lines symmetrical to p_s and r_s in O, and by Π_s the parallelogram bounded by $\mp p_s$ and $\mp r_s$. Let further H_s^* and H_t^* be the two hexagons that consist of those points of Π_s for which $|y| \leq s$ and $|y| \leq t$, respectively. Then $K_s \subset H_s^*$ and $K_t \subset H_t^*$. Again, as in the first case, Λ_s has remained a critical lattice of H_s, and it is trivial that a critical lattice of H_t is K_t-admissible.

Let $V(H_s^*)$ and $V(H_t^*)$ be the two-dimensional volumes, i.e. the areas of H_s^* and H_t^*, respectively. As is well-known, the lattice determinants of these two hexagons are given by

$$\Delta(H_s^*) = \frac{1}{4} V(H_s^*), \ \Delta(H_t^*) = \frac{1}{4} V(H_t^*),$$

and therefore also

$$\Delta(s) = \frac{1}{4} V(H_s^*), \ \Delta(t) \leq \frac{1}{4} V(H_t^*).$$

Further the hexagon H_s^* consists of all those points (x, y) of H_t for which $|y| \leq s$. Therefore, by the property (1) of convex sets,

$$\frac{1}{s} V(H_s^*) \geq \frac{1}{t} V(H_t^*),$$

whence

$$\frac{1}{s} \Delta(s) = \frac{1}{4} \cdot \frac{1}{s} V(H_s^*) \geq \frac{1}{4} \cdot \frac{1}{s} V(H_t^*) \geq \frac{1}{t} \Delta(t),$$

as asserted.

4. Case C: Just two points of Λ_s are inner points of L_s.

We may assume that this are the two points Q_s and R_s and that P_s lies on A_s; for it is clearly impossible for P_s and R_s to be both on L_s. Denote by p_s a tac-line of K at P_s, by $-p_s$ the parallel tac-line at $-P_s$, and by Π_s^* and Π_t^* the two parallelograms bounded by the four lines $\mp p_s$ and $y = \mp s$, and $\mp p_s$ and $y = \mp t$, respectively. Then Λ_s is still a critical lattice of Π_s^*, and every critical lattice of Π_t is obviously K_t-admissible. It follows therefore that

$$\Delta(\Pi_s^*) = \Delta(K_s), \qquad \Delta(\Pi_t^*) \geq \Delta(K_t).$$

On the other hand, by the form of the two parallelograms,

$$\frac{1}{s}\,\Delta(\Pi_s^*) = \frac{1}{t}\,\Delta(\Pi_t^*)\,,$$

and so finally

$$\frac{1}{s}\,\Delta\,(K_s) = \frac{1}{s}\,\Delta(\Pi_s^*) = \frac{1}{t}\,\Delta(\Pi_t^*) \geq \frac{1}{t}\,\Delta\,(K_t)\,.$$

This completes the proof since there are no further cases.

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