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An Interpolation Series for Continuous Functions of a p-adic Variable.

Meinem Lehrer C. L. Siegel zu seinem 60. Geburtstag gewidmet.

By *K. Mahler* in Manchester.

The theory of analytic functions of a p-adic variable (i. e. of functions defined by power series) is much simpler than that of complex analytic funktions and offers few surprises. On the other hand, the behaviour of *continuous functions of a p-adic variable* is quite distinct from that of real continuous functions, and many basic theorems of real analysis have no p-adic analogues. Thus there is no simple analogue to the mean value theorem of differential calculus, even for polynomials like $\binom{x}{p}$; there exist infinitely many linearly independent non-constant functions the derivative of which vanishes identically; and if a series $f(x) = \sum f_n(x)$ converges and the derived series $g(x) = \sum f_n'(x)$ converges uniformly, $g(x)$ still need not be the derivative of $f(x)$; etc.

The main paper on the subject is that by *J. Dieudonné*, Sur les fonctions continues p-adiques, Bull. Sci. Math. (2) **68** (1944), 79—95. I mention, in particular, his p-adic analogue to Weierstrass's theorem on the approximation of continuous functions by polynomials, and his existence theorem for differential equations. Most of his paper deals with the more general class of p-adic valued continuous functions on compact totally discontinuous spaces and falls outside the subject of this note.

I had already become interested in the subject before I learned of his paper. Earlier this year, J. F. Koksma (who then also did not know of Dieudonné's work) suggested to me that there should be a p-adic analogue to Weierstrass's approximation theorem. The solution which I obtained finally proved to be very different from that by Dieudonné.

There is no great loss of generality in restricting oneself to functions $f(x)$ on the set I of all p-adic integers. The subset J of the non-negative integers is dense on I. Hence a continuous function $f(x)$ on I is already determined by its values on J, hence also by the numbers

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k) \qquad (n = 0, 1, 2, \ldots).$$

I prove that $\{a_n\}$ is a p-adic null sequence, and that

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

for all $x \in I$. Thus $f(x)$ can be approximated by means of polynomials.

I further study conditions for the a_n under which $f(x)$ is differentiable at a point, or has a continuous derivative everywhere on I. Thus, by way of example, $\sum_{r=0}^{\infty} p^r \binom{x}{p^r}$ is continuous, but nowhere differentiable, on I (an entirely different example was given by Dieudonné); and $\sum_{r=0}^{\infty} p^r \binom{x}{p^r-1}$ has a continuous derivative for $x \neq -1$, but is not differentiable at $x = -1$. Two problems on differentiation are stated which I have not succeeded in solving; they seem well worth of further study. I conclude the paper with a result on a special infinite system of linear equations.

1. Throughout this paper, p is a fixed prime; R is the field of all p-adic numbers; $|x|_p$ is the p-adic value normed such that $|p|_p = 1/p$; $I = \{x;\ |x|_p \leq 1\}$ is the ring of all p-adic integers; and J is the subset of all non-negative rational integers. *Thus J lies everywhere dense in I.*

Limits both of real and of p-adic numbers will occur, but it will in each case be clear from the context which kind of limit is meant.

All functions $f(x)$ will be defined for all $x \in I$ and have values in R. We shall mainly be concerned with functions that are continuous at all points of I, or that have a continuous derivative on I.

2. With each function $f(x)$ we associate the infinite sequence of coefficients

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k) \qquad (n = 0, 1, 2, \ldots)$$

and the *formal* interpolation series

$$f^*(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}.$$

These coefficients a_n are the successive differences at $x = 0$ of the sequence $\{f(0), f(1), f(2), \ldots\}$, and they may also be defined by the recursive formulae

$$f^*(n) = f(n) \qquad (n = 0, 1, 2, \ldots),$$

in which $f^*(n)$ reduces to a finite sum.

3. Lemma 1. *The series* $f^*(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ *converges for all $x \in I$ if and only if*

$$\lim_{n \to \infty} a_n = 0.$$

Proof. (a) The condition is necessary because e. g. the series

$$f^*(-1) = \sum_{n=0}^{\infty} (-1)^n a_n$$

does not converge unless its terms $\mp a_n$ tend to zero.

(b) Assume that $\lim a_n = 0$. For every $x \in I$ select a $y \in J$ such that

$$\left| \frac{x-y}{n!} \right|_p \leq 1.$$

Then $\binom{y}{n-k}$ is a positive integer, hence

$$\left| \binom{y}{n-k} \right|_p \leq 1 \qquad (k = 0, 1, 2, \ldots, n).$$

Further $\binom{x-y}{0} = 1$, and

$$\binom{x-y}{k} = \frac{(x-y)(x-y-1)\cdots(x-y-k+1)}{k!} = \frac{x-y}{n!} \lambda_k \qquad (k = 1, 2, \ldots, n)$$

where λ_k denotes a p-adic integer; therefore also

$$\left|\binom{x-y}{k}\right|_p \leq 1 \qquad (k = 0, 1, 2, \ldots, n).$$

The identity

$$\binom{x}{n} = \sum_{k=0}^{n} \binom{x-y}{k} \binom{y}{n-k}$$

implies then that

$$\left|\binom{x}{n}\right|_p \leq 1 \text{ if } x \in I,\ n \in J,$$

and so

$$\left|a_n \binom{x}{n}\right|_p \leq |a_n|_p \to 0 \text{ as } n \to \infty,$$

giving the convergence of $f^*(x)$.

This proof shows, moreover, that $f^*(x)$ converges *uniformly* for $x \in I$, hence that its sum is a *continuous* function because the terms are polynomials and therefore continuous.

Lemma 2. *Let* $\lim_{n\to\infty} a_n = 0$. *Then* $f^*(x) = f(x)$ *if* $x \in I$.

Proof. Both $f(x)$ and $f^*(x)$ are continuous on I, and they are equal when $x \in J$. Since J is dense in I, every $x \in I$ is the limit of a sequence $\{y_n\}$ of elements of J. Then

$$f^*(y_n) = f(y_n) \qquad (n = 1, 2, 3, \ldots),$$

and so, by continuity,

$$f^*(x) = \lim_{n\to\infty} f^*(y_n) = \lim_{n\to\infty} f(y_n) = f(x).$$

4. Theorem 1. *Let $f(x)$ be continuous on I. Then*

$$\lim_{n\to\infty} a_n = 0, \textit{ and therefore } f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \textit{ if } x \in I.$$

Proof. As a continuous function on a compact set, $f(x)$ is both *bounded* and *uniformly continuous* on I. As we may, if necessary, multiply $f(x)$ by a power of p, there is no restriction in assuming that

$$|f(x)|_p \leq 1 \text{ if } x \in I.$$

Further, if s is any positive integer, there is a second positive integer $t = t(s)$ such that

$$|f(x) - f(y)|_p \leq p^{-s} \text{ if } x, y \in I,\ |x - y|_p \leq p^{-t}.$$

In the remainder of the proof x and y may be restricted to the set J. For every $x \in J$ there is a unique integer $g(x) \in J$ satisfying

$$|f(x) - g(x)|_p \leq p^{-s},\ 0 \leq g(x) \leq p^s - 1.$$

This function $g(x)$ on J is periodic,

$$g(x) = g(y) \text{ if } x, y \in J,\ x \equiv y \pmod{p^t}.$$

For the congruence is equivalent to $|x - y|_p \leq p^{-t}$, and so

$$\begin{aligned} |g(x) - g(y)|_p &= |(g(x) - f(x)) + (f(x) - f(y)) + (f(y) - g(y))|_p \\ &\leq \max(|g(x) - f(x)|_p,\ |f(x) - f(y)|_p,\ |f(y) - g(y|_p) \leq p^{-s}, \end{aligned}$$

whence $g(x) = g(y)$ because distinct values of this function are, by definition, incongruent (mod p^s).

In analogy to a_n define now

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(n-k) \qquad (n = 0, 1, 2, \ldots)$$

so that b_n is a rational integer. Evidently

$$|a_n - b_n|_p \leq p^{-s} \qquad (n = 0, 1, 2, \ldots).$$

Next let ω be a fixed primitive p^t-th root of unity; thus

$$\sum_{n=0}^{p^t-1} \omega^{mn} = \begin{cases} p^t \text{ if } m \equiv 0 \pmod{p^t}, \\ 0 \text{ if } m \not\equiv 0 \pmod{p^t}. \end{cases}$$

Further put

$$\lambda_m = p^{-t} \sum_{n=0}^{p^t-1} \omega^{-mn} g(n) \qquad (m = 0, 1, 2, \ldots, p^t - 1).$$

Then, conversely,

$$\sum_{m=0}^{p^t-1} \lambda_m \omega^{mn} = p^{-t} \sum_{m=0}^{p^t-1} \sum_{r=0}^{p^t-1} \omega^{mn-mr} g(r) = p^{-t} \sum_{r=0}^{p^t-1} g(r) \sum_{m=0}^{p^t-1} \omega^{m(n-r)} = g(n)$$

if $n = 0, 1, 2, \ldots, p^t - 1$. Here $g(n)$ is periodic in n with the period p^t, and so is the sum on the left-hand side. Hence

$$g(n) = \sum_{m=0}^{p^t-1} \lambda_m \omega^{mn} \text{ for all } n \in J,$$

whence

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{m=0}^{p^t-1} \lambda_m \omega^{m(n-k)} = \sum_{m=0}^{p^t-1} \lambda_m \sum_{k=0}^{n} (-1)^k \binom{n}{k} \omega^{m(n-k)}$$

and finally

$$b_n = \sum_{m=0}^{p^t-1} \lambda_m (\omega^m - 1)^n \text{ for all } n \in J.$$

Let now K be the cyclotomic field generated by ω, and let $\mathfrak{o}$ be the ring of all algebraic integers in K. Not only ω, but also the quotients

$$\frac{\omega^m - 1}{\omega - 1} = \omega^{m-1} + \omega^{m-2} + \cdots + \omega + 1 \qquad (m = 0, 1, 2, \ldots, p^t - 1)$$

and the products

$$p^t \lambda_m = \sum_{n=0}^{p^t-1} \omega^{-mn} g(n) \qquad (m = 0, 1, 2, \ldots, p^t - 1)$$

are elements of $\mathfrak{o}$. The expression for b_n implies therefore that

$$p^t (\omega - 1)^{-n} b_n \in \mathfrak{o} \text{ if } n \in J.$$

It is well-known that the two principal ideals (p) and $(\omega - 1)$ in $\mathfrak{o}$ satisfy the relation

$$(p) = (\omega - 1)^{p^{t-1}(p-1)}$$

which expresses (p) as the power of a prime ideal. Put

$$N = [n p^{-(t-1)} (p-1)^{-1}]$$

where, as usual, [a] is the integral part of a. Then p^N is a divisor of $(\omega - 1)^n$. The rational numbers $p^{-N} b_n$ are therefore algebraic integers and so are rational integers. Hence

$$|b_n|_p \leq p^{t-N},$$

whence

$$|b_n|_p \leq p^{-s} \text{ if } n \geq p^{t-1}(p-1)(s+t), = n_0 \text{ say,}$$

because $N \geq s+t$ if $n \geq n_0$.

On combining this with the earlier inequality for $a_n - b_n$ we obtain the result that

$$|a_n|_p = |(a_n - b_n) + b_n|_p \leq \max(|a_n - b_n|_p, |b_n|_p) \leq p^{-s} \text{ if } n \geq n_0.$$

Here s may be arbitrarily large, and n_0 depends only on s because t is a function of s. Therefore

$$\lim_{n\to\infty} a_n = 0,$$

giving the assertion.

5. Lemma 3. *Let $f(x)$ be continuous on I, and let $x, y \in I$. Then all series*

$$a_n(y) = \sum_{k=0}^{\infty} a_{n+k} \binom{y}{k} \qquad (n = 0, 1, 2, \ldots)$$

converge, and further

$$\lim_{n\to\infty} a_n(y) = 0, \quad f(x+y) = \sum_{n=0}^{\infty} a_n(y) \binom{x}{n}.$$

Proof. The convergence of $a_n(y)$ follows from $\lim_{k\to\infty} a_{n+k} = 0$, since $\left|\binom{y}{k}\right|_p \leq 1$. Next $\{a_n(y)\}$ forms a null sequence because

$$|a_n(y)|_p \leq \max_{k=0,1,2,\ldots} |a_{n+k}|_p \to 0 \text{ as } n \to \infty.$$

Finally

$$f(x+y) = \sum_{m=0}^{\infty} a_m \binom{x+y}{m} = \sum_{m=0}^{\infty} a_m \sum_{n=0}^{m} \binom{x}{n}\binom{y}{m-n}$$

$$= \sum_{n=0}^{\infty} \binom{x}{n} \sum_{m=0}^{\infty} a_m \binom{y}{m-n} = \sum_{n=0}^{\infty} \binom{x}{n} \sum_{k=0}^{\infty} a_{n+k} \binom{y}{k} = \sum_{n=0}^{\infty} a_n(y) \binom{x}{n}.$$

Here the reordering of the terms is allowed since we are dealing with p-adic series, and since $\{a_n\}$ is a null sequence.

6. We next establish necessary and sufficient conditions, in terms of the coefficients a_n, for the existence of a derivative of $f(x)$. The proof will be based on the following Tauberian theorem.

Theorem 2. *Let $\{a_n\}$ be a p-adic null sequence. If the p-adic limit*

$$\lambda = \lim_{|x|_p \to 0} \sum_{n=1}^{x} \frac{a_n}{n} \binom{x-1}{n-1}$$

extended over all elements $x \neq 0$ of J exists, then

$$\text{(i)} \quad \lim_{n\to\infty} \frac{a_n}{n} = 0; \text{ (ii) } \lambda = \sum_{n=1}^{\infty} \frac{a_n}{n} \binom{-1}{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}.$$

The proof of the assertion (i) is rather long and involved and is indirect. It will be carried out in several steps.

7. As a first step assume that λ exists, but that

$$\limsup_{n\to\infty} \left|\frac{a_n}{n}\right|_p = \infty.$$

There are then infinitely many integers $n_1, n_2, n_3, \ldots$ such that

$$0 < n_1 < n_2 < n_3 < \cdots,$$

$$a_{n_r} \neq 0 \text{ for } r = 1, 2, 3, \ldots,$$

$$\lim_{r\to\infty} \left|\frac{a_{n_r}}{n_r}\right|_p = \infty.$$

Since, by hypothesis,

$$\lim_{n\to\infty} |a_n|_p = \lim_{r\to\infty} |a_{n_r}|_p = 0,$$

necessarily

$$\lim_{r\to\infty} |n_r|_p = 0.$$

The sequence $\{n_r\}$ may be replaced by any infinite subsequence. Hence there is no loss of generality in further assuming that

$$\left|\frac{a_n}{n}\right|_p < \left|\frac{a_{n_r}}{n_r}\right|_p \text{ for } n = 1, 2, \ldots, n_r - 1 \qquad (r = 1, 2, 3, \ldots).$$

In the limit defining λ we may allow x to tend to zero over the sequence $\{n_r\}$; thus

$$\lambda = \lim_{r\to\infty} \sum_{n=1}^{n_r} \frac{a_n}{n}\binom{n_r - 1}{n - 1} = \lim_{r\to\infty}\left\{\frac{a_{n_r}}{n_r} + \sum_{n=1}^{n_r - 1} \frac{a_n}{n}\binom{n_r - 1}{n - 1}\right\}.$$

Here, by the construction of n_r,

$$\left|\frac{a_n}{n}\binom{n_r - 1}{n - 1}\right|_p < \left|\frac{a_{n_r}}{n_r}\right|_p \qquad (n = 1, 2, \ldots, n_r - 1)$$

and therefore

$$|\lambda|_p = \lim_{r\to\infty} \left|\frac{a_{n_r}}{n_r}\right|_p = \infty,$$

contrary to hypothesis.

8. As a second step, assume that λ exists and that $\left\{\frac{a_n}{n}\right\}$ is a bounded sequence, but not a null sequence. As we may multiply the coefficients a_n by a fixed power of p and may further change finitely many of these coefficients arbitrarily, without affecting the assertion, there is no loss of generality in assuming that

$$|a_n|_p \leq 1, \quad \left|\frac{a_n}{n}\right|_p \leq 1 \qquad (n = 1, 2, 3, \ldots),$$

$$\limsup_{n\to\infty} \left|\frac{a_n}{n}\right|_p = 1.$$

The existence of the limit λ now implies that there is a positive integer s such that

$$\left|\sum_{n=1}^{x} \frac{a_n}{n}\binom{x - 1}{n - 1} - \lambda\right|_p \leq \frac{1}{p} \text{ if } x \in J,\ 0 < |x|_p \leq p^{-s},$$

and this inequality remains valid if s is increased. We satisfy the condition for x by putting

$$x = p^s(\xi + 1) \text{ where } \xi \in J.$$

Next, since $\{a_n\}$ is a null sequence, evidently

$$\lim_{\substack{n\to\infty \\ p^s \nmid n}} \frac{a_n}{n} = 0.$$

Therefore, on increasing s, if necessary, there is also no loss of generality in further assuming that

$$\left|\frac{a_n}{n}\right|_p \leq \frac{1}{p} \text{ if } p^s \nmid n,\ n \geq p^s.$$

Hence, by $\left|\binom{x-1}{n-1}\right|_p \leq 1$, the inequality for λ implies that

$$\left|\sum_{n=1}^{p^s-1} \frac{a_n}{n}\binom{x-1}{n-1} + \sum_{\substack{n=p^s \\ p^s|n}}^{x} \frac{a_n}{n}\binom{x-1}{n-1} - \lambda\right|_p \leq \frac{1}{p} \text{ if } x = p^s(\xi+1),\ \xi \in J.$$

9. We introduce now a simple congruence for binomial coefficients $\binom{M}{N}$ where M is a positive and N a non-negative integer. Let

$$M = g_0 + g_1 p + \cdots + g_r p^r,\quad N = h_0 + h_1 p + \cdots + h_r p^r$$

be the representations of M and N, respectively, to the basis p; here the digits g_j and h_j assume only the values $0, 1, \ldots, p-1$. It is easily proved that

$$\binom{M}{N} \equiv \binom{g_0}{h_0}\binom{g_1}{h_1}\cdots\binom{g_{s-1}}{h_{s-1}} \pmod{p}.$$

We apply this formula to $\binom{x-1}{n-1}$ where $x = p^s(\xi+1)$ and $\xi \in J$, and either $n \leq p^s - 1$, or $n \geq p^s$ and $p^s \mid n$. In the second case n may be written as

$$n = p^s(\nu+1),\ \text{where } \nu \in J.$$

Then $x-1$ has the representation

$$x-1 = \{(p-1) + (p-1)p + \cdots + (p-1)p^{s-1}\} + g_s p^s + g_{s+1} p^{s+1} + \cdots + g_r p^r;$$

and $n-1$ has in the first case the representation

$$n-1 = \{h_0 + h_1 p + \cdots + h_{s-1} p^{s-1}\} + 0 \cdot p^s + 0 \cdot p^{s+1} + \cdots + 0 \cdot p^r,$$

and in the second case the representation

$$n-1 = \{(p-1) + (p-1)p + \cdots + (p-1)p^{s-1}\} + h_s p^s + h_{s+1} p^{s+1} + \cdots + h_r p.$$

Here $g_s, g_{s+1}, \ldots, g_r;\ h_1, h_2, \ldots, h_{s-1};\ h_s, h_{s+1}, \ldots, h_r$ are again certain digits $0, 1, \ldots, p-1$.

From the congruence above it follows at once that for $n \leq p^s - 1$

$$\binom{x-1}{n-1} \equiv \binom{p-1}{h_0}\binom{p-1}{h_1}\cdots\binom{p-1}{h_{s-1}} \equiv (-1)^{h_0+h_1+\cdots+h_{s-1}} \pmod{p},$$

and for $n = p^s(\nu+1)$

$$\binom{x-1}{n-1} \equiv \binom{g_s}{h_s}\binom{g_{s+1}}{h_{s+1}}\cdots\binom{g_r}{h_r} \equiv \binom{\xi}{\nu} \pmod{p};$$

for in the second case ξ and ν allow the representations

$$\xi = g_s p^s + g_{s+1} p^{s+1} + \cdots + g_r p^r,\quad \nu = h_s p^s + h_{s+1} p^{s+1} + \cdots + h_r p^r.$$

Since $\left|\frac{a_n}{n}\right|_p \leq 1$, we thus obtain the formulae

$$\sum_{n=1}^{p^s-1} \frac{a_n}{n}\binom{x-1}{n-1} = \sum_{n=1}^{p^s-1} \chi(n)\frac{a_n}{n} + \varrho(x)$$

and

$$\sum_{\substack{n=p^s \\ p^s|n}}^{x} \frac{a_n}{n}\binom{x-1}{n-1} = \sum_{\nu=0}^{\xi} \alpha_\nu \binom{\xi}{\nu} + \sigma(x).$$

Here the sign

$$\chi(n) = (-1)^{h_0+h_1+\cdots+h_{s-1}}$$

depends only on n and not on x. Further we have put

$$\frac{a_n}{n} = \alpha_\nu \text{ if } n = p^s(\nu+1),$$

and we denote by $\varrho(x)$ and $\sigma(x)$ two p-adic functions of x such that

$$|\varrho(x)|_p \leq \frac{1}{p}, \quad |\sigma(x)|_p \leq \frac{1}{p}.$$

10. The estimate for λ takes now the form

$$\left| \sum_{n=1}^{p^s-1} \chi(n)\frac{a_n}{n} + \varrho(x) + \sum_{\nu=0}^{\xi} \alpha_\nu \binom{\xi}{\nu} + \sigma(x) - \lambda \right|_p \leq \frac{1}{p} \text{ if } \xi \in J$$

or

$$\sum_{\nu=0}^{\xi} \alpha_\nu \binom{\xi}{\nu} = \mu + \tau(\xi).$$

Here μ denotes the new constant

$$\mu = \lambda - \sum_{n=1}^{p^s-1} \chi(n)\frac{a_n}{n},$$

and $\tau(\xi)$ is a p-adic function of ξ such that

$$|\tau(\xi)|_p \leq \frac{1}{p}.$$

Hence, on putting successively $\xi = 0, 1, 2, \ldots$, we obtain the infinite system of equations

$$\alpha_0 = \mu + \tau(0), \quad \alpha_0 + \alpha_1 = \mu + \tau(1), \quad \alpha_0 + 2\alpha_1 + \alpha_2 = \mu + \tau(2),$$
$$\alpha_0 + 3\alpha_1 + 3\alpha_2 + \alpha_3 = \mu + \tau(3), \ldots$$

and deduce at once that

$$|\alpha_\nu|_p \leq \frac{1}{p} \text{ if } \nu = 1, 2, 3, \ldots.$$

On the other hand, it was assumed that

$$\left|\frac{a_n}{n}\right|_p < 1 \text{ if } p^s \nmid n, \ n \geq p^s,$$

$$\limsup_{n\to\infty} \left|\frac{a_n}{n}\right|_p = 1.$$

Hence there are infinitely many suffixes n for which

$$p^s \mid n \text{ and } \left|\frac{a_n}{n}\right|_p = 1,$$

and so there exist also infinitely many suffixes ν satisfying

$$|\alpha_\nu|_p = 1,$$

contrary to what has just been proved.

Thus the hypothesis at the beginning of § 8 likewise leads to a contradiction. This proves the assertion (i) of Theorem 2.

11. Instead of the assertion (ii) of Theorem 2 we prove now a slightly stronger result.

Lemma 4. *Let* $\left\{\frac{a_n}{n}\right\}$ *be a p-adic null sequence. Then the limit*

$$\lambda = \lim_{x \to 0} \sum_{n=1}^{\infty} \frac{a_n}{n} \binom{x-1}{n-1}$$

extended over all $x \in I$ *exists and is equal to*

$$\lambda = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}.$$

Proof. Let s be any given positive integer. There exists a positive integer N such that $\left|\frac{a_n}{n}\right|_p \leq p^{-s}$ if $n > N$, hence

$$\left|\sum_{n=N+1}^{\infty} (-1)^{n-1} \frac{a_n}{n}\right|_p \leq p^{-s} \text{ and } \left|\sum_{n=N+1}^{\infty} \frac{a_n}{n} \binom{x-1}{n-1}\right|_p \leq p^{-s};$$

here both series converge as their terms tend to zero.

The finite sum

$$\sum_{n=1}^{N} \frac{a_n}{n} \binom{x-1}{n-1}$$

is a polynomial in x, hence is a continuous function, and so

$$\lim_{x \to 0} \sum_{n=1}^{N} \frac{a_n}{n} \binom{x-1}{n-1} = \sum_{n=1}^{N} \frac{a_n}{n} \binom{-1}{n-1} = \sum_{n=1}^{N} (-1)^{n-1} \frac{a_n}{n}.$$

Therefore a positive integer $t = t(s)$ exists such that

$$\left|\sum_{n=1}^{N} \frac{a_n}{n} \binom{x-1}{n-1} - \sum_{n=1}^{N} (-1)^{n-1} \frac{a_n}{n}\right|_p \leq p^{-s} \text{ if } |x|_p \leq p^{-t}.$$

On combining these estimates, we find that

$$\left|\sum_{n=1}^{\infty} \frac{a_n}{n} \binom{x-1}{n-1} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n}{n}\right|_p \leq p^{-s} \text{ if } |x|_p \leq p^{-t}.$$

Since s may be arbitrarily large and t depends only on s, the assertion follows at once.

12. Theorem 3. *Let* $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ *be continuous on* I, *and let* $a_n(y)$ *be defined as in Lemma* 3. *The function* $f(x)$ *is differentiable at a point* $y \in I$ *if, and only if,*

$$\lim_{n \to \infty} \frac{a_n(y)}{n} = 0,$$

and then

$$f'(y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a_n(y)}{n}.$$

Proof. By Lemma 3, $\{a_n(y)\}$ is a null sequence, and

$$f(x+y) = \sum_{n=0}^{\infty} a_n(y) \binom{x}{n}.$$

Therefore

$$\frac{f(x+y) - f(y)}{x} = \frac{1}{x} \sum_{n=1}^{\infty} a_n(y) \binom{x}{n} = \sum_{n=1}^{\infty} \frac{a_n(y)}{n} \binom{x-1}{n-1}.$$

The assertion follows therefore immediately from the definition of the derivative and from Theorem 2 and Lemma 4.

13. Theorem 4. *Let* $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ *be continuous on* I. *If the derivative* $f'(x)$ *exists and is continuous for all* $x \in I$, *then*

(i) *all series* $a'_n = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{a_{k+n}}{k}$ $(n = 0, 1, 2, \ldots)$ *converge*;

(ii) *the sequence* $\{a'_n\}$ *is a null sequence*; *and*

(iii) $$f'(x) = \sum_{n=0}^{\infty} a'_n \binom{x}{n} \quad \textit{if } x \in I.$$

Proof. Assume, first, that $f'(y)$ exists for all $y \in J$. By Theorem 3, the sequence

$$\left\{\frac{a_k(y)}{k}\right\} = \left\{\sum_{n=0}^{y} \frac{a_{k+n}}{k} \binom{y}{n}\right\}$$

is a null sequence. As this holds for each $y = 0, 1, 2, \ldots$, the simpler sequences

$$\left\{\frac{a_{1+n}}{1}, \frac{a_{2+n}}{2}, \frac{a_{3+n}}{3}, \ldots\right\}$$

are likewise null sequences when $n = 0, 1, 2, \ldots$. The series a'_n therefore all converge, and $f'(y)$ is given by

$$f'(y) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{a_k(y)}{k} = \sum_{n=0}^{y} a'_n \binom{y}{n} \qquad \text{if } y \in J.$$

Hence the formal interpolation series

$$f^{**}(x) = \sum_{n=0}^{\infty} a'_n \binom{x}{n},$$

which for $x \in J$ reduces to a finite sum, satisfies the equations

$$f^{**}(x) = f'(x) \text{ if } x \in J.$$

Secondly, let $f'(x)$ exist and be continuous for all $x \in I$. By Theorem 1, $f'(x)$ can then be developed into a convergent interpolation series, and this must be exactly the series $f^{**}(x)$ because $f^{**}(x)$ coincides with $f'(x)$ for $x \in J$. Therefore, again by Theorem 1, the assertions (ii) and (iii) follow at once.

14. By way of example, let us consider two special functions. First, let

$$f(x) = \sum_{r=0}^{\infty} p^r \binom{x}{p^r} = \sum_{n=0}^{\infty} a_n \binom{x}{n} \text{ where } a_n = \begin{cases} p^r & \text{if } n = p^r, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$a_n(x) = \sum_{k=0}^{\infty} a_{k+n} \binom{x}{k} = \sum_{p^r \geq n} p^r \binom{x}{p^r - n},$$

and so, in particular,

$$\frac{a_{p^s}(x)}{p^s} = \sum_{r=s}^{\infty} p^{r-s} \binom{x}{p^r - p^s} = 1 + p\binom{x}{p^{s+1} - p^s} + p^2 \binom{x}{p^{s+2} - p^s} + \cdots = 1 + \alpha_s(x),$$

where

$$|\alpha_s(x)|_p \leq \frac{1}{p} \qquad (s = 0, 1, 2, \ldots).$$

Therefore $\left\{\frac{a_n(x)}{n}\right\}$ is not a null sequence, and so, by Theorem 3, $f'(x)$ does not exist. Thus while $f(x)$ evidently is continuous, it is nowhere differentiable on I.

In his paper, Dieudonné constructed already a function of the same kind by an entirely different method. Let $x = g_0 + g_1 p + g_2 p^2 + \cdots$ be the p-adic development of $x \in I$; the digits $g_0, g_1, g_2, \ldots$ assume only the values $0, 1, \ldots, p-1$. Then the function $f(x)$ defined by

$$f(x) = g_0^2 + g_1^2 p + g_2^2 p^2 + \cdots$$

is continuous but non-differentiable on I provided that $p \geqq 3$.

15. As a second example take

$$f(x) = \sum_{r=0}^{\infty} p^r \binom{x}{p^r - 1} = \sum_{n=0}^{\infty} a_n \binom{x}{n} \text{ where } a_n = \begin{cases} p^r \text{ if } n = p^r - 1, \\ 0 \text{ otherwise}, \end{cases}$$

so that

$$a_n(x) = \sum_{p^r \geqq n+1} p^r \binom{x}{p^r - n - 1}.$$

First let $x = -1$. Evidently

$$a_n(-1) = \sum_{p^r \geqq n+1} p^r \binom{-1}{p^r - n - 1} = (-1)^{p-n-1} \sum_{p^r \geqq n+1} p^r,$$

and therefore

$$\frac{a_{p^s}(-1)}{p^s} = -\sum_{r=s+1}^{\infty} p^r = -\frac{p}{1-p}.$$

Hence $\left\{\frac{a_n(-1)}{n}\right\}$ is not a null sequence, and $f'(-1)$ does not exist.

Assume next that $x \neq -1$. Then $\frac{a_n(x)}{n}$ may be written as

$$\frac{a_n(x)}{n} = \frac{1}{x+1} \sum_{p^r \geqq n+1} p^r \frac{p^r - n}{n} \binom{x+1}{p^r - n}.$$

Here the summation extends over all suffixes $r \geqq s+1$ where s is the integer defined by $p^s \leqq n < p^{s+1}$. Now it is obvious that

$$\left|\frac{p^r - n}{n}\right|_p = 1 \text{ if } r \geqq s+1,$$

and $\binom{x+1}{p^r - n}$ is a p-adic integer. Therefore it follows from the series that

$$\left|\frac{a_n(x)}{n}\right|_p \leqq \frac{p^{-(s+1)}}{|x+1|_p} < \frac{1}{n\,|x+1|_p},$$

and hence $\left\{\frac{a_n(x)}{n}\right\}$ is a null sequence; thus $f'(x)$ exists. It is not difficult to show that $f'(x)$ is in fact continuous if $x \neq -1$.

One can also easily verify that all series a_n' converge, but that $\{a_n'\}$ ist not a null sequence.

16. I have not succeeded in solving the following problems which deserve further study[1]).

Problem A. *Let* $\{a_n\}$ *be a null sequence, so that* $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ *is continuous on* I. *Further assume that,* (i) *all series*

$$a_n' = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{a_{k+n}}{k} \qquad (n = 0, 1, 2, \ldots)$$

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[1]) I. W. S. Cassels has just shown, by means of a very beautiful counter-example, that both problems A and B have negative answers. The problem of finding necessary and sufficient conditions, in terms of the a_n, for the continuity of $f'(x)$ remains therefore still open. (16 November, 1956)

converge, and (ii) $\{a'_n\}$ *is a null sequence. Does this hypothesis imply that* $f'(x)$ *exists and is continuous on* I?

Problem B. *Let* $\{a_n\}$ *satisfy the same hypothesis as in Problem* A. *Is it true that then*

$$\lim_{n\to\infty} n \mid a_n \mid_p = 0\,?$$

If this limit is zero, then the conditions (i) and (ii) are satisfied, and $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ has a continuous derivative on I, as is proved without difficulty.

17. I conclude this paper with an application of a theorem by Dieudonné and Theorem 4 to a special infinite system of linear equations.

In his paper, Dieudonné established a general existence theorem for differential equations in the p-adic field. The simplest case of this theorem states:

If $g(x)$ *is continuous on* I, *then for every* $\varepsilon > 0$ *there exists a function* $f(x)$ *continuous and continuously differentiable on* I *which is such that* $f'(x) = g(x)$ *and* $\mid f(x) \mid_p < \varepsilon$ *for all* $x \in I$.

For write again x as a p-adic series $x = g_0 + g_1 p + g_2 p^2 + \cdots$ and put

$$x_n = g_0 + g_1 p + \cdots + g_{n-1} p^{n-1}.$$

Further let s be any fixed positive integer. The sequence of functions

$$f_n(x) = \sum_{k=s}^{n-1} (x_{k+1} - x_k)\, g(x_k) + (x - x_n)\, g(x_n) \qquad (n = s, s+1, s+2, \ldots)$$

can then be shown to tend to a limit function $f(x)$ with the required properties, provided s exceeds a certain bound which depends only on ε and the given function $g(x)$.

With the help of this theorem, we show the

Theorem 5. *Let* $\{a'_n\}$ *be any null sequence, and let* ε *be an arbitrary positive constant. There exists a second null sequence* $\{a_n\}$ *such that*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{a_{k+n}}{k} = a'_n, \quad \mid a_n \mid_p < \varepsilon \qquad (n = 0, 1, 2, \ldots).$$

Proof. The function $g(x) = \sum_{n=0}^{\infty} a'_n \binom{x}{n}$ is continuous on I. Let $f(x)$ be the function of Dieudonné satisfying $f'(x) = g(x)$ and $\mid f(x) \mid_p < \varepsilon$ for $x \in I$. This function can itself be expanded into an interpolation series $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ with coefficients a_n that likewise form a null sequence. Since

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k),$$

these coefficients satisfy the inequalities $\mid a_n \mid_p < \varepsilon$. Since further $f(x)$ has the continuous derivative $g(x)$, it follows from Theorem 4 that the coefficients a_n also satisfy the linear equations of Theorem 5.

The result so proved suggests that there may be an interesting general theory of infinite systems of linear equations in infinitely many p-adic unknowns.

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