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BY

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In his Topics in Number Theory, vol. 2, chapter 2 (Reading, Mass., 1956) W. J. LeVeque proved an important generalisation of Roth's theorem (K. F. Roth, Mathematika 2, 1955, 1—20).

Let ξ be a fixed algebraic number, σ a positive constant, and K an algebraic number field of degree n. For $\kappa \in K$ denote by $\kappa^{(1)}, \cdots, \kappa^{(n)}$ the conjugates of κ relative to K, by $h(\kappa)$ the smallest positive integer such that the polynomial

$$g(x) = h(\kappa) \prod_{j=1}^{n} (x - \kappa^{(j)}) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n$$

has rational integral coefficients, and by $q(\kappa)$ the quantity

$$q(\kappa) = \max(|b_0|, |b_1|, \cdots, |b_n|).$$

LeVeque's theorem states that the inequality

$$\text{(I)} \qquad |\kappa - \xi| \leqq q(\kappa)^{-\sigma}$$

can only then have infinitely many distinct solutions κ in K when $\sigma \leqq 2$. When K is the rational field, this exactly is Roth's theorem.

In the present paper I generalise LeVeque's theorem and, for $1 \leqq N \leqq n$, study the simultaneous approximation of N given algebraic numbers $\xi_1, \cdots, \xi_N$ by the conjugates $\kappa^{(1)}, \cdots, \kappa^{(N)}$ of those elements κ of K that satisfy the inequality

$$\text{(II)} \qquad h(\kappa) \leqq c'' q(\kappa)^{\tau}.$$

Here τ is a constant in the interval $0 \leqq \tau \leqq 1$, and c'' is an arbitrary positive constant. In the case $N = 1$ the result is that both (I) and (II) can only then hold for infinitely many distinct κ in K when $\sigma \leqq 1 + \tau$.

Of particular interest is the special case when $\tau = 0$ and $c'' = 1$. The problem then becomes that of approximating $\xi_1, \cdots, \xi_N$ by the conjugates $\kappa^{(1)}, \cdots, \kappa^{(N)}$ of an algebraic *integer* in K. Naturally this problem is only then non-trivial when K is neither the rational field nor an imaginary quadratic field; for, with the exception of these special fields, the integers of K lie dense on the real axis or in the whole complex plane. By considering the approximation by integers, one arrives at results on *non-homogeneous* Diophantine approximations for algebraic numbers. Such problems do not seem to have been studied before.

Theorems 1 and 2 contain the main results of this paper, and the paper ends with a few simple applications.

More general theorems can be proved, and I have, without proof, stated several possible generalisations in the Appendix C of my book Lectures on Diophantine Approximations, I (University of Notre Dame Press, Notre Dame, Indiana, 1961). Regrettably, the text of this appendix is disfigured by several bad misprints.

1. Throughout this paper K denotes a fixed algebraic number field, say of the (finite) degree n over the rational field R. The n fields $K^{(1)}, \cdots, K^{(n)}$ conjugate to K are considered as embedded in the complex field. Thus, if κ is any element of K, its n conjugates $\kappa^{(1)}, \cdots, \kappa^{(n)}$ relative to K are real or complex numbers.

There exists to every κ in K a smallest positive integer

$$h = h(\kappa) \geqq 1$$

such that

$$h(x-\kappa^{(1)}) \cdots (x-\kappa^{(n)}), = g(x) = b_0x^n+b_1x^{n-1}+ \cdots +b_n \qquad \text{say,}$$

is a polynomial with rational integral coefficients. We put

$$q = q(\kappa) = \max (|b_0|, |b_1|, \cdots, |b_n|)$$

and call q the height of κ relative to K. Then

$$1 \leqq h \leqq q$$

because $b_0 = h$. According as to whether κ generates K, or a subfield of K, $g(x)$ is irreducible over R, or it is at least the second power of such an irreducible polynomial.

Denote by

$$\xi_1, \cdots, \xi_n$$

n fixed algebraic numbers that need not lie in K and may be chosen completely arbitrarily. Further put

$$\Delta = \Delta(\kappa) = \prod_{j=1}^{n} \min(1, |\kappa^{(j)}-\xi_j|)$$

so that

$$0 \leqq \Delta \leqq 1 \qquad \text{for all } \kappa \text{ in } K.$$

Then Δ vanishes for at most finitely many elements of K, and it is exactly then less than 1 when at least one of the inequalities

$$|\kappa^{(j)}-\xi_j| < 1 \qquad (j = 1, 2, \cdots, n)$$

is satisfied.

2. Let

$$\Sigma = \{\kappa(1), \kappa(2), \kappa(3), \cdots\}$$

be an infinite sequence of distinct elements $\kappa(l)$ of K. For shortness put

$$h(l) = h(\kappa(l)), \quad q(l) = q(\kappa(l)), \quad \Delta(l) = \Delta(\kappa(l)),$$

and denote by $\kappa^{(1)}(l), \cdots, \kappa^{(n)}(l)$ the conjugates of $\kappa(l)$. Then

$$(1) \qquad 1 \leqq h(l) \leqq q(l) \qquad (l = 1, 2, 3, \cdots)$$

and

$$(2) \qquad \lim_{l\to\infty} q(l) = \infty;$$

the latter formula holds because at most finitely many elements of K have heights less than any given number.

It is also clear that the products

$$h(l)\kappa(l) \qquad (l = 1, 2, 3, \cdots)$$

are integers in K. In fact, the following stronger result holds.

(3) *If $j_1, \cdots, j_N$ are indices such that*

$$1 \leqq j_1 < j_2 < \cdots < j_N \leqq n,$$

then, for each l, the product

$$h(l)\kappa^{(j_1)}(l) \cdots \kappa^{(j_N)}(l)$$

is an algebraic integer.

For a proof of this classical theorem see, e.g. W. J. LeVeque, Topics in Number Theory, vol. 2, p. 64.

Finally, for all l,

$$(4) \qquad \prod_{j=1}^{n} \max\left(1, \kappa^{(j)}(l)\right) \leqq (n+1)\frac{q(l)}{h(l)}.$$

For a proof see, e.g. my note Mathematika 7 (1960), pp. 98—100.

3. The sequence Σ is said to be admissible if

$$0 < \Delta(l) < 1 \qquad \text{for all } l.$$

Let Σ be such an admissible sequence. It is obvious that then also

$$(n+1)\frac{q(l)}{h(l)} > 1 \qquad \text{for all } l.$$

Therefore, for each l, there exist $2n$ non-negative numbers

$$A_1(l), \cdots, A_n(l), B_1(l), \cdots, B_n(l)$$

such that, simultaneously,

$$\left.\begin{aligned} \min(1, |\kappa^{(j)}(l)-\xi_j|) &= \Delta(l)^{A_j(l)} \\ \max(1, |\kappa^{(j)}(l)|) &= \left\{(n+1)\frac{q(l)}{h(l)}\right\}^{B_j(l)} \end{aligned}\right\} \qquad (j = 1, 2, \cdots, n).$$

Here

$$(5) \qquad \sum_{j=1}^{n} A_j(l) = 1, \quad \sum_{j=1}^{n} B_j(l) \leqq 1 \qquad \text{for all } l,$$

where the inequality follows from (4).

Denote now by η an arbitrarily small positive constant, and by ω a constant satisfying

$$\omega > n\eta^{-1}.$$

Further, for each pair j, l, let

$$A_j^*(l), B_j^*(l)$$

be the $2n$ integers defined by the inequalities

$$A_j^*(l) \leqq \omega A_j(l) < A_j^*(l)+1, \qquad B_j^*(l)-1 \leqq \omega B_j(l) < B_j^*(l).$$

Then these integers are likewise non-negative, and by (5)

$$\sum_{j=1}^{n} A_j^*(l) \leqq \omega < \sum_{j=1}^{n} A_j^*(l)+n, \qquad \sum_{j=1}^{n} B_j^*(l) < \omega+n.$$

Hence, for all j and l, these integers are bounded and so have at most finitely many possibilities.

Since Σ is an infinite sequence, it contains then an infinite subsequence Σ' for the elements of which the $2n$ integers

$$A_j^*(l) = A_j^* \quad \text{and} \quad B_j^*(l) = B_j^* \qquad (j = 1, 2, \cdots, n)$$

assume fixed values independent of l. On putting

$$\alpha_j^* = \omega^{-1} A_j^* \quad \text{and} \quad \beta_j = \omega^{-1} B_j^* \qquad (j = 1, 2, \cdots, n)$$

these constants are again non-negative, and it is obvious that

$$1-\eta < \sum_{j=1}^{n} \alpha_j^* \leqq 1, \qquad \sum_{j=1}^{n} \beta_j < 1+\eta$$

and

$$\alpha_j^* \leqq A_j(l), \qquad \beta_j \geqq B_j(l) \quad \text{for} \quad \kappa(l) \in \Sigma' \text{ and all } j.$$

Hence the following result has been obtained.

Lemma 1: *Let Σ be an admissible sequence, and let η be an arbitrarily small positive constant. There exist an infinite subsequence Σ' of Σ and a set of $2n$ non-negative constants $\alpha_1^*, \cdots, \alpha_n^*, \beta_1, \cdots, \beta_n$ satisfying*

$$1-\eta < \sum_{j=1}^{n} \alpha_j^* \leqq 1, \qquad \sum_{j=1}^{n} \beta_j < 1+\eta$$

with the property that

$$\left.\begin{aligned} \min\left(1, |\kappa^{(j)}(l)-\xi_j|\right) &\leqq \Delta(l)^{\alpha_j^*} \\ \max\left(1, |\kappa^{(j)}(l)|\right) &\leqq \left\{(n+1)\frac{q(l)}{h(l)}\right\}^{\beta_j} \end{aligned}\right\} \qquad (j=1, 2, \cdots, n)$$

for all elements $\kappa(l)$ of Σ'.

4. If σ is a positive constant, the sequence Σ is said to have the property $P(\sigma)$ if a further positive constant c' exists such that

$$\Delta(l) \leqq c' q(l)^{-\sigma} \qquad \text{for all } l.$$

Thus, if σ' is any constant satisfying

$$0 < \sigma' < \sigma,$$

then all but finitely many elements of Σ have the weaker property that

$$\Delta(l) \leqq q(l)^{-\sigma'}$$

because $q(l)$ tends to infinity with l.

Next, if τ is a constant in the interval

$$0 \leqq \tau \leqq 1,$$

Σ is said to have the property $Q(\tau)$ if a further positive constant c'' exists such that

$$h(l) \leqq c'' q(l)^{\tau} \qquad \text{for all } l.$$

For any constant τ' satisfying

$$\tau' > \tau$$

this implies then again that all but finitely many elements of Σ have the weaker property

$$h(l) \leqq q(l)^{\tau'}.$$

We note that, by (1), Σ always trivially has the property $Q(1)$ with $c'' = 1$.

More exactly, put

$$\tau_0 = \liminf_{l\to\infty} \frac{\log h(l)}{\log q(l)};$$

again, by (1),

$$0 \leqq \tau_0 \leqq 1.$$

From the definition of the lower limit there exists now an infinite subsequence of Σ with the property $Q(\tau)$ when $\tau > \tau_0$, but no such subsequence can exist when $\tau < \tau_0$. Furthermore, if τ_1 and τ_2 are constants such that

$$\tau_1 < \tau_0 < \tau_2$$

there is an infinite subsequence Σ'' of Σ with the property

$$q(l)^{\tau_1} \leqq h(l) \leqq q(l)^{\tau_2} \quad \text{for} \quad \kappa(l) \in \Sigma''.$$

Here we may choose

$$\tau_1 = 0 \quad \text{if} \quad \tau_0 = 0, \qquad \tau_2 = 1 \quad \text{if} \quad \tau_0 = 1,$$

and we may in addition assume that $\tau_2 - \tau_1$ is less than a prescribed positive constant.

5. Let Σ be a sequence with the property $P(\sigma)$, and let $\varepsilon > 0$ be an arbitrarily small positive constant. As we found, if

$$0 < \sigma' < \sigma,$$

all but finitely many elements of Σ have the property

$$\Delta(l) \leqq q(l)^{-\sigma'}. \tag{6}$$

By (2), this implies in particular that

$$\lim_{l \to \infty} \Delta(l) = 0,$$

hence that Σ becomes admissible if at most finitely many elements are omitted.

Without loss of generality, let already Σ itself be admissible and have the property (6). We apply Lemma 1 to Σ and, in the notation of this lemma, find that

$$\min(1, |\kappa^{(j)}(l) - \xi_j|) \leqq \Delta(l)^{\alpha_j^*} \leqq q(l)^{-\alpha_j^* \sigma'} \qquad (j = 1, 2, \cdots, n).$$

For shortness put

$$\alpha_1 = \alpha_1^* \sigma', \cdots, \alpha_n = \alpha_n^* \sigma'.$$

Then $\alpha_1, \cdots, \alpha_n$ are non-negative, and by the lemma,

$$(1-\eta)\sigma' < \sum_{j=1}^{n} \alpha_j \leqq \sigma' < \sigma.$$

Here the difference

$$\sigma - (1-\eta)\sigma' = (\sigma - \sigma') + \eta\sigma' < (\sigma - \sigma') + \eta\sigma$$

can be made less than ε by choosing both $\sigma - \sigma'$ and η sufficiently small, say with $\eta < \varepsilon$. Hence the following result is obtained.

Lemma 2: *Let Σ be a sequence with the property $P(\sigma)$ where $\sigma > 0$, and let $\varepsilon > 0$ be arbitrarily small. There exist an infinite subsequence Σ_1 of Σ and a set of $2n$ non-negative constants $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ satisfying*

$$\sigma-\varepsilon < \sum_{j=1}^{n} \alpha_j < \sigma, \qquad \sum_{j=1}^{n} \beta_j < 1+\varepsilon,$$

such that

$$\left.\begin{aligned} \min(1, |\kappa^{(j)}(l)-\xi_j|) &\leqq q(l)^{-\alpha_j} \\ \max(1, |\kappa^{(j)}(l)|) &\leqq \left\{(n+1)\,\frac{q(l)}{h(l)}\right\}^{\beta_j} \end{aligned}\right\} \qquad (j = 1, 2, \cdots, n)$$

for all elements $\kappa(l)$ of Σ_1.

6. In what follows, we shall be concerned with polynomials in several variables, of the form

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1 \cdots i_m} x_1^{i_1} \cdots x_m^{i_m}.$$

We use then the abbreviated notation

$$A_{j_1 \cdots j_m}(x_1, \cdots, x_m) = \frac{\partial^{j_1+\cdots+j_m} A(x_1, \cdots, x_m)}{j_1! \cdots j_m! \partial x_1^{j_1} \cdots \partial x_m^{j_m}},$$

where $j_1, \cdots, j_m$ denote arbitrary non-negative integers.

Two results on such polynomials are required. The first result is a special case of a lemma due to LeVeque which generalises Roth's Lemma.

Lemma 3: *Let $m, r_1, \cdots, r_m, q_1, \cdots, q_m$ be positive integers, and let t be a positive number, such that*

$$0 < t < (2^{m+2}m)^{-1}, \quad r_m > 10t^{-1}, \quad \frac{r_j}{r_{j-1}} < t \quad \text{for} \quad j = 2, 3, \cdots, m,$$

$$\log q_1 > 2m(2m+1)t^{-1}, \quad r_j \log q_j \geqq r_1 \log q_1 \quad \text{for} \quad j = 2, 3, \cdots, m.$$

Let further

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1 \cdots i_m} x_1^{i_1} \cdots x_m^{i_m} \not\equiv 0$$

be a polynomial with rational integral coefficients satisfying

$$|a_{i_1 \cdots i_m}| \leqq q_1^{r_1 t} \text{ for all } i_1, \cdots, i_m.$$

Let finally $\kappa_1, \cdots, \kappa_m$ be m elements of K of the heights $q_1, \cdots, q_m$, respectively. Then m non-negative suffixes $J_1, \cdots, J_m$ exist such that

$$A_{J_1 \cdots J_m}(\kappa_1, \cdots, \kappa_m) \neq 0, \qquad \sum_{l=1}^{m} \frac{J_l}{r_l} \leqq 10^m t^{1/2^m}.$$

For the proof see W. J. LeVeque, Topics in Number Theory, vol. 2, pp. 124—142. (Choose for K the rational field and put $N = 1$.)

The second result needed is an existence theorem.

LEMMA 4: *Let*

$$F(x) = F_0x^f + F_1x^{f-1} + \cdots + F_f, \quad \text{where} \quad f \geqq 1,\ F_0 \neq 0,$$

be a polynomial with rational integral coefficients which has no multiple zeros. Put

$$c = 80 \max(|F_0|, |F_1|, \cdots, |F_f|).$$

Let $r_1, \cdots, r_m$ be arbitrary positive integers, and let s be a positive number not less than $4f(2m)^{\frac{1}{2}}$. Then there exists a polynomial

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1\cdots i_m} x_1^{i_1} \cdots x_m^{i_m} \not\equiv 0$$

with the following properties.

(A) *Its coefficients are rational integers such that*

$$|a_{i_1\cdots i_m}| \leqq c^{r_1+\cdots+r_m} \quad \textit{for all } i_1, \cdots, i_m,$$

and they vanish unless

$$\sum_{l=1}^{m} \frac{i_l}{r_l} < \tfrac{1}{2}(m+s).$$

(B) *$A_{j_1\cdots j_m}(x, \cdots, x)$ is divisible by $F(x)$ whenever*

$$\sum_{l=1}^{m} \frac{j_l}{r_l} \leqq \tfrac{1}{2}(m-s).$$

(C) *The derivatives of the polynomial have the majorants*

$$A_{j_1\cdots j_m}(x_1, \cdots, x_m) \ll c^{r_1+\cdots+r_m}(1+x_1)^{r_1} \cdots (1+x_m)^{r_m}.$$

For a proof see my Lectures on Diophantine Approximations, I, pp. 98—105. (The assumption made in this proof that F_f does not vanish is used nowhere and may be discarded.)

7. The main result of this paper is as follows.

THEOREM 1: *Let Σ be an infinite sequence of distinct elements $\kappa(l)$ of K with the two properties $P(\sigma)$ and $Q(\tau)$ where*

$$\sigma > 0, \qquad 0 \leqq \tau \leqq 1.$$

Then

$$\sigma \leqq 1+\tau.$$

As may be expected, the proof of this theorem is somewhat involved, although in its basic ideas it is quite simple. It is indirect: It will be assumed that the assertion of the theorem is false, thus that

(7) $$\sigma > 1+\tau,$$

and from this assumption a contradiction will be deduced.

We first replace (7) by a stronger assumption.

Since Σ has the property $P(\sigma)$, for every positive constant σ' less than σ it trivially has also the property $P(\sigma')$. Therefore, if $\tau = 0$, we may without loss of generality assume that

$$\sigma = 1+20\varepsilon = 1+\tau+20\varepsilon \quad \text{where} \quad 0 \leqq \varepsilon \leqq \tfrac{1}{10}.$$

Next let $0 < \tau \leqq 1$. By the discussion in § 4, there exist now an infinite subsequence Σ'' of Σ and a pair of constants τ_1, τ_2 with arbitrarily small difference $\tau_2 - \tau_1$ satisfying

$$0 \leqq \tau_1 < \tau_2 \leqq \tau \leqq 1, \quad \text{hence} \quad 1+\tau \geqq 1+\tau_2,$$

such that

$$q(l)^{\tau_1} \leqq h(l) \leqq q(l)^{\tau_2} \quad \text{for all} \quad \kappa(l) \in \Sigma''.$$

Here we can identify τ_2 with τ and then, without loss of generality, replace Σ by Σ''. After a small change of notation we therefore find that, if Theorem 1 is false, the following assumption may be made.

HYPOTHESIS. *There exist three constants ε, σ, τ, a fourth constant c'' for $\tau = 0$, and an infinite sequence Σ with the property $P(\sigma)$, such that either*

$$\tau = 0, \qquad 0 < \varepsilon \leqq \tfrac{1}{10}, \qquad \sigma = 1+20\varepsilon = 1+\tau+20\varepsilon, \qquad c'' \geqq 1,$$

and

$$1 \leqq h(l) \leqq c'' \qquad \textit{for all } l;$$

or

$$0 < \tau \leqq 1, \qquad 0 < \varepsilon \leqq \min(\tfrac{1}{10}, \tau), \qquad \sigma = 1+\tau+20\varepsilon,$$

and

$$q(l)^{\tau-\varepsilon} \leqq h(l) \leqq q(l)^{\tau} \qquad \textit{for all } l.$$

This hypothesis will finally lead to a contradiction.

8. By Lemma 2, there exist an infinite subsequence Σ_1 of Σ and a set of $2n$ non-negative constants $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ satisfying

(8) $$\sigma - \varepsilon < \sum_{j=1}^{n} \alpha_j < \sigma, \qquad \sum_{j=1}^{n} \beta_j < 1+\varepsilon$$

such that

(9) $$\left\{ \begin{array}{l} \min\left(1, |\kappa^{(j)}(l) - \xi_j|\right) \leqq q(l)^{-\alpha_j} \\ \max\left(1, |\kappa^{(j)}(l)|\right) \leqq \left\{(n+1)\dfrac{q(l)}{h(l)}\right\}^{\beta_j} \end{array} \right\} \qquad (j = 1, 2, \cdots, n)$$

for all elements $\kappa(l)$ of Σ_1. To this subsequence Σ_1 we shall apply Lemmas 3 and 4. It is, however, necessary first to fix the parameters that occur in these lemmas.

For $F(x)$ in Lemma 4 we take the primitive polynomial

$$F(x) = F_0 x^f + F_1 x^{f-1} + \cdots + F_f, \quad \text{where} \quad F_0 > 0,$$

with rational integral coefficients and of lowest degree $f \geqq 1$, for which

$$F(\xi_1) = \cdots = F(\xi_n) = 0. \tag{10}$$

Such a polynomial exists because the numbers $\xi_1, \cdots, \xi_n$ are algebraic. While $F(x)$ may possibly be reducible over R, it certainly cannot have any multiple zeros.

This choice of $F(x)$ fixes the two numbers f and c. Next we put

$$m = \left[\frac{32f^2}{\varepsilon^2}\right] + 1, \qquad s = \varepsilon m, \tag{11}$$

so that

$$m > \frac{32f^2}{\varepsilon^2} \quad \text{and therefore} \quad s \geqq 4f(2m)^{\frac{1}{2}},$$

as is required in Lemma 4.

For the further parameter t we choose any constant such that

$$0 < t < \min(\varepsilon, (2^{m+2}m)^{-1}), \qquad 10^m t^{1/2^m} \leqq \varepsilon m. \tag{12}$$

We now select m distinct elements of Σ_1,

$$\kappa(l_1) = \kappa_1, \cdots, \kappa(l_m) = \kappa_m \quad \text{say},$$

and for shortness write

$$q(l_1) = q_1, \cdots, q(l_m) = q_m \quad \text{and} \quad h(l_1) = h_1, \cdots, h(l_m) = h_m.$$

Since Σ_1 is an infinite sequence of distinct elements, the numbers $\kappa_1, \cdots, \kappa_m$ can be chosen so as to satisfy the inequalities

$$\log q_1 > \max(2m(2m+1)t^{-1}, \log T), \tag{13}$$

$$\log q_l \geqq \frac{2}{t} \log q_{l-1} \qquad (l = 2, 3, \cdots, m). \tag{14}$$

Here T is to denote a sufficiently large positive constant that will be fixed later in terms of f, c and ε, as well as of c'' if $\tau = 0$.

The inequalities (9) now take the simpler form,

$$\left\{\begin{array}{l} \min(1, |\kappa_l^{(j)} - \xi_j|) \leqq q_l^{-\alpha_j} \\ \max(1, |\kappa_l^{(j)}|) \leqq \left\{(n+1)\dfrac{q_l}{h_l}\right\}^{\beta_j} \end{array}\right\} \quad \begin{pmatrix} j = 1, 2, \cdots, n \\ l = 1, 2, \cdots, m \end{pmatrix}. \tag{15}$$

In addition, by the *Hypothesis*,

$$(16) \qquad \begin{cases} 1 \leqq h_l \leqq c'' & \text{if} \quad \tau = 0 \\ q_l^{\tau-\varepsilon} \leqq h_l \leqq q_l^{\tau} & \text{if} \quad \tau > 0 \end{cases} \qquad (l = 1, 2, \cdots, m).$$

We finally choose m positive integers $r_1, \cdots, r_m$ so as to satisfy the inequalities

$$(17) \qquad r_1 > \frac{10}{t} \cdot \frac{\log q_m}{\log q_1}$$

and

$$(18) \qquad r_l \geqq r_1 \frac{\log q_1}{\log q_l} > r_l - 1 \qquad (l = 2, 3, \cdots, m).$$

Then, in particular,

$$r_m \geqq r_1 \frac{\log q_1}{\log q_m} > \frac{10}{t}.$$

Since t trivially is less than 1, it follows from (13) and (14) that

$$(19) \qquad 2 < q_1 < q_2 < \cdots < q_m.$$

Hence, for all l,

$$r_l \geqq r_1 \frac{\log q_1}{\log q_l} \geqq r_1 \frac{\log q_1}{\log q_m} > \frac{10}{t} > 2.$$

The formula

$$0 < t \leqq \varepsilon \leqq \tfrac{1}{10}$$

implies then that

$$r_l - 1 = r_l \left(1 - \frac{1}{r_l}\right) > r_l \left(1 - \frac{t}{10}\right) \geqq r_l \left(1 - \frac{\varepsilon}{10}\right) > \frac{r_l}{1+\varepsilon}$$

because

$$\left(1 - \frac{\varepsilon}{10}\right)(1+\varepsilon) = 1 + \frac{\varepsilon}{10}(9-\varepsilon) > 1.$$

Hence, by (18),

$$(20) \qquad r_{l-1} \log q_{l-1} \geqq r_1 \log q_1 > (r_l - 1) \log q_l > \frac{r_l}{1+\varepsilon} \log q_l > \frac{r_l}{2} \log q_l$$

$$(l = 2, 3, \cdots, m).$$

Therefore

$$(21) \qquad q_1^{r_1} \leqq q_l^{r_l} < q_1^{r_1(1+\varepsilon)} \qquad (l = 1, 2, \cdots, m),$$

and by (14),

$$r_{l-1} > \frac{1}{t} r_l \qquad (l = 2, 3, \cdots, m),$$

so that in particular

(22) $$r_1 > r_2 > \cdots > r_m > 2, \qquad r_1+r_2+\cdots+r_m < mr_1.$$

9. The formulae in the last section show that $F(x)$, $m, s, r_1, \cdots, r_m$ satisfy the conditions of Lemma 4. There hence exists a polynomial

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1 \cdots i_m} x_1^{i_1} \cdots x_m^{i_m} \not\equiv 0$$

with the properties (A), (B), and (C), of Lemma 4. In particular, for all suffixes $i_1, \cdots, i_m$,

$$|a_{i_1 \cdots i_m}| \leqq c^{r_1+r_2+\cdots+r_m} < c^{mr_1} \leqq q_1^{r_1 t}$$

provided that

$$q_1 \geqq c^{m/t}.$$

By (13) this condition is satisfied if we demand from now on that

(23) $$T \geqq c^{m/t}.$$

Now $m, t, \kappa_1, \cdots, \kappa_m, r_1, \cdots, r_m$ have also the properties required in Lemma 3. It follows then from this lemma and from the formulae (13) for t that there exist m non-negative suffixes $J_1, \cdots, J_m$ such that

(24) $$A_{J_1 \cdots J_m}(\kappa_1, \cdots, \kappa_m) \neq 0, \qquad \sum_{l=1}^{m} \frac{J_l}{r_l} \leqq \varepsilon m.$$

For shortness put

$$\gamma = A_{J_1 \cdots J_m}(\kappa_1, \cdots, \kappa_m).$$

Since the polynomial $A_{J_1 \cdots J_m}(x_1, \cdots, x_m)$ has rational coefficients, γ is a number in the algebraic number field K, and its conjugates with respect to K have the values

$$\gamma^{(j)} = A_{J_1 \cdots J_m}(\kappa_1^{(j)}, \cdots, \kappa_m^{(j)}) \qquad (j = 1, 2, \cdots, n).$$

As the norm of γ, the product of all these conjugates,

$$\Gamma = \gamma^{(1)} \gamma^{(2)} \cdots \gamma^{(n)},$$

is a rational number. By (24), γ and so also all its conjugates $\gamma^{(j)}$ are distinct from zero, leading to the important inequality

(25) $$\Gamma \neq 0.$$

The next aim will be to establish lower and upper estimates for $|\Gamma|$.

10. A lower estimate for $|\Gamma|$ can be obtained by determining an upper bound for the denominator of this rational number. This is done by considering the product

$$\Pi = h_1^{r_1} \cdots h_m^{r_m} \Gamma = h_1^{r_1} \cdots h_m^{r_m} \prod_{j=1}^{n} A_{J_1 \cdots J_m}(\kappa_1^{(j)}, \cdots, \kappa_m^{(j)}).$$

On substituting the explicit expressions as polynomials for the factors

$$A_{J_1 \cdots J_m}(\kappa_1^{(j)}, \cdots \kappa_m^{(j)}),$$

Π becomes a sum of finitely many terms of the form

$$G = g h_1^{r_1} \cdots h_m^{r_m} \prod_{l=1}^{m} \prod_{j=1}^{n} \kappa_l^{(j) i_{jl}}.$$

Here g is a product of certain n coefficients of $A_{J_1 \cdots J_m}(x_1, \cdots, x_m)$ and so is a rational integer, and the exponents i_{jl} are integers such that

$$0 \leqq i_{jl} \leqq r_l \qquad \begin{pmatrix} j = 1, 2, \cdots, n \\ l = 1, 2, \cdots, m \end{pmatrix}.$$

By the property (3) in § 2, the m factors

$$h_l^{r_l} \prod_{j=1}^{n} \kappa_l^{(j) i_{jl}} \qquad (l = 1, 2, \cdots, m)$$

of G are algebraic integers. It follows that G is likewise an algebraic integer. As the sum of the terms G, the rational number Π is then also an algebraic integer and hence a rational integer. By (25), it is distinct from zero, and so its absolute value is at least 1. It follows that

$$|\Gamma| \geqq (h_1^{r_1} \cdots h_m^{r_m})^{-1}.$$

We finally apply the formulae (16) and (21), so finding that

$$(26) \qquad |\Gamma| \geqq \begin{cases} c''^{-(r_1 + \cdots + r_m)} > c''^{-mr_1} & \text{if} \quad \tau = 0, \\ (q_1^{r_1} \cdots q_m^{r_m})^{-\tau} > q_1^{-mr_1(1+\varepsilon)\tau} & \text{if} \quad \tau > 0. \end{cases}$$

11. To find an upper bound for $|\Gamma|$ we first determine such bounds for each of the numbers $|\gamma^{(j)}|$.

Denote by S the set of all suffixes $j = 1, 2, \cdots, n$ for which

$$\alpha_j > 0$$

and by S' the set of such suffixes for which

$$\alpha_j = 0.$$

By the *Hypothesis* and by the first formula (8), S contains at least one element; on the other hand, S' may or may not be the null set.

We begin by establishing a rather weak upper bound for $|\gamma^{(j)}|$ which is valid for all j, whether in S or in S'.

From the definition, $\gamma^{(j)}$ has the explicit value

$$\gamma^{(j)} = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1\cdots i_m} \binom{i_1}{J_1} \cdots \binom{i_m}{J_m} \kappa_1^{(j)i_1-J_1} \cdots \kappa_m^{(j)i_m-J_m}.$$

Here, by the assertion (A) of Lemma 4 and by (22),

$$|a_{i_1\cdots i_m}| \leqq c^{r_1+\cdots+r_m} < c^{mr_1} \qquad \text{for all } i_1, \cdots, i_m,$$

and

$$a_{i_1\cdots i_m} = 0 \quad \text{unless} \quad \sum_{l=1}^{m} \frac{i_l}{r_l} < \tfrac{1}{2}(m+s).$$

Denote by I the set of all systems of m integers $i_1, \cdots, i_m$ where

$$J_1 \leqq i_1 \leqq r_1, \cdots, J_m \leqq i_m \leqq r_m, \qquad \sum_{l=1}^{m} \frac{i_l}{r_l} < \tfrac{1}{2}(m+s).$$

It is evident that the term

$$a_{i_1\cdots i_m} \binom{i_1}{J_1} \cdots \binom{i_m}{J_m} \kappa_1^{(j)i_1-J_1} \cdots \kappa_m^{(j)i_m-J_m}$$

of $\gamma^{(j)}$ can only then be distinct from zero when $(i_1, \cdots, i_m)$ lies in I. It follows that

$$|\gamma^{(j)}| \leqq C^* C_j^{**}$$

where C^* and C_j^{**} denote the expressions

$$C^* = \sum_{(i_1,\cdots,i_m)\in I} |a_{i_1\cdots i_m}| \binom{i_1}{J_1} \cdots \binom{i_m}{J_m}$$

and

$$C_j^{**} = \max_{(i_1,\cdots,i_m)\in I} \{\max(1, |\kappa_1^{(j)}|)\}^{i_1-J_1} \cdots \{\max(1, |\kappa_m^{(j)}|)\}^{i_m-J_m}.$$

The sum C^* has not more than

$$(r_1+1)\cdots(r_m+1) \leqq 2^{r_1}\cdots 2^{r_m} < 2^{mr_1}$$

terms, and these terms are not greater than

$$c^{r_1+\cdots+r_m} \cdot 2^{r_1}\cdots 2^{r_m} < (2c)^{mr_1}.$$

Hence

$$C^* < 2^{mr_1} \cdot (2c)^{mr_1} = (4c)^{mr_1}.$$

Next, by the second line of (15), C_j^{**} does not exceed

$$C_j^{**} \leqq \max_{(i_1,\cdots,i_m)\in I} \left\{(n+1)\frac{q_1}{h_1}\right\}^{\beta_j i_1} \cdots \left\{(n+1)\frac{q_m}{h_m}\right\}^{\beta_j i_m}$$

$$\leqq (n+1)^{\beta_j(r_1+\cdots+r_m)} \max_{(i_1,\cdots,i_m)\in I} \left\{\left(\frac{q_1}{h_1}\right)^{i_1} \cdots \left(\frac{q_m}{h_m}\right)^{i_m}\right\}^{\beta_j}.$$

Here, by (16) and (21),

$$\left(\frac{q_l}{h_l}\right)^{r_l} \leqq \begin{cases} q_l^{r_l} \leqq q_1^{r_1(1+\varepsilon)} & \text{if} \quad \tau = 0 \\ q_l^{r_l(1-\tau+\varepsilon)} \leqq q_1^{r_1(1+\varepsilon)(1-\tau+\varepsilon)} & \text{if} \quad \tau > 0 \end{cases} \qquad (l = 1, 2, \cdots, m).$$

Therefore, by (22), for $\tau = 0$,

$$|C_j^{**}| < (n+1)^{mr_1\beta_j} \max_{(i_1, \cdots, i_m) \in I} q_1^{r_1(1+\varepsilon)\beta_j \sum_{l=1}^m i_l/r_l} < (n+1)^{mr_1\beta_j} q_1^{r_1(1+\varepsilon)\beta_j (m+s)/2},$$

and similarly, for $\tau > 0$,

$$|C_j^{**}| < (n+1)^{mr_1\beta_j} q_1^{r_1(1+\varepsilon)(1-\tau+\varepsilon)B_j(m+s)/2}.$$

Here, by (11),

$$s = \varepsilon m.$$

Therefore, on combining these estimates for C^* and C_j^{**}, finally

$$(27) \qquad |\gamma^{(j)}| < \begin{cases} (4c)^{mr_1}(n+1)^{mr_1\beta_j} q^{\frac{1}{2}mr_1(1+\varepsilon)^2\beta_j} & \text{if } \tau = 0, \\ (4c)^{mr_1}(n+1)^{mr_1\beta_j} q^{\frac{1}{2}mr_1(1+\varepsilon)^2(1-\tau+\varepsilon)\beta_j} & \text{if } \tau > 0. \end{cases}$$

12. A much better upper bound for $|\gamma^{(j)}|$ can be obtained when j lies in S because we may then apply the inequalities

$$(28) \qquad |\kappa_l^{(j)} - \xi_j| \leqq q_l^{-\alpha_j} \qquad (j \in S; l = 1, 2, \cdots, m),$$

which are implied by the first line of (15).

We use the identity

$$A_{J_1 \cdots J_m}(x_1, \cdots, x_m)$$
$$= \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} A_{j_1, \cdots, j_m}(x, \cdots, x) \binom{j_1}{J_1} \cdots \binom{j_m}{J_m} (x_1 - x)^{j_1 - J_1} \cdots (x_m - x)^{j_m - J_m},$$

which follows on applying Taylor's formula to $A_{J_1 \cdots J_m}(x_1, \cdots, x_m)$. Substitute the following values for the variables,

$$x_1 = \kappa_1^{(j)}, \cdots, x_m = \kappa_m^{(j)}, x = \xi_j.$$

Then we find that

$$\gamma^{(j)} = \sum_{j_1=0}^{r_1} \cdots \sum_{j_m=0}^{r_m} A_{j_1 \cdots j_m}(\xi_j, \cdots, \xi_j) \times$$
$$\times \binom{j_1}{J_1} \cdots \binom{j_m}{J_m} (\kappa_1^{(j)} - \xi_j)^{j_1 - J_1} \cdots (\kappa_m^{(j)} - \xi_j)^{j_m - J_m},$$

and here the last factors on the right-hand side may be replaced by (28).

Since each algebraic number ξ_j is a zero of $F(x)$, by the assertion (B) of Lemma 4,

$$A_{j_1 \cdots j_m}(\xi_j, \cdots, \xi_j) = 0 \quad \text{if} \quad \sum_{l=1}^{m}$$

Next, by the assertion (C) of the lemma,

$$|A_{j_1 \cdots j_m}(\xi_j, \cdots, \xi_j)| \leqq c^{r_1 + \cdots + r_m}(1+|\xi_j|)^{r}$$

It is further again obvious that

$$\binom{j_l}{J_l} = 0 \quad \text{if} \quad j_l < J_l, \qquad \binom{j_1}{J_1} \cdots \binom{j_m}{J_m}$$

and that $\gamma^{(j)}$ is a sum of not more than

$$(r_1+1) \cdots (r_m+1) \leqq 2^{r_1} \cdots$$

terms.

Denote by J the set of all systems of m i

$$J_1 \leqq j_1 \leqq r_1, \cdots, J_m \leqq j_m \leqq r_m,$$

From what has been said, only those terms of
that correspond to systems $(j_1, \cdots, j_m)$ in J

$$C_j = \max_{(j_1, \cdots, j_m) \in J} |\kappa_1^{(j)} - \xi_j|^{j_1 - J_1} \cdots$$

we find that

$$|\gamma^{(j)}| < 2^{mr_1} \cdot \{c(1+|\xi_j|)\}^{mr_1} \cdot 2^{mr_1} \cdot C_j$$

By (28) and by the definition of J,

$$C_j \leqq \Big(\min_{(j_1, \cdots, j_m) \in J} q_1^{j_1 - J_1} \cdots$$

since all exponents $j_1 - J_1, \cdots, j_m - J_m$ are no

$$q_1^{j_1 - J_1} \cdots q_m^{j_m - J_m} = q_1^{r_1(j_1 - J_1)/r_1} \cdots q_m^{r_m(j_m -}$$

Here

$$\sum_{l=1}^{m} \frac{j_l}{r_l} > \tfrac{1}{2}(m-s) = \tfrac{1}{2}m(1-\varepsilon) \quad \text{if}$$

and

$$\sum_{l=1}^{m} \frac{J_l}{r_l} \leqq \varepsilon m$$

since $s = \varepsilon m$, and therefore

$$\sum_{l=1}^{m} \frac{j_l - J_l}{r_l} > \tfrac{1}{2}m(1-\varepsilon) - \varepsilon m = \tfrac{1}{2}m(1-3\varepsilon) \quad \text{if} \quad (j_1, \cdots, j_m) \in J.$$

C_j has then the upper bound

$$C_j < q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)\alpha_j}.$$

Therefore, finally,

$$(29) \qquad |\gamma^{(j)}| < \{4c(1+|\xi_j|)\}^{mr_1} q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)\alpha_j} \quad \text{if} \quad j \in S.$$

13. In the equation

$$(30) \qquad |\Gamma| = \prod_{j \in S} |\gamma^{(j)}| \cdot \prod_{j \in S'} |\gamma^{(j)}|$$

the factors $|\gamma^{(j)}|$ will now be replaced by the upper bounds (29) when $j \in S$ and by the upper bounds (27) when $j \in S'$.

The set S has not more than n elements, and by Lemma 2

$$\sum_{j \in S} \alpha_j = \sum_{j=1}^{n} \alpha_j > \sigma - \varepsilon.$$

Hence

$$(31) \qquad \prod_{j \in S} |\gamma^{(j)}| < c_1^{mr_1} q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)(\sigma-\varepsilon)}$$

where c_1 denotes the constant

$$c_1 = (4c)^n \prod_{j=1}^{n} (1+|\xi_j|).$$

Next, also the set S' has not more than n elements, and again by Lemma 2

$$\sum_{j \in S'} \beta_j \leqq \sum_{j=1}^{n} \beta_j < 1+\varepsilon < 2.$$

Therefore

$$(32) \qquad \prod_{j \in S'} |\gamma^{(j)}| < \begin{cases} (4c)^{mnr_1}(n+1)^{2mr_1} q^{\frac{1}{2}mr_1(1+\varepsilon)^3} & \text{if} \quad \tau = 0, \\ (4c)^{mnr_1}(n+1)^{2mr_1} q^{\frac{1}{2}mr_1(1+\varepsilon)^3(1-\tau+\varepsilon)} & \text{if} \quad \tau > 0. \end{cases}$$

Put

$$c_2 = (4c)^n (n+1)^2 c_1.$$

By combining the formulae (30), (31), and (32), we arrive at the upper bounds

$$(33) \qquad |\Gamma| < \begin{cases} c_2^{mr_1} q^{-\frac{1}{2}mr_1(1-3\varepsilon)(\sigma-\varepsilon)+\frac{1}{2}mr_1(1+\varepsilon)^3} & \text{if} \quad \tau = 0, \\ c_2^{mr_1} q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)(\sigma-\varepsilon)+\frac{1}{2}mr_1(1+\varepsilon)^3(1-\tau+\varepsilon)} & \text{if} \quad \tau > 0. \end{cases}$$

14. The lower and upper estimates (26) and (33) for $|\Gamma|$ imply that

$$c''^{-mr_1} < c_2^{mr_1} q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)(\sigma-\varepsilon)+\frac{1}{2}mr_1(1+\varepsilon)^3} \qquad \text{if} \quad \tau = 0,$$

and

$$q_1^{-mr_1(1+\varepsilon)\tau} < c_2^{mr_1} q_1^{-\frac{1}{2}mr_1(1-3\varepsilon)(\sigma-\varepsilon)+\frac{1}{2}mr_1(1+\varepsilon)^3(1-\tau+\varepsilon)} \qquad \text{if} \quad \tau > 0.$$

After a slight simplification these inequalities take the form

$$q_1^{E(\tau)} < C(\tau), \tag{34}$$

with the abbreviations

$$E(\tau) = \begin{cases} \frac{1}{2}(1-3\varepsilon)(\sigma-\varepsilon)-\frac{1}{2}(1+\varepsilon)^3 & \text{if} \quad \tau = 0, \\ \frac{1}{2}(1-3\varepsilon)(\sigma-\varepsilon)-(1+\varepsilon)\tau-\frac{1}{2}(1+\varepsilon)^3(1-\tau+\varepsilon) & \text{if} \quad \tau > 0, \end{cases}$$

and

$$C(\tau) = \begin{cases} c''c_2 & \text{if} \quad \tau = 0, \\ c_2 & \text{if} \quad \tau > 0. \end{cases}$$

Now, by the *Hypothesis*,

$$\sigma = 1+\tau+20\varepsilon, \qquad 0 < \varepsilon \leqq \tfrac{1}{10}, \qquad 0 \leqq \tau \leqq 1.$$

Hence, for $\tau = 0$,

$$\begin{aligned} E(0) &= \tfrac{1}{2}(1-3\varepsilon)(1+19\varepsilon)-\tfrac{1}{2}(1+\varepsilon)^3 = \tfrac{1}{2}(13\varepsilon-60\varepsilon^2-\varepsilon^3) \\ &\geqq \frac{\varepsilon}{2}\left(13-\tfrac{60}{10}-\tfrac{1}{100}\right) > \varepsilon, \end{aligned}$$

and for $\tau > 0$,

$$\begin{aligned} E(\tau) &= \tfrac{1}{2}(1-3\varepsilon)(1+\tau+19\varepsilon)-(1+\varepsilon)\tau-\tfrac{1}{2}(1+\varepsilon)^3(1-\tau+\varepsilon) \\ &= \tfrac{1}{2}\{(12\varepsilon-63\varepsilon^2-4\varepsilon^3-\varepsilon^4)-\tau(2\varepsilon-3\varepsilon^2-\varepsilon^3)\} \\ &\geqq \frac{\varepsilon}{2}\left\{\left(12-\tfrac{63}{10}-\tfrac{4}{100}-\tfrac{1}{1000}\right)-1(2-0)\right\} > \varepsilon. \end{aligned}$$

Thus the relation (34) certainly does not hold if

$$q_1 \geqq C(\tau)^{1/\varepsilon},$$

and this will be the case if we impose on the parameter T in (13) in addition to (23) the further condition that

$$T \geqq C(\tau)^{1/\varepsilon}. \tag{35}$$

Thus, if we define T by the formula

$$T = \max(c^{m/t}, C(\tau)^{1/\varepsilon}),$$

then the *Hypothesis* leads to a contradiction. This concludes the proof of Theorem 1.

14. By specialising Theorem 1, we obtain a number of results that have some interest in themselves.

As the simplest case, consider an infinite sequence

$$\Sigma = \{\kappa(1), \kappa(2), \kappa(3), \cdots\}$$

of distinct elements of K with the property

$$|\kappa^{(1)}(l)-\xi_1| \leqq q(l)^{-\sigma} \qquad \text{for all } l,$$

where σ is some positive constant. Since evidently

$$\lim_{l\to\infty} |\kappa^{(1)}(l)-\xi_1| = 0,$$

all except at most finitely many elements of Σ satisfy the inequality

$$\prod_{j=1}^{n} \min(1, |\kappa^{(j)}(l)-\xi_j|) \leqq |\kappa^{(1)}-\xi_1| \leqq q(l)^{-\sigma}.$$

Hence, by Theorem 1,

$$\sigma \leqq 1+\tau \leqq 2$$

because Σ always has the property $Q(1)$. Thus LeVeque's generalisation of Roth's theorem is a special case of Theorem 1.

More generally, assume that the elements of Σ satisfy the inequalities

$$|\kappa^{(j)}(l)-\xi_j| \leqq q(l)^{-\alpha_j} \qquad \text{for all } l, \text{ and } 1 \leqq j \leqq N,$$

where $1 \leqq N \leqq n$, and

$$\alpha_1, \alpha_2, \cdots, \alpha_N$$

are N positive constants. Then again for all but at most finitely many elements of Σ,

$$\prod_{j=1}^{n} \min(1, |\kappa^{(j)}(l)-\xi_j|) \leqq \prod_{j=1}^{N} |\kappa^{(j)}(l)-\xi_j| \leqq q(l)^{-(\alpha_1+\cdots+\alpha_N)}.$$

Hence, by Theorem 1,

$$\alpha_1+\cdots+\alpha_N \leqq 2.$$

This result remains true if one or more of the constants $\alpha_1, \cdots, \alpha_N$ are equal to zero.

15. It has its advantages to change the notation by expressing the elements of Σ in terms of a field basis.

Let $\omega_1, \cdots, \omega_n$ be a field basis of K which need *not* be an integral basis. The letters $c_1, c_2, c_3, \cdots$ will be used to denote positive constants that are independent of the suffixes j and l. In particular, c_1 is to denote the smallest positive integer such that the n products

$$c_1\omega_1, \cdots, c_1\omega_n$$

are algebraic integers. As usual, $\omega_k^{(j)}$ is the j-th conjugate of ω_k.

We shall again be concerned with an infinite sequence

$$\Sigma = \{\kappa(1), \kappa(2), \kappa(3), \cdots\}$$

of distinct elements of K. In terms of the basis, these numbers can now be written as

$$\kappa(l) = \frac{x_1(l)\omega_1 + \cdots + x_n(l)\omega_n}{y(l)}$$

where $x_1(l), \cdots, x_n(l), y(l) \neq 0$ are $n+1$ rational integers which we assume to be relatively prime. The conjugates of $\kappa(l)$ similarly have the form

$$\kappa^{(j)}(l) = \frac{x_1(l)\omega_1^{(j)} + \cdots + x_n(l)\omega_n^{(j)}}{y(l)}.$$

For shortness, put

$$X(l) = \max(|x_1(l)|, \cdots, |x_n(l)|), \quad Y(l) = |y(l)|$$

so that $X(l)$ is a non-negative integer, and $Y(l)$ is a positive integer. Since the elements of Σ are all distinct, the larger one of these two integers tends to infinity with l.

From now on assume that

$$(38) \qquad |\kappa^{(j)}(l)| \leqq c_2 \qquad \text{for} \quad j = 1, 2, \cdots, N \text{ and for all } l,$$

and

$$(39) \qquad |\kappa^{(1)}(l)| \geqq c_3 \qquad \text{for all } l.$$

Here N again is an integer such that $1 \leqq N \leqq n$. From the equation

$$|x_1(l)\omega_1^{(j)} + \cdots + x_n(l)\omega_n^{(j)}| = |\kappa^{(j)}(l)|Y(l)$$

it follows then that

(40) $|x_1(l)\omega_1^{(j)}+\cdots+x_n(l)\omega_n^{(j)}| \leqq c_2 Y(l)$ for $j=1,2,\cdots,N$ and for all l,

and

$$|x_1(l)\omega_1^{(1)}+\cdots+x_n(l)\omega_n^{(1)}| \geqq c_3 Y(l) \qquad \text{for all } l.$$

On the other hand, it is obvious that

(41) $$|x_1(l)\omega_1^{(j)}+\cdots+x_n(l)\omega_n^{(j)}| \leqq c_4 X(l) \qquad \text{for all } j \text{ and all } l,$$

and hence it follows that

(42) $$Y(l) \leqq c_5 X(l) \qquad \text{for all } l.$$

Thus, in particular, the assumptions (38) and (39) have the consequence that $X(l)$ tends to infinity when $\kappa(l)$ runs over the elements of Σ.

We form now the polynomial in x,

$$\begin{aligned} g^*(x) &= \{c_1 y(l)\}^n \prod_{j=1}^{n} (x-\kappa^{(j)}(l)) \\ &= \prod_{j=1}^{n} \{c_1 y(l)\cdot x-(x_1(l)c_1\omega_1^{(j)}+\cdots+x_n(l)c_1\omega_n^{(j)})\} \\ &= b_0^* x^n+b_1^* x^{n-1}+\cdots+b_n^* \quad \text{say, where } b_0^* = c_1^n y(l)^n. \end{aligned}$$

The coefficients of $g^*(x)$ are symmetric functions in the conjugates of $\kappa(l)$ and so are rational numbers. Further the linear polynomials in x,

(43) $$c_1 y(l)\cdot x-(x_1(l)c_1\omega_1^{(j)}+\cdots+x_n(l)c_1\omega_n^{(j)}) \qquad (j=1,2,\cdots,n),$$

have integral algebraic coefficients. Therefore the coefficients b_k^* of $g^*(x)$ are algebraic integers and so are rational integers.

Denote by $d(l)$ the greatest common divisor of these coefficients $b_0^*, b_1^*, \cdots, b_k^*$ and put

$$g(x) = \frac{1}{d(l)}\, g^*(x) = b_0 x^n+b_1 x^{n-1}+\cdots+b_n,$$

so that

$$b_k = \frac{1}{d(l)}\, b_k^* \qquad (k=0,1,\cdots,n).$$

Hence, by $d(l) \geqq 1$, in the former notation

$$h(l) = |b_0| \leqq |b_0^*|, \quad q(l) = \max(|b_0|,|b_1|,\cdots,|b_n|) \leqq \max(|b_0^*|,|b_1^*|,\cdots,|b_n^*|).$$

By the formulae (40) and (41), the coefficients of the linear factors (43) of $g^*(x)$ are of the order $O(X(l))$ for all values of j, and of the possibly lower order $O(Y(l))$ when $j = 1, 2, \cdots, N$. On forming the product of these factors, we obtain then the estimates,

$$h(l) \leqq c_6 Y(l)^n, \quad q(l) \leqq c_7 X(l)^{n-N} Y(l)^N \qquad \text{for all } l. \tag{44}$$

16. The sequence Σ will now be specialised so as to simplify the final result. Denote by $\lambda_1, \cdots, \lambda_N$ non-negative constants, by $C_1, \cdots, C_N$, and C positive constants, and by μ a constant in the interval

$$0 \leqq \mu \leqq 1.$$

Assume from now on that Σ has the following properties.

$$1 \leqq Y(l) \leqq C X(l)^{\mu} \qquad \text{for all } l, \tag{45}$$

$$|\kappa^{(j)}(l) - \xi_j| \leqq C_j X(l)^{-\lambda_j} \qquad \text{for } j = 1, 2, \cdots, N \text{ and for all } l, \tag{46}$$

$$\xi_1 \neq 0, \qquad \lambda_1 > 0. \tag{47}$$

Since $X(l) \geqq 1$ from (45), the former assumption (38) follows from (46). Next, by (45), $X(l)$ tends to infinity with l because the larger one of the two integers $X(l)$, $Y(l)$ has this property. It follows from the case $j = 1$ of (46) and from (47) that

$$\lim_{l \to \infty} \kappa^{(1)}(l) = \xi_1 \neq 0.$$

The former assumption (39) therefore is satisfied for all sufficiently large l, say for $l \geqq l_0$.

Since then both (38) and (39) hold when $l \geqq l_0$, we may apply the estimates (44). By (45), they imply that

$$h(l) \leqq c_8 X(l)^{n\mu}, \qquad q(l) \leqq c_9 X(l)^{n-(1-\mu)N}, \qquad \text{for } l \geqq l_0. \tag{48}$$

As we know, $q(l)$ tends to infinity with l. The quantity

$$n - (1-\mu)N, = \nu \qquad \text{say,}$$

is therefore a positive number; i.e. the special case when

$$N = n \quad \text{and} \quad \mu = 1$$

is excluded.

Put

$$\omega = \liminf_{l \to \infty} \frac{\log q(l)}{\log X(l)}.$$

The second formula (48) shows that

$$0 \leqq \omega \leqq \nu. \tag{49}$$

It becomes now necessary to distinguish two cases.

Case 1: $\omega > 0$. Denote by $\varepsilon > 0$ an arbitrarily small constant which is less than ω. From the definition of ω, there exists an infinite subsequence Σ_1 of Σ consisting only of elements $\kappa(l)$ with $l \geqq l_0$, such that

$$X(l)^{\omega-\varepsilon} \leqq q(l) \leqq X(l)^{\omega+\varepsilon} \qquad \text{if} \quad \kappa(l) \in \Sigma_1. \tag{50}$$

By the first formula (48),

$$h(l) \leqq c_8 q(l)^{n\mu/(\omega-\varepsilon)} \qquad \text{if} \quad \kappa(l) \in \Sigma_1.$$

Hence, in the notation of § 4, Σ_1 has the property $Q(\tau)$ where

$$\tau = \min\left(\frac{n\mu}{\omega-\varepsilon}, 1\right),$$

because it certainly has the property $Q(1)$, so that τ need not be chosen greater than 1.

By (46) and (50), the elements of Σ_1 have the property

$$|\kappa^{(j)}(l)-\xi_j| \leqq C_j q(l)^{-\lambda_j/(\omega+\varepsilon)} \qquad (j = 1, 2, \cdots, N)$$

and hence also the property

$$\prod_{j=1}^{n} \min(1, |\kappa^{(j)}(l)-\xi_j|) \leqq \prod_{j=1}^{N} |\kappa^{(j)}(l)-\xi_j| \leqq c_{10} q(l)^{-(\lambda_1+\cdots+\lambda_N)/(\omega+\varepsilon)}.$$

Thus, in the notation of § 4, Σ_1 has also the property $P(\sigma)$ where,

$$\sigma = \frac{\lambda_1+\cdots+\lambda_N}{\omega+\varepsilon}.$$

On applying Theorem 1, it follows that

$$\frac{\lambda_1+\cdots+\lambda_N}{\omega+\varepsilon} \leqq 1+\min\left(\frac{n\mu}{\omega-\varepsilon}, 1\right)$$

or

$$\lambda_1+\cdots+\lambda_N \leqq \omega+\varepsilon+\min\left(\frac{\omega+\varepsilon}{\omega-\varepsilon}\cdot n\mu, \omega+\varepsilon\right).$$

This inequality has been proved for arbitrarily small ε. On allowing ε to tend to zero, it implies that

$$\lambda_1+\cdots+\lambda_N \leqq \omega+\min(n\mu, \omega).$$

Here, by (49), ω cannot be larger than ν, and hence we find that

(51) $$\lambda_1+\cdots+\lambda_N \leqq \nu+\min(n\mu, \nu).$$

Case 2: $\omega = 0$. Denote again by $\varepsilon > 0$ an arbitrarily small constant. There exists now an infinite subsequence Σ_2 of Σ such that

$$q(l) \leqq X(l)^{\varepsilon} \qquad \text{if} \quad \kappa(l) \in \Sigma_2.$$

Hence, by (46), the elements of Σ_2 satisfy the inequalities

$$|\kappa^{(j)}(l)-\xi_j| \leqq C_j q(l)^{-\lambda_j/\varepsilon} \qquad (j = 1, 2, \cdots, N)$$

and so also the inequality

$$\prod_{j=1}^{n} \min(1, |\kappa^{(j)}(l)-\xi_j|) \leqq \prod_{j=1}^{N} |\kappa^{(j)}(l)-\xi_j| \leqq c_{10} q(l)^{-(\lambda_1+\cdots+\lambda_N)/\varepsilon}.$$

Therefore Σ_2 has the property $P(\sigma)$ where

$$\sigma = \frac{\lambda_1+\cdots+\lambda_N}{\varepsilon}.$$

By LeVeque's theorem, σ does not exceed 2, and hence

$$\lambda_1+\cdots+\lambda_N \leqq 2\varepsilon.$$

Here ε is arbitrarily small and may be allowed to tend to zero, proving that all N constants $\lambda_1, \cdots, \lambda_N$ are equal to zero. As this is contrary to the second formula (47), we obtain a contradiction, and it follows that the *Case* 2 cannot arise.

We have then the following result.

THEOREM 2: *Let K be an algebraic number field of degree n, with the field basis $\omega_1, \cdots, \omega_n$, and let*

$$\Sigma = \{\kappa(1), \kappa(2), \kappa(3), \cdots\}$$

be an infinite sequence of distinct elements of K. Write in terms of this basis

$$\kappa(l) = \frac{x_1(l)\omega_1+\cdots+x_n(l)\omega_n}{y(l)}$$

where $x_1(l), \cdots, x_n(l), y(l) \neq 0$ are rational integers that are relatively prime; further put

$$X(l) = \max(|x_1(l)|, \cdots, |x_n(l)|), \qquad Y(l) = |y(l)|.$$

Denote by $\xi_1 \neq 0, \xi_2, \cdots, \xi_N$ *any* N *algebraic numbers where* $1 \leqq N \leqq n$, *and by* $\lambda_1, \lambda_2, \cdots, \lambda_N, C_1, \cdots, C_N, C$, *and* μ *a set of* $2N+2$ *constants such that*

$$\lambda_1 > 0, \lambda_2 \geqq 0, \cdots, \lambda_N \geqq 0, C_1 > 0, \cdots, C_N > 0, C > 0, 0 \leqq \mu \leqq 1.$$

Also assume that the constant

$$\nu = n-(1-\mu)N$$

is positive, i.e. exclude the case when both $N = n$ *and* $\mu = 1$.

If the elements of Σ *satisfy the inequalities*

$$1 \leqq Y(l) \leqq CX(l)^{\mu}$$

and

$$|\kappa^{(j)}(l)-\xi_j| \leqq C_j X(l)^{-\lambda_j} \qquad (j = 1, 2, \cdots, N),$$

then

$$\lambda_1+\cdots+\lambda_N \leqq \nu+\min(n\mu, \nu).$$

17. Several special cases of Theorem 2 are worth mentioning. We first deal with the case $N = 1$ of a single inequality and in this case omit the suffix 1 of the conjugate. On first choosing $\mu = 1$, the theorem may be put in the following form.

(A) *If* $\omega_1, \cdots, \omega_n$ *is a field basis of the algebraic number field K, if* $\xi \neq 0$ *is an arbitrary algebraic number, and if* ε *is any positive constant, there are at most finitely many distinct sets of* $n+1$ *rational integers* $x_1, \cdots, x_n, y \neq 0$ *such that*

$$\left|\frac{x_1\omega_1+\cdots+x_n\omega_n}{y}-\xi\right| \leqq \{\max(|x_1|, \cdots, |x_n|, 1)\}^{-(2n+\varepsilon)}.$$

When $n = 1$, this is exactly Roth's theorem. On the other hand, this result can be further generalised when $n \geqq 2$, and it takes then the following form.

(B) *If* $\omega_1, \cdots, \omega_n$ *is a field basis of the algebraic number field K, if* $\xi \neq 0$ *is an arbitrary algebraic number, and if* ε *is any positive constant, there are at most finitely many distinct sets of* $2n$ *rational integers* $x_1, \cdots, x_n, y_1, \cdots, y_n$ *such that at least one of the integers* $y_1, \cdots, y_n$ *is distinct from zero and that*

$$\left|\frac{x_1\omega_1+\cdots+x_n\omega_n}{y_1\omega_1+\cdots+y_n\omega_n}-\xi\right| \leqq \{\max(|x_1|, \cdots, |x_n|, |y_1|, \cdots, |y_n|)\}^{-(2n+\varepsilon)}.$$

For on putting

$$\kappa = \frac{x_1\omega_1+\cdots+x_n\omega_n}{y_1\omega_1+\cdots+y_n\omega_n}, \qquad Z = \max(|x_1|, \cdots, |x_n|, |y_1|, \cdots, |y_n|),$$

a calculation similar to that in § 15 shows that κ has the height $O(Z^n)$. The assertion (B) is therefore an immediate consequence of LeVeque's theorem.

It is clear that the restriction $\xi \neq 0$ in (A) is essential and may not be omitted. The same restriction in (B) may, however, be disregarded.

Very similar results hold when N is greater than 1, when Theorem 2 gives the condition

$$\lambda_1 + \cdots + \lambda_N \leqq 2n.$$

18. Let again $N = 1$, but assume now that $\mu = 0$, so that the denominator $y(l)$ is bounded, say is equal to 1. Theorem 2 may now be expressed as follows.

(C) *If $\omega_1, \cdots, \omega_n$ is a field basis of the algebraic number field K, if $\xi \neq 0$ is an arbitrary algebraic number, and if ε is any positive constant, there are at most finitely many distinct sets of n rational integers $x_1, \cdots, x_n$ not all zero such that*

$$|x_1\omega_1 + \cdots + x_n\omega_n - \xi| \leqq \{\max(|x_1|, \cdots, |x_n|)\}^{-(n-1+\varepsilon)}.$$

As is easily seen, this result remains valid for $\xi = 0$, even with $\varepsilon = 0$.

Two simple applications of (C) have some interest in themselves. Assume that $n \geqq 2$, and denote by ϑ an arbitrary generating element of K. The n powers $\vartheta^{n-1}, \vartheta^{n-2}, \cdots, \vartheta, 1$ of ϑ form then a field basis of K, and it is clear that $\vartheta \neq 0$. We identify ξ in (C) with $-x_0\vartheta^n$ where $x_0 \neq 0$ is an arbitrary rational integer. With a slight change of notation, we then find the following corollary.

(D) *Let ϑ be an algebraic number of exact degree $n \geqq 2$; let ε be a positive constant; and let $x_0 \neq 0$ be a fixed rational integer. There exist at most finitely many sets of n rational integers $x_1, \cdots, x_n$ not all zero such that*

$$0 < |x_0\vartheta^n + x_1\vartheta^{n-1} + \cdots + x_{n-1}\vartheta + x_n| \leqq \{\max(|x_1|, \cdots, |x_n|)\}^{-(n-1+\varepsilon)}.$$

Next let $n = 2$; let K be a real quadratic field, and let α be any generating element of K. Hence α is a real quadratic irrationality. Instead of ξ we write now β. The result (C) implies then the following statement.

(E) *If α is any real quadratic irrationality; if β is an arbitrary real algebraic number; and if ε is any positive constant; then there exist at most finitely many pairs of rational integers $x \neq 0$, y such that*

$$|x\alpha - y - \beta| \leqq |x|^{-(1+\varepsilon)}.$$

This result may be compared with the well-known theorem by Čebyšev which states:

If α is any real irrational number; if β is an arbitrary real number; and if

$c > 0$ *is a certain positive constant; then there exist infinitely many pairs of integers* $x \neq 0$, y *such that*

$$|x\alpha - y - \beta| \leqq c|x|^{-1}.$$

We see that (E) is nearly best-possible. The restriction that α should be of the second degree is due to the method of proof, and it would have great interest to decide whether (E) remains true when α is at least of the third degree.

The result (C) can be generalised to systems of more than one inequality. It will suffice to give here one such consequence of Theorem 2.

(F) *Let* $\omega_1, \cdots, \omega_n$ *be a field basis of* K; *let* $\xi_1 \neq 0, \xi_2, \cdots, \xi_N$ *where* $1 \leqq N \leqq n-1$ *be* N *arbitrary algebraic numbers, and let* ε *be an arbitrary positive constant. There exist at most finitely many systems of* n *rational integers* $x_1, \cdots, x_n$ *not all zero such that*

$$|x_1\omega_1^{(j)} + \cdots + x_n\omega_n^{(j)} - \xi_j| \leqq \{\max(|x_1|, \cdots, |x_n|)\}^{-((n/N)-1+\varepsilon)} \qquad (j = 1, 2, \cdots, N).$$

Also here the restriction that $\xi_1 \neq 0$ can easily be removed.

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