Reprinted from ILLINOIS JOURNAL OF MATHEMATICS
Vol. 8, No. 1, March 1964
Printed in U.S.A.

A REMARK ON A PAPER OF MINE ON POLYNOMIALS

BY
KURT MAHLER

1. The measure $M(f)$ of a polynomial

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

with real or complex coefficients is defined by

$$\begin{aligned} M(f) &= \exp\left\{\int_0^1 \log|f(e^{2\pi it})|\,dt\right\} \quad &&\text{if} \quad f(x) \not\equiv 0, \\ &= 0 &&\text{if} \quad f(x) \equiv 0. \end{aligned}$$

Denote by S_{mn} the set of all polynomial vectors

$$\mathbf{f}(x) = (f_1(x), \cdots, f_n(x))$$

with components $f_h(x)$ that are polynomials at most of degree m and that do not all vanish identically. Further put

$$M(\mathbf{f}) = \sum_{h=1}^n M(f_h), \qquad N(\mathbf{f}) = \sum_{h=1}^n \sum_{k=1}^n M(f_h - f_k),$$
$$Q(\mathbf{f}) = N(\mathbf{f})/M(\mathbf{f}).$$

In my paper *On Two Extremum Properties of Polynomials*[1] I proved that the least upper bound

$$C_{mn} = \sup_{\mathbf{f} \in S_{mn}} Q(\mathbf{f})$$

satisfies the nearly trivial inequality

$$(1) \qquad C_{mn} \leqq 2^{m+1}(n-1)$$

and is attained for a polynomial vector

$$\mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$$

in S_{mn} with the following properties:

(2) *Those components $F_h(x)$ of $\mathbf{F}(x)$ that do not vanish identically all have the exact degree m, and all their zeros lie on the unit circle.*

It does not seem to be easy to determine the exact value of C_{mn}. In this note I shall replace (1) by an inequality (9) which is slightly better when m is large relative to $\log n$.

2. Let $\mathbf{F}(x)$ be defined as before. Without loss of generality,

$$F_1(x) \not\equiv 0, \quad \cdots, \quad F_p(x) \not\equiv 0, \qquad F_{p+1}(x) \equiv \cdots \equiv F_n(x) \equiv 0,$$

Received October 7, 1963.

[1] Illinois Journal of Mathematics, vol. 7 (1963), pp. 681–701.

where p is a certain integer satisfying

$$1 \leqq p \leqq n.$$

By (2), each of the first p components of $\mathbf{F}(x)$ allows a factorisation

$$F_h(x) = c_h(x - \gamma_{h1}) \cdots (x - \gamma_{hm}),$$

where

$$c_h \neq 0, \qquad |\gamma_{h1}| = \cdots = |\gamma_{hm}| = 1.$$

Therefore

$$M(F_h) = |c_h| \prod_{l=1}^{m} \max(1, |\gamma_{hl}|) = |c_h| \qquad (h = 1, 2, \cdots, p),$$

and it follows, in particular, that

$$M(\mathbf{F}) = \sum_{h=1}^{p} |c_h|. \tag{3}$$

The problem is to obtain an upper estimate for

$$N(\mathbf{F}) = \sum_{h=1}^{n} \sum_{k=1}^{n} M(F_h - F_k) = 2 \sum\sum_{1 \leqq h < k \leqq n} M(F_h - F_k).$$

Here, from the hypothesis,

$$\begin{aligned} M(F_h - F_k) &= |c_h| \quad \text{if} \quad 1 \leqq h \leqq p, \quad p+1 \leqq k \leqq n, \\ &= 0 \quad \text{if} \quad p+1 \leqq h \leqq n, \quad p+1 \leqq k \leqq n. \end{aligned}$$

Thus it suffices to estimate the remaining terms

$$M(F_h - F_k) \qquad (1 \leqq h \leqq p, \quad 1 \leqq k \leqq p)$$

of $N(\mathbf{F})$.

3. As usual, put for positive s

$$\log^+ s = \max(0, \log s),$$

so that

$$\begin{aligned} &\log s \leqq \log^+ s; \qquad \log^+(st) \leqq \log^+ s + \log^+ t; \\ &\log^+ |s \mp t| \leqq \log^+ s + \log^+ t + \log 2. \end{aligned} \tag{4}$$

If further $f(x)$ is again any polynomial, put

$$\begin{aligned} M^+(f) &= \exp\left\{\int_0^1 \log^+ |f(e^{2\pi i t})| \, dt\right\} \quad \text{if} \quad f(x) \not\equiv 0, \\ &= 0 \quad \text{if} \quad f(x) \equiv 0. \end{aligned}$$

Then, by (4), $M^+(f)$ has the properties:

$$\begin{aligned} &M(f) \leqq M^+(f), \\ &M^+(fg) \leqq M^+(f)M^+(g), \\ &M^+(f \mp g) \leqq 2M^+(f)M^+(g). \end{aligned} \tag{5}$$

If further a is any constant,

$$(6) \qquad M^+(a) = \max(1, |a|).$$

4. From these formulae (5) and (6),

$$M(F_h - F_k) \leqq M^+(F_h - F_k) \leqq 2M^+(F_h)M^+(F_k),$$

where, e.g.

$$M^+(F_h) \leqq \max(1, |c_h|) \prod_{l=1}^{m} M^+(x - \gamma_{hl}).$$

Moreover,

$$M^+(x - \gamma_{hl}) = M^+(x - 1)$$

because γ_{hl} has the absolute value 1.

For shortness therefore put

$$\theta = M^+(x - 1) = \exp\left\{\int_0^1 \log^+ |e^{2\pi it} - 1| \, dt\right\}.$$

It follows then that

$$M^+(F_h) \leqq \max(1, |c_h|)\theta^m,$$

whence, for $1 \leqq h \leqq p, \quad 1 \leqq k \leqq p$,

$$(7) \quad M(F_h - F_k) \leqq M^+(F_h - F_k) \leqq 2\max(1, |c_h|)\max(1, |c_k|)\theta^{2m}.$$

Now, for any constant $a \neq 0$,

$$Q(a\mathbf{f}) = Q(\mathbf{f}).$$

Thus there is no loss of generality in assuming that

$$M(\mathbf{F}) = \sum_{h=1}^{p} |c_h| = 1,$$

and hence that

$$\max(1, |c_h|) = 1 \qquad (h = 1, 2, \cdots, p).$$

The estimate (7) takes then the simpler form

$$M(F_h - F_k) \leqq 2\theta^{2m} \qquad (1 \leqq h \leqq p, \quad 1 \leqq k \leqq p),$$

and it follows that

$$\sum_{h=1}^{p}\sum_{k=1}^{p} M(F_h - F_k) \leqq 2(p^2 - p)\theta^{2m}$$

because the p terms with $h = k$ vanish. Therefore, by

$$N(\mathbf{F}) = \sum_{h=1}^{p}\sum_{k=1}^{p} M(F_h - F_k) + 2(n - p)\sum_{h=1}^{p} M(F_h),$$

we obtain the inequality

$$(8) \qquad Q(\mathbf{F}) = N(\mathbf{F}) \leqq 2(p^2 - p)\theta^{2m} + 2(n - p).$$

5. An approximate value of θ is now easily obtained. Write

$$j = \log\theta = \int_0^1 \log^+ |e^{2\pi it} - 1| \, dt = \int_{-1/2}^{1/2} \log^+ |e^{2\pi it} - 1| \, dt.$$

The function

$$| e^{2\pi it} - 1 | = 2 |(e^{\pi it} - e^{-\pi it})/2i | = 2 | \sin \pi t |$$

is even and of period 1, and

$$\begin{aligned} | e^{2\pi it} - 1 | &\leqq 1 \quad \text{if} \quad 0 \leqq t \leqq \tfrac{1}{6}, \\ &> 1 \quad \text{if} \quad \tfrac{1}{6} < t \leqq \tfrac{1}{2}. \end{aligned}$$

It follows that

$$j = 2 \int_{1/6}^{1/2} \log (2 \sin \pi t) \, dt = \frac{1}{3} \log 4 + 2 \int_{1/6}^{1/2} \log \sin \pi t \, dt.$$

Here, by numerical integration (for which I am much indebted to my colleague Professor K. J. Le Couteur),

$$\int_{1/6}^{1/2} \log \sin \pi t \, dt = -0.069516 \cdots,$$

and hence

$$j = 0.32307 \cdots .$$

Therefore

$$\theta^2 = e^{2j} < 1.91.$$

Since $p \leqq n$, it follows then finally from (8) that

$$Q(\mathbf{F}) \leqq 2(n^2 - n)\theta^{2m} + 2(n - n),$$

and therefore that

$$C_{mn} \leqq 2(n^2 - n)\lambda^m \quad \text{where} \quad \lambda < 1.91. \tag{9}$$

Apart from the value of the constant λ, this result is best possible for fixed n and increasing m. This is easily seen for $n = 2$ on taking for $\mathbf{f}(x)$ the polynomial vector

$$\mathbf{f}(n) = ((x + 1)^m, (x - 1)^m).$$

AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, AUSTRALIA