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H. FEHR et **C.-A. LAISANT**

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A REMARK ON KRONECKER'S THEOREM

by K. Mahler

Kronecker's theorem on the inhomogeneous simultaneous approximations can be obtained in many different ways. Perhaps the simplest proof is based on the geometry of numbers, as I shall show in this note.

The basic facts from the geometry of numbers can be found in the book *An Introduction to the Geometry of Numbers* by J. W. S. Cassels, Berlin 1959. We follow the notation of this book; references to it will bear the letters IGN and the page number.

1. The n-dimensional space of all points

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n), \qquad 0 = (0, \ldots, 0)$$

with real coordinates is denoted by R^n. Points in R^n are considered also as vectors. The sum of two points x and y is then defined by

$$x + y = (x_1 + y_1, \ldots, x_n + y_n),$$

the product of the point x with the real number t by

$$tx = (tx_1, \ldots, tx_n),$$

and the inner product of the points x and y by

$$xy = x_1 y_1 + \ldots + x_n y_n .$$

The real-valued function $F(x)$ of x in R^n is called a convex distance function if

$$F(0) = 0; \qquad F(x) > 0 \text{ } if \text{ } x \neq 0 . \tag{1}$$

$$F(tx) = |t| F(x) . \tag{2}$$

$$F(x+y) \leqq F(x) + F(y) . \tag{3}$$

When $F(x)$ is such a convex distance function, the point set K defined by

$$K: \quad F(x) \leqq 1$$

is convex, bounded, closed, symmetric in 0, and it contains 0 as an interior point. The volume of K defined by

$$V = \int_F \cdots \int dx_1 \ldots dx_n$$

is a positive number (IGN 108 f.).

It can be shown that with every convex distance function $F(x)$ there is associated a second convex distance function

$$F^*(y) = \sup_{x \neq o} \frac{xy}{F(x)} = \sup_{F(x) = 1} xy,$$

and then, conversely,

$$F(x) = \sup_{y \neq o} \frac{xy}{F^*(y)} = \sup_{F^*(y) = 1} xy.$$

The two convex bodies

$$K: \; F(x) \leqq 1 \quad \text{and} \quad K^*: \; F^*(y) \leqq 1$$

are polar reciprocal with respect to the unit sphere

$$E: \quad xx = 1$$

in the sense that every point y on the frontier of K^* has as its polar hyperplane relative to E the tangential hyperplane (or rather, tac-hyperplane)

$$xy = F^*(y)$$

of K. The analogous relation holds when K and K^* and hence also $F(x)$ and $F^*(y)$ are interchanged (IGN 112 f.).

2. A lattice point in R^n is a point with rational integral coordinates. In terms of these lattice points the n successive minima of the convex body K are defined as follows.

The first minimum $m^{(1)}$ is the smallest value of $F(x)$ at any lattice point $x \neq 0$; denote by $x^{(1)} \neq 0$ a lattice point satisfying

$$F(x^{(1)}) = m^{(1)}.$$

Next let $2 \leqq k \leqq n$, and assume that the first $k-1$ successive minima $m^{(1)}, \ldots, m^{(k-1)}$ and $k-1$ corresponding independent lattice points $x^{(1)}, \ldots, x^{(k-1)}$ satisfying

$$F(x^{(h)}) = m^{(h)} \qquad (h = 1, 2, \ldots, k-1)$$

have already been defined. The $k-th$ minimum $m^{(k)}$ is then the smallest value of $F(x)$ in any lattice point x that is linearly independent of $x^{(1)}, \ldots, x^{(k-1)}$, and we denote by $x^{(k)}$ such a lattice point for which

$$F(x^{(k)}) = m^{(k)}.$$

It can be proved that the n successive minima $m^{(1)}, \ldots, m^{(n)}$ are unique. They satisfy the inequalities

$$0 < m^{(1)} \leqq m^{(2)} \leqq \ldots \leqq m^{(n)}; \quad \frac{2^n}{n!} \leqq V m^{(1)} m^{(2)} \ldots m^{(n)} \leqq 2^n. \qquad (4)$$

The corresponding n lattice points $x^{(1)}, \ldots, x^{(n)}$ are not unique. They are linearly independent and so form a base of R^n, but they need not form a base of the set of all lattice points (IGN, 215 f.).

There naturally exist also n successive minima $m^{*(1)}, \ldots, m^{*(n)}$ of the polar reciprocal couvex body K^* and a set of n linearly independent lattice points $y^{(1)}, \ldots, y^{(n)}$ such that

$$F^*(y^{(k)}) = m^{*(k)} \qquad (k = 1, 2, \ldots, n).$$

These minima satisfy inequalities analogous to (4).

By a general theorem in the geometry of numbers (IGN 213 f. and 219 f.) the two sets of successive minima are related to one another by the system of n inequalities

$$1 \leqq m^{(k)} m^{*(n-k+1)} \leqq n! \qquad (k = 1, 2, \ldots, n). \qquad (5)$$

The two sets of successive minima of K and K^* are also connected with the problem of inhomogeneous approximations. Let x^0 be an arbitrary points in R^n. There exists then a lattice point x such that (IGN 313 f.).

$$F(x - x^0) \leqq \frac{n m^{(n)}}{2} \leqq \frac{n \cdot n!}{2\, m^{*(1)}}. \qquad (6)$$

3. We give now two kinds of applications of the results just stated to inhomogeneous problems; both can be generalised.

First let $n = N + 1$ where $N \geqq 1$. We number the suffixes of the coordinates $0, 1, \ldots, N$ rather than $1, 2, \ldots, n$ as before. Denote by $\alpha_1, \ldots, \alpha_N$ a set of N arbitrary fixed real numbers and by A and B two positive parameters. Then the expressions

$$F(x) = A(|x_1 - \alpha_1 x_0| + \ldots + |x_N - \alpha_N x_0|) + B\,|\,x_0\,|$$

and

$$F^*(y) = max\left(\frac{|\,y_1\,|}{A}, \ldots, \frac{|\,y_N\,|}{A}, \frac{|\,y_0 + \alpha_1 y_1 + \ldots + \alpha_N y_N\,|}{B}\right)$$

form a pair of polar reciprocal distance functions. With a slight change of notation, let $m^{(0)}, m^{(1)}, \ldots, m^{(N)}$ and $m^{*(0)}, m^{*(1)}, \ldots, m^{*(N)}$ be the two corresponding sets of $N + 1$ successive minima.

It will be assumed that

$$F^*(y) \geqq 1 \quad \textit{for all lattice points } y \neq 0.$$

This is equivalent to the hypothesis that

$$m^{*(0)} \geqq 1\,.$$

Therefore, with a trivial change of notation, the formula (6) implies that for any given point x^0 there exists a lattice point x satisfying

$$F(x - x_0^0) \leqq \frac{(N+1)(N+1)\,!}{2}\,.$$

These estimates can be expressed in a more explicit form. Put

$$\beta_0 = x_0^0, \; \beta_1 = \alpha_1 x_0^0 - x_1^0, \ldots, \beta_N = \alpha_N x_0^0 - x_N^0,$$

where x^0 has the coordinates

$$x^0 = (x_0^0, x_1^0, \ldots, x_N^0)\,.$$

For any given $N + 1$ constants $\beta_0, \beta_1, \ldots, \beta_N$ there is always a unique point x^0 satisfying these equations.

The result obtained may now be expressed in the following form.

THEOREM 1. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_N$, $A > 0$, *and* $B > 0$ *be real numbers such that*

$$|\,y_0 + \alpha_1 y_1 + \ldots + \alpha_N y_N\,| \geqq B$$

for all lattice points y satisfying

$$y \neq 0, \quad max\,(|y_1|, |y_2|, \ldots, |y_N|) \leqq A\,.$$

Then for every choice of the real numbers $\beta_0, \beta_1, \ldots, \beta_N$ *there exists a lattice point x such that*

$$|x_0 - \beta_0| \leqq \frac{(N+2)!}{2B}, \quad |\alpha_1 x_0 - x_1 - \beta_1| + \ldots + |\alpha_N x_0 - x_N - \beta_N| \leqq \frac{(N+2)!}{2A}.$$

Assume, in particular, that the $N + 1$ numbers

$$1, \alpha_1, \alpha_2, \ldots, \alpha_N$$

are linearly independent over the rational field. Then to any arbitrarily large number $A > 0$ there exists a number $B > 0$ satisfying the hypothesis of the Theorem. Therefore the N quantities

$$|\alpha_k x_0 - x_k - \beta_k| \qquad (k = 1, 2, \ldots, N)$$

can simultaneously be made less than any prescribed number $\varepsilon > 0$ by a suitable choice of the lattice point x. In addition, if J is any interval of length $\dfrac{(N+2)!}{B}$, then there is a lattice point x with this property for which $x_0 \in J$.

4. For the second application put $n = N$ and denote by $\alpha_1, \alpha_2, \ldots, \alpha_N$ a set of N real numbers where $\alpha_N \neq 0$. Let further A and B be again two positive parameters. The two expressions

$$F(x) = \frac{A}{|\alpha_N|}(|\alpha_N x_1 - \alpha_1 x_N| + \ldots + |\alpha_N x_{N-1} - \alpha_{N-1} x_N|) + \frac{B}{|\alpha_N|}|x_N|$$

and

$$F^*(y) = max\left(\frac{|y_1|}{A}, \ldots, \frac{|y_{N-1}|}{A}, \frac{|\alpha_1 y_1 + \ldots + \alpha_N y_N|}{B}\right)$$

form then a pair of polar reciprocal distance functions. Denote by $m^{(1)}, \ldots, m^{(N)}$ and $m^{*(1)}, \ldots, m^{*(N)}$ the two corresponding sets of successive minima.

It will again be assumed that

$$F^*(y) \geqq 1 \qquad \text{for all lattice points} \qquad y \neq 0,$$

so that

$$m^{*(1)} \geqq 1\,.$$

By the formula (6) there exists then to every point x^0 a lattice point x satisfying the inequality

$$F(x - x^0) \leqq \frac{N \cdot N!}{2}. \tag{7}$$

Assume x satisfies the inequality (7). Then

$$\sum_{k=1}^{N-1} |\alpha_N (x_k - x_k^0) - \alpha_k (x_N - x_k^0)| \leqq \frac{(N+1)!\,|\alpha_N|}{2A}, \tag{8}$$

$$|x_N - x_N^0| \leqq \frac{(N+1)!\,|\alpha_N|}{2B}.$$

Denote by $x_0, \beta_1, \beta_2, \ldots, \beta_N$ a set of $N + 1$ real numbers and put

$$\lambda_k = \alpha_k x_0 - x_k - \beta_k \qquad (k = 1, 2, \ldots, N).$$

Then

$$x_k - x_k^0 = \alpha_k x_0 - x_k^0 - \beta_k - \lambda_k$$

and hence

$$\alpha_N (x_k - x_k^0) - \alpha_k (x_N - x_N^0) = \alpha_k (x_N^0 + \beta_N) - \alpha_N (x_k^0 + \beta_k) + (\alpha_k \lambda_N - \alpha_N \lambda_k).$$

In order to establish a simple result we finally fix $x_1^0, x_2^0 \ldots, x_{N-1}^0$ in terms of $x_0, \beta_1, \beta_2, \ldots, \beta_N$ by putting

$$x_k^0 = \frac{\alpha_k}{\alpha_N}(x_N^0 + \beta_N) - \beta_k \qquad (k = 1, 2, \ldots, N-1)$$

and defining then x_0 by the equation

$$\lambda_N = \alpha_N x_0 - x_N - \beta_N = 0.$$

It follows that

$$x_N - x_N^0 = \alpha_N x_0 - x_N^0 - \beta_N$$

and

$$\alpha_N (x_k - x_k^0) - \alpha_k (x_N - x_N^0) = -\alpha_N \lambda_k \qquad (k = 1, 2, \ldots, N-1).$$

By $\lambda_N = 0$ the formulae (8) imply therefore that

$$\sum_{k=1}^{N} |\lambda_k| = \sum_{k=1}^{N} |\alpha_k x_0 - x_k - \beta_k| \leqq \frac{(N+1)!}{2A}$$

and

$$|\alpha_N x_0 - x_N^0 - \beta_N| \leqq \frac{(N+1)!\,|\alpha_N|}{2B}.$$

In the first of these inequalities the parameters x_N^0 and β_N are still at our disposal. We choose their values such that the quantity

$$d = \frac{x_N^0 + \beta_N}{\alpha_N}$$

is equal to any prescribed number. Hence the final result is as follows.

THEOREM 2. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_N, A > 0$, *and* $B > 0$, *be real numbers such that* $\alpha_N \neq 0$, and that

$$| \alpha_1 y_1 + \ldots + \alpha_N y_N | \geqq B$$

for all lattice points y satisfying

$$y \neq 0 , \qquad max\ (|y_1|, |y_2|, \ldots, |y_{N-1}|) \leqq A .$$

Then for every choice of the $N+1$ *real numbers* $\beta_1, \beta_2, \ldots, \beta_N$, *and d, there exists a lattice point x and a real number* x_0 *for which*

$$d \leqq x_0 \leqq d + \frac{(N+1)\,!}{B}, \qquad \sum_{k=1}^{N} | \alpha_k x_0 - x_k - \beta_k | \leqq \frac{(N+1)\,!}{2A} .$$

Assume, in particular, that the N numbers

$$\alpha_1, \alpha_2, \ldots, \alpha_N$$

are linearly independent over the rational field. Then $\alpha_N \neq 0$, and to any arbitrarily large number $A > 0$ there exists a number $B > 0$ satisfying the hypothesis of the Theorem. Therefore the N quantities

$$| \alpha_k x_0 - x_k - \beta_k | \qquad (k = 1, 2, \ldots, N)$$

can simultaneously be made less than any prescribed number $\varepsilon > 0$ by a suitable choice of the lattice point x and the real number x_0. Moreover, for every interval J of length $\frac{(N+1)\,!}{B}$ there is such a solution x, x_0 with $x_0 \in J$.

The proofs of Theorems 1 and 2 explain very clearly why different conditions have to be imposed on $\alpha_1, \alpha_2, \ldots, \alpha_N$ according as to whether we require a solution with integral x_0 or only with real x_0.

Australian National University
Canberra, A.C.T.

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