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Applications of a theorem by A. B. Shidlovski

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To L. J. Mordell on his 80th birthday

Shidlovski's deep theorem on Siegel E-functions satisfying systems of linear differential equations is applied in this paper to the study of the arithmetic properties of the partial derivatives

$$C_k(z) = \frac{1}{k!}\left\{\frac{\partial}{\partial\nu}\right\}^k J_\nu(z)|_{\nu=0} \quad (k = 0, 1, 2, 3)$$

of the Bessel function $J_0(z)$. As a by-product, expressions involving Euler's constant γ and the constant $\zeta(3)$ are obtained for which the transcendency can be established.

Let $w_1 = f_1(z), \ldots, w_m = f_m(z)$ be a finite set of Siegel E-functions (see Siegel 1949, p. 33) which satisfies a system of linear differential equations

$$w'_h = q_{h0} + \sum_{h=1}^{m} q_{hk} w_k \quad (h = 1, 2, \ldots, m),$$

where the coefficients q_{h0} and q_{hk} are rational functions of z. Denote by $\alpha \neq 0$ any algebraic number such that all coefficients q_{h0} and q_{hk} are regular at the point $z = \alpha$. A beautiful and deep theorem by Shidlovski (1962) states that the maximum number of function values

$$f_1(\alpha), \ldots, f_m(\alpha)$$

that are algebraically independent over the rational number field, is equal to the maximum number of functions

$$f_1(z), \ldots, f_m(z)$$

that are algebraically independent over the field of rational functions of z.

In a number of papers, Shidlovski and his students have applied this theorem to the study of special E-functions. The present paper gives a further application of his theorem to such functions.

My main aim is to construct certain expressions which involve Euler's constant γ and the constant $\zeta(3)$ and which can be proved to be transcendental numbers. The simplest transcendental expression containing γ is given by

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma,$$

where, as usual, $J_0(z)$ and $Y_0(z)$ denote the Bessel functions of the first and the second kinds of suffix 0.

In a certain sense the results proved in this paper are quite trivial consequences of Shidlovski's work, and they do not even imply the irrationality of γ or of $\zeta(3)$. However, they deserve perhaps a little interest because, up to now, nothing was known about the arithmetic behaviour of these constants.

Chapter 1

1. Let z and ν be two complex variables. Differentiations with respect to these variables will be denoted by a dash, and by the symbol $\partial/\partial\nu$, respectively. The letter **C** denotes the complex number field; **C**(z) is the field of rational functions of z with coefficients in **C**; **M** is the field of meromorphic functions; and **E** is the ring of entire functions of z.

In Siegel's notation, let

$$K_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z^2/4)^n}{n!\,(\nu+1)(\nu+2)\ldots(\nu+n)}. \tag{1}$$

Then $K_\nu(z)$ is an entire function of z, and a meromorphic function of ν, and it is related to the Bessel function of the first kind $J_\nu(z)$ by the equation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} K_\nu(z). \tag{2}$$

It follows that $K_\nu(z)$ satisfies the linear differential equation

$$w'' + \frac{2\nu+1}{z} w' + w = 0. \tag{3}$$

If ν is not an integer, two further integrals of this differential equation are given by the functions

$$K_\nu^*(z) = z^{-2\nu} K_{-\nu}(z)$$

and

$$L_\nu(z) = \frac{K_\nu(z) - z^{-2\nu} K_{-\nu}(z)}{2\nu}. \tag{4}$$

The Wronski determinant

$$\mathbf{W}(u, v) = uv' - u'v$$

of any two integrals u and v of (3) has the explicit form

$$\mathbf{W}(u, v) = \exp\left(-\int \frac{2\nu+1}{z}\,\mathrm{d}z\right) = cz^{-2\nu-1},$$

where c does not depend on z. Since the integrals $u = K_\nu(z)$ and $v = K_\nu^*(z)$ and their derivatives allow series in ascending powers of z of the form:

$$K_\nu(z) = 1 + \ldots, \quad K_\nu'(z) = -\frac{z}{2(\nu+1)} + \ldots,$$

$$K_\nu^*(z) = z^{-2\nu} + \ldots, \quad K_\nu^{*\prime}(z) = -2\nu z^{-2\nu-1} + \ldots,$$

it follows that

$$\mathbf{W}(K_\nu, K_\nu^*) = K_\nu(z)\,K_\nu^{*\prime}(z) - K_\nu'(z)\,K_\nu^*(z) = -2\nu z^{-2\nu-1}.$$

Hence, by (4),

$$\mathbf{W}(K_\nu, L_\nu) = K_\nu(z)\,L_\nu'(z) - K_\nu'(z)\,L_\nu(z) = z^{-2\nu-1}. \tag{5}$$

2. This paper is concerned mainly with the functions

$$A_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k K_\nu(z)\Big|_{\nu=0}, \quad B_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k L_\nu(z)\Big|_{\nu=0} \quad (k = 0, 1, 2, \ldots), \tag{6}$$

and in particular with those that belong to the suffixes

$$k = 0, 1, 2, 3.$$

Frequently, when the actual value of the variable z is immaterial, we shall write A_k and B_k instead of $A_k(z)$ and $B_k(z)$ and do similarly for other functions of z.

Each B_k can be expressed in terms of the functions A_k. For let

$$Z = 2\log z.$$

Then, if $|\nu|$ is sufficiently small, we have the convergent series

$$K_\nu(z) = \sum_{k=0}^{\infty} A_k(z)\,\nu^k, \quad L_\nu(z) = \sum_{k=0}^{\infty} B_k(z)\,\nu^k, \quad z^{-2\nu} = \sum_{k=0}^{\infty} \frac{(-Z)^k}{k!}\,\nu^k,$$

from which it follows, by (4), that

$$\sum_{k=0}^{\infty} B_k(z)\,\nu^k = \frac{1}{2\nu}\left(\sum_{k=0}^{\infty} A_k(z)\,\nu^k - \left\{\sum_{k=0}^{\infty} \frac{(-Z)^k}{k!}\,\nu^k\right\}\left\{\sum_{k=0}^{\infty} A_k(z)\,(-\nu)^k\right\}\right).$$

Since for each k the coefficients of ν^k on both sides of this equation must be the same, we obtain the general formula

$$B_k(z) = \frac{1}{2}\left(A_{k+1}(z) + (-1)^k \sum_{h=0}^{k+1} \frac{Z^h}{h!}\,A_{k-h+1}(z)\right). \tag{7}$$

Thus, for the lowest values of k,

$$B_0(z) = A_1(z) + A_0(z)\log z,$$
$$B_1(z) = -[A_1(z)\log z + A_0(z)\,(\log z)^2],$$
$$B_2(z) = A_3(z) + A_2(z)\log z + A_1(z)\,(\log z)^2 + \tfrac{2}{3}A_0(z)\,(\log z)^3,$$
$$B_3(z) = -[A_3(z)\log z + A_2(z)\,(\log z)^2 + \tfrac{2}{3}A_1(z)\,(\log z)^3 + \tfrac{1}{3}A_0(z)\,(\log z)^4].$$

3. More generally, let

$$w = w(z,\nu) = \sum_{k=0}^{\infty} w_k\,\nu^k$$

be any integral of the linear differential equation (3) which, for sufficiently small $|\nu|$, can be expanded into a convergent series in powers of ν with coefficients w_k that are functions of z. Then

$$\sum_{k=0}^{\infty} w_k''\,\nu^k + \frac{2\nu+1}{z}\sum_{k=0}^{\infty} w_k'\,\nu^k + \sum_{k=0}^{\infty} w_k\,\nu^k = 0,$$

and therefore $$\sum_{k=0}^{\infty}\left(w_k'' + \frac{1}{z}\,w_k' + w_k\right)\nu^k + \frac{2}{z}\sum_{k=0}^{\infty} w_k'\,\nu^{k+1} = 0.$$

Here the coefficients of all the different powers of ν necessarily vanish identically as functions of z.

Hence the system of functions $w_k = w_k(z)$ satisfies the following infinite system of differential equations:

$$w_0'' + \frac{1}{z}\,w_0' + w_0 = 0,$$

$$w_k'' + \frac{1}{z}\,w_k' + w_k + \frac{2}{z}\,w_{k-1}' = 0 \quad (k = 1, 2, 3, \ldots).$$

We shall be concerned here not with this infinite system, but only with the finite subsystem Q of its first four equations

$$Q:\begin{cases} w_0''+\frac{1}{z}w_0'+w_0=0, \\ w_k''+\frac{1}{z}w_k'+w_k+\frac{2}{z}w_{k-1}'=0 \quad (k=1,2,3). \end{cases}$$

The solutions of Q will be written as row vectors

$$\mathbf{w}=(w_0,w_1,w_2,w_3),$$

where the usual definitions hold for sums and scalar products of such vectors.

From the series in powers of ν of the functions

$$K_\nu(z)\,\nu^k \quad\text{and}\quad L_\nu(z)\,\nu^k \quad (k=0,1,2,3),$$

it follows at once that Q possesses the following eight special solutions,

$$\mathbf{A}_k(z)=(\overbrace{0,0,\ldots,0}^{k\text{ zeros}},A_0(z),A_1(z),\ldots,A_{3-k}(z)) \quad (k=0,1,2,3),$$

$$\mathbf{B}_k(z)=(\overbrace{0,0,\ldots,0}^{k\text{ zeros}},B_0(z),B_1(z),\ldots,B_{3-k}(z)) \quad (k=0,1,2,3).$$

We assert that *these eight solutions of Q are linearly independent over* $\mathbf{C}$. For, on putting $\nu=0$ in (5) and applying (6), we find that

$$A_0B_0'-A_0'B_0=\frac{1}{z};$$

therefore the vectors $\mathbf{A}_0$ and $\mathbf{B}_0$ certainly are linearly independent. The assertion for the eight vectors follows now from the triangular form of the two square matrices

$$\begin{bmatrix}\mathbf{A}_0\\ \mathbf{A}_1\\ \mathbf{A}_2\\ \mathbf{A}_3\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}\mathbf{B}_0\\ \mathbf{B}_1\\ \mathbf{B}_2\\ \mathbf{B}_3\end{bmatrix}.$$

We immediately deduce from the linear independence of the vectors $\mathbf{A}_k$ and $\mathbf{B}_k$ that every solution $\mathbf{w}$ of Q allows a unique representation

$$\mathbf{w}=\sum_{h=0}^{3}(a_h\mathbf{A}_h+b_h\mathbf{B}_h), \tag{8}$$

where the coefficients a_h and b_h lie in $\mathbf{C}$.

4. Let $\mathbf{w}=(w_0,w_1,w_2,w_3)$ be again any solution of Q. It is evident from the differential equations of Q that every derivative

$$w_h^{(j)} \quad \begin{pmatrix} h=0,1,2,3; \\ j=0,1,2,\ldots \end{pmatrix}$$

of the components of $\mathbf{w}$ can be written as a linear form in the eight functions

$$w_h,\ w_h' \quad (h=0,1,2,3), \tag{9}$$

with coefficients in $\mathbf{C}(z)$ that have denominators which are powers of z.

Surprisingly, *these eight functions* (9) *are not algebraically independent over* $\mathbf{C}(z)$. For denote by $L_1\{\mathbf{w}\}$ and $L_2\{\mathbf{w}\}$ the two differential operators

$$L_1\{\mathbf{w}\} = z(w_0 w_1' - w_0' w_1) + w_0^2$$

and
$$L_2\{\mathbf{w}\} = z(w_0 w_3' - w_1 w_2' + w_2 w_1' - w_3 w_0') + (2w_0 w_2 - w_1^2).$$

On differentiating with respect to z and eliminating the second derivatives of the components by means of the equations of Q, it is found that all terms cancel out. Hence

$$\frac{d}{dz} L_1\{\mathbf{w}\} = \frac{d}{dz} L_2\{\mathbf{w}\} = 0.$$

There exist then two complex numbers

$$C_1 = C_1\{\mathbf{w}\} \quad \text{and} \quad C_2 = C_2\{\mathbf{w}\},$$

which are independent of the variable z *but depend on the vector* $\mathbf{w}$, *such that*

$$L_1\{\mathbf{w}\} = C_1 \tag{10}$$

and
$$L_2\{\mathbf{w}\} = C_2. \tag{11}$$

The eight functions (9) are thus connected by *two independent algebraic relations* (10) *and* (11). These relations allow to express

$$w_1' \quad \text{and} \quad w_3'$$

as rational functions of the six functions

$$w_0,\ w_1,\ w_2,\ w_3,\ w_0',\ w_2', \tag{12}$$

where the coefficients of these rational functions lie in $\mathbf{C}(z)$ and their denominators are powers of zw_0.

Equation (10) is due to Belogrivov (1967, p. 56).

5. The two constants $C_1\{\mathbf{w}\}$ and $C_2\{\mathbf{w}\}$ can be expressed in terms of the coefficients a_h and b_h that occur in (8). This may be done as follows.

By (8),

$$w_h = \sum_{j=0}^{h} (a_j A_{h-j} + b_j B_{h-j}) \quad (h = 0, 1, 2, 3). \tag{13}$$

Here, for each suffix h, A_h and zA_h' are elements of $\mathbf{E}$, while B_h and zB_h' by (7) are polynomials in $\log z$ with coefficients in $\mathbf{E}$.

If p is any such polynomials in $\log z$ with coefficients in $\mathbf{E}$, denote by $[p]$ its 'constant' part, i.e. that term which has no factor $\log z$. The formulae (7) show that

$$[B_0] = A_1, \quad [B_1] = 0, \quad [B_2] = A_3, \quad [B_3] = 0,$$

$$[zB_0'] = zA_1' + A_0, \quad [zB_1'] = -A_1, \quad [zB_2'] = zA_3' + A_2, \quad [zB_3'] = -A_3.$$

Therefore, from (13),

$$[w_0] = a_0 A_0 + b_0 A_1,$$
$$[w_1] = a_1 A_0 + (a_0 + b_1) A_1,$$
$$[w_2] = a_2 A_0 + (a_1 + b_2) A_1 + a_0 A_2 + b_0 A_3,$$
$$[w_3] = a_3 A_0 + (a_2 + b_3) A_1 + a_1 A_2 + (a_0 + b_1) A_3,$$

and

$$[zw_0'] = z(a_0 A_0' + b_0 A_1') + b_0 A_0,$$
$$[zw_1'] = z\big(a_1 A_0' + (a_0 + b_1) A_1'\big) + (b_1 A_0 - b_0 A_1),$$
$$[zw_2'] = z\big(a_2 A_0' + (a_1 + b_2) A_1' + a_0 A_2' + b_0 A_3'\big) + (b_2 A_0 - b_1 A_1 + b_0 A_2),$$
$$[zw_3'] = z\big(a_3 A_0' + (a_2 + b_3) A_1' + a_1 A_2' + (a_0 + b_1) A_3'\big) + (b_3 A_0 - b_2 A_1 + b_1 A_2 - b_0 A_3).$$

It is also obvious that $L_1\{\mathbf{w}\}$ and $L_2\{\mathbf{w}\}$ are polynomials in $\log z$ with coefficients in $\mathbf{E}$ and that the differences

$$L_1\{\mathbf{w}\} - [L_1\{\mathbf{w}\}] \quad \text{and} \quad L_2\{\mathbf{w}\} - [L_2\{\mathbf{w}\}]$$

are of the form $p \log z$ where p is a polynomial in $\log z$ with coefficients in $\mathbf{E}$. It follows that the two equations (10) and (11) cannot hold unless

$$L_1\{\mathbf{w}\} = [L_1\{\mathbf{w}\}] \quad \text{and} \quad L_2\{\mathbf{w}\} = [L_2\{\mathbf{w}\}].$$

The left-hand sides in these two equations are independent of z. Their values can thus be determined by putting $z = 0$ on the right-hand sides.

Now $[L_1\{\mathbf{w}\}]$ and $[L_2\{\mathbf{w}\}]$ evidently are the same quadratic forms in the expressions $[w_h]$ and $[zw_h']$ as $L_1\{w\}$ and $L_2\{w\}$ are in w_h and zw_h'. It is further obvious from the definition (6) of $A_k(z)$ that

$$A_0(0) = 1, \quad A_1(0) = A_2(0) = A_3(0) = 0,$$

hence from the explicit formulae for $[w_h]$ and $[zw_h']$ that

$$[w_h]_{z=0} = a_h, \quad [zw_h']_{z=0} = b_h \quad (h = 0, 1, 2, 3).$$

On substituting these values in $[L_1\{\mathbf{w}\}]_{z=0}$ and $[L_2\{\mathbf{w}\}]_{z=0}$, it follows then finally, by (10) and (11), that

$$C_1\{\mathbf{w}\} = (a_0 b_1 - a_1 b_0) + a_0^2 \tag{14}$$

and

$$C_2\{\mathbf{w}\} = (a_0 b_3 - a_1 b_2 + a_2 b_1 - a_3 b_0) + (2a_0 a_2 - a_1^2). \tag{15}$$

By way of example,

$$C_1\{\mathbf{A}_0\} = 1, \quad C_2\{\mathbf{A}_0\} = 0.$$

6. It is convenient at this point to prove a simple lemma.

A field of analytic functions $f(z)$ of the variable z is said to be *closed under differentiation* if with $f(z)$ also its derivative $f'(z)$ belongs to the field.

LEMMA 1. *Let* $\mathbf{F}$ *be an extension field of* $\mathbf{C}(z)$ *which is closed under differentiation. Let* f *be an element of some extension field* $\mathbf{F}^*$ *of* $\mathbf{F}$ *such that* f *is algebraic over* $\mathbf{F}$, *while its derivative* f' *lies in* $\mathbf{F}$ *itself. Then* f *is an element of* $\mathbf{F}$.

Proof. The hypothesis implies that there exists a polynomial

$$P(x) = x^n + P_1 x^{n-1} + \dots + P_n$$

in $\mathbf{F}[x]$ of smallest possible degree $n \geqslant 1$ such that

$$P(f) = 0. \tag{16}$$

We derive from $P(x)$ a second polynomial

$$P^*(x) = [nx^{n-1} + (n-1)P_1 x^{n-2} + \dots + 1\,.\,P_{n-1}]f' + [P_1' x^{n-1} + \dots + P_n'].$$

By the hypothesis, also $P^*(x)$ lies in $\mathbf{F}[x]$, and further

$$P^*(\phi) = \frac{\mathrm{d}P(\phi)}{\mathrm{d}z} \quad \text{if } \phi \text{ satisfies } \phi' = f'. \tag{17}$$

Now, explicitly,

$$P^*(x) = (nf' + P_1')x^{n-1} \quad \text{plus terms in lower powers of } x,$$

so that $P^*(x)$ has lower degree than $P(x)$. Moreover, by (16) and (17),

$$P^*(f) = 0.$$

The definition of $P(x)$ implies then that $P^*(x)$ is identically zero, hence that its highest coefficient

$$nf' + P_1' = 0.$$

On integrating this equation, it follows that

$$f = -\frac{1}{n}P_1 \quad \text{plus a constant,}$$

which proves the assertion that f lies in $\mathbf{F}$.

7. The two equations (10) and (11) implied that

$$w_1' \quad \text{and} \quad w_3'$$

were rational functions of

$$w_0,\ w_1,\ w_2,\ w_3,\ w_0',\ w_2', \tag{12}$$

with coefficients in $\mathbf{C}(z)$ and with denominators that are powers of zw_0.

It is essential for the later application to find when these six functions (12) are algebraically independent over $\mathbf{C}(z)$. That this is not the case when $w_0 \equiv 0$ and hence $L_1\{\mathbf{w}\} = 0$ is trivial. A simple result is, however, obtained when $L_1\{\mathbf{w}\} \neq 0$.

THEOREM 1. *Let* $\mathbf{w}$ *be any solution of* Q *such that* $L_1\{\mathbf{w}\} \neq 0$. *Then the six functions*

$$w_0,\ w_1,\ w_2,\ w_3,\ w_0',\ w_2'$$

are algebraically independent over $\mathbf{C}(z)$.

COROLLARY: *Since* $L_1\{\mathbf{A}_0\} \neq 0$, *the six functions*

$$A_0,\ A_1,\ A_2,\ A_3,\ A_0',\ A_2'$$

are algebraically independent over $\mathbf{C}(z)$.

The proof of theorem 1 is split into a number of separate steps which we state as lemmas. For the first two of these steps we can refer to the literature.

LEMMA 2. *The two functions*

$$w_0 \quad and \quad w_0'$$

are algebraically independent over $\mathbf{C}(z)$ *if* $w_0 \not\equiv 0$.

This assertion is a very special case of a theorem on Bessel functions which goes back to Liouville. For a proof see, for example, the book by Siegel (1949, pp. 60–65).

LEMMA 3. *The three functions*

$$w_0, w_0', \quad and \quad w_1$$

are algebraically independent over $\mathbf{C}(z)$ *if* $w_0 \not\equiv 0$.

A more general result which contains this assertion as a special case is proved in the paper by Belogrivov (1967, pp. 56–58).

For the remaining three steps, detailed proofs will be established in the next sections.

8. LEMMA 4. *The four functions*

$$w_0, w_0', w_1, \quad and \quad z(w_0 w_2' - w_0' w_2) + w_0 w_1$$

are algebraically independent over $\mathbf{C}(z)$ *if* $w_0 \not\equiv 0$.

Proof. Assume that this assertion is false. Put

$$s = z(w_0 w_2' - w_0' w_2) + w_0 w_1,$$

and denote by $\mathbf{F}_1$ the extension field

$$\mathbf{F}_1 = \mathbf{C}(z, w_0, w_0', w_1)$$

of $\mathbf{C}(z)$. Here, by (10),

$$w_1' = \frac{z w_0' w_1 - w_0^2 + C_1}{z w_0}. \tag{13}$$

Hence $\mathbf{F}_1$ is identical with the larger extension field

$$\mathbf{F}_1 = \mathbf{C}(z, w_0, w_0', w_1, w_1'),$$

and therefore, by Q, $\mathbf{F}_1$ *is closed under differentiation*.

By lemma 3, and by our hypothesis, the *three* functions w_0, w_0', and w_1, but *not* also the *four* functions w_0, w_0', w_1, and s, are algebraically independent over $\mathbf{C}(z)$. This implies then that s *is algebraic over* $\mathbf{F}_1$.

On the other hand, by the equations of Q,

$$\frac{ds}{dz} = (w_0 w_2' - w_0' w_2) + z\left\{w_0\left(-\frac{1}{z}w_2' - w_2 - \frac{2}{z}w_1'\right) - w_2\left(-\frac{1}{z}w_0' - w_0\right)\right\} + w_0' w_1 + w_0 w_1'$$
$$= w_0' w_1 - w_0 w_1',$$

so that, by (13),

$$\frac{ds}{dz} = \frac{w_0^2 - C_1}{z}. \tag{14}$$

Thus s' lies in $\mathbf{F}_1$. It follows then from lemma 1 that s *itself is an element of* $\mathbf{F}_1$.

There exists then a rational function

$$R = R(z, x_0, y_0, x_1)$$

in $\mathbf{C}(z, x_0, y_0, x_1)$ such that

$$s = R(z, w_0, w_0', w_1) \tag{15}$$

identically in z.

Put
$$R^*(z, x_0, y_0, x_1) = \frac{\partial R}{\partial z} + \frac{\partial R}{\partial x_0} y_0 + \frac{\partial R}{\partial y_0} u_0 + \frac{\partial R}{\partial x_1} u_1$$

where u_0 and u_1 denote the expressions

$$u_0 = -\frac{1}{z} y_0 - x_0, \quad u_1 = \frac{z y_0 x_1 - x_0^2 + C_1}{z x_0}.$$

Let further
$$\overline{\mathbf{w}} = (\overline{w}_0, \overline{w}_1, \overline{w}_2, \overline{w}_3)$$

be any second solution of Q for which

$$L_1\{\overline{\mathbf{w}}\} = C_1 = L_1\{\mathbf{w}\}.$$

If we choose

$$x_0 = \overline{w}_0, \quad y_0 = \overline{w}_0', \quad x_1 = \overline{w}_1,$$

it follows immediately from Q and from the analogue of (13) for $\overline{\mathbf{w}}$ that

$$u_0 = \overline{w}_0'', \quad u_1 = \overline{w}_1'.$$

The definition of R^* leads therefore to the equation

$$R^*(z, \overline{w}_0, \overline{w}_0', \overline{w}_1) = \frac{\mathrm{d}}{\mathrm{d}z} R(z, \overline{w}_0, \overline{w}_0', \overline{w}_1). \tag{16}$$

In particular,
$$R^*(z, w_0, w_0', w_1) = \frac{\mathrm{d}}{\mathrm{d}z} R(z, w_0, w_0', w_1),$$

whence, by (14) and (15),

$$R^*(z, w_0, w_0', w_1) = \frac{w_0^2 - C_1}{z} \tag{17}$$

identically in z.

In this equation, w_0, w_0', and w_1 are by lemma 3 algebraically independent over $\mathbf{C}(z)$; the equation implies therefore that also

$$R^*(z, x_0, y_0, x_1) = \frac{x_0^2 - C_1}{z} \tag{18}$$

identically in the indeterminates z, x_0, y_0 and x_1.

Put now

$$\overline{s} = z(\overline{w}_0 \overline{w}_2' - \overline{w}_0' \overline{w}_2) + \overline{w}_0 \overline{w}_1$$

and repeat for $\overline{s}$ the calculation which lead to the formula (14) for s. We find then that also

$$\frac{\mathrm{d}\overline{s}}{\mathrm{d}z} = \frac{\overline{w}_0^2 - C_1}{z},$$

hence, by (18), that

$$R^*(z, \overline{w}_0, \overline{w}_0', \overline{w}_1) = \frac{\mathrm{d}\overline{s}}{\mathrm{d}z}.$$

Finally, by (16), this implies that

$$\frac{d}{dz} R(z, \overline{w}_0, \overline{w}_0', \overline{w}_1) = \frac{d\bar{s}}{dz}$$

identically in z. On integrating, we obtain therefore the relation

$$z(\overline{w}_0 \overline{w}_2' - \overline{w}_0' \overline{w}_2) + \overline{w}_0 \overline{w}_1 = R(z, \overline{w}_0, \overline{w}_0', \overline{w}_1) + c\{\overline{\mathbf{w}}\}, \tag{19}$$

where $c\{\overline{\mathbf{w}}\}$ denotes a quantity that depends on the special solution $\overline{\mathbf{w}}$ of Q, but is independent of z.

9. The proof of lemma 4 will now be concluded by deducing a contradiction from this relation (19).

For this purpose, it suffices to choose the special solution $\overline{\mathbf{w}}$ suitably. We take

$$\overline{\mathbf{w}} = \overline{a}_0 \mathbf{A}_0 + \overline{b}_1 \mathbf{B}_1,$$

where $\overline{a}_0$ and $\overline{b}_1$ are complex numbers on which, for the present, we impose only the conditions

$$\overline{a}_0 \neq 0, \quad L_1\{\overline{\mathbf{w}}\} = \overline{a}_0^2 + \overline{a}_0 \overline{b}_1 = C_1, \quad \overline{b}_1 \neq 0. \tag{20}$$

Here the expression (14) for $L_1\{\overline{\mathbf{w}}\}$ has been applied.

The choice of $\overline{w}$ means that

$$\overline{w}_0 = \overline{a}_0 A_0, \quad \overline{w}_1 = \overline{a}_0 A_1 + \overline{b}_1 B_0, \quad \overline{w}_2 = \overline{a}_0 A_2 + \overline{b}_1 B_1.$$

Thus the left-hand side (l.h.s.) of (19) becomes

$$\text{l.h.s.} = \overline{a}_0^2\{z(A_0 A_2' - A_0' A_2) + A_0 A_1\} + \overline{a}_0 \overline{b}_1\{z(A_0 B_1' - A_0' B_1) + A_0 B_0\}.$$

Here, by § 2,

$$B_0 = A_1 + A_0 \log z, \quad B_1 = -\{A_1 \log z + A_0 (\log z)^2\},$$

and $$B_0' = A_1' + A_0' \log z + \frac{1}{z} A_0, \quad B_1' = -\left\{A_1' \log z + A_0'(\log z)^2 + \frac{1}{z} A_1 + \frac{2}{z} A_0 \log z\right\}.$$

Therefore, after a trivial calculation,

$$z(A_0 B_1' - A_0' B_1) + A_0 B_0 = -\{z(A_0 A_1' - A_0' A_1) + A_0^2\} \log z = -\log z,$$

because, by § 5,

$$z(A_0 A_1' - A_0' A_1) + A_0^2 = L_1\{\mathbf{A}_0\} = C_1\{\mathbf{A}_0\} = 1.$$

Hence, finally, $$\text{l.h.s.} = \overline{a}_0^2 E(z) - \overline{a}_0 \overline{b}_1 \log z,$$

where $$E(z) = z(A_0 A_2' - A_0' A_2) + A_0 A_1$$

denotes an entire function of z.

Next, the right-hand side (r.h.s.) of (19) has the explicit form

$$\text{r.h.s.} = R(z, \overline{a}_0 A_0, \overline{a}_0 A_0', (\overline{a}_0 + \overline{b}_1) A_1 + \overline{b}_1 A_0 \log z) + c\{\overline{w}\}.$$

Here, by lemma 3, A_0, A_0', and A_1 are entire functions which are algebraically independent over $\mathbf{C}(z)$; hence also the four functions

$$A_0, A_0', A_1, \quad \text{and} \quad \log z$$

are algebraically independent over $\mathbf{C}(z)$. Since R is a rational function of its arguments, equation (19): l.h.s. = r.h.s., cannot hold unless R is an *entire linear function* of its last argument. This means that R has the form

$$R(z, x_0, y_0, x_1) = r_0(z, x_0, y_0) + r_1(z, x_0, y_0)\, x_1,$$

where r_0 and r_1 are rational functions of z, x_0 and y_0, which do not depend on x_1.

The equation (19) becomes now

$$\bar{a}_0^2 E(z) - \bar{a}_0 \bar{b}_1 \log z =$$
$$= r_0(z, \bar{a}_0 A_0, \bar{a}_0 A_0') + r_1(z, \bar{a}_0 A_0, \bar{a}_0 A_0')\{(\bar{a}_0 + \bar{b}_1) A_1 + \bar{b}_1 A_0 \log z\} + c\{\bar{\mathbf{w}}\}$$

and requires that

$$r_1(z, \bar{a}_0 A_0, \bar{a}_0 A_0') \bar{b}_1 A_0 \log z = -\bar{a}_0 \bar{b}_1 \log z,$$

hence, by $\bar{b}_1 \neq 0$, that

$$r_1(z, \bar{a}_0 A_0, \bar{a}_0 A_0') = -\frac{\bar{a}_0}{A_0} = -\frac{\bar{a}_0^2}{w_0}.$$

Since, by lemma 2, A_0 and A_0' are algebraically independent over $\mathbf{C}(z)$, r_1 has then the explicit form

$$r_1(z, x_0, y_0) = -\frac{\bar{a}_0^2}{x_0}.$$

However, it follows from their definitions that R and r_1 do not depend on the special choice of the constant $\bar{a}_0$, while, on the other hand, it is possible to satisfy the conditions (20) for every choice of $\bar{a}_0$ distinct from 0 and $\sqrt{C_1}$. Hence a contradiction arises, so proving the truth of lemma 4.

10. LEMMA 5. *The five functions*

$$w_0,\ w_0',\ w_1,\ w_2, \quad \textit{and} \quad w_2'$$

are algebraically independent over $\mathbf{C}(z)$ *if* $L_1\{\mathbf{w}\} \neq 0$ *and hence* $w_0 \not\equiv 0$.

Proof. Assume, on the contrary, that these five functions are algebraically dependent over $\mathbf{C}(z)$, hence also the five functions

$$w_0,\ w_0',\ w_1,\ s = z(w_0 w_2' - w_0' w_2) + w_0 w_1, \quad \text{and} \quad w_2.$$

By lemma 4, the first four of these functions are algebraically independent over $\mathbf{C}(z)$. Therefore, if $\mathbf{F}_2$ denotes the extension field

$$\mathbf{F}_2 = \mathbf{C}(z, w_0, w_0', w_1, s)$$

of $\mathbf{C}(z)$, then w_2 and hence also

$$q = w_2 / w_0$$

are algebraic over $\mathbf{F}_2$.

By the relations Q, (13), and (14), $\mathbf{F}_2$ is closed under differentiation. Further, from the definitions of q and s,

$$q' = \frac{s - w_0 w_1}{z w_0^2}. \tag{21}$$

Hence q' lies in $\mathbf{F}_2$. On applying lemma 1 once more, it follows that q *itself is an element of* $\mathbf{F}_2$.

There exists then a rational function

$$S = S(z, x_0, y_0, x_1, t)$$

in $\mathbf{C}(z, x_0, y_0, x_1, t)$ such that

$$q = S(z, w_0, w_0', w_1, s) \tag{22}$$

identically in z.

Similarly, as in § 8, put

$$S^*(z, x_0, y_0, x_1, t) = \frac{\partial S}{\partial z} + \frac{\partial S}{\partial x_0} y_0 + \frac{\partial S}{\partial y_0} u_0 + \frac{\partial S}{\partial x_1} u_1 + \frac{\partial S}{\partial t} u_2,$$

where u_0, u_1, and u_2 denote the expressions

$$u_0 = -\frac{1}{z} y_0 - x_0, \quad u_1 = \frac{z y_0 x_1 - x_0^2 + C_1}{z x_0}, \quad u_2 = \frac{x_0^2 - C_1}{z}.$$

Let further

$$\overline{\mathbf{w}} = (\overline{w}_0, \overline{w}_1, \overline{w}_2, \overline{w}_3)$$

be an arbitrary second solution of Q for which again

$$L_1\{\overline{\mathbf{w}}\} = C_1 = L_1\{\mathbf{w}\}.$$

With the choice

$$x_0 = \overline{w}_0, \quad y_0 = \overline{w}_0', \quad x_1 = \overline{w}_1, \quad t = \overline{s} = z(\overline{w}_0 \overline{w}_2' - \overline{w}_0' \overline{w}_2) + \overline{w}_0 \overline{w}_1,$$

the relations Q, (13) and (14) imply that

$$u_0 = \overline{w}_0'', \quad u_1 = \overline{w}_1', \quad u_2 = \overline{s}'.$$

Therefore, from the definition of S^*,

$$S^*(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{s}) = \frac{\mathrm{d}}{\mathrm{d}z} S(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{s}). \tag{23}$$

In particular, $$S^*(z, w_0, w_0', w_1, s) = \frac{\mathrm{d}}{\mathrm{d}z} S(z, w_0, w_0', w_1, s),$$

whence, by (21) and (22),

$$S^*(z, w_0, w_0', w_1, s) = \frac{s - w_0 w_1}{z w_0^2}.$$

In this equation, the functions w_0, w_0', w_1, and s, are by lemma 4 algebraically independent over $\mathbf{C}(z)$. It therefore implies that

$$S^*(z, x_0, y_0, x_1, t) = \frac{t - x_0 x_1}{z x_0^2} \tag{24}$$

identically in the indeterminates z, x_0, y_0, x_1 and t.

Assume that also $\overline{w}_0 \neq 0$, and in analogy to q put

$$\overline{q} = \frac{\overline{w}_2}{\overline{w}_0},$$

so that evidently also

$$\overline{q}' = \frac{\overline{s} - \overline{w}_0 \overline{w}_1}{z \overline{w}_0^2}. \tag{25}$$

It follows now from the formulae (23), (24) and (25) that

$$\frac{d}{dz} S(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{s}) = \frac{d\overline{q}}{dz},$$

whence, on integrating with respect to z,

$$\overline{q} = S(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{s}) + \gamma\{\overline{\mathbf{w}}\}. \tag{26}$$

Here $\gamma\{\overline{\mathbf{w}}\}$ denotes a quantity that depends on the special solution $\overline{\mathbf{w}}$ of Q, but is independent of the variable z.

11. To conclude the proof of lemma 5, we proceed now as in § 9. We choose

$$\overline{\mathbf{w}} = \overline{a}_0 \mathbf{A}_0 + \overline{b}_2 \mathbf{B}_2,$$

where $\overline{a}_0$ and $\overline{b}_2$ are complex numbers satisfying

$$\overline{a}_0 = \sqrt{C_1}, \quad \overline{b}_2 \neq 0.$$

This choice obviously is compatible with the condition $L_1\{\overline{\mathbf{w}}\} = C_1$ and has the consequence that $\overline{a}_0 \neq 0$ because $C_1 = L_1\{\mathbf{w}\} \neq 0$ by hypothesis. Further

$$\overline{w}_0 = \overline{a}_0 A_0, \quad \overline{w}_1 = \overline{a}_0 A_1, \quad \overline{w}_2 = \overline{a}_0 A_2 + \overline{b}_2 B_0,$$

where $$B_0 = A_1 + A_0 \log z,$$

so that evidently $$z(A_0 B_0' - A_0' B_0) = 1.$$

Hence, on putting again

$$E(z) = z(A_0 A_2' - A_0' A_2) + A_0 A_1,$$

we find that $$\overline{s} = \overline{a}_0^2 E(z) + \overline{a}_0 \overline{b}_2.$$

Equation (26) takes now the form

$$\overline{a}_0 A_2 + \overline{b}_2 A_1 + \overline{b}_2 A_0 \log z = \\ = \{S[z, \overline{a}_0 A_0, \overline{a}_0 A_0', \overline{a}_0 A_1, \overline{a}_0^2 E(z) + \overline{a}_0 \overline{b}_2] + \gamma\{\overline{\mathbf{w}}\}\} \overline{a}_0 A_0.$$

Here the l.h.s. has a logarithmic singularity at $z = 0$, while the r.h.s. is a meromorphic function of z. Thus a contradiction is obtained, so proving the assertion.

12. We come now at last to the proof of theorem 1 itself. Denote by $\mathbf{F}_3$ the extension field

$$\mathbf{F}_3 = \mathbf{C}(z, w_0, w_0', w_1, w_2, w_2')$$

of $\mathbf{C}(z)$. By lemma 5, $\mathbf{F}_3$ is a purely transcendental extension of $\mathbf{C}(z)$; and by Q and by the equation (13), $\mathbf{F}_3$ is closed under differentiation.

Assume now that theorem 1 is false, hence that

$$w_0, w_1, w_2, w_3, w_0', w_2'$$

are algebraically dependent over $\mathbf{C}(z)$. By lemma 5, this hypothesis evidently implies that w_3 and hence also the quotient

$$\varphi = w_3/w_0$$

are algebraic over $\mathbf{F}_3$.

Now it was proved in § 4 that

$$L_2\{\mathbf{w}\} = C_2,$$

where $C_2 = C_2\{\mathbf{w}\}$ likewise is independent of z, but depends on the special choice of the solution $\mathbf{w}$ of Q. This equation is equivalent to

$$zw_0^2\varphi' = z(w_1w_2' - w_1'w_2) + (w_1^2 - 2w_0w_2) + C_2, \tag{26'}$$

and it implies that φ' lies in $\mathbf{F}_3$. On applying once more lemma 1, it follows then that φ and hence also w_3 are elements of $\mathbf{F}_3$.

This means that there exists a rational function

$$T = T(z, x_0, y_0, x_1, x_2, y_2)$$

in $C(z, x_0, y_0, x_1, x_2, y_2)$ such that

$$\varphi = \frac{w_3}{w_0} = T(z, w_0, w_0', w_1, w_2, w_2') \tag{27}$$

identically in z.

Put $\quad T^*(z, x_0, y_0, x_1, x_2, y_2) = \frac{\partial T}{\partial z} + \frac{\partial T}{\partial x_0}y_0 + \frac{\partial T}{\partial y_0}u_0 + \frac{\partial T}{\partial x_1}u_1 + \frac{\partial T}{\partial x_2}y_2 + \frac{\partial T}{\partial y_2}u,$

where u_0, u_1 and u denote the expressions

$$u_0 = -\frac{1}{z}y_0 - x_0, \quad u_1 = \frac{zy_0x_1 - x_0^2 + C_1}{zx_0}, \quad u = -\frac{1}{z}y_2 - x_2 - \frac{2}{z}u_1.$$

Let further $\qquad \overline{\mathbf{w}} = (\overline{w}_0, \overline{w}_1, \overline{w}_2, \overline{w}_3)$

be an arbitrary second solution of Q satisfying the two additional conditions

$$L_1\{\overline{\mathbf{w}}\} = C_1 = L_1\{\mathbf{w}\} \quad \text{and} \quad L_2\{\overline{\mathbf{w}}\} = C_2 = L_2\{\mathbf{w}\}.$$

On choosing now in the last formulae

$$x_0 = \overline{w}_0, \quad y_0 = \overline{w}_0', \quad x_1 = \overline{w}_1, \quad x_2 = \overline{w}_2, \quad y_2 = \overline{w}_2',$$

by Q and (13) $\qquad u_0 = \overline{w}_0'', \quad u_1 = \overline{w}_1', \quad u = \overline{w}_2''.$

It follows therefore that

$$T^*(z, \overline{w}_0, \overline{w}_0', w_1, \overline{w}_2, \overline{w}_2') = \frac{\mathrm{d}}{\mathrm{d}z}T(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{w}_2, \overline{w}_2'). \tag{28}$$

In particular,

$$T^*(z, w_0, w_0', w_1, w_2, w_2') = \frac{\mathrm{d}}{\mathrm{d}z}T(z, w_0, w_0', w_1, w_2, w_2'),$$

whence, by (26) and (27),

$$T^*(z, w_0, w_0', w_1, w_2, w_2') = (zw_0^2)^{-1}[z(w_1w_2' - w_1'w_2) + (w_1^2 - 2w_0w_2) + C_2].$$

In this equation, the functions w_0, w_0', w_1, w_2, w_2' are by lemma 5 algebraically independent over $\mathbf{C}(z)$; it therefore requires that

$$T^*(z, x_0, y_0, x_1, x_2, y_2) = (zx_0^2)^{-1}[z(x_1y_2 - y_1x_2) + (x_1^2 - 2x_0x_2) + C_2] \tag{29}$$

identically in the indeterminates z, x_0, y_0, x_1, x_2, y_2.

In analogy to φ, put

$$\overline{\varphi} = \frac{\overline{w}_3}{\overline{w}_0},$$

so that, similar to (26′),

$$z\overline{w}_0^2\overline{\varphi}' = z(\overline{w}_1\overline{w}_2' - \overline{w}_1'w_2) + (\overline{w}_1^2 - 2w_0w_2) + C_2.$$

This equation, together with the formulae (28) and (29), has then the consequence that

$$\frac{d}{dz}T(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{w}_2, \overline{w}_2') = \frac{d\overline{\varphi}}{dz},$$

and so, after integrating with respect to z, we find that

$$\overline{\varphi} = T(z, \overline{w}_0, \overline{w}_0', \overline{w}_1, \overline{w}_2, \overline{w}_2') + \theta\{\overline{\mathbf{w}}\}, \tag{31}$$

where $\theta\{\overline{\mathbf{w}}\}$ again denotes a certain complex quantity which depends on the special solution $\overline{\mathbf{w}}$ of Q, but is independent of the variable z.

Choose here $$\mathbf{w} = \overline{a}_0\mathbf{A}_0 + \overline{a}_2\mathbf{A}_2 + \overline{b}_3\mathbf{B}_3,$$

where $\overline{a}_0$, $\overline{a}_2$, and $\overline{b}_3$ are three complex numbers such that

$$\overline{a}_0 = \sqrt{C_1} \neq 0, \quad \overline{a}_0\overline{b}_3 + 2\overline{a}_0\overline{a}_2 = C_2, \quad \overline{b}_3 \neq 0;$$

such a choice is evidently possible, and it implies, by (14) and (15), that

$$L_1\{\overline{\mathbf{w}}\} = C_1, \quad L_2\{\overline{\mathbf{w}}\} = C_2.$$

Now $\overline{w}_0 = \overline{a}_0 A_0$, $\overline{w}_1 = \overline{a}_0 A_1$, $\overline{w}_2 = \overline{a}_0 A_2 + \overline{a}_2 A_0$, $\overline{w}_3 = \overline{a}_0 A_3 + \overline{a}_2 A_1 + \overline{b}_3 B_0$.

Since $B_0 = A_1 + A_0 \log z$, it follows that the quotient

$$\overline{\varphi} = \frac{\overline{w}_3}{\overline{w}_0} = \frac{\overline{a}_0 A_3 + (\overline{a}_2 + \overline{b}_3) A_1}{\overline{a}_0 A_0} + \frac{\overline{b}_3}{\overline{a}_0} \log z$$

on the l.h.s. of (31) has a logarithmic singularity at the point $z = 0$. On the other hand, the expression

$$T(z, \overline{a}_0 A_0, \overline{a}_0 A_0', \overline{a}_0 A_1, \overline{a}_0 A_2 + \overline{a}_2 A_0, \overline{a}_0 A_2' + \overline{a}_2 A_0') + \theta\{\overline{\mathbf{w}}\}$$

on the r.h.s. of (31) is a meromorphic function of z.

Hence a contradiction arises which proves that our hypothesis was false and that theorem 1 is true.

Chapter 2

13. The functions $A_k(z)$ were defined by

$$A_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k K_\nu(z)\big|_{\nu=0},$$

where $$K_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z^2/4)^n}{n!\,(\nu+1)(\nu+2)\dots(\nu+n)}.$$

We now use this definition to establish arithmetic properties of $A_k(z)$.

For this purpose let a quantity $p_k(\nu, n)$ be defined by the formula

$$\frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k \frac{1}{(\nu+1)(\nu+2)\dots(\nu+n)} = \frac{p_k(\nu,n)}{(\nu+1)(\nu+2)\dots(\nu+n)},$$

and then let further $$p_k(n) = p_k(0, n).$$

Then evidently $$p_0(\nu, n) = 1, \quad p_1(\nu, n) = -\left(\frac{1}{\nu+1} + \frac{1}{\nu+2} + \dots + \frac{1}{\nu+n}\right),$$

and $$p_{k+1}(\nu, n) = p_1(\nu, n)\, p_k(\nu, n) + \frac{\partial}{\partial\nu} p_k(\nu, n).$$

This recursive formula leads for small values of k to the result that

$$p_0(n) = 1, \quad p_1(n) = -(1^{-1}+2^{-1}+\ldots+n^{-1}),$$
$$p_2(n) = (1^{-1}+2^{-1}+\ldots+n^{-1})^2+(1^{-2}+2^{-2}+\ldots+n^{-2}),$$
$$p_3(n) = -(1^{-1}+2^{-1}+\ldots+n^{-1})^3-3(1^{-1}+2^{-1}+\ldots+n^{-1})(1^{-2}+2^{-2}+\ldots+n^{-2})-$$
$$-2(1^{-3}+2^{-3}+\ldots+n^{-3}), \quad \text{etc.}$$

In terms of the expressions $p_k(n)$, $A_k(z)$ allows now the representation

$$A_0(z) = 1+\sum_{n=1}^{\infty}\frac{(-z^2/4)^n}{n!\,n!}, \quad A_k(z) = \sum_{n=1}^{\infty} p_k(n)\frac{(-z^2/4)^n}{n!\,n!} \quad (k = 1, 2, 3, \ldots). \tag{32}$$

14. The theorems of Siegel and Shidlovski which are to be applied deal with the transcendency of the values of *Siegel E-functions*. For the general definition of such functions we refer to the book by Siegel (1949, p. 33). For our purpose the following special case of such functions suffices.

A power series

$$f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}$$

is a Siegel E-function if it has rational coefficients f_n with the following property. There exists a positive integer C such that both

$$|f_n|$$

and the least common denominator of

$$f_0, f_1, \ldots, f_n$$

are for sufficiently large n not greater than C^n.

That $A_0(z) = K_0(z)$ has this property was shown by Siegel (1949, pp. 56–58). But, as we now prove, $A_k(z)$ is an E-function also for suffixes $k \geqslant 1$.

For denote by D_n the least common multiple of the integers $1, 2, \ldots, n$, and put

$$s_h(n) = 1^{-h}+2^{-h}+\ldots+n^{-h} \quad (h = 1, 2, \ldots, k).$$

Then $p_k(z)$ evidently can be written as a polynomial in these sums, of the form

$$p_k(n) = \Sigma c_{i_1 i_2 \ldots i_k} s_1(n)^{i_1} s_2(n)^{i_2} \ldots s_k(n)^{i_k},$$

where the summation extends over all sets of k integers $i_1, i_2, \ldots, i_n$ satisfying

$$i_1 \geqslant 0, \quad i_2 \geqslant 0, \ldots, i_k \geqslant 0, \quad 1.i_1+2.i_2+\ldots+k.i_k = k.$$

It follows that

$$D_n^k p_k(n)$$

is an integer for all $k \geqslant 1$ and $n \geqslant 1$. Now, for large n,

$$s_k(n) = \begin{cases} O(\log n) & \text{if} \quad k = 1, \\ O(1) & \text{if} \quad k \geqslant 2, \end{cases}$$

and it is also known from number theory that there exists a positive integer c_0 such that for large n

$$D_n = O(c_0^n).$$

It follows that for each suffix $k \geqslant 1$ there exists a positive integer c_k such that, for all sufficiently large n, both $p_k(n)$ and the least common denominator of the numbers $p_k(1), p_k(2), \ldots, p_k(n)$ are of the order $O(c_k^n)$. This estimate evidently implies that also the functions $A_k(z)$, where $k \geqslant 1$, are E-functions.

15. Generalizing Siegel's work, Shidlovski established the following beautiful theorem (1962, pp. 898–899).

Let $w_1 = f_1(z), \ldots, w_m = f_m(z)$ be finitely many E-functions satisfying a system of linear differential equations

$$w_h' = q_{h0} + \sum_{k=1}^{m} q_{hk} w_k \quad (h = 1, 2, \ldots, m),$$

where the coefficients q_{h0} and q_{hk} are rational functions in $\mathbf{C}(z)$, say with the least common denominator $d(z)$. Let α be any algebraic number such that

$$\alpha d(\alpha) \neq 0.$$

Assume that N_1, but no more, of the functions $f_1(z), \ldots, f_m(z)$ are algebraically independent over $\mathbf{C}(z)$, the field of rational functions of z, and that N_2, but no more, of the function values $f_1(\alpha), \ldots, f_m(\alpha)$ are algebraically independent over $\mathbf{Q}$, the field of rational numbers. Then

$$N_1 = N_2.$$

16. Let us apply Shidlovski's theorem to the system of eight functions

$$A_k(z), A_k'(z) \quad (k = 0, 1, 2, 3). \tag{33}$$

As we have seen, the four functions $A_k(z)$ form a solution of the system Q. Here each equation of Q is a differential equation of the second order. Therefore Q is equivalent to the following system Q_8 of eight differential equations of the first order,

$$Q_8: \begin{cases} w_0' = W_0, \quad W_0' = -\dfrac{1}{z} W_0 - w_0, \\[2ex] w_k' = W_k, \quad W_k' = -\dfrac{1}{z} W_k' - w_k - \dfrac{2}{z} W_{k-1} \quad (k = 1, 2, 3). \end{cases}$$

Denote by Q_2, Q_4 and Q_6 the subsystems of the first two, four, and six of these differential equations, respectively.

The coefficients of all four systems are rational functions in $\mathbf{C}(z)$, and in each system the least common denominator of the coefficients is the same function

$$d(z) = z.$$

As was proved in lemmas 2, 3 and 5, and in theorem 1, respectively, the following four sets of functions occurring in the successive systems are algebraically independent over $\mathbf{C}(z)$,

(Q_2): $w_0 = A_0(z)$ and $W_0 = A_0'(z)$;

(Q_4): $w_0 = A_0(z)$, $W_0 = A_0(z)$, and $w_1 = A_1(z)$;

(Q_6): $w_0 = A_0(z)$, $W_0 = A_0'(z)$, $w_1 = A_1(z)$, $w_2 = A_2(z)$, and $W_2 = A_2'(z)$

(Q_8): $w_0 = A_0(z)$, $W_0 = A_0'(z)$, $w_1 = A_1(z)$, $w_2 = A_2(z)$, $W_2 = A_2'(z)$, and $w_3 = A_3(z)$.

On the other hand, the four functions

$$w_0, W_0, w_1 \quad \text{and} \quad W_1 = A_1'(z),$$

and the seven functions

$$w_0, W_0, w_1, w_2, W_2, w_3 \quad \text{and} \quad W_3 = A_3'(z),$$

certainly are algebraically dependent over $\mathbf{C}(z)$. Hence the integer N_1 in Shidlovski's theorem has for the four systems Q_2, Q_4, Q_6, and Q_8 the values

$$N_1 = 2, \quad N_1 = 3, \quad N_1 = 5 \quad \text{and} \quad N_1 = 6,$$

respectively.

Let now α be any algebraic number distinct from zero, so that

$$\alpha d(\alpha) \neq 0.$$

We apply Shidlovski's Theorem to each of the four systems Q_2, Q_4, Q_6 and Q_8, and find that for these the second integer N_2 is equal to

$$N_2 = 2, \quad N_2 = 3, \quad N_2 = 5 \quad \text{and} \quad N_2 = 6,$$

respectively.

Thus, to begin with, the two function values $A_0(\alpha)$ and $A_0'(\alpha)$ are algebraically independent over $\mathbf{Q}$, and so, in particular, both are transcendental and therefore distinct from 0.

Now, by the identity (10) and by the equation $C_1\{\mathbf{A}_0\} = 1$,

$$\alpha[A_0(\alpha)A_1'(\alpha) - A_0'(\alpha)A_1(\alpha)] + A_0(\alpha)^2 - 1 = 0.$$

Hence if the elements of either of the two sets of three function values

$$A_0(\alpha),\ A_0'(\alpha) \quad \text{and} \quad A_1(\alpha), \tag{34}$$

and

$$A_0(\alpha),\ A_0'(\alpha) \quad \text{and} \quad A_1'(\alpha) \tag{35}$$

were algebraically dependent over $\mathbf{Q}$, so would the elements of the other set. But then the value of N_2 for the system Q_4 would be only 2, and not 3 as has just been proved.

Since $N_2 = 5$ for the system Q_6, this result implies immediately that also the elements of the two sets of five function values

$$A_0(\alpha),\ A_0'(\alpha),\ A_1(\alpha),\ A_2(\alpha) \quad \text{and} \quad A_2'(\alpha), \tag{36}$$

and

$$A_0(\alpha), A_0'(\alpha),\ A_1'(\alpha),\ A_2(\alpha) \quad \text{and} \quad A_2'(\alpha) \tag{37}$$

are algebraically independent over $\mathbf{Q}$.

We come finally to the system Q_8 for which we found that $N_1 = N_2 = 6$. By the identity (11), and by the formula $L_2\{\mathbf{A}_0\} = 0$,

$$\alpha[A_0(\alpha)A_3'(\alpha) - A_1(\alpha)A_2'(\alpha) + A_2(\alpha)A_1'(\alpha) - A_3(\alpha)A_0'(\alpha)] \\ + [2A_0(\alpha)A_2(\alpha) - A_1(\alpha)^2] = 0;$$

and we also found already that $A_0(\alpha) \neq 0$ and $A_0'(\alpha) \neq 0$. It follows that each of the function values $A_3(\alpha)$ and $A_3'(\alpha)$ can be expressed rationally in terms of the other one, where the coefficients involve only the five function values (36), or, equivalent to this, the five function values (37).

By Shidlovski's theorem, we arrive therefore at the following result.

THEOREM 2. *Let α be any algebraic number distinct from zero, and let Σ_1, Σ_2, Σ_3 and Σ_4 denote the four sets of six function values each,*

$$\Sigma_1 = \{A_0(\alpha), A_0'(\alpha), A_1(\alpha), A_2(\alpha), A_2'(\alpha), A_3(\alpha)\},$$
$$\Sigma_2 = \{A_0(\alpha), A_0'(\alpha), A_1'(\alpha), A_2(\alpha), A_2'(\alpha), A_3(\alpha)\},$$
$$\Sigma_3 = \{A_0(\alpha), A_0'(\alpha), A_1(\alpha), A_2(\alpha), A_2'(\alpha), A_3'(\alpha)\},$$
and
$$\Sigma_4 = \{A_0(\alpha), A_0'(\alpha), A_1'(\alpha), A_2(\alpha), A_2'(\alpha), A_3'(\alpha)\}.$$

Then the elements of each of these four sets are algebraically independent over the rational number field $\mathbf{Q}$. *In particular, the eight function values*

$$A_k(\alpha),\ A_k'(\alpha) \quad (k = 0,\ 1,\ 2,\ 3)$$

are transcendental numbers and are therefore distinct from zero.

CHAPTER 3

17. In analogy to $A_k(z)$ and $B_k(z)$, we define further functions $C_k(z)$ by the formula

$$C_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k J_\nu(z)\big|_{\nu=0} \quad (k = 0, 1, 2, \ldots), \tag{38}$$

where $J_\nu(z)$ denotes the Bessel function of the first kind. Then, in particular,

$$C_0(z) = J_0(z), \quad C_1(z) = \frac{2}{\pi} Y_0(z),$$

where $Y_\nu(z)$ denotes the Bessel function of the second kind.

These new functions $C_k(z)$ can be expressed in terms of the $A_k(z)$ by means of the relation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} K_\nu(z). \tag{2}$$

For assume from now on that $z \neq 0$. Then $J_\nu(z)$ is an entire function of ν, and hence, for all values of ν,

$$J_\nu(z) = \sum_{k=0}^{\infty} C_k(z)\, \nu^k. \tag{39}$$

We had further proved in § 2 that

$$K_\nu(z) = \sum_{k=0}^{\infty} A_k(z)\, \nu^k \tag{40}$$

for all sufficiently small $|\nu|$. Next, on putting

$$\zeta = \log\left(\tfrac{1}{2}z\right),$$

for all ν,

$$(z/2)^\nu = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} \nu^k. \tag{41}$$

Finally, $1/\Gamma(\nu+1)$ is an entire function of ν which is real for real ν. Hence this function allows for all ν a power series development

$$\frac{1}{\Gamma(\nu+1)} = \sum_{k=0}^{\infty} \gamma_k \nu^k, \tag{42}$$

where the coefficients $\gamma_0 = 1, \gamma_1, \gamma_2, \ldots$ are real numbers.

Put
$$\zeta_k = \sum_{h=0}^{\infty} \gamma_{k-h} \frac{\zeta_h}{h!} \quad (k = 0, 1, 2, \ldots).$$

Then, for all ν,
$$\frac{(z/2)^\nu}{\Gamma(\nu+1)} = \sum_{k=0}^{\infty} \zeta_k \nu^k. \tag{43}$$

For small suffixes k,

$$\zeta_0 = 1, \quad \zeta_1 = \zeta + \gamma_1, \quad \zeta_2 = \frac{\zeta^2}{2!} + \gamma_1 \zeta + \gamma_2, \quad \zeta_3 = \frac{\zeta^3}{3!} + \gamma_1 \frac{\zeta^2}{2!} + \gamma_2 \zeta + \gamma_3, \quad \text{etc.} \tag{44}$$

On substituting the series (43) in (2), applying the two developments (39) and (40), and after multiplying out comparing the coefficients of the different powers of ν, it follows finally that

$$C_k(z) = \sum_{h=0}^{k} \zeta_{k-h} A_h(z) \quad (k = 0, 1, 2, \ldots). \tag{45}$$

18. The coefficients γ_k and therefore also the sums ζ_k can further be expressed in terms of well-known simpler constants.

Denote by $\Psi(s)$ as usual the logarithmic derivative of the Gamma function $\Gamma(s)$. It is proved in the theory of the Gamma function (see Nielsen 1906, Kapitel 3) that for $|\nu| < 1$,

$$\Psi(\nu+1) = \sum_{k=0}^{\infty} (-1)^{k+1} s_{k+1} \nu^k.$$

Here
$$s_1 = \gamma$$

denotes *Euler's constant*, while for suffixes $k \geqslant 2$

$$s_k = \zeta(k) = \sum_{n=1}^{\infty} n^{-k}$$

is a value of the *Riemann Zeta function*. It is further proved that the coefficients γ_k in (42) are connected with the coefficients s_k by the recursive formulae

$$(k+1)\gamma_{k+1} = \sum_{h=0}^{k} (-1)^h s_{h+1} \gamma_{k-h} \quad (k = 0, 1, 2, \ldots). \tag{46}$$

Therefore, for the smallest values of k,

$$\gamma_1 = \gamma, \quad \gamma_2 = \tfrac{1}{2}(\gamma^2 - s_2), \quad \gamma_3 = \tfrac{1}{6}(\gamma^3 - 3\gamma s_2 + 2s_3), \quad \text{etc.} \tag{47}$$

We introduce the abbreviation
$$\chi = \zeta + \gamma,$$

and we further note that $\chi = \log(\tfrac{1}{2}z) + \gamma$ has the derivative

$$\chi' = 1/z.$$

By the formulae (44) and (47) we obtain then for $0 \leqslant k \leqslant 3$ the explicit results,

$$\zeta_0 = 1, \quad \zeta_1 = \chi, \quad \zeta_2 = \tfrac{1}{2}(\chi^2 - s_2), \quad \zeta_3 = \tfrac{1}{6}(\chi^3 - 3s_2\chi + 2s_3),$$

$$\zeta_0' = 0, \quad \zeta_1' = \frac{1}{z}, \quad \zeta_2' = \frac{1}{z}\chi, \quad \zeta_3' = \frac{1}{2z}(\chi^2 - s_2).$$

For such suffixes k the functions $C_k(z)$ and $C_k'(z)$ allow then the representations

$$\left.\begin{aligned} C_0(z) &= A_0(z), \\ C_1(z) &= A_1(z) + \chi A_0(z), \\ C_2(z) &= A_2(z) + \chi A_1(z) + \tfrac{1}{2}(\chi^2 - s_2)A_0(z), \\ C_3(z) &= A_3(z) + \chi A_2(z) + \tfrac{1}{2}(\chi^2 - s_2)A_1(z) + \tfrac{1}{6}(\chi^3 - 3s_2\chi + 2s_3)A_0(z), \end{aligned}\right\} \quad (48)$$

and

$$\left.\begin{aligned} C_0'(z) &= A_0'(z), \\ C_1'(z) &= A_1'(z) + \chi A_0'(z) + (1/z)C_0(z), \\ C_2'(z) &= A_2'(z) + \chi A_1'(z) + \tfrac{1}{2}(\chi^2 - s_2)A_0'(z) + (1/z)C_1(z), \\ C_3'(z) &= A_3'(z) + \chi A_2'(z) + \tfrac{1}{2}(\chi^2 - s_2)A_1'(z) + \\ &\qquad + \tfrac{1}{6}(\chi^3 - 3s_2\chi + 2s_3)A_0'(z) + (1/z)C_2(z). \end{aligned}\right\} \quad (49)$$

These formulae can be solved for the functions $A_k(z)$ and their derivatives, and they lead then to the equivalent relations,

$$\left.\begin{aligned} A_0(z) &= C_0(z), \\ A_1(z) &= C_1(z) - \chi C_0(z), \\ A_2(z) &= C_2(z) - \chi C_1(z) + \tfrac{1}{2}(\chi^2 + s_2)C_0(z), \\ A_3(z) &= C_3(z) - \chi C_2(z) + \tfrac{1}{2}(\chi^2 + s_2)C_1(z) - \tfrac{1}{6}(\chi^3 + 3s_2\chi + 2s_3)C_0(z), \end{aligned}\right\} \quad (50)$$

and

$$\left.\begin{aligned} A_0'(z) &= C_0'(z), \\ A_1'(z) &= C_1'(z) - \chi C_0'(z) - (1/z)A_0(z), \\ A_2'(z) &= C_2'(z) - \chi C_1'(z) + \tfrac{1}{2}(\chi^2 + s_2)C_0'(z) - (1/z)A_1(z), \\ A_3'(z) &= C_3'(z) - \chi C_2'(z) + \tfrac{1}{2}(\chi^2 + s_2)C_1'(z) - \\ &\qquad - \tfrac{1}{6}(\chi^3 + 3s_2\chi + 2s_3)C_0'(z) - (1/z)A_2(z). \end{aligned}\right\} \quad (51)$$

19. Let us now replace the functions $A_k(z)$ and $A_k'(z)$ in theorem 2 by their expressions (50) and (51) in $C_k(z)$ and $C_k'(z)$. In the assertions on algebraic independence we evidently may omit the last terms

$$-\frac{1}{\alpha}A_0(\alpha), \quad -\frac{1}{\alpha}A_1(\alpha), \quad -\frac{1}{\alpha}A_2(\alpha),$$

which by (51) occur in the derivatives $A_1'(\alpha)$, $A_2'(\alpha)$, and $A_3'(\alpha)$, because α is assumed distinct from zero, and $A_0(\alpha)$, $A_1(\alpha)$, and $A_2(\alpha)$ are elements of all four sets Σ_1, Σ_2, Σ_3, and Σ_4.

We arrive in this way at the following final result.

THEOREM 3. *Let*

$$C_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial\nu}\right)^k J_\nu(z)\Big|_{\nu=0} \quad (k = 0, 1, 2, 3);$$

let α be any algebraic number distinct from zero; let γ be Euler's constant; and let $s_2 = \zeta(2) = \frac{1}{6}\pi^2$, and $s_3 = \zeta(3)$. Put

$$\chi_1 = \log(\tfrac{1}{2}\alpha) + \gamma, \quad \chi_2 = \tfrac{1}{2}(\chi_1^2 + s_2), \quad \chi_3 = \tfrac{1}{6}(\chi_1^3 + 3s_2\chi_1 + 2s_3),$$

and further define numbers A_k and A_k^* by the equations

$$A_0 = C_0(\alpha), \quad A_1 = C_1(\alpha) - \chi_1 C_0(\alpha), \quad A_2 = C_2(\alpha) - \chi_1 C_1(\alpha) + \chi_2 C_0(\alpha),$$
$$A_3 = C_3(\alpha) - \chi_1 C_2(\alpha) + \chi_2 C_1(\alpha) - \chi_3 C_0(\alpha),$$
$$A_0^* = C_0'(\alpha), \quad A_1^* = C_1'(\alpha) - \chi_1 C_0'(\alpha), \quad A_2^* = C_2'(\alpha) - \chi_1 C_1'(\alpha) + \chi_2 C_0'(\alpha),$$
$$A_3^* = C_3'(\alpha) - \chi_1 C_2'(\alpha) + \chi_2 C_1'(\alpha) - \chi_3 C_0'(\alpha).$$

Then the elements of any one of the following four sets of six numbers each,

$$\{A_0, A_0^*, A_1, A_2, A_2^*, A_3\}, \quad \{A_0, A_0^*, A_1^*, A_2, A_2^*, A_3\},$$
$$\{A_0, A_0^*, A_1, A_2, A_2^*, A_3^*\}, \quad \{A_0, A_0^*, A_1^*, A_2, A_2^*, A_3^*\},$$

are algebraically independent over the rational number field $\mathbf{Q}$. *In particular, all eight numbers A_k and A_k^*, where $0 \leqslant k \leqslant 3$, are transcendental.*

One can specialize this theorem and deduce some consequences that have perhaps some interest in themselves. Thus, for algebraic $\alpha \neq 0$,

$$\frac{\pi}{2}\frac{Y_0(\alpha)}{J_0(\alpha)} - \{\log(\tfrac{1}{2}\alpha) + \gamma\}$$

is transcendental, hence in particular the value

$$\frac{\pi}{2}\frac{Y_0(2)}{J_0(2)} - \gamma.$$

Another transcendental expression involving Euler's constant is given by

$$\{C_0(\alpha)C_2'(\alpha) - C_0'(\alpha)C_2(\alpha)\} - \{C_0(\alpha)C_1'(\alpha) - C_0'(\alpha)C_1(\alpha)\}\{\log(\tfrac{1}{2}\alpha) + \gamma\}.$$

The transcendency of s_2 is, of course, due to Lindemann. For s_3 nothing is yet known. However, both expressions A_3 and A_3^* involve this number and are transcendental.

20. Theorem 3 is based on the algebraic result of theorem 1. The problem arises whether the latter can be strengthened. One can easily prove that for every *odd* suffix k the function $A_k'(z)$ can be expressed rationally in the functions

$$A_0(z), A_1(z), \ldots, A_k(z), \quad A_0'(z), A_1'(z), \ldots, A_{k-1}'(z),$$

with coefficients in $C(z)$. One may therefore conjecture that *any finite set* of the functions

$$A_0(z), A_1(z), A_2(z), \ldots, A_0'(z), A_2'(z), A_4'(z), \ldots$$

is algebraically independent over $\mathbf{C}(z)$.

It would have some interest to decide whether this conjecture is true. If it is, then theorem 3 can immediately be generalized so as to assert the algebraic independence of any finite number of sums

$$A_k = C_k(\alpha) - \chi_1 C_{k-1}(\alpha) + \chi_2 C_{k-2}(\alpha) - + \ldots + (-1)^k \chi_k C_0(\alpha)$$

and

$$A_k^* = C_k'(\alpha) - \chi_1 C_{k-1}'(\alpha) + \chi_2 C_{k-2}'(\alpha) - + \ldots + (-1)^k \chi_k C_0'(\alpha),$$

provided that for each odd suffix k at most one of the two numbers A_k and A_k^* is included. Here the general coefficient χ_k is a polynomial of degree k in χ_1, with coefficients that are themselves polynomials in the values $\zeta(2), \zeta(3), \ldots, \zeta(k)$ of the Riemann Zeta function with rational numerical coefficients.

21. Theorem 3 is only one of infinitely many analogous theorems.

For instance, let λ be an arbitrary rational number which is not an integer. We consider now the Siegel E-functions

$$A_k(z, \lambda) = \frac{1}{k!}\left(\frac{\partial}{\partial \nu}\right)^k K_\nu(z)|_{\nu=\lambda} \quad (k = 0, 1, 2, \ldots),$$

and the functions $C_k(z, \lambda) = \dfrac{1}{k!}\left(\dfrac{\partial}{\partial \nu}\right)^k J_\nu(z)|_{\nu=\lambda} \quad (k = 0, 1, 2, \ldots)$

which are easily seen to be connected by linear relations analogous to (45). However, the coefficients γ_k will now involve the values $\Gamma(\lambda+1)$ and $\Psi^{(h)}(\lambda+1)$, where the latter, for $h \geqslant 2$, are expressible in terms of the values of Dirichlet L-series at $s = 2, 3, 4, \ldots$.

Assume that one can determine all systems of functions

$$A_k(z, \lambda), \quad A_k'(z, \lambda)$$

that are algebraically independent over $\mathbf{C}(z)$. We obtain then immediately a theorem on the algebraic independence over $\mathbf{Q}$ of the corresponding function values

$$A_k(\alpha, \lambda), \quad A_k'(\alpha, \lambda).$$

This theorem naturally implies in its turn one on the algebraic independence of linear expressions in the function values

$$C_k(\alpha, \lambda), \quad C_k'(\alpha, \lambda)$$

where the coefficients of these linear expressions involve now the values

$$\Gamma(\lambda+1), \quad \Psi^{(h)}(\lambda+1).$$

In particular, we can establish the transcendency of expressions in which these function values occur. Whether these function values themselves are transcendental remains of course an open question. The cases when $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ are particularly interesting.

22. So far only functions related to the Bessel functions have been considered. However, the method reaches much further.

Let $\mu_1, \mu_2, \ldots, \mu_r$ and $\nu_1, \nu_2, \ldots, \nu_s$ be finitely many parameters where

$$0 \leqslant r < s,$$

and let
$$f(z) = \sum_{n=0}^{\infty} \frac{[\mu, n] \ldots [\mu_r, n]}{[\nu_1, n] \ldots [\nu_s, n]} (z/t)^{nt},$$

where
$$t = s - r,$$

be the corresponding hypergeometric E-function (Siegel 1949, pp. 54–58). Here

$$[\rho, 0] = 1, \quad [\rho, n] = \rho(\rho+1)\ldots(\rho+n-1).$$

Let similarly $$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu_1+n)\ldots\Gamma(\mu_r+n)}{\Gamma(\nu_1+n)\ldots\Gamma(\nu_s+n)} (z/t)^{nt+\tau},$$

where we have put $$\tau = (\nu_1+\ldots+\nu_s)-(\mu_1+\ldots+\mu_r).$$

The two functions $f(z)$ and $F(z)$ are then connected by the identity

$$F(z) = \frac{\Gamma(\mu_1)\ldots\Gamma(\mu_r)}{\Gamma(\nu_1)\ldots\Gamma(\nu_s)} (z/t)^{\tau} f(z). \tag{52}$$

This identity plays for $f(z)$ and $F(z)$ a role analogous to that of the relation (2) for the functions $J_\nu(z)$ and $K_\nu(z)$ considered in chapters 1–3. From its definition, $f(z)$ satisfies a linear differential equation of order s with respect to the variable z (Siegel 1949, pp. 55–56), where the coefficients are polynomials in the parameters μ_ρ and ν_σ, but are rational functions of z.

Next denote by $\mu_1^0, \ldots, \mu_r^0, \nu_1^0, \ldots, \nu_s^0$ a fixed set of rational numbers distinct from $0, -1, -2, \ldots$, and by $i_1, \ldots, i_r, j_1, \ldots, j_s$ a set of non-negative integers. Then put

$$f_{[i,j]}(z) = \prod_{\rho=1}^{r} \frac{\partial^{i_\rho}}{i_\rho!\,\partial\mu_\rho^{i_\rho}} \prod_{\sigma=1}^{s} \frac{\partial^{j_\sigma}}{j_\sigma!\,\partial\nu_\sigma^{j_\sigma}} f(z)\Big|_{\mu_1=\mu_1^0,\ldots,\mu_r=\mu_r^0,\,\nu_1=\nu_1^0,\ldots,\nu_s=\nu_s^0},$$

and define partial derivatives $F_{[i,j]}(z)$ analogously. There is no difficulty in proving that the derivatives $f_{[i,j]}(z)$ are Siegel E-functions and that they satisfy an infinite system of linear differential equations of order s.

The investigation is now started by determining a full set of derivatives $f_{[i,j]}(z)$ that are algebraically independent over $\mathbf{C}(z)$; this may, of course, not be an easy problem. In this set of derivatives, one may next select finite subsets that again satisfy a system of linear differential equations. To this subset, Shidlovski's theorem can then be applied and establishes the algebraic independence over $\mathbf{Q}$ of its functions at all non-trivial algebraic points $z = \alpha$. As a final step, these algebraically independent function values $f_{[i,j]}(\alpha)$ are, by means of (52), expressed as linear forms in function values $F_{[i,j]}(\alpha)$. Here the coefficients of these linear forms evidently depend on the function values

$$\Gamma(\mu_\rho), \quad \Psi^{(h)}(\mu_\rho), \quad \Gamma(\nu_\sigma), \quad \Psi^{(h)}(\nu_\sigma).$$

Thus also these much more general assumptions lead to transcendental expressions that involve values at rational points of the Gamma function and of its derivatives.

It would be of interest to carry this program out in detail for the special case of the *Kummer functions* when $r = 1$, $s = 2$, and $\nu_2 = 1$.

23. Instead of doing so, let us consider a much simpler case. The functions

$$f(z,\nu) = \sum_{n=0}^{\infty} \frac{1}{[\nu, n]} z^n \quad \text{and} \quad F(z,\nu) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu)} z^{n+\nu-1}$$

correspond to the hypergeometric E-function with $r = 0$, $s = 1$, and they are connected by the identity

$$F(z,\nu) = \frac{z^{\nu-1}}{\Gamma(\nu)} f(z,\nu).$$

For simplicity, we allow ν to tend to the value $\nu^0 = 1$. We obtain then the derivatives

$$f_k(z) = \frac{1}{k!}\left(\frac{\partial}{\partial \nu}\right)^k f(z,\nu)\big|_{\nu=1} \quad \text{and} \quad F_k(z,\nu) = \frac{1}{k!}\left(\frac{\partial}{\partial \nu}\right)^k F(z,\nu)\big|_{\nu=1} \quad (k = 0, 1, 2, \ldots),$$

where, in particular, $$f_0(z) = F_0(z) = e^z.$$

The functions $w_k = f_k(z)$, where $k = 0, 1, 2, \ldots$, satisfy the following infinite system of linear differential equations of the first order,

$$w_0' = w_0, \quad w_1' = w_1 - \frac{1}{z}w_0 + \frac{1}{z}, \quad w_k' = w_k - \frac{1}{z}w_{k-1} \quad (k = 2, 3, 4, \ldots).$$

It can quite easily be proved that, for every integer $m \geqslant 0$, *the* $m+1$ *functions*

$$f_0(z), f_1(z), \ldots, f_m(z)$$

are algebraically independent over $\mathbf{C}(z)$.

Hence, if $\alpha \neq 0$ is an algebraic number, it follows from Shidlovski's theorem that the corresponding function values

$$f_0(\alpha), f_1(\alpha), \ldots, f_m(\alpha)$$

are algebraically independent over $\mathbf{Q}$. Here $f_0(\alpha) = e^\alpha$.

Let now $\phi_k(z)$ be the coefficient of ν^k in the power series

$$\Gamma(\nu+1)z^{-\nu} = \sum_{k=0}^{\infty} \phi_k(z)\,\nu^k;$$

then $\phi_k(z)$ is a polynomial in $\log z$, with numerical coefficients that involve Euler's constant as well as the values $\zeta(2), \zeta(3), \ldots, \zeta(k)$ of the Zeta function. Then evidently

$$f_k(z) = \sum_{h=0}^{k} \phi_h(z)\,F_{k-h}(z).$$

We arrive therefore at the result that, if $\alpha \neq 0$ is an algebraic number, then any finite number of expressions

$$\sum_{h=0}^{k} \phi_h(\alpha)\,F_{k-h}(\alpha) \quad (k = 0, 1, 2, \ldots)$$

are algebraically independent over $\mathbf{Q}$, so that in particular all these expressions are transcendental.

A similar theorem is obtained if the parameter ν is made to tend to any other rational number ν^0 distinct from $0, -1, -2, \ldots$. In this way one finds in particular that for algebraic $\alpha \neq 0$ any finite number of integrals

$$\int_0^1 x^{\nu^0-1}(\log x)^k\, e^{-\alpha x}\, dx \quad (k = 0, 1, 2, \ldots)$$

are algebraically independent over $\mathbf{Q}$.

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